Diagrams in Intra-Configurational Analysis

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Abstract

In this paper we would like to attempt to shed some light, at least tentatively, on the way in which diagrams enter into the practice of ancient Greek geometrical analysis. To this end, we will first distinguish two main forms of this practice, i.e. trans-configurational and intra-configurational. We will then argue that, while in the former diagrams enter in the proof essentially in the same way (*mutatis mutandis*) they enter in canonical synthetic demonstrations, in the latter, they take part in the analytic argument in a specific way, which has no correlation in other aspects of classical geometry. In intra-configurational analysis, diagrams represent in fact the result of a purely material gesture, which is not codified by any construction canon, but permitted only by the (theoretical) practice of the method of analysis and synthesis.

1 Introduction

A common question that many works on mathematical diagrams have tried and still try to answer is whether there exist pieces of mathematical activity that indispensably require diagrams to achieve or justify some relevant result (Giaquinto 2007; Carter 2010; De Toffoli 2017). In this context, one of the most studied domains is Euclidean (synthetic) geometry—the first four books of Euclid's *Elements* in particular. By contrast, diagrams in ancient Greek geometrical analysis represent an almost unexplored territory. As pointed out by Kenneth Manders (2008, 77)

> The role of diagram use in ancient geometric analysis is highly non-trivial, and 20th-century discussions have shed little light on this. [...] one as yet undescribed factor in geometrical analysis is the sense of what-determines-what that one only obtains by actually constructing diagrams.

In this paper, we would like to attempt to shed some light, at least tentatively, on how diagrams enter the practice of the method of analysis and synthesis in ancient Greek geometry. To this end, following Panza (2007), we will distinguish two main forms of analysis, i.e., trans-configurational and intra-configurational. We will then argue that, while in the former diagrams enter in the proof essentially in the same way (*mutatis mutandis*) they enter in canonical synthetic demonstrations, in the latter, they take part in the analytic argument in a specific way, which does not correlate with other aspects of classical geometry. The thesis we want to put forward is that, while diagrams in trans-configurational and synthetic arguments can be identified as the material counterpart of the κατασκευή (i.e., a canon of construction), in the intra-configurational analysis, they represent the result of a purely material gesture, which is not codified by any constructive canon but permitted only by the (theoretical) practice of the method of analysis and synthesis.

The paper is structured in three sections. We will first examine some examples of the interaction between text and diagram in trans-configurational analysis, both in modern and ancient times. In the second section, we will deal with diagrams in intra-configurational analysis. In this part, we will try to explain how diagrams take part in these types of arguments and how this is somewhat different from synthetic and trans-configurational proofs. Finally, in the third section, by examining the manuscript tradition of Greek mathematical texts, we will provide some historical and philological evidence in support of our thesis. In particular, we will highlight that, at least in the classic geometric corpus, the writing of the intra-configurational analyses has a peculiarity which is absent in the synthetic demonstrations: in a significant number of cases, we find two somewhat visually identical diagrams, one for the analysis and the other for the synthesis. As we will see, this specificity, absent in synthetic proofs, finds an explanation in the theses set out in the second section.

2 Trans-Configurational Analysis

Trans-configurational analysis is typical of the use of algebra in geometry.[1] Viète's *Zeteticorum Libri* (Viète 1591b), a collection of problems both stated and solved employing the language of algebra, provides several examples of this kind of reasoning. Let us consider,

[1]Within classical mathematics, algebra was in no way opposed to geometry as a different branch of mathematics. It was rather a common "art" for solving problems both in arithmetic and geometry, going together with a common language, admitting different interpretations in the two cases. A crucial aspect of the deep innovation promoted by Descartes's *Géométrie* pertains to the introduction of a new way to interpret this language in geometry (also going together with a new and particularly performant notation).

for instance, proposition 17 of the second book (Viète 1591b, 54):

> Data differentia laterum, & differentia cuborum: invenire latera.[2]

Viète takes B to be the difference of sides ("*differentia laterum*"), D the difference of cubes ("*differentia cuborum*") and E the sum of sides. In his notations, capital consonants denote known quantities, and capital vowels unknown ones. Solving a problem within his collection requires making the latter known in terms of the former. In the case at issue, the problem requires making the sides known in terms of B and D. Though making E known is still not that, it makes the task quite easier, and it is, then, an important stage in the solution.

By adapting Viète's language and notation to our modern ones (for the sake of simplicity), his argument goes as follows. Let x and y be the two (unknown) sides, then:

$$x - y = B \quad x^3 - y^3 = D \quad x + y = E$$

$$E + B = 2x \quad E - B = 2y$$

$$(E + B)^3 - (E - B)^3 = 6E^2B + 2B^3 = 8(x^3 - y^3) = 8D$$

$$E^2 = \frac{4D - B^3}{3B}$$

This reduces the given problem to another, already solved, one (Viète 1591b, 2):

> *Zeteticum* I.1: Data differentia laterum & aggregato eorundem invenire latera.[3]

If the difference of sides is B and their sum ("*aggregato eorundem*") is $\sqrt{\frac{4D-B^3}{3B}}$, the two sides are of course

$$x = \frac{B + \sqrt{\frac{4D-B^3}{3B}}}{2} \qquad \text{and} \qquad y = \frac{\sqrt{\frac{4D-B^3}{3B}} - B}{2},$$

which makes them known in terms of B and D.

By its nature, analysis supposes the problem solved and works on the supposed solution to indicate a way for actually solving it. Here, supposing the problem solved consists in expressing an unknown quantity (the sum of sides), whose knowledge makes

[2] "Given the difference of the sides and the difference between their cubes, to find the sides". For a slightly different translation, see Viète (1983, 108).

[3] "Given the difference of the sides and their sum, to find the sides".

the actual solution easier, through a given letter, namely 'E'. Though this is a vowel rather than a consonant, the difference is purely typographic and does not forbid working on it just as on the consonants expressing the known quantities. In other words, what analysis does, here, is supposing the problem solved and working on the supposed solution in order to modify the configuration of data to transform the given problem into another and simpler one (one, by the way, that have been already solved). This is just what makes it trans-configurational: it does not operate within a given configuration of data but, instead, transforms it into a new configuration, corresponding to a more straightforward problem.

