Louis Bachelier's Théorie de la spéculation : The missing piece in Walras' general equilibrium

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Louis Bachelier's *Théorie de la spéculation*:

The missing piece in Walras' general equilibrium

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Abstract: (150 words)

We propose a revisited view of Louis Bachelier's contribution to economic analysis. Conventional wisdom presents Bachelier as the founding father of modern financial theory. We show that Bachelier's work is constructed to respond to a gap in the Walrasian general equilibrium, where the options market is verbosely introduced but not modeled. By providing a price formation theory for the missing options market, Bachelier undoubtedly presents himself as the heir apparent of the mathematical economics tradition founded by Walras. Indeed, Bachelier's methodological stance is clearly formed on the "rational method" of Walras, proceeding by mathematical demonstration from postulates that we make explicit. We show additionally how Walras and Bachelier in pre-WW2 France reached to the same audience. We propose to name this augmented general equilibrium model the Walras-Bachelier model of intertemporal general equilibrium in the presence of risk. This theory prefigures the Arrow-Debreu model, with some differences which we make clear.

Keywords: general equilibrium, financial markets, option pricing, Bachelier, Walras

JEL Codes :

Introduction:

Over the past fifty years, the rediscovery of Louis Bachelier's work led to a retrospective conception of financial mathematics, markets and products: by considering Bachelier as a "precursor" of modern financial theory, one naturally comes to think that the Paris stock market of the 1900s "prefigured" contemporary options markets. Although commentators have been careful to point out the differences between Bachelier's formulations and those of Black & Scholes, notably Schachermayer-Teichmann (2008) and Haug-Taleb (2010) recalled that Bachelier's representation made dynamic hedging inconceivable, these authors still consider that Bachelier wrote to provide his contemporaries with a ready-to-use valuation model. Another modality of retrospective reading appears among academics, for example Jovanovic and Numa (2021) are surprised by the apparent contradiction between the fact that "the ideas of the French pioneers were taught in France until the 1950s" and the fact that "French academics failed to take advantage of their national scientific knowledge when they reintroduced the ideas of financial economics in the 1970s and 1980s." The key of this paradox may lie in Bachelier's work rather than institutions: reading Bachelier (1900) without prejudice about his project may reveal motivations very different from what market finance has become decades after Bachelier's death. The last sentence of his *Theory of Speculation* testifies to this: "If, in respect of several questions treated in this study, I have compared the results of observation to those of the theory, this was not to verify the formulae established by mathematical methods, but to demonstrate only that the market, unwittingly, obeys a law which governs it: the Law of Probability[1].". This makes clear that the author is turning a blind eye toward the project of constituting an *empirical* science of market prices, but his own project does not appear clearly. Is it pragmatic - recognizing the law of probability in order to profit from it - or purely aesthetic? Since Bachelier's theory does not seem to have found immediate application, it is difficult to infer the author's intentions from the use that would have been made of his writings. The present text proposes to start from the mathematics of the *Theory of Speculation* in order to shed light on the *original design* of its author.

The thesis of this article is that Bachelier has been read as a financier, whereas he is first and foremost a mathematical economist, a disciple and follower of Walras. We show how Bachelier provides the missing piece to the Walrasian general equilibrium puzzle by proposing a partial equilibrium model of the options market that completes Walras' *unfinished* general equilibrium.

First of all, Bachelier published a thesis in science, under the direction of Henri Poincaré, a mathematician, in 1900 in the *Annales de l'Ecole Normale Supérieure*. At this time, there was an effervescence in the theory of integration that led to Lebesgue's thesis in 1902 (Bru et al. 2004) and to Borel's work linking integration and probability (Mazliak et al. 2014). Another remarkable feature of scientific life at the turn of the century was the shift to the modern axiomatic method by Hilbert (1899). Louis Bachelier's thesis is a further manifestation of this scientific effervescence in these two directions: he develops the stochastic integral as a tool to identify the process of option price formation; this stochastic integral is built on postulates that foreshadows the axiomatic method in economics. We prove that working as mathematician, Bachelier completed Walras' general equilibrium system by adding the option market, which was mentioned by Walras but eventually missing in his equations. This addition is made as an extension of Walras' hypothetical approach: postulates are introduced to prove that the expected

[1] Quotations from Bachelier are taken from D. May translation, see References.

price of the option clears the option market. Finally, and spectacularly, Bachelier’s option market theory solves the case solemnly designated by Walras (1889) as the most emblematic case of the functioning of the “well-organised market”, that of the French *rente*.

The rest of the paper is organized as follows: in section 1 we review Bachelier’s postulates, in section 2 we introduce his method of option valuation, in section 3 we establish the connection with Walras’ *Eléments d’économie politique pure*, in section 4 we show that his vision of economics as a mathematical science was shared by his readers, in section 5 we analyse how the general equilibrium model that we call “Walras-Bachelier” prefigures the economics of Arrow and Debreu.

1. Bachelier’s postulates

Louis Bachelier’s theory of speculation contains a set of postulates that we restore in the order of Bachelier's demonstration in the appendix. In this section we present these same postulates grouped into three thematically-coherent subsets. Bachelier’s postulates do not correspond to the reality of the Paris options market, and can be classified into (1.1.) assumptions on the organization of the market, (1.2.) assumptions on the agents, and (1.3.) “technical” representation assumptions.

