The Network Origin of Slow Labor Reallocation

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Abstract

What explains slow worker reallocation after trade or technological shocks? In this paper, I explore the idea that imperfect skill transferability constrains the reallocation of workers between occupations. I model these frictions as forming a network (the "occupational network"), whose nodes are occupations and whose edges represent the possible occupational transitions. I show that the structure of the occupational network matters for the speed of worker reallocation and I make two contributions. First, I find that this network is very sparse, hinting at large frictions to occupational mobility, and that there exists central bottleneck occupations which can block the reallocation process. Second, I extend the search and matching model of the labor market with an occupational network. I find that asymmetric shocks can generate transition dynamics which are two orders of magnitude slower than in the standard model. Moreover, I show that the intensity of network bottlenecks is a key determinant of slow worker reallocation dynamics after asymmetric shocks. In other words, central occupations have a granular effect on worker reallocation speed.

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1 Introduction

Trade liberalization, automation, or any type of structural shock, have a massive, asymmetric effect on labor markets. They destroy jobs in some occupations (e.g. manufacturing employment, routine occupations), while simultaneously creating jobs in others (e.g. non-tradable sector, non-routine occupations). This generates a process of worker reallocation between occupations, as workers move from declining occupations to expanding ones. For a long time, the common view was that worker reallocation is fast, and hence the effects of trade or new technologies on the labor markets are short-lived.[1] However, a growing empirical literature now questions this narrative and suggests instead that in certain contexts worker reallocation was a lot more sluggish than previously thought (e.g. Autor et al. (2013), Autor et al. (2014), Dix-Carneiro and Kovak (2017), Dix-Carneiro and Kovak (2019), Adão et al. (2021), Autor et al. (2021)). What explains the sluggishness of worker reallocation during these episodes? Will future worker reallocation episodes, e.g. due the green transition, be fast or slow? This matters because the speed of worker reallocation controls the magnitude of welfare gains from trade or new technologies.[2]

In this paper, I propose a new empirical and theoretical framework to address these questions. Building on the idea that skills are imperfectly transferable across occupations, I construct a network summarizing the architecture of skill frictions between occupations. I label it the occupational network. The nodes of the occupational network are occupations, and the edges represent the possible occupational transitions. For instance, the textile worker and assembly operator occupations are connected in the occupational network, because their skills are similar enough for a transition to be possible, whereas the textile worker and engineer occupations are not. The structure of the occupational network captures in reduced-form the constraints affecting the movement of workers between occupations. This paper studies how the structure of the occupational network shapes the speed of worker reallocation dynamics.

My contributions are twofold. First, I identify the occupational network using a French occupational database and I study its architecture using tools from network theory. I provide evidence that the occupational network is very sparse, hinting at large occupational mobility frictions, and that there exists bottleneck occupations which can block the worker reallocation dynamics. Second, I extend the canonical search and matching model of the labor market with

[1] The argument is that the average job finding and average job destruction rates are at high levels in the US, entailing large churns in the labor markets: that is, workers reallocate at a high pace across jobs (Davis and Haltiwanger (1992)).

[2] In Autor et al. (2016), the authors write: "Establishing the speed of regional labor-market adjustment to trade shocks should capture considerably more attention from trade and labor economists." (p. 235) This is precisely what this papers aims at doing.

heterogeneous occupations connected by an occupational network. The key new ingredient is that unemployed workers can search for jobs in adjacent occupations in the occupational network. I find that asymmetric shocks generate very sluggish transition dynamics, around two orders of magnitude slower than predicted by the standard search and matching model. Moreover, I show that the larger the intensity of network bottlenecks, the slower the worker reallocation dynamics after asymmetric shocks. I turn now to describe each part in more detail.

In the first part of the paper, I provide new evidence on the structure of the occupational network. In order to identify the structure of the occupational network, I rely on a French database on occupational characteristics called ROME, which was elaborated by experts from the French employment agency. This database proposes a nomenclature of the economy's different occupations based on their task and skill, and contains textual information about each occupation (e.g. list of skill requirements, diploma, etc.). Here, I use a unique feature of the ROME database: for each occupation, the set of occupations towards which professional transitions are deemed possible by experts is indicated. This directly gives me the edges of the occupational network.