There is no diagram here. But there is quite a codified formalism at work. When presented in the notation we used above, this formalism is not so dissimilar from the elementary algebraic one still used today. It is true that Viète uses another notation, quite unlike our present one, and justifies the formalism in a different way than we would do (and Descartes himself will do, only a few decades later). Though quite convenient for the purpose of trans-configurational analysis, this formalism is not indispensability required by it, however.

To see why, we need first understand what makes it so convenient. This is not only a question of its deductive agility. It is also, and even overall, a question of its neutrality regarding the possible interpretations of the problem at issue and the analysis itself. Are Viète's "*latera*", and "*cubi*", just geometrical segments and solids? They can certainly be taken as such. But not necessarily, and Viète is quite explicit on it in his *In artem anayticem Isagoge* (Viète 1591a), a sort of foundational essay from which *Zeteticorum Libri* openly depends on. They are just scalar quantities (Bos 2001, 143) admitting different possible instantiations: as numbers, or as the mentioned geometrical objects, or even as other magnitudes sharing with those some crucial metric properties. Viète's formalism and notation are just conceived to render this generality explicit.

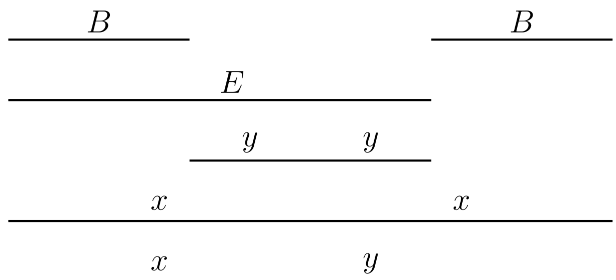
At this level, diagrams (or, at least, the usual geometrical diagrams, which are the only ones we are considering here) can at most help offering palpable illustrations of the general arguments. But they are certainly not indispensable as such and potentially misleading. All that is relevant here is the capacity to fix the formalism appropriately to make the required inferences licensed. But when one comes back from this level to the particular case of geometrical segments and solids, the formalism that makes them both dispensable and equivocal can be replaced by classical geometry's ordinary language without making trans-configurational analysis impossible at all.

To show this,[4] let us consider four segments, B, E, x and y (with B and E repre-

[4] We adopt here a convent notation, but we do it only for reasons of brevity. The argument we present here does in no way require this notation and can perfectly be restated in the (codified but) discursive

senting, respectively, the difference and the sum of x and y) so that

$$E + B = (x + y) + (x - y) = 2x \quad \text{and} \quad E - B = (x + y) - (x - y) = 2y$$



Notice that the diagram includes two distinct representations for B and three for x and y. This is possible because of the purely quantitative nature of the problem at issue. This means that what the problem is asking for is not (the construction of) a pair of particular segments, but rather the identification of the metric relations they must have with the given segment B and the given difference of cubs D. The metric nature of these relations makes the only relevant property of the given and searched for segments be their length, which, in classical geometry, was not conceived as a real number (as we do today) but as the property that a segment shares with all other segments equal to it. In modern terms, this means that what is relevant here are not single segments but equivalent classes of equal segments. Hence, what 'B', 'x' and 'y' properly denote are not single segments but circumstantial representatives of these equivalence classes.

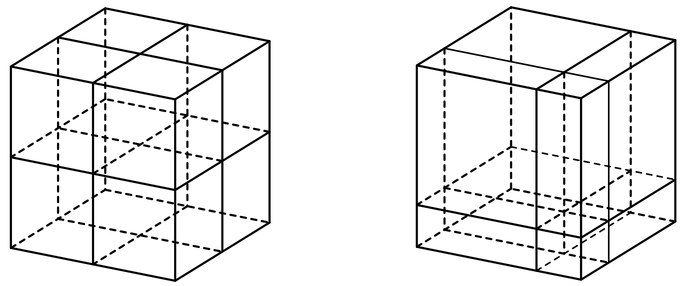
Now, for whatever segments a, b, and c, let $\mathcal{C}(a)$ be the cube of the side a, and $\mathcal{P}(a, b, c)$ the parallelepiped with sides a, b, and c. Hence, from the elementary properties of cubes and parallelepipeds, immediately made manifest by the following diagrams, and taking D to be the difference between $\mathcal{C}(x)$ and $\mathcal{C}(y)$, we will have that

$$\mathcal{C}(E + B) = \mathcal{C}(2x) = 8\,\mathcal{C}(x)$$

$$\mathcal{C}(E - B) = \mathcal{C}(2y) = 8\,\mathcal{C}(y)$$

$$\mathcal{C}(E + B) - \mathcal{C}(E - B) = 8\,[\mathcal{C}(x) - \mathcal{C}(y)] = 8\,D$$

language of the Euclidean geometry, merely appealing to the composition and decomposition of cubes and parallelepipeds, indispensably supplemented by appropriate diagrams (possibly only imagined, but not for this less indispensable in this intellectual activity).



and

$$\mathcal{C}\,(E{+}B) = \mathcal{C}\,(E) + \mathcal{C}\,(B) + 3\,\mathcal{P}\,(E,B,B) + 3\,\mathcal{P}\,(E,E,B)$$

$$\mathcal{C}\,(E{-}B) = \mathcal{C}\,(E) - \mathcal{C}\,(B) - 3\,\mathcal{P}\,(B,B,E{-}B) - 3\,\mathcal{P}\,(B,E{-}B,E{-}B)$$

$$\mathcal{P}\,(B,E{-}B,E{-}B) = \mathcal{P}\,(E,E,B) - \mathcal{C}\,(B) - 2\mathcal{P}\,(B,B,E{-}B)$$

$$\mathcal{P}\,(B,B,E{-}B) = \mathcal{P}\,(E,B,B) - \mathcal{C}\,(B)$$

so that

$$\mathcal{C}\,(E{-}B) = \mathcal{C}\,(E) - 3\mathcal{P}\,(E,E,B) + 3\mathcal{P}\,(E,B,B) - \mathcal{C}\,(B)$$

and

$$\mathcal{C}\,(E{+}B) - \mathcal{C}\,(E{-}B) = 6\mathcal{P}\,(E,E,B) + 2\mathcal{C}\,(B) = 8D$$

which is just the geometrical interpretation of the equality

$$(E+B)^3 - (E-B)^3 = 6E^2B + 2B^3 = 8D$$

already obtained through Viète's formalism. There is, however, no need to appeal to this last equality and this formalism to go ahead and conclude that

$$\mathcal{P}\,(E,E,B) = \frac{1}{3}\left[4D - \mathcal{C}\,(B)\right].$$

This conclusion does not directly correspond to that obtained through Viète's formalism, namely:

$$E^2 = \frac{4D - B^3}{3B}$$

since nothing allow to decompose $\mathcal{P}(E, E, B)$ in the square $\mathcal{S}(E)$ with side E and the segment B. But it still makes the original problem reduced to a new simpler one, easily solvable by ruler and compass.