1.1. Assumptions on the organisation of the market

Bachelier’s representation of a continuous price process is often taken for granted, but this assumption does not correspond to the institutional reality of the Paris market, which operates in a discrete manner since the operating range is very limited[2] and the prices displayed correspond to the daily *fixing*. In Paris, continuous quotation did not begin until 1987, more than forty years after the death of Louis Bachelier.

Furthermore, Bachelier introduces a previously unnoticed hypothesis, albeit he explicitly wrote

> *“In all that will follow, it will be* ***assumed*** *[e. a.] that the adjustment price coincides with the price at the declaration of options. This hypothesis can be justified, for nothing prevents liquidation of operations at the declaration of options.” (1900 p. 27)*

All commentators have deduced that Bachelier was describing *European* options, which must be exercised at maturity. However, there is no doubt that the option contract sold by the Parisian stockbrokers is an *American* option contract, as all the specialized works of the time testify (Ricard 1722 p. 57, Bizet 1821 p. 75, Bresson 1826 p. 21, Peuchet 1829 pp. 161-162 etc. until Buchère 1892 p. 357.). These authors also show that options are rarely exercised before

[2] Market opening hours were extremely limited in time. François-Marsal (1931) has listed the regulations that provided for the opening hours of the stock exchange between 1801 and 1931. This survey shows that the opening hours of the stock exchange range from 1 to 3 hours per day at most, with no definite trend (i. e. no steady increase in time).

maturity because early exercise is costly. However, Bachelier does not model precisely this early exercise mechanism with an additional transaction cost, he makes the *simplifying* assumption that options are exercised at maturity.

1.2. Assumptions about agents' beliefs and preferences

Bachelier's assumptions do not correspond to the reality of (beliefs and preferences of) market participants either. With regard to beliefs, Bachelier writes:

> *"It seems that the market, that is to say, the totality of speculators, must believe at a given instant neither in a price rise nor in a price fall, since, for each quoted price, there are as many buyers as sellers." (pp. 31-32)*

However, it is enough to read only a few stock market chronicles or literary works of the time to be convinced that "the market" sometimes believes in the rise and sometimes in the fall, and moreover that there may be *insiders* with privileged information. The market that Bachelier refers to does not correspond to this reality:

> *"But, while the market believes neither in a rise nor a fall in the true price, some movements of a certain amplitude may be supposed to be more or less probable.*
> *The determination of the law of probability that the market admits at a given instant will be the object of this study." (p.32)*

As this quote indicates, the distribution of traders' beliefs about future price changes is the distribution of price changes itself. Every market participant has a random belief, without anyone having an informational advantage. It is therefore an ideal type where traders are themselves stylized in their beliefs.

In terms of preference, Bachelier considers that the price of options corresponds to a *mathematical expectation*, which therefore refers to risk neutrality. But the press of the time presented option buyers as having a taste for risk: rational sellers would therefore price options at a level *higher* than the mathematical expectation in order to take advantage of their clients' taste for risk.

Note, however, that Bachelier does not justify expectation as being a representation of the operators' preferences, but by a principle of justice:

> *"It is evident that a gambler will be neither advantaged, nor disadvantaged if his total mathematical expectation is zero.*
>
> *The game is then said to be* fair*" (p. 32)*

Jovanovic (2001) has already mentioned what he calls an “ethical choice”, which he presents in the context of Bachelier’s time. However, it should be recalled that the notion of *fair game* is as old as probability theory, and that it is from this notion of fair play game Pascal and Fermat develop mathematical expectation as a solution to the problem of parties:

> *In Pascal’s games, there is, in fact, equality of stakes, equality of the amounts that accrue to each player in the event of loss or gain, and equality of chances (they are games of “pure chance” and in which, in each game, there is “as much chance for one as for the other). Jallais-Pradier (1997) p. 12 quoting Pascal (1654)*

If Bachelier were to describe the *actual* stock market, it would be simply amazing to consider that *justice* stands as its organizing principle. This assumption of justice is a mathematical assumption in the sense that it is an assumption in a model that does not describe reality but makes it possible to find a solution, *i. e.* to value the option. Indeed, if the price of the option is equal to its expected gain, then the mathematical expectation of both parties (buyer and seller) is zero: Montessus (1908) §83 p. 101 calls this result “Bachelier’s theorem”. Calling Bachelier’s contribution a theorem emphasizes the mathematical nature of Bachelier’s contribution, which is not a description of market practices but a mathematical transposition. More precisely, it emphasizes the axiomatic nature of Bachelier’s work, where assumptions form the hypothesis of a theorem.

1.3. “technical” representation assumptions

Beyond the obvious deviations from the reality of the object he was describing, Bachelier made assumptions that enabled him to represent his object in mathematical terms. The idea that prices follow a continuous process was a fundamentally innovative hypothesis in 1900. This hypothesis can be considered as intrinsically mathematical because it will enable the price process to be identified.