I highlight three stylized facts on the structure of the occupational network, using tools from network theory. First, the occupational network is very sparse, meaning there exist few connections compared to the maximum possible number of connections. This suggests that occupational mobility frictions are very large. Second, occupations differ widely in their number of direct and indirect connections. Most occupations have a small number of connections, but a minority has many connections, hence suggesting that occupational mobility frictions affect occupations very heterogeneously. Third, there exist bottlenecks in the occupational network. Indeed, paths in the network must disproportionally often pass through a small number of central nodes. These central nodes can block the flows of workers between otherwise non-connected clusters of occupations, i.e. they can act as bottlenecks on the worker reallocation process.

In the second part of the paper, I propose a tractable model of worker reallocation across occupations, in order to characterize how the occupational network shapes worker reallocation dynamics. I extend the standard search and matching model of the labor market, in the spirit of Diamond (1982), Mortensen (1982) and Pissarides (1985), with heterogeneous occupations connected by an occupational network.[3] The key departure from the standard model is the

[3]The search & matching framework is particularly well suited for the study of transition dynamics, because it introduces in a tractable manner the time dimension into macroeconomic models. It builds on the idea that it takes time for unemployed workers to find jobs, since search frictions prevent workers and firms to match immediately.

assumption that unemployed workers can search for jobs in occupations which are adjacent in the occupational network. For example, textile workers can search for jobs in the assembly operator occupation. When unemployed workers match with a firm in an adjacent occupation, they move across the network.

I study the effect of asymmetric productivity shocks on the process of worker reallocation between occupations. In the model, productivity shocks change the distribution of vacancy posting by firms across occupations, and therefore the distribution of job finding rates. This affects the transition matrix, which governs how the distribution of workers across occupations evolves over time. After the shock, the worker distribution gradually converges towards the new steady-state distribution implied by the new transition matrix. Put differently, workers reallocate between occupations. I am interested in the time it takes for workers to reallocate between occupations, i.e. the transition time. Formally, the transition time is measured as the time taken for the worker distribution to converge towards the new steady-state distribution. Building on a recent literature in macroeconomics on distributional dynamics, I propose a new measure of transition time using the information contained in the eigenvectors and eigenvalue of the transition matrix.

Next, I investigate the determinants of the transition time, and in particular how it is shaped by the structure of the occupational network. To build intuition, I focus on a simple occupational network structure with three occupations connected by a line network. For concreteness, let refer to them as the textile occupation, the retail sales occupation and the social worker occupation. The retail sales occupation is central, and connected to both the textile and social worker occupations, whereas the textile and social worker occupation are only connected to the central retail sales occupation. Now, consider a trade liberalization episode (or automation shock): the textile worker occupation experiences a negative productivity shock, whereas the social worker occupation experiences a positive productivity shock. This requires a reallocation of workers from the textile worker occupation to the social worker occupation. However, workers must transit through the central occupation (retail sales), which can therefore act as a bottleneck on the reallocation process.

I perform a simple calibration of the stylized model and study the reallocation dynamics after the asymmetric shock. This gives me two main results. First, the network search and matching model can generate reallocation dynamics which are two orders of magnitude slower than predicted by the standard search and matching model. To fix ideas, the transition time predicted by the standard search and matching model is approximately 1.5 months, while it is approximately 150 months in the calibrated network model.

Second, I show that the intensity of network bottleneck is a key determinant of worker reallocation speed after asymmetric shocks. I measure the intensity of network bottleneck intensity as the inverse level of productivity in the central bottleneck occupation, and I show that the transition time reacts strongly to the level of bottleneck intensity after asymmetric shocks: a 10% decrease of productivity in the central bottleneck occupation raises transition time by 50%. This means that central bottleneck occupations have a granular effect on worker reallocation dynamics after asymmetric shocks. A key take-away for empirical analysis is that transition rates in central bottleneck occupations, not average transition rates, are informative about worker reallocation speed after asymmetric shocks.[4]

Towards deriving my main result on the reallocation dynamics, I also solve for the equilibrium distributions of wages and tightness ratios across occupations, and for the steady-state worker distribution. These results are of interest in themselves and show that the network approach can speak to other topics such as labor market power and wage inequality. But they also matter for the reallocation dynamics, since they determine the distribution of equilibrium transition probabilities across occupations. As discussed above, the interaction between job finding rates and occupations' centrality is a key determinant of worker reallocation speed. This gives me three additional minor results.