For the sake of simplicity, let us take the difference of cubes D to be a parallelepiped $\mathcal{P}(B, B, a)$ constructed on the the square $\mathcal{S}(B)$, where, according to *El.* XI.32 and VI.1, a is such that $\mathcal{C}(B) : D : B : a$. Insofar as B is given, admitting that this is so also for D is the same as admitting that the segment a is given. This makes $4D$ equal to $\mathcal{P}(B, B, 4a)$, $4D - \mathcal{C}(B)$ equal to $\mathcal{P}(B, B, 4a - B)$, and $\frac{1}{3}[4D - \mathcal{C}(B)]$ equal to $\mathcal{P}\left(B, B, \frac{4a-B}{3}\right)$, all of which will also be given by quite easy constructions. Let this last parallelepiped be C. The new problem is, then, the following:

> On a given side (B), construct a square-based parallelepiped equal to a given parallelepiped (C).

This is nothing but a relatively simple particular case of *El.* XI.27, and can then be solved through the construction spelt out there.

The reduction of the given problem—i.e., *Zeteticum*, II.17—to *El.* XI.27 is a clear case of trans-configurational analysis performed with no help of Viète's formalism or of any other similar algebraic formalism. It shows, then, that this form of analysis does in no way require such a formalism.

Insofar as it is a form of analysis, it involves, of course, also in this version, some work on unknown quantities. In this case, this work consists of reasoning on the unknown segments E, x and y, together with the known segment B and solid D, without making any difference among them, that is dealing with the former just as with the latter. This includes representing both the former and the latter by appropriate diagrams, where no difference is at work among them. These diagrams undoubtedly play an indispensable role in the argument, perfectly in line with the essential role they play in the whole EPG, as accounted for in Panza (2012). However, this role is relatively marginal in this case—or better—not so explicitly manifest. The most explicit role diagrams have, in this context, is keeping fixed the reference of the names of the relevant geometrical objects or equivalence classes, and to allow proving the primary results about the appropriate segments, cubes and parallelepiped appealed to, by the consideration of how they mutually compose and decompose.

More important for our purpose is what diagrams do not do here. Considering the last two diagrams—representing a cube decomposed into eight cubes and into two cubes and six (non–cubic) parallelepipeds—makes this immediately manifest. Though visually and intuitively different, these diagrams are, in fact, perfectly equivalent in relation to the role they have within the previous argument: either can be replaced by the other. This is because all they do is representing how a cube can be decomposed when all of its sides are cut into two parts. All that the argument requires to the diagram(s) is showing how the cut of sides propagates on the whole solid, which is just the same regardless these sides are cut in equal parts or not, and even if they are not all cut in the same proportion. The relevant information to decide whether the parts that are so obtained in the whole solids are cubes or (non-cubic) parallelepipeds is neither provided nor reflected by the diagrams but rather by the problems' assumption.

This is an obvious consequence of the fact the equality is not a diagrammatic relation. In a cursory way, we could say that it is in no way accounted for in diagrammatic terms. Still, this consequence has, here, a crucial effect on the logical nature of the argument: it allows avoiding requiring that the relevant unknown segments be properly supposed to be constructed; it makes inessential, to let the diagrams play their role, whether these segments are known or not, and actually, satisfy the conditions of the problem; all that is relevant is that they are taken as sides of a cube or a parallelepiped, or as parts of such sides. In other terms, all that is relevant here is that we reason on cubes or parallelepipeds, and their respective parts and sides, whatever they might be given, one with respect to another, both in magnitude, position, and ratio (to use the three modalities in which geometrical objects can be given according to Euclid's *Data*).

This is what makes diagrams play their role in the argument. But the argument does not only go ahead because of diagrams. It also depends on additive relations and rules of replacements proper to any magnitudes, for example to the fact that

$$(\alpha+\beta)-(\alpha-\beta)=2\beta \quad \text{and} \left(\begin{array}{l} \alpha=\beta-\gamma-3\delta-3\varepsilon \\ \varepsilon=\zeta-\gamma-2\delta \\ \delta=\eta-\gamma \end{array} \right) \Rightarrow \alpha=\beta-3\zeta+3\eta-\gamma$$

whatever quantities α, β, γ, δ, ε, ζ, η might be. These relations and rules are essential for making analysis proceeds, but they do not depend on any constructive clause, and diagrams have no role in fixing them (or, at least, not a direct role).

The analytical argument justifying the reduction to the given problem to another is, then, perfectly independent of the way the relevant segments, cubes, and parallelepipeds are given or supposed to be given, provided they enduringly belong to the same equivalence classes, and they bear the relevant additive relations and obey the relevant substitution rules. There is no need to suppose that they have been constructed since they are not identified in the force of their actual or supposed constructions, but rather of their staying to each another in the relevant additive relations and of the fact that the segments are taken to be as sides or parts of sides of the relevant cubes and parallelepipeds.[5]

[5]For further clarifications of trans-configurational (problematic) analysis in connection with the classical

Mutatis mutandis, this happens in any instance of geometrical trans-configurational analysis, even in theorematic cases.[6] Some clear examples are provided in the alternative proofs of *El.* XIII.1–5 (*EOO*, IV, 364.18–376.22), usually attributed to Heron of Alexandria (*EE*, 390–400), who probably lived in the first century AD (Masià 2015). All of them are preceded by an explicit analysis, which is openly trans-configurational indeed. Let us consider the case of *El.* XIII.1 (*EOO*, IV, 366.5–24):[7]