The first operation necessary to identify the price process is to clean the observations of the accrued interest, which is why Bachelier considers what he calls “true prices”, i.e. net of accrued interest:

> *“So far we have spoken only of quoted spread, the only ones with which we are ordinarily concerned. However, they are not the ones that will be introduced in our theory, but rather true spreads, that is to say, the spreads between prices of options and true prices corresponding to declaration of options.” (1900 p. 30)*

In order to further identify the price change process, a number of additional assumptions are necessary, which Bachelier details from p. 35 of his *Theory of speculation*. Thus, time is not only continuous but also consistent on average, in that the density of price changes remains unchanged over time (1.) and changes are time-separable (2.). To write this double hypothesis of *stationarity* and *time-separability*, let us modernize Bachelier's notations by indicating by $\rho_t(x)dx$ the probability that in the time interval between 0 and t, the true price varies by an amount included in the interval $[x, x + dx]$ (we will say in the following that *the price change* between 0 and t is included in $[x, x + dx]$). The double hypothesis of *stationarity* and *time separability* is then written:

> *By virtue of the Principle of Compound Probabilities, the desired probability will be equal to the product of the probability that x be the quoted price at epoch t_1, that is to say, $\rho_{t_1}(x)dx$, multiplied by the probability that x be the price quoted at epoch t, the current price z being quoted at epoch $t_1 + t_2$, that is to say, multiplied by $\rho_{t_2}(z - x)dz$.*
>
> *The desired probability is therefore:*
>
> $$\rho_{t_1+t_2}(z)dz = \left(\int_{-\infty}^{+\infty} \rho_{t_1}(x)\rho_{t_2}(z - x)dx\right)dz \qquad (1)$$

This equation does not correspond to the observed reality of price variations, because there can be periods of greater or lesser agitation of prices (which is opposed to the idea of *stationarity* of the distribution of price changes) and statistical dependence between periods. Although it does not correspond to observations, the convolution equation chosen by Bachelier is solved by the heat kernel, as it has been published by Fourier (1822). Bachelier then only has to reproduce Fourier's calculations to show that the heat kernel:

$$\rho_t(x) = \frac{1}{2\pi k\sqrt{t}} e^{-\frac{x^2}{4\pi k^2 t}}$$

is a solution of equation (1). The solution is then interpreted as "the law of probability of the market", *i.e.* both the density of price changes and beliefs about price changes among market participants. The parameter k characterizes what Bachelier calls the *instability of the market*[3].

It is important to understand that Bachelier *does not start by picking a probability density of price changes* as subsequent authors do (and as he did himself in his later works (1914) and (1938)[4]), he starts from postulates which allow him to follow in Fourier's footsteps, identifying the probability distribution of price changes with the heat kernel.

[3] It should be noted here that market instability is relative to the market itself, it is not time-varying, in line with the assumption of time consistency (stationarity).

[4] Bachelier (1914) begins his presentation of the theory of speculation as follows: "In the theory of speculation it is assumed that the variations in the price of a *rente* security, for example, are due to chance and the laws of these variations are studied, that is to say, the probability is sought that at a given time the price will differ from the current price by a given amount" (p. 176). Only one assumption is then sufficient to justify the normality of this law of price changes: "By saying that the variations in prices are due to chance, we want to express that, as a result of the excessive complexity of the causes that produce these variations, everything happens, in reality, as if chance were acting alone" (p. 176). The "excessive complexity" of causes can rightly be considered as an infinite sum of random variables, which are proven to converge toward a normal distribution by the central limit theorem. Bachelier (1938) uses exactly the same terms on pp. 4-5.

Thinking as a mathematician, Bachelier chooses three sets of assumptions that do not correspond to the legal nature of option contracts, nor to the beliefs or preferences of the parties to these contracts, nor even to the statistical properties of the prices of these contracts. It cannot be considered that he chooses such "unrealistic" assumptions to make a *descriptive* theory of the market. On the contrary, the notion of *fair game* corresponds to a *normative* theory, which allows Bachelier to calculate the fair value of a call option as a mathematical expectation.

2. Option valuation by Bachelier

In the previous section, we showed that Bachelier posed the assumptions that made it possible to write the value of the option as the mathematical expectation of its price. In order to clarify the modalities of this calculation, we will recall the parameters, whose terminology is clearly different from our uses, as shown in the following table:

Options terminology in Bachelier (1900)

Notation	Current English name	French name in Bachelier (1900)
T	Maturity (in days)	échéance
S_0^T	Forward price of asset in 0	cours à terme
S_T^T	Settlement price of the option	cours de la réponse des primes
K	Strike price	pied de la prime
c	Call price	dont
$K + c$	Strike price + option price	cours à prime
$K + c - S_0^T$	(*spread*, litteral translation)	écart = cours à prime moins cours à terme
σ	Volatility of the underlying	Bachelier considère le *coefficient d'instabilité du marché* $k = \frac{\sigma}{\sqrt{2}}$

With these notations, we can write that the value of the option is equal, by definition, to:

$$E(S_T^T - K | S_T^T \geq K)$$

Bachelier showed (see previous section) that the distribution of *true price* changes is Gaussian, i.e. $S_t^T = S_0^T(1 + \sigma W_t)$ where W_t is a geometric Brownian process. Following Bachelier, we have:

$$E(S_T^T - S_0^T | S_T^T \geq S_0^T) = \int_0^{+\infty} \frac{x}{2\pi k\sqrt{t}} e^{-\frac{x^2}{4\pi k^2 t}} dx$$

Bachelier considers only the case where the option is sold *at the money*[5] hence $K = S_0^T$, which allows him to write:

[5] Bachelier (1900) p. 51 wrote « on assuming $m = 0$ », 'i.e. « the case (…) where the amount of the premium for an option is equal to its spread », or in symbolical notation

$$c = K + c - S_0^T$$

which implies $K = S_0^T$, *i.e.* the strike price is equal to the price of the underlying asset or the option is sold *at the money*.

$$c = E(S_T^T - K | S_T^T \geq K) = E(S_T^T - S_0^T | S_T^T \geq S_0^T) = \int_0^{+\infty} \frac{x}{2\pi k\sqrt{t}} e^{-\frac{x^2}{4\pi k^2 t}} dx$$

Sachermayer-Teichmann (2008) p. 3 have shown that this defining equation[6], even if it does not have the usual form of the option pricing formula, allows for a modern option pricing formula.