First, I show that equilibrium wages are proportional to their index of Katz-Bonacich centrality in the network. Katz-Bonacich centrality is a popular measure of node centrality, which is defined recursively: a node is central if it is connected to other central nodes. Here, equilibrium wages solve a similar recursion in the network: the wage in an occupation is high if the wages in adjacent occupations are high, because this improves the workers' outside option in bargaining. This means that, everything else being equal, central occupations have higher bargaining power, and therefore higher wages. It also implies that, everything else being equal, the labor market tightness ratios and job finding rates are lower in central occupations, because high wages discourages vacancy posting by firms. This contributes to slow down the reallocation dynamics, since it becomes more difficult to transit through central occupations.

Second, I show that the tightness ratios in central occupations are negatively related to the tightness ratios in adjacent occupations. In other words, a higher tightness ratio in a central occupation creates a negative spillover for the neighboring occupations. This comes from the fact that a higher labor market tightness ratios in central occupations increases the bargaining power of unemployed workers in adjacent occupations - since it becomes easier to find jobs - which in turn discourages firm vacancy posting and lowers labor market tightness

[4]In the empirical literature, the intensity of reallocation is measured as the sum of average job creation and job destruction rates. This is precisely the speed of transition in the standard search and matching model. Given that the standard model substantially overestimates reallocation speed, this means that the standard measure of reallocation intensity is not informative about reallocation speed. In other words, high measured reallocation intensity might co-exist with very slow worker reallocation dynamics.

ratios in the adjacent occupations. This negative spillover in fact contributes to speed up the reallocation dynamics, because depressed low-tightness labor markets will *ceteris paribus* tend be located close to dynamic high-tightness labor markets. It shortens the distance for reallocation and hence lowers reallocation time.

Third, I demonstrate that two network statistics matter for the level of the steady-state aggregate unemployment rate: (i) the correlation between occupations' number of connections and their job finding rate, and (ii) the correlation between job finding rates in adjacent occupations. In economic terms, the first network statistics is an inverse measure of bottleneck intensity in the occupational network: a low correlation value indicates that well-connected occupations have low job finding rates, implying that few workers can transit through central occupations. The second network statistics reflects the degree of assortativity in labor market conditions between adjacent occupations: a high value indicates that neighboring occupations have similar values of job finding rates, implying that tight (or conversely slack) labor markets are clustered together. This result has two key implications: in order to decrease aggregate unemployment, policy-makers should (i) concentrate subsidies towards central occupations, rather than equally subsidize every occupation, and (ii) concentrate subsidies in certain cluster of occupations.

Relation to the literature. My paper connects to two strands of the literature. First, it connects to the literature on the adjustment of labor markets to trade or technology shocks. A growing body of papers provides reduced-form evidence, both at the local labor market-level and worker-level, that economies adjust slowly to trade liberalization episodes (e.g. Autor et al. (2013), Autor et al. (2014), Dix-Carneiro and Kovak (2017), Dix-Carneiro and Kovak (2019), Autor et al. (2021)). In particular, they find that the decrease in local manufacturing employment is not matched by a proportional increase in local non-manufacturing employment or migration outside of the employment zone, and that most of the adjustment to the shock played out through higher unemployment and exit of the labor force. Adão et al. (2021) show that the worker reallocation dynamics after technological shocks was very slow too. Motivated by these evidence, a series of papers build structural models of dynamic occupational choice with switching costs, in order to estimate the consequences of sluggish worker reallocation after trade of technological shocks for welfare (e.g. Artuç et al. (2010), Dix-Carneiro (2014), Caliendo et al. (2019), Traiberman (2019), Rodríguez-Clare et al. (2020), Humlum (2019), Dix-Carneiro et al. (2021)). However, with the exception of Rodríguez-Clare et al. (2020) and Dix-Carneiro et al. (2021), these papers abstract from involuntary unemployment, which the empirical literature has found to be an important channel of adjustment to trade or technological shocks. Rodriguez-Clare and co-authors generate involuntary unemployment by assuming downward nominal wage rigidity, but this has the counterfactual implication that the rise in involuntary unemployment is only temporary. Dix-Carneiro and co-authors generate involuntary unemployment by assuming labor market frictions within sectors, together with switching costs between sectors, but they rely on the simplifying assumption of segmented labor markets.[5] I contribute to this literature in three dimensions. First, I propose a new model of frictional worker reallocation, with a combination of search and skill frictions between occupations, suggesting an alternative micro-foundation of switching costs.[6] Second, I propose to view the architecture of bilateral frictions, may they come from switching costs or skill frictions, through the lenses of network theory. Third, I use this very tractable framework to characterize the determinants of worker reallocation speed.