> Let some straight line, AB, be cut in extreme and mean ratio at Γ, and let there be a greater segment, $A\Gamma$, and let an equal to half of AB be set out, $A\Delta$. I say that <the square> on $\Gamma\Delta$ is quintuple of <the square> on $A\Delta$. For since <the square> on $\Gamma\Delta$ is quintuple of <the square> on $A\Delta$, and <the square> on $\Gamma\Delta$ is <the squares> on $A\Gamma$, $A\Delta$ plus twice <the rectangle contained> by $A\Gamma$, $A\Delta$, therefore <the squares> on $A\Gamma$, $A\Delta$ plus twice <the rectangle contained> by $A\Gamma$, $A\Delta$ are quintuples of <the square> on $A\Delta$. Therefore, *separando* (διελόντι),[8] <the square> on $A\Gamma$ plus twice <the rectangle contained> by $A\Gamma$, $A\Delta$ is quadruple of <the square> on $A\Delta$. But <the rectangle contained> by BA, $A\Gamma$ is equal to twice <the rectangle contained> by $A\Gamma$, $A\Delta$—for BA <is> double of $A\Delta$—and <the rectangle contained> by AB, $B\Gamma$ is equal to <the square> on $A\Gamma$—for AB has been cut in extreme and mean ratio. Therefore <the rectangle contained> by BA, $A\Gamma$ plus <the rectangle contained> by AB, $B\Gamma$ is quadruple of <the square> on $A\Delta$. But <the rectangle contained> by BA, $A\Gamma$ plus <the rectangle contained> by AB, $B\Gamma$ is <the square> on AB. Therefore <the square> on AB is quadruple of <the square> on $A\Delta$. And it is (ἔστι δέ): for AB is double of $A\Delta$.

Using the previous notations the theorem can be expressed in this way:[9]

$$(b + c = a \;\&\; a : b = b : c) \Rightarrow \mathcal{S}\left(b + \frac{a}{2}\right) = 5\,\mathcal{S}\left(\frac{a}{2}\right)$$

and Arabic mathematical tradition, see Panza (2007).

[6]Problematic analysis is usually opposed to theorematic one: while the former aims to construct a geometrical object satisfying certain given conditions, the latter is aimed to prove the truth of theorems. This distinction, and that between problems and theorems come back to ancient sources, like Proclus's *Commentary on the First Book of Euclid's Elements* (*iEE*, 77–80), and the Seventh Book of Pappus's *Collectio* (*PAC*, VII.1, 634–6).

[7]Unless otherwise specified, the translations from ancient Greek are done by G. Longa. A special thank to A. Jones and N. Sidoli and for their kind help in this regard.

[8]We read this "separation" (διελόντι) as a way of applying the ratio operation of *El.* V.17, different from a simple division.

[9]With respect to the text, a stands for AB, b for $A\Gamma$ and c for $B\Gamma$ (therefore, $A\Delta$ is $1/2$ of a).

The analysis goes, then, as follows:

Let's suppose that	$\mathcal{S}\left(\frac{a}{2}+b\right)=5\,S\left(\frac{a}{2}\right)$.	1
For *El.* II.4	$\mathcal{S}(b)+\mathcal{R}(a,b)=\mathcal{S}\left(\frac{a}{2}+b\right)-\mathcal{S}\left(\frac{a}{2}\right)$.	2
Hence	$\mathcal{S}(b)+\mathcal{R}(a,b)=4\,\mathcal{S}\left(\frac{a}{2}\right)$.	3
For *El.* VI.17	$\mathcal{S}(b)=\mathcal{R}(a,c)$.	4
Hence	$\mathcal{R}(a,c)+\mathcal{R}(a,b)=4\,\mathcal{S}\left(\frac{a}{2}\right)$.	5
For *El.* II.2	$\mathcal{R}(a,c)+\mathcal{R}(a,b)=\mathcal{S}(a)$.	6
Hence	$\mathcal{S}(a)=4\,\mathcal{S}\left(\frac{a}{2}\right)$.	7

As this last equality is an obvious particular case of *El.* II.4, the proof can start from it and go ahead by merely reversing the analysis.

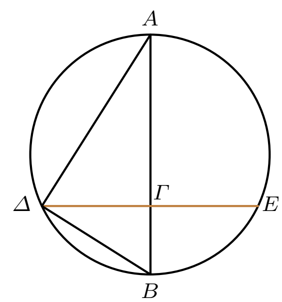
Here all the segments, squares, and rectangles are given. So, the analysis does not work on any unknown object as if it were known. It rather starts from a statement that is not (yet) proved and deduces an already estabished theorem from it suggesting that the required proof starts from it. Diagrams can at most enter it, then, insofar as they are required to perform the deduction or, indirectly, to prove the theorems that the deduction has recourse to. As it happens for trans-configurational problematic analysis, also trans-configurational theorematic analysis does neither require that the unknown geometrical objects be supposed to have been constructed nor does any specific use of diagrams, apart from that which classical geometry requires to conduct non-analytical arguments.

3 Intra-Configurational Analysis and the Purely Material Role of Diagrams

A crucial difference arises in intra-configurational analysis. Unlike the trans-configurational one, this form of analysis cannot avoid assuming the relevant unknown objects as constructed and, thus, have a specific constructive relation to the given ones. At issue here are not equivalence classes of (equal) segments and the additive relation of the figures that have them as sides, but rather particular points or segments in an identified position, to be obtained by a specific construction. This construction is to be disclosed since it makes the searched points or segments known and characterizes them as points and segments.

Consider a simple example: proposition II.3 of Archimedes's *On the Sphere and Cylinder* asks (*AOO*, I 184.2–4):

> To cut the given sphere by a plane, so that the surfaces of the segments have to each other a ratio the same as the given one.