For his part, Bachelier develops his result by proving that

$$\int_0^{+\infty} \frac{x}{2\pi k\sqrt{t}} e^{-\frac{x^2}{4\pi k^2 t}} dx = k\sqrt{t}$$

which he interprets in a sentence distinguished by *italics* :

« The value of a simple option must be proportional to the square root of the elapsed time. » (p. 52)

Bru (2001) and Jovanovic-Le Gall (2001) recalled that this relationship of the option price to the square root of its duration, known as the "law of the square root", was discovered by Jules Regnault (1863). Émile Dormoy (1873) mentions Regnault and the square root law, in a volume that also features a pioneering article by Henri Lefèvre (1873), introducing the canonical graphical representation of option payoffs, which is then found in Bachelier (1900). Because he reproduced Lefèvre's graphs and singled out the square root law expounded by Regnault, Bachelier has been included in their filiation and more generally in a stream of research prefiguring "modern market finance" (Jovanovic-Le Gall 2001). We now show that Bachelier's text is undoubtedly part of the mathematical economics of Léon Walras.

3. Bachelier and Walras' *Elements*

Bachelier's conclusion (1900) will seem cryptic to those who see him exclusively as a financier:

"A final remark will perhaps not be superfluous. If, in respect of several questions treated in this study, ***I have compared the results of observation to those of the theory, this was not to verify the formulae established by mathematical methods*** *[e. a.], but to demonstrate only that the market, unwittingly, obeys a law which governs it: the Law of Probability." (p. 86)*

[6] More precisely, Sachermayer-Teichmann (2008) write that Bachelier proved

$$c = \int_{K-S_0}^{+\infty} (S_0 + x - K).\frac{1}{S_0\sigma\sqrt{2\pi T}} \exp\left(-\frac{x^2}{2\sigma^2 S_0^2 T}\right) dx$$

Where Bachelier does not divide by S_0.

The last proposition of the quotation makes very clear why Bachelier did not bother to disseminate his theory to operators of a market that was already applying it "unwittingly". The central proposition is a strong epistemological statement, and it is doubtlessly reminiscent of Walras (1889):

> "… the mathematical sciences properly speaking, do go beyond experience as soon as they have borrowed their type concepts from it. (…) they construct a priori the whole framework of their theorems and proofs. **They re-enter after that into experience not to confirm but to apply their conclusions**[7]." (§30 p. 27)

The peculiar status of experience in Bachelier's work corresponds word for word to what Walras wrote: it is not a matter of testing the theory but of applying it directly. As it is not the habit of these authors to cite their sources, one might think of this homology as a *pure coincidence*. However, Bachelier "go[es] beyond experience as soon as he borrows his type concepts", as we explained in the first section when we showed how far Bachelier's assumptions are from a description of the market. Bachelier is therefore in line with the epistemological continuity of Walras (1889), who considers that:

> "The mathematical method is not an experimental method; it is the rational method. (…) the physico-mathematical sciences, like the mathematical sciences properly speaking, do go beyond experience as soon as they have borrowed their type concepts from it. They abstract from these real types ideal types that they define, and on the basis of these definitions they construct a priori the whole framework of their theorems and proofs." (§30 p. 27)

The very construction of Bachelier's theory, from the choice of assumptions as "abstractions of ideal types", to the proof of theorems, in the words of Montessus, and up to the "re-entry into experience" follows in an exemplary manner the description of the mathematical method of Léon Walras. Not only did Bachelier apply the Walras method, he chose a subject designated by Walras himself, that of the French annuity at 3%.

Bachelier's choice of subject is sometimes considered by commentators as a consequence of his biography, although there is no positive evidence that the author actually worked at or near the stock exchange. On the other hand, it should be noted that Walras (1889), whose general equilibrium model covers all markets, takes as the only paradigmatic example of a "well-organised market" the very one that will be the subject of Bachelier's research, the French 3% *rente* market:

[7] Quotations from Walras are taken from Donald Walker and Jan van Daal's translation of the 1896 edition (see references), except when the 1896 edition differed from the 1889 edition apparently read by Bachelier.