Second, my paper connects to the labor literature on mismatch unemployment. Sahin et al. (2014) and Barnichon and Figura (2015) show that the the misalignment of the distribution of vacancies and unemployed workers across labor markets generates unemployment. However, their measure of mismatch relies on the assumption of segmented labor markets: unemployed workers only search for jobs in their own employment zone and occupation. This assumption is consequential, since the level of aggregation matters for the measure of mismatch: too much aggregation will result in little level of mismatch. This motivated subsequent research to drop the assumption of segmented labor market, by allowing workers to search in different locations or occupations. For example, Marinescu and Rathelot (2018) quantify the level of mismatch in a directed search model, where workers can apply to jobs in any location but have a distaste for distance.[7] Another strand of this literature develops a different but complementary concept of mismatch, where individuals are mismatched to jobs if their bundle of skills does not correspond to the bundle of skills required by the job (Baley et al. (2018), Guvenen et al. (2020), Lise and Postel-Vinay (2020)). These papers explicitly model the multidimensional

[5]This approach builds on a recent literature in macroeconomics, which combines labor market frictions within sectors and occupational choice with switching costs between sectors. Prominent examples include Chodorow-Reich and Wieland (2020) or Carrillo-Tudela and Visschers (2020).

[6]Schmutz and Sidibé (2019) build a partial equilibrium model of worker reallocation between geographical locations, featuring both mobility costs and search frictions between labor markets. They find that the size of mobility costs are one order of magnitude lower, once search frictions are accounted for.

[7]This paper is related to a new and growing literature on non-segmented labor markets, to which this paper also contributes. In a seminal paper, Manning and Petrongolo (2017) show that the geographical boundaries of labor markets substantially overlap between each other, leading local stimulus to diffuse across labor markets. Similarly, Nimczik (2017) and Schubert et al. (2021) use labor market flows to identify the local relevant labor market size of workers, along either sectoral or occupational dimensions. By contrast, this paper does not identify the occupational network using observed transition flows, because the former depend on endogenous variables such as wage differentials, tightness ratios, etc. Instead, it relies on data on occupational characteristics, which is supposedly exogenous to economic conditions.

skill space, typically accounting for three different skills: manual, intellectual and social. By contrast, my paper focuses only on the distance between occupations in the multi-dimensional skill space - which is, eventually, what matters for mobility frictions. This simplification allows me to characterize in a richer way how the distribution of jobs in the multidimensional skill space affects the allocation of workers to jobs, by using tools from network theory. Furthermore, this literature is interested in different outcomes: they study the effects of skill frictions on mismatch, while my focus is on the speed of worker reallocation.

The papers by Adão et al. (2021) and Schmutz and Sidibé (2021) are the closest to mine. Adão et al. (2021) augment a dynamic occupational choice model with a forward-looking entry/exit decision, and argue that the degree of skill similarity between old and new technologies is a key determinant of worker reallocation speed. I share with them their focus on the role played by skill frictions in slowing-down worker reallocation dynamics and their goal of deriving an analytical characterization of worker reallocation speed. However, my paper builds on the tractability of the search and matching model to allow for richer reallocation dynamics between heterogeneous occupations, rather than between two classes of production technologies. Schmutz and Sidibé (2021) build a general-equilibrium model of migration flows between cities, with search frictions between each cities located on a two-dimensional urban network. I share with them the network view of the labor markets. However, their focus is on geographical mobility and in explaining how it affects the distribution of city size, while mine is on explaining the speed of worker reallocation between occupations. In addition, their model is less tractable and prevents them from providing an analytical characterization of the equilibrium in terms of the network structure.

Outline Section 2 documents the network structure of the occupational network. Section 3 presents a stylized model of search & matching extended with an occupational network. I review the main analytical results derived from the stylized model in section 4. Section 5 concludes.

2 Descriptive evidence

In this section, I present descriptive evidence on the architecture of occupational mobility frictions. The idea that skill dissimilarity prevents workers from doing certain occupational transitions is standard. My contribution is to view the architecture of occupational mobility frictions through the lenses of network theory: the possible occupational transitions form the edges of a network, whose nodes are occupations. This network is called the occupational network.