The analysis goes as follows:

Consider a great circle, $A\Delta BE$, of the sphere with diameter AB. 1

Let the problem supposed to be solved by cutting the sphere with a plane perpendicular to AB, which cuts $A\Delta BE$ in Δ and E and the diameter in Γ. 2

Then, the surface of the spherical segment ΔBE will be to the surface of the spherical segment ΔAE in the given ratio. 3

According to *On the Sphere and Cylinder*, I.42-43, these surfaces are equal to the circles of radii ΔB and ΔA, respectively. 4

By *El.* XII.2, then, these circles are to one other as the squares $\mathcal{S}(\Delta B)$ and $\mathcal{S}(\Delta A)$ on the radii. 5

According to the Pythagorean theorem, these squares are to one other as the segments ΓB and $A\Gamma$. 6

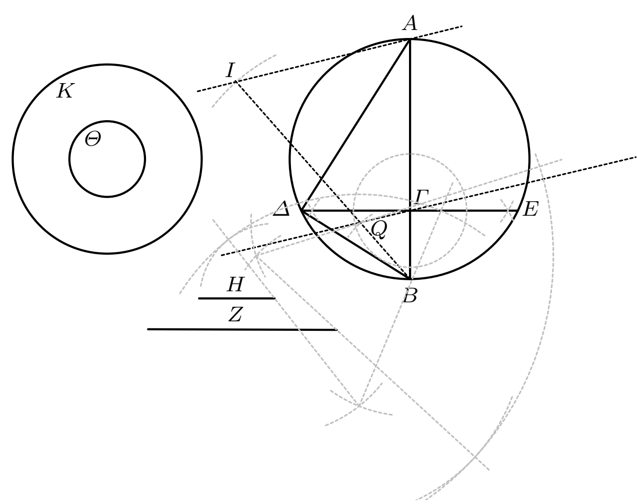
Hence, these segments are to one another as the surfaces of spherical segments ΔBE and ΔAE. 7

The synthesis becomes, then, obvious (*AOO*, 184.21–186.3):

> And it will be put together (συντεθήσεται) thus: let there be a sphere with great circle $AB\Delta E$ and diameter AB, and <let> the given ratio <be> that of Z to H, and let AB be cut at Γ so as to be: as $A\Gamma$ to $B\Gamma$, so Z to H, and let the sphere be cut by a plane through Γ at right <angles> to the line

> AB, and let there be a common section, ΔE, and let $A\Delta$, ΔB be joined, and let there be set out two circles, Θ, K: Θ having the radius equal to $A\Delta$, K having the radius equal to ΔB.

To construct the object solving the problem (and its diagrammatic representation), one has to cut the diameter AB at a certain point Γ, so that $A\Gamma$ and ΓB be to one another in the given ratio (that of Z to H). Since this ratio is given insofar the two segments Z and H are given (*Data* 1), then the construction can easily be carried out by applying the procedure for finding the fourth proportional between $Z+H$, AB and Z (or H). This is quite obvious (following *El.* I.2, I.11, I.31 and VI.9), and Archimedes does not detail it. Here is as it might go.



In agreement to *El.* I.2 or I.3, construct a segment BQ, starting from B and equal to Z. Then, by Post. 2 and *El.* I.2 or I.3 draw a segment QI, starting from Q, collinear to BQ, and equal to H. Join I and A (Post. 1). Construct the parallel $Q\Gamma$ to IA passing through Q (*El.* I.31). The intersection point of this parallel and the diameter AB—i.e., the point Γ—cuts AB in such a way that as $A\Gamma$ is to ΓB so H is to Z. Finally, draw the perpendicular ΔE to AB through Γ (*El.* I.11). Therefore, the plane perpendicular to AB passing through ΔE cuts the sphere so that the surface of the two spherical segments ΔBE and ΔAE have to one another the given ratio of Z to H. Here the diagram takes part in the argument just as it participates in the synthesis of our previous examples of trans-configurational analysis, but with an essential difference. In the latter, the given segments are whatever elements of the relevant equivalence class of equal segments, working as incidental representatives of these classes. By contrast, here, the given segments are particular individual objects, working as incidental representatives or whatever segments, and they provide a basis for the construction of other particular segments and points forming a geometrical system. This system works, as a whole, as an incidental representation of any other (topologically) equivalent geometrical system. In other terms, in these last two cases, the focus is no more on equivalence classes of equal segments but on a(n equivalence class of) particular system(s) of geometrical objects.

What makes this (equivalence class of) system(s) represented by a diagram is that the diagram is drawn, step by step, in agreement to some constructive clauses belonging to a constructive canon which is, as such, an essential component of the theory within which the argument is conducted. Even the given segments, squares, and circles are given just since they are supposed to have been constructed in advance through a licensed constructive procedure, which is simply not detailed here (since the focus is not on it but on a further construction): their drawing is nothing but a (material) counterpart of a constructive argument, which details, step by step, the whole process of this drawing; each diagrammatic gesture that is here at issue obeys a constructive clause licensed by the relevant canon; and it is just this, rather than the actual features of the material drawing, that makes the diagrams represent the relevant objects and play, then, their role in the geometrical argument.

However, this is what happens in Archimedes's synthesis, not in his analysis. What makes the latter an analysis is just that its starting point—the act of taking $\mathit{\Gamma}$ as a given particular point on AB—is not the outcome of a licensed construction. This makes the dot representing this point in the diagram not drawn in agreement to any constructive clause, but rather freehand, so to say. The diagrammatic gesture making this point represented is a purely material act, which enters the argument as such, rather than as the counterpart of a licensed constructive step. It is just this material act—this free diagrammatic gesture, as it were—that makes the analysis possible, by allowing taking the searched for point as given; it is this act that triggers an effective (but conditional) construction, making, all along with the analytical argument itself, other diagrams constructed in agreement with appropriate constructive clauses. In short, we could say that diagrams indispensably play, in intra-configurational analysis, a purely material role.

Whereas in a geometrical trans-configurational analysis, what makes it possible to work on unknown objects is, as said above, the fact that analytical arguments do not require these objects to be supposed to have been constructed, since these objects are not identified in the force of their actual or supposed construction, this is what happens here, as well as in any other instance of intra-configurational geometrical analysis: the point $\mathit{\Gamma}$ is here properly supposed to have been constructed, and solving the problem just consists in detailing its construction, which is, then, what identifies it as such. A crucial role of diagrams is, then, here, that of supporting the very possibility of working on a non-constructed point as if it were provided, by displaying it in a discretionary position:

a role that can only be played by diagrams drawn freehand, or, as it were, thanks to a purely material act.