> “Let us therefore see how competition works in a well-organized market, and to do so, let us go into the stock exchange of a large capital market like Paris or London. (…) Let us take, for example, the trading activities in 3% French government bonds on the Paris Bourse (…).” (§42 p. 43)

For Walras, the functioning of the competitive market, whose archetype is the 3% rent, is the guarantee of economic efficiency:

> *“Production and exchange in a market ruled by free competition is an operation through which services can be combined into products of the kinds and quantities that give greatest possible satisfaction of wants within the limits imposed by the double condition that each service as well as each product has only a single price in the market, the one for which supply and demand are equal, and that the selling price of products is equal to their cost in services.*[8]” *(Walras 1889 §217 p. 261, whole paragraph in italics)*

It is precisely because the market makes it possible to achieve an economic optimum (“the greatest possible satisfaction of wants”) that Walras sets out to show that supply and demand are equal when the selling price is equal to the cost. The market example he chose, the market for French 3% *rente* securities, is particularly interesting because it is not just a spot market:

> “Suppose that the same operation [=market-clearing equilibrium] that is made in that way in the market for 3% French government bonds is made at the same time in the markets for the bonds of all countries […] and suppose that besides operations of sale and purchase on cash terms, there are operations of sale and purchase of futures, some firm and others optional, then the tumult of the Bourse becomes a veritable concert in which everyone plays his part.” (§42 p. 69)

The exemplary market which Walras uses to establish the proof of the functioning of the mechanics of supply and demand in fact comprises “firm futures” and “optional futures” (= options). If Walras intends to demonstrate the existence of a general equilibrium of an exchange economy (26th lesson of the fourth edition) and then of a single-period production economy

[8] Authors’ translation of the French text of the second edition, Walras 1889 §217 p. 261. In this case only we did not consider the 1896 edition, which provides a simple condition and reads as follows: “Production and exchange in a market ruled by free competition is an operation through which services can be combined into products of the kinds and quantities that give greatest possible satisfaction of wants within the limits imposed by the condition that each service as well as each product has only a single price in the market.” (§217 p. 247, Walras applied italics to the whole paragraph)

(27th lesson), the 28th lesson of the EEPP introduces time through the equilibrium of futures markets: (§275 p. 280), Walras writes that the priceP_k of a security k is equal to:

$$P_k = \frac{p_k}{i + \mu_k + \nu_k}$$

Where p_k is the gross return of asset k over the period, i the *macroeconomic* interest rate[9], which is equal to the growth rate of the whole economy over the period, μ_k the "amortization premium" and ν_k the "insurance premium". In other words, if an asset k' is more risky than k then its insurance premium $\nu_{k'} > \nu_k$ hence for the same return, k' has a lower market value that k. One can interpret $i + \mu_k + \nu_k$ as the specific interest rate on k, since

$$p_k = (i + \mu_k + \nu_k)P_k$$

If we use an index t to denote time, hence $r_{k,t} = i_t + \mu_{k,t} + \nu_{k,t}$, then we have for instance

$$P_{k,t+1} = (1 + r_{k,t})P_k$$

hence the price of the asset in $t + 1$ as the result of capitalization at rate r_t of a capital stock equal to the price in t. This equation thus defines $P_{k,t+1}$ as a *forward price* and $r_{k,t}$ as a forward *rate determined* by general equilibrium. It is then important to recall in this respect that

> "… the rate of interest (…) is determined (…) in the capital goods market, that is to say, in the stock market." §251 pp. 289.

That is to say, the stock market, by determining the forward prices of financial securities, sets the interest rates for them (including forward interest rates and future interest rates if we consider multiple future periods as Walras does).

Walras thought to show that all spot and forward markets were in equilibrium through the interplay of supply and demand, but the last category of markets he mentions, *i.e.* option markets, are excluded from his demonstration. And for good reason: while Walras analyses the *scarcity* of goods traded in both spot and forward markets, he fails to express the *scarcity* of contingent contracts such as option. This is precisely the problem that Bachelier solved with his *theory of speculation*. We will now show that the continuity between Walras and Bachelier did not escape the attention of their readers.

4. Bachelier's readers were mathematical economists

Although Bachelier may have appeared to be a lonely and misunderstood figure, the dissemination of his work now seems to be well understood. On the one hand, Bernard Bru (2001) has searched the university libraries of France and Navarre, and on the other hand Jovanovic (2012) has undertaken a systematic search of JSTOR and Web of Science for all occurrences in periodicals: before 1960, the latter found only references to the *Calcul des*

[9] Walras calls it « taux de revenu » or "income rate".

Probabilités (Bachelier 1912), with the exception of a mention of Bachelier's reception (1938) by the *American Mathematical Monthly*. Bernard Bru (Taqqu 2001) indicates that Montessus (1908) and Barriol (1908) refer explicitly to Bachelier. Montessus (1908) is a popular science book, as Bachelier (1914) and Boll (1936). All these works, like Julien Laferrière's law course, which Jovanovic (2002) has shown to have been inspired by Bachelier (1900) and (1914), have had no proven echo. On the other hand, the Barriol track is more interesting: Bru et al. (2012) indicate that the latter had organized the training of actuaries. He presented Bachelier's theory to auditors and readers equipped to understand the method and scope of his theory. Alfred Barriol constitutes an indirect but undeniable link between Bachelier and Walras: at the same time as he contributed to the dissemination of Bachelier's ideas in his book and course (1908), he helped Léon Walras' daughter to decipher her father's manuscripts, as Potier and Walker (2004) write:

> *"As Léon Walras' drafts were very difficult to decipher, she [Aline Walras] contacted the numerous correspondents or their descendants in order to recover the letters. Alfred Barriol advised her, as far as France was concerned, to contact the Société de statistique and the Société d'économie politique de Paris. Aline envisages a publication in French [...] The Parisian publisher could be Larose et Tenin. Aline enlisted the help of Alfred Barriol and Etienne Antonelli in this task of deciphering and translating...". (p. 2)*

Léon Walras' daughter turned to Alfred Barriol because, shortly before the father's death, he had shown a sustained interest in his pure economics, as evidenced by a letter published by Jaffé (1965):

> *I have the honour of enclosing the programme of the courses that were taught in Paris. Last year I outlined a course in Rational Economics and I thought that this year my friend, Aupetit, would be able to give this course - he is overloaded with work and I must, for another year, give up publishing and popularising your fine work!*
>
> *Please accept, Sir and dear Master, with my best wishes for good health, the expression of my deepest respect*

This letter from Barriol is crucial since it shows that he had included in the training programme for the actuarial students for whom he was responsible both the "rational economics" of Léon Walras, which he would have liked Aupetit to teach but which he had to teach himself, and Bachelier's theory of speculation. Barriol (1908) takes up Bachelier's (1900) hypotheses and theory in a chapter devoted to premium markets. He considers the valuation of options sold on the Paris market under the assumption, already posed by Bachelier (1900)[10], that "the settlement price of premiums equals the liquidation price of the forward" (p. 357). Like Bachelier, he takes

[10] See above, § 1.1. "the adjustment price coincides with the price at the declaration of options."

into account only the exercise of options at maturity (European options). And if he does not mention the notion of *fair game*, he nevertheless makes, like Bachelier, the hypothesis that the option price is equal to the expectation of its cost (for the seller), and that the distribution of these costs at maturity is normal:

> *the probability of a loss* $x = f - l$
>
> $x = f - l$ *can be equated with the probability of a deviation (in the mathematical sense of the word) and is therefore:*
>
> $$\frac{h}{\sqrt{\pi}} e^{-h^2 x^2}$$
>
> *p. 357*

Indeed, the quantity denoted $x = f - l$ by Barriol corresponds in the above notations (section 2) to $S_T^T - S_0^T$. The term "deviation" refers to the centred normal distribution whose scale parameter is here expressed as the "modulus of precision" $h = \frac{1}{\sigma\sqrt{2}}$ (where σ is the standard error, which was introduced by R.A. Fisher). Like Bachelier (1900), Barriol (1908) therefore evaluates the option as the expectation of the quantity $S_T^T - S_0^T$ when it is positive, i.e.

$$c = E(S_T^T - S_0^T | S_T^T \geq S_0^T)$$

We established as a fact that French actuaries of the 1900s were taught "rational economics" which combined the theories of Léon Walras and Louis Bachelier. It is therefore not a misnomer to speak of the *Walras-Bachelier model* to designate a general equilibrium model with forward and options markets. It seems natural to ask now about the relationship between such a model and the Arrow-Debreu model.

5. Walras-Bachelier vs. Arrow-Debreu

It is sometimes considered that the Walrasian general equilibrium is single-period and does not take time into account. This is the interpretation of Maurice Allais (1943):

> *"In pure economics, the determination of the economic general equilibrium has so far been limited to the determination of the various parameters at a given moment. It has never yet been done by explicitly involving time." p. 22 §11*

Allais's words are not anecdotal because practically all French mathematical economists of the post-war period (notably Marcel Boiteux, Gérard Debreu, Jacques Lesourne, Edmond Malinvaud) attended Maurice Allais's seminar (Drèze 1989, Lévy-Lambert 2020) and read the lines we have just quoted. Allais's reading of Walras became the dominant interpretation at that time, in contrast to the vision that Barriol taught pre-war to apprentice actuaries: insofar as he consolidates the Walrasian general equilibrium, Bachelier (1900) reveals the temporal dimension and the consideration of risk in Walras that went unnoticed by readers like Allais.

5.1. Time in the Walras-Bachelier general equilibrium model

According to the interpretation that became established after Allais, time in the Walrasian model is only a fiction, indeed Allais (1943) continues:

> *"The studies therefore should include desirability for money and savings, and depreciation and insurance coefficients for physical capital. These elements were absolutely arbitrary and remained undetermined. Walras and Pareto had thus [left] the theory of the interest mechanism completely out of the picture." p. 22 §11*

It should be pointed out here that Allais uses exactly the terminology of Walras in the 28th lesson of the PPEC, where the latter speaks respectively of an "amortization premium" and an "insurance premium" for the coefficients μ_k et ν_k which are *determined* by the equilibrium of the forward market (in general equilibrium). Allais considers that pricing does not allow for the simultaneous determination of these two coefficients. But for Walras, depreciation - the first manifestation of time - refers to the physical wear and tear of capital and can therefore be measured objectively, whereas the insurance premium is *effectively* determined by the equilibrium of the forward market as a measure of what we would call today the risk premium of the asset. Now, as Bachelier considers futures prices as processes, as he describes their trajectories, there is an undeniable expression of time in this Walras-Bachelier model, which is therefore *effectively* a model of intertemporal general equilibrium.

In contrast to Maurice Allais, d'Autume (1982) perceived the presence of time in Walras, in particular in the "temporary equilibria" which he distinguishes from a "long-term stationary regime":

> *"Ex post expectations are likely to be disappointed and actual net income rates may differ. It is only in the long run, in a steady state, that they even out." (p. 124)*

For d'Autume (1982), in a stationary regime, the Walrasian model is a perfect forecasting model, but outside the stationary regime, it is a model of rational expectations. This representation of temporary equilibria where agents form rational expectations is compatible with the Bachelier model. But if there are options and "insurance premiums" in the Walras model, it is because we are outside the long-run stationary regime, in an economy where risk is irreducible, and this is why we speak of the Walras-Bachelier model. Allais did not grasp this dimension of the Walras-Bachelier model because he himself represents an economy without a financial market, where risks are only exchanged via lotteries and mutual insurance companies.