It has often been argued that geometrical analysis, although certainly present, is not very frequent in the mathematical works of the Hellenistic period.[10] For instance, it has been maintained that analysis is missing from Euclid's *Elements*—interpolations apart, of course. This depends on a restricted way of understanding geometrical analysis, which we think is not appropriate, conceptually speaking. If geometrical analysis is concerned with some sort of item that is not at hand, but is nevertheless supposed to be at our disposal, or with a statement not justified, but anyway taken for granted, then an instance of intra-configurational geometrical analysis is at work any time a certain geometrical object is supposed to be constructed, though not being so. This happens, in particular, when a geometrical object is proved to be not constructible by means of a demonstration showing that, if it were so, a contradiction would arise, that is, any time a *reductio ad absurdum* is performed in geometry. If this is the case, then intra-configurational geometric analysis is largely present, if not pervasive, in the mathematical works of the Hellenistic Period and, of course, also in Euclid's *Elements*.

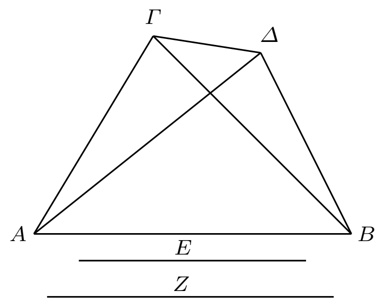
Here is a simple example. It concerns a case of intra-configurational analysis drawn from Euclid's proof of *El.* I.7. This is properly a proof, since this proposition is properly a theorem, not a problem. Still, strictly speaking, the analysis is problematic, even if it concludes that proof, in fact. Since what is to be proved is that a certain construction cannot be performed: a certain triangle is supposed to have been constructed, and it is shown that this supposition leads to a contradiction. What makes the analysis provide a proof is just a last meta-geometric step, in which it is implicitly admitted that no licensed constructions can lead to a contradiction. It is, then, concluded that no such construction could make the relevant triangle constructed.

Euclid's theorem is the following (*EOO*, I, 24.12–16):

> On the same straight-line, there will not be constructed (οὐ συσταθήσονται) two other straight-lines towards different points on the same side, equal, respectively, to the same two straight-lines and having the same limits as the initial straight-lines.

Euclid takes three given segments—AB, $A\Gamma$, and $B\Gamma$—so as to form a triangle $AB\Gamma$, and proves, by a *reductio ad absurdum*, that no other point Δ, which is not symmetric to Γ with respect to AB, can be constructed, such that $A\Delta = A\Gamma$ and $B\Delta = B\Gamma$. More generally, his argument actually proves that if whatever three distinct segments AB, E, Z, and a triangle $AB\Gamma$ are given, such that $A\Gamma = E$ and $B\Gamma = Z$,

[10] See, for example, Netz (2001, 140–1).



then, up to symmetry, no other triangle can be constructed on AB, whose other sides are respectively equal to E and Z. Hence, three segments being given, only one triangle, can be constructed, up to symmetry, such that its sides are equal to these segments. Here is Euclid's argument.

> Let a point $\varDelta$ other than $\varGamma$ supposed given "on the same side" of AB than 1
> $\varGamma$, such that $A\varDelta = A\varGamma$ and $B\varDelta = B\varGamma$.
>
> In agreement to *El.* Post. 1, let the segment $\varGamma\varDelta$ be traced. 2
>
> The isosceles triangles $A\varDelta\varGamma$ and $B\varDelta\varGamma$ are, then, given. 3
>
> According to *El.* I.5, the angles $A\varGamma\varDelta$ and $A\varDelta\varGamma$ are equal. 4
>
> As the angle $B\varGamma\varDelta$ is part of the angle $A\varGamma\varDelta$, in agreement to *El.* Com. 5
> Not. 5, the latter is greater than the former, which also makes the angle $A\varDelta\varGamma$ greater than the angle $B\varGamma\varDelta$.
>
> As the angle $A\varDelta\varGamma$ is part of the angle $B\varDelta\varGamma$, in agreement to *El.* Com. 6
> Not. 5, the latter is greater than the former, which makes the angle $B\varGamma\varDelta$ greater than the angle $B\varDelta\varGamma$.
>
> But according to *El.* I.5, these last two angles are equal. 7
>
> Contraddiction. 8
>
> Hence, no point $\varDelta$ other than $\varGamma$ can be given "on the same side" of AB 9
> than $\varGamma$, such that $A\varGamma = A\varGamma$ and $B\varDelta = B\varGamma$.

This argument can be extended to prove that, up to symmetry, no triangle other than $AB\varGamma$ can be constructed on AB, whose other sides are equal to E and Z, which is obvious, and there is no need to detail it.

The steps *1–7* of this argument clearly form a problematic intra-configurational analysis. If, on the one side, this is made possible by the drawing of the diagram representing the two segments $A\Delta$ and $B\Delta$, sharing the extremity Δ, which allows supposing that the triangles $A\Delta\Gamma$ and $B\Delta\Gamma$ be given, this same argument proves that these segments are not only supposed to be constructed without being actually so, but even can actually not be constructed. This diagram is, then, not only traced freehand, but it cannot but be so traced. Tracing it is necessarily a free diagrammatic gesture, that is, as we have said below, a purely material act.

Such a gesture or act is an indispensable component of intra-configurational geometric analysis, which entails, in turn, that diagrams indispensably enter Greek geometry not only as counterparts of licensed constructions, but, in the case of intra-configurational analysis, also as material devices, independent of any construction, but still belonging to a perfectly codified theoretical procedure.

4 Some Historical Confirmations

In this section, we intend to provide a historical and philological argument in favour of the thesis we have presented in the previous section. If we examine, even superficially, the manuscript tradition of the Greek mathematical texts, we notice that the arguments we have called 'intra-configurational' exhibit often a peculiarity that is almost absent in synthetic proofs and, to a minor extent, in trans-configurational analyses, i.e., the presence of two diagrams, one for analysis and one for synthesis.