5.2. Risk in the Walras-Bachelier general equilibrium model

We have shown that the Walras-Bachelier world includes time and randomness, no less than the Arrow-Debreu world. How are these worlds different? Arrow and Debreu (1954) achieve general equilibrium by equalizing supply and demand in all markets (at all times and in all states of nature). Walras and Bachelier's model differs in two respects:

1. There is no partition of the future into states of nature and thus no complete system of markets in Walras-Bachelier. Instead of a different equilibrium price vector in each state of nature, Walras and Bachelier consider a single price vector per period which corresponds to the decisions of agents in response to the expectation of random future quantities (with the distribution of price changes being equal to the distribution of expected price changes).
2. Whereas in Arrow and Debreu the demands for all goods correspond to the maximization of agents' utility, the Walras-Bachelier equilibrium regime involves a partition of markets. In all markets except options markets, the equilibrium price is the one that equals supply and demand. In the option markets, the mathematical expectation of the option value equals supply and demand *by assumption*. The option price is determined as *fair*, supply and demand play no role in its determination.

This equilibrium price of the option market corresponds to the expected gains of the option for the buyers and the expected losses for the sellers. The option price is then equal to the demand price for risk-neutral investors and the supply price for risk-neutral sellers. In contrast, in Arrow-Debreu, it is to guarantee the uniqueness of the solution that the demand results from the convex preferences of the agents, i.e. they must be risk averse.

To conclude this parallel between the Walras-Bachelier and Arrow-Debreu models, we recall that in their article demonstrating the existence of a competitive general equilibrium, Arrow and Debreu (1954) considered that if he designed the analytical framework:

> *"Walras did not, however, give any conclusive arguments to show that the equations, as given, have a solution." (p. 265)*

This is obviously true for the Walras model alone. The Walras-Bachelier model includes a demonstration of the existence and uniqueness of the partial equilibrium of options markets. In an astonishing twist of fate, Allais made four "observations" in his discussion of Arrow (1953) to which Bachelier had already responded, namely:

1. Arrow considers a countable number of states of nature. "Such a simplifying hypothesis can naturally be made, but it seems to me to have the major disadvantage of being very far from reality." Allais proposes to consider "as a first approximation" Gaussian distributions, which was already done by Bachelier (1900).

2. The convexity of preferences is not necessary for the existence of an equilibrium, which corresponds in particular to the case envisaged by Bachelier of linear preferences in probabilities (risk neutrality).

3. "In the present state of the theory, a formulation that is too general seems to me to have more disadvantages than advantages, because it does not allow us to push the calculations to the end."

4. "I don't think it can be said that only if individuals are risk averse can we be sure that competitive risk allocation is viable."

It has been shown (1.) that Bachelier's distribution of payoffs at option expiration was normal, (2.) that the option was valued as an expectation, which corresponds to (4.) linear preferences that can be interpreted as risk neutrality of the parties to the exchange; eventually (3.) that Bachelier chooses his hypotheses precisely to "push the calculation to the limit" and explicitly determine the option price.

*

Conclusion

We have shown that Louis Bachelier's work is in line with Léon Walras' hypothetico-deductive model. This "rational method" consists of set of postulates that permits deriving a theory of the *fair price* of options. The *Théorie of speculation* is not a description of the practices of the Parisian stock market. Moreover, Bachelier responds to a shortcoming of the Walrasian theory by making it possible to demonstrate the existence and uniqueness of an equilibrium price-vector for the family of markets missing in the demonstration of Walras, who after having mentioned premium markets as an archetype of the competitive market (Walras 1889 p. 67), forgets them in the continuation of the *Eléments d'Economie Politique Pure*. More than the isolated attempt of a solitary researcher, Bachelier's theory provides the missing piece to the Walrasian puzzle, and the complete Walras-Bachelier model is a model of intertemporal general equilibrium in the presence of risk.

This perspective of Walras and Bachelier allows us to establish the theoretical coherence of this Walras-Bachelier model but also to understand Bachelier's oversight: if mathematical economists had remained in line with Barriol (1908), they would have continued to link Walras and Bachelier. By theorizing an economy without financial markets, Allais (1943) obliterates both the paradigmatic role of the securities market in Walras and the contribution of Bachelier (1900).

At the end of this path, one grasps the full significance of the last sentence of the Theory of Speculation (Bachelier 1900): "the market, unwittingly, obeys a law which governs it: the Law of Probability". By putting empirical prices into the equations of pure economics, Bachelier fulfils Walras' scientific program "re-enter[ing] after that into experience not to confirm but to apply [his] conclusions".

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Appendix – Bachelier's postulates

H1 European options

> "In all that will follow, it will be assumed that the adjustment price coincides with the price at the declaration of options. This hypothesis can be justified, for nothing prevents liquidation of operations at the declaration of options." (1900 p. 27)

Let us denote 0 the current time and T the time of settlement ("declaration of options"), S_0^T is the current forward price of the underlying asset for settlement ("declaration") in T, S_T^T is the (forward) "price at the declaration of options" and K is the strike price, it follows from H1 that the profit from the options is $(S_T^T - K | S_T^T - K \geq K)$.

H2 True prices

> "So far we have spoken only of quoted spread, the only ones with which we are ordinarily concerned. However, they are not the ones that will be introduced in our theory, but rather true spreads, that is to say, the spreads between prices of options and true prices corresponding to declaration of options." (1900 p. 30)

H3 Continuous time distribution of price changes

> "But, while the market believes neither in a rise nor a fall in the true price, some movements of a certain amplitude may be supposed to be more or less probable.
>
> The determination of the law of probability that the market admits at a given instant will be the object of this study." (1900 p. 32)

> $\rho_t(x)dx$ is the probability that in the time interval between 0 and t, the true price varies by an amount included in the interval $[x, x + dx]$ (1900 p. 35)

In our notations, $\rho_T(x)dx$ is the density of price changes between the issuance of the option (0) and the settlement (T).

H4 Market equilibrium condition

> "It seems that the market, that is to say, the totality of speculators, must believe at a given instant neither in a price rise nor in a price fall, since, for each quoted price, there are as many buyers as sellers." (p. 32)

The idea here is that every buyer is matched by one seller, as

> "[The] conclusions [of every speculator] are completely personal, since his counter-party necessarily has the opposite opinion." (p. 31)

H5 Nature of expectations

The distribution of beliefs about future price changes among market participants is the true distribution of future price changes

> "… while the market believes neither in a rise nor a fall in the true price, some movements of a certain amplitude may be supposed to be more or less probable.
>
> The determination of the law of probability that the market admits at a given instant will be the object of this study." (p. 32)

> "[every speculator] analyses the reasons which may influence rises or falls in prices and the amplitude of price movements. His conclusions are completely personal, since his counter-party necessarily has the opposite opinion." (p. 31)

H6 Fair game

> "It is evident that a gambler will be neither advantaged, nor disadvantaged if his total mathematical expectation is zero.
>
> The game is then said to be fair." (p. 32)

H7 Principle of mathematical expectation

> "The mathematical expectation of a speculator is nil.
>
> The generality of this principle needs to be appreciated: it signifies that the market, at a given instant, considers as having nil expectation not only the current trading operations, but also those that would be based on a subsequent movement of prices. » (p. 34)

As a consequence, the value of the option, c, is equal to the mathematical expectation of the gain:

$$c = E(S_T^T - K | S_T^T - K \geq 0)$$

In order to compute this mathematical expectation, we need the probability distribution of the price of the underlying on maturity, i. e. the probability distribution of price changes between the current date and option maturity. This is the probability distribution defined but not determined in H3 and H5. The next assumptions H8 and H9 solve the determination of the probability distribution of price changes between the current date (0) and option maturity (T), $\rho_T(x)dx$,

H8 Stationarity of the density of price changes over time

$\forall t$, the probability distribution of price changes between t and t+T is

$$\rho_T(x)dx$$

H9 Time-separability of the density of price changes

> The probability law can be determined from the Principle of Compound Probabilities.
>
> (…) By virtue of the Principle of Compound Probabilities, the desired probability will be equal to the product of the probability that x be the quoted price at epoch t_1, that is to say, $\rho_{t_1}(x)dx$, multiplied by the probability that x be the price quoted at epoch t_1, the current price z being quoted at epoch $t_1 + t_2$, that is to say, multiplied by $\rho_{t_2}(z - x)dz$. (1900, p. 35)

$$\forall (t_1, t_2), \rho_{t_1+t_2}(z)dz = \left(\int_{-\infty}^{+\infty} \rho_{t_1}(x) \rho_{t_2}(z - x) dx \right) dz$$

Under assumptions, H3, H5, H8, the heat kernel

$$\rho_t(x) = \frac{1}{2\pi k\sqrt{t}} e^{-\frac{x^2}{4\pi k^2 t}}$$

is a solution of the convolution equation in H9.

Hence, the value of the option can be computed as a mathematical expectation

$$c = \int_{-\infty}^{+\infty} \rho_T(x)\,(x - K|x - K \geq 0)dx$$

H10 the option sold is sold at the money

The simplest case from the above equations is that where $[K - S_0] = 0$, that is to say, the one where the amount of the premium for an option is equal to its spread. (1900 p. 51)

$$K = S_0^T$$

By integration, Bachelier finds the solution

$$c = \int_{K-S_0^T}^{+\infty} (x - K).\frac{1}{2k\sqrt{2\pi T}} \exp\left(-\frac{x^2}{4k^2 {S_0^T}^2 T}\right) dx$$

H11 approximation for options sold either in-the-money or out-of-the-money option
Bachelier then proposes an approximation for either in-the-money or out-of-the-money option, without providing a closed-form analytical solution. He does so by positing that

$$c + (K - S_0^T) \int_{K-S_0^T}^{+\infty} \frac{1}{2\pi(S_0^T + c)} \exp\left(-\frac{x^2}{4\pi(S_0^T + c)^2}\right) dx$$
$$= \int_{K-S_0^T}^{+\infty} \frac{x}{2\pi(S_0^T + c)} \exp\left(-\frac{x^2}{4\pi(S_0^T + c)^2}\right) dx$$

(Bachelier 1938 p. 36)

(This equation is only alluded to in 1900)

Which he expands in Fourier series to find an approximation of c. $(c + K - S_0^T) = k^2 T$, hence

“the spread [$c + (K - S_0^T)$] for options for a rise [call options] is approximately proportional to the term to expiration [T] and the square of the coefficient of instability [k^2].” (1900 p. 55)