Let us take the example of the Archimedean corpus:[11] there, we find only six problems solved through the method of analysis and synthesis, all included in the second book of *On the Sphere and Cylinder* (II.1 and II.3–7).[12] With regards to diagrams, of the six propositions mentioned, four have two diagrams (II.4–7),[13] one placed at the end of the analysis and one at the end of the synthesis—or, to be more precise, one at the beginning of the synthesis (that of the analysis) and one at the beginning of the subsequent proposition (that of the synthesis).[14] Consider, for instance, proposition II.7: it requires to cut a given sphere $\Gamma\Delta$ in such a way that the spherical segment $AB\Gamma$ has a given ratio (that of ΘK to $K\Lambda$) to the cone $AB\Gamma$, i.e., the cone having the base and the height equal to those of the spherical segment.

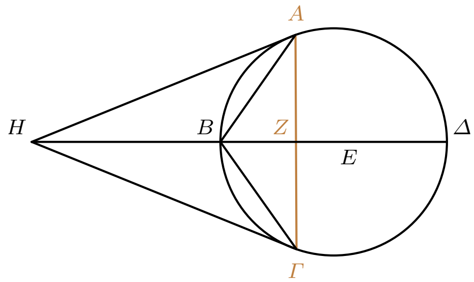
The diagram of the analysis and that of the synthesis are partially different: in the latter, there is a graphical representation, besides of the base of the cone or the spherical

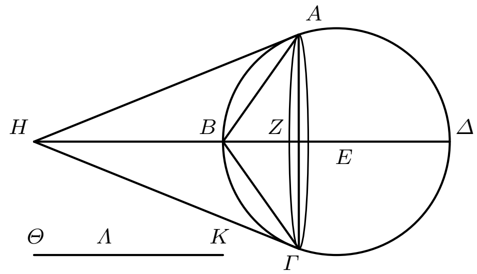
[11] Archimedean diagrams are reproduced according to their critical edition included in *WA*. Likewise, the reproduction of Apollonius' diagrams will follow the critical edition included in *DF*. Different choices, due to a different reading of the manuscripts, will be indicated whenever necessary.

[12] We have already encountered proposition II.3 of the Archimedean treatise in the previous section.

[13] *AOO*, I, 190.11–194.10, 206.1–234.30.

[14] In almost every Greek mathematical text, diagrams are placed after the proposition (Acerbi 2020).





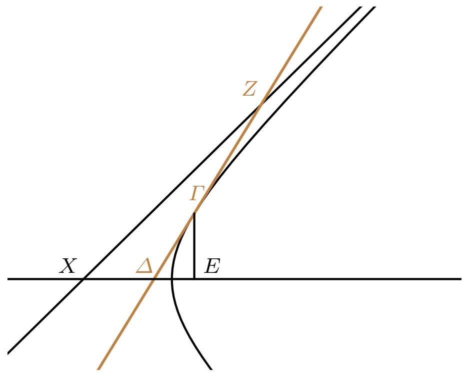
segment $AB\Gamma$, also of the ratio of ΘK to $K\Lambda$ (through a segment $\Theta\Lambda K$). According to the thesis we have stated in the previous paragraph, this difference has a crucial importance: in the analysis, in fact, the assumption that the spherical segment $AB\Gamma$ has a certain given ratio to the cone $AB\Gamma$ is only hypothetical. The given ratio of ΘK to $K\Lambda$ does not figure there because it is not necessary to the analytic argument, which consists only in showing that the plane solving the problem is given, that is, constructible.[15] In the synthesis, instead, this ratio is required to carry out the constructive procedure leading to the determination of the point Z on AB such that, tracing a plane through it and perpendicular to the diameter AB, the spherical segment $AB\Gamma$ has, with the cone $AB\Gamma$, the same ratio as the given one (that of ΘK to $K\Lambda$). By mirroring the analysis (Berggren and Brummelen 2001, 13) the problem is then transformed in that of finding the fourth proportional ΔZ between segments $\Theta\Lambda$, ΛK, and ΔE (a straightforward application of *El.* VI.10). But, to apply the constructive procedure exposed in *El.* VI.10 one needs three segments in input (so to say). For this reason, $\Theta\Lambda K$ has to be represented in the diagram of the synthesis, where the constructive procedure has to be actually exhibited and not simply assumed hypothetically as realized. In this sense, in the diagram of the analysis, Z cannot but be drawn freehand and is not intended to result from the application of a constructive procedure. In the synthesis, instead, Z is determined by a construction procedure (that of *El.* VI.10)[16] and the diagram is drawn accordingly.

In the Apollonian corpus preserved in direct tradition, we find the same kind of phenomenon, albeit a bit more complex from a philological point of view. Here too, there exists only an isolated group of six propositions proved by analysis and synthesis, in

[15]For the meaning of the term ‘given’ as constructible in Greek mathematics see Sidoli (2018, 381–2).

[16]This examination is identical to that we have proposed for *SC* II.3, even though in that proposition we find only one diagram in the manuscript tradition. As a matter of fact, also in that case the given ratio, as in *SC* II.7, is never used in the analysis but is employed only in the synthesis. In a certain sense, therefore, we can see the diagram of II.3 as being like two different diagrams.

a sort of appendix to the second book (propositions II.44, 46, 47, 49–51).[17] Concerning the presence of double diagrams, it only occurs in the proposition II.50b (*DF*, 122.12–126.24), in which a tangent to a given hyperbola has to be constructed, making with the axis an angle equal to the given acute angle:



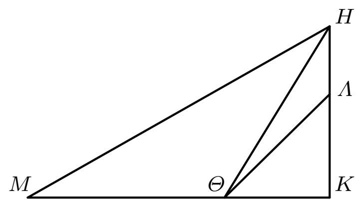
In the analysis, it is assumed that a tangent $\Lambda\Delta$ to the given hyperbola has already been drawn (γεγονέτω, καὶ ἔστω ἐφαπτομένη ἡ $\Lambda\Delta$), making an angle with the axis equal to a given acute angle (not represented in the diagram). The ordinate ΓE of point Γ and an asymptote XZ are then drawn, with Z as intersecting point between the asymptote and the tangent $\Gamma\Delta$. Finally, the center X and Γ are joined. This is sufficient to see the possibility of applying the fundamental result of *Conics* I.37 (*DF*, 87): Since $\Gamma\Delta$ is assumed to be a tangent (analytical assumption), ΓE is the ordinate (Γ) and EX is a diameter, then:

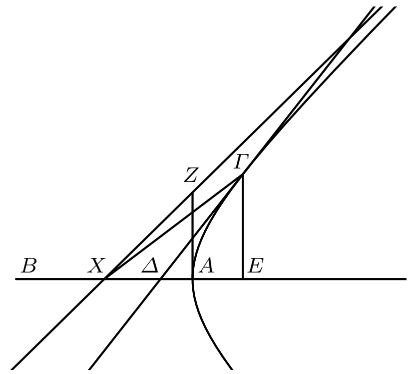
$$\mathcal{R}\,(E\Delta, EX) : \mathcal{S}(E\Gamma) :: \textit{latus transversum } (=2XA) : \textit{latus rectum}$$

On the other hand, the ratio of the *latus transversum* to the *latus rectum* is obviously given (since the hyperbola is given). Therefore the ratio of the rectangle contained by $E\Delta$ and EX to the square on the ordinate $E\Gamma$ is also given (*Data* def. 2). From here, by means of a chain of givens not difficult to decode,[18] Apollonius demonstrates that the object solving the problem, the tangent $\Gamma\Delta$, is indeed given in position (θέσει ἄρα ἡ $\Gamma\Delta$).

Leaving aside the diorism,[19] in the synthesis, we see that the given angle (presented in the enunciation and never used in the analysis) comes to be actively involved and, there-

[17] On the problem of the authenticity of these propositions see Federspiel (2008).
[18] By using *Data* def.2 and props. 8, 25, 29, 40, and 50.
[19] Which establishes that the problem is soluble if and only if the given angle is greater than half the angle between the asymptotes.





fore, it is represented in the diagram, in this case by $K\Theta$.[20] This allows, through means of a manipulation of proportions and the application of some procedures of construction taken from the sixth book of Euclid's *Elements*, to reach the actual construction of the tangent $\Gamma\Delta$ making with the axis of the hyperbola an angle equal to the given acute angle $K\Theta$. Beyond the demonstrative details, here we only aim to point out that, also in this case, the presence of the two diagrams is explicable in the same way we have proposed for the previous examples taken from Archimedes. Here too, we have an action free from constructive constraints (the drawing of the tangent $\Gamma\Delta$ in the analysis) in the first diagram and an effective construction only in the synthesis.

Although *Conics* II.51b is the only explicit case, in the Apollonian corpus, of the introduction of two diagrams, it is possible to suppose that other propositions, at least a certain stage of their transmission, had the same feature. This could have been the case for *Conics* II.44 and II.46, for example. We find an argument favouring this in an (interesting) mistake of the copyist of the manuscript *Vat. gr.* 206 (siglum V), the archetype from which all the later manuscript tradition and the modern editions of the Apollonian treatise derive directly or indirectly (Decorps-Foulquier 2000, 180–3). Here is how Federspiel (*DF*, II.3, 159) accounts for this error in his remarks on the manuscript tradition of the *Conics*:

> On a ainsi un espace réservé pour les propositions 45V et 48V, qui sont les synthèses de nos propositions 44 et 46, ce qui n'offre pas d'intérêt en soi, puisque les figures sont les mêmes que celles des analyses correspondantes. En tout cas, l'existence de ces espaces a favorisé l'erreur d'un copiste antérieur à celui de V, puisque l'emplacement réservé pour la figure de la proposition 45V, au début de la proposition 46V, n'a pas été rempli par

[20] Greater than the half-angle between the asymptotes, in compliance with the diorism just mentioned, i.e., greater than $K\Theta\Lambda = AXZ$.

la figure attendue (la même figure que la prop. 44), mais par les figures de la prop. $46V$. De ce fait, l'espace réservé pour les figures de la proposition $46V$ (au début de la prop. $47V$) est resté vide, et le copiste de V écrit au milieu de l'emplacement (cela devait être dans son modèle) : ἐγράφη τὸ σχῆμα ἄνω (« La figure a été représentée plus haut »).

Without entering into the details of the discrepancies between V and the modern editions of the *Conics*,[21] for our purposes, it is enough to say that the copyist's error shows indirectly that, in the manuscript at his disposal, for the two propositions in question, there were two diagrams,[22] not subsequently copied because one of them was considered as a mere (useless) repetitions of the other. This is also confirmed by the fact that in the Arabic version of the *Conics*, an independent witness of the Apollonian text from our Eutocian edition, we find a double diagram for both propositions (*DF*, II.1, 199–203).[23]

From this brief overview of the available sources, we can therefore reasonably conclude that, with regard to the classic corpus, the practice of inserting two diagrams was present in a not insignificant number of cases in the analysis of problems ($7/12, 58\%$). This particularity can be explained by the different logical statute that the object solving the problem (of which the diagram is a representation) assumes in analysis, on the one hand, and in synthesis, on the other. As we have explained in the previous paragraph, and reiterated in the examples cited here, in the former, such an object is not presented as the result of an actual construction procedure, but simply as hypothetically existing (by recourse to the classical expression γεγονέτω). On the other hand, in the synthesis, the same object takes on a different connotation: it is no longer an assumption, but rather the result of a construction procedure that reproduces the resolution, or chain of givens (Acerbi 2011), as a series of constructive actions/steps. In this sense, the diagrams of the analysis and synthesis, although almost identical, have a completely different status. Hence, the need to reproduce them twice.

[21] Decorps-Foulquier (2000) provides a detailed examination of this intricate issue.

[22] Or, at least, a diagram and an empty indentation intended to contain the diagram of the syntheses of propositions 44 and 46.

[23] In the Arabic tradition, we find another possible confirmation of the presence of two diagrams in the writing of intra-configurational analyses. In Apollonius' *De Sectione Rationis*, a work entirely written using analysis and synthesis, all propositions (or cases) present a double diagram (in many cases even a triple diagram, the extra diagram being reserved for diorism), one for analysis and the other for synthesis. See Saito and Sidoli (2010) for a careful examination of these cases.

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