In this section, I begin by presenting the database allowing me to identify the occupational network. Then, I discuss the methodological choices made to construct the occupational network. Finally, I highlight three stylized facts on the structure of the network.

2.1 Data

In order to identify the occupational network, I use French administrative database on occupational characteristics called ROME. The acronym ROME stands for *Répertoire Opérationnel des Métiers et Emplois*, which can be roughly translated as "Operational Nomenclature of Occupations and Jobs". This database provides information on the occupational structure of the economy, based on the occupations' skills and tasks contents. It was constructed in 1989 by experts from the French employment agency Pole Emploi. Historically, its goal was to help the employment agency better match job applicants to job vacancies based on their skill similarity, at a time where labor markets underwent substantial structural transformations. The database has been revised many times since its creation. Here, I use the third and latest available version, which was released in 2009.

The ROME database contains a nomenclature of occupations at different levels of aggregation. I consider the 5-digit level of aggregation, because the information of interest only exist at this level of aggregation. This represents 531 different occupations. For instance, the 5-digit level occupation "higher education", which comprises the sub-occupations "assistant professor" or full professor", is encoded as K2018. The letter K refers to the family of professions, here "social service" ; the two following numbers refer to the professional field, here "education" and the last two numbers enable to exactly identify the "higher education" occupation.

Each occupation at the 5-digit level is endowed with a set of textual information, which can be divided into five categories: (i) a list of sub-occupations or alternative denominations of the occupations ; (ii) a general description of the occupation ; (iii) a list of the skills required to perform the occupation ; (iv) the diploma and professional certification necessary to access the occupation ; and (v) the set of occupations towards which an occupational transition is deemed possible by experts. The set of occupations towards which an occupational transition is possible is further split into two categories: the close occupations, towards which mobility is possible right away, and the distant occupations, which are only accessible after a phase of adaptation or training. This last information is a unique feature of the ROME database, and is useful for constructing the occupational network.

2.2 The occupational network

I build the occupational network using two sets of information from the ROME database. The nomenclature of occupations at the 5-digit level gives me the set of nodes of the occupational network $\mathcal{N}$. The information about the occupational transitions deemed possible by experts gives me the set of edges of the occupational network $\mathcal{E}$. The occupational network $\mathcal{G}$ is simply the collection of these nodes and edges $\mathcal{G} = (\mathcal{N}, \mathcal{E})$.

I make two simplifying assumptions to construct the edges of the occupational network. First, I assume that close and distant occupations are equally accessible. This simplification allows me to build an unweighted network, where edges' weights take binary values $\{0, 1\}$. I could build a network with weighted edges $\{0, \lambda, 1\}$, where $\lambda < 1$ is the weight of distant occupational transitions and 1 is the weight of close occupational transitions, but it is unclear how to calibrate the weight λ at this stage. Note that the occupations are always connected with themselves, i.e. there are self-loops in the occupational network.

Second, I assume that occupational mobility frictions are symmetric. If it is possible to move from occupation A to B, then the reverse occupational transition from B to A must also be possible. This allows me to build a network with undirected edges. Admittedly, this assumption partially abstracts from the vertical, hierarchical nature of the occupational structure. For example, the team's manager is able to do her subordinates' work, but the reserve might not be true. But I make this assumption for three reasons: (i) the ROME nomenclature of occupations represents movements along the job ladder as within-occupation transitions, since sub-occupations with different ranks but similar skills are pooled together in the same occupations. For example, the sub-occupations lecturer, assistant professor and full professor are all comprised in the higher education occupation. (ii) this assumption is consistent with the rest of the literature on skill frictions, which measures skill frictions between occupations as the distance between between their respective vector of skill requirements - which, by definition, is always symmetric. (iii) the occupational network becomes disconnected if edges are allowed to be directed, meaning that some occupations are not reachable from other occupations anymore. This poses problems for the estimation of the theoretical model.

Figure 1 plots the occupational network, constructed with the simplifying assumptions presented above. For large networks, the choice of layout is key.[8] Here, I use a very standard layout method called the spring layout. The idea is intuitive: nodes repel each others, but edges act as springs which constrain adjacent nodes to stay close. Nodes belonging to

[8] Indeed, plotting a network necessarily entails an information loss: it projects a highly multi-dimensional object, such as a network, on a two-dimensional space. The higher the number of nodes and edges to plot, the larger is the information loss. As a consequence, visual inspection can often lead to misleading interpretations and the choice of the layout method is crucial - and the more so for large networks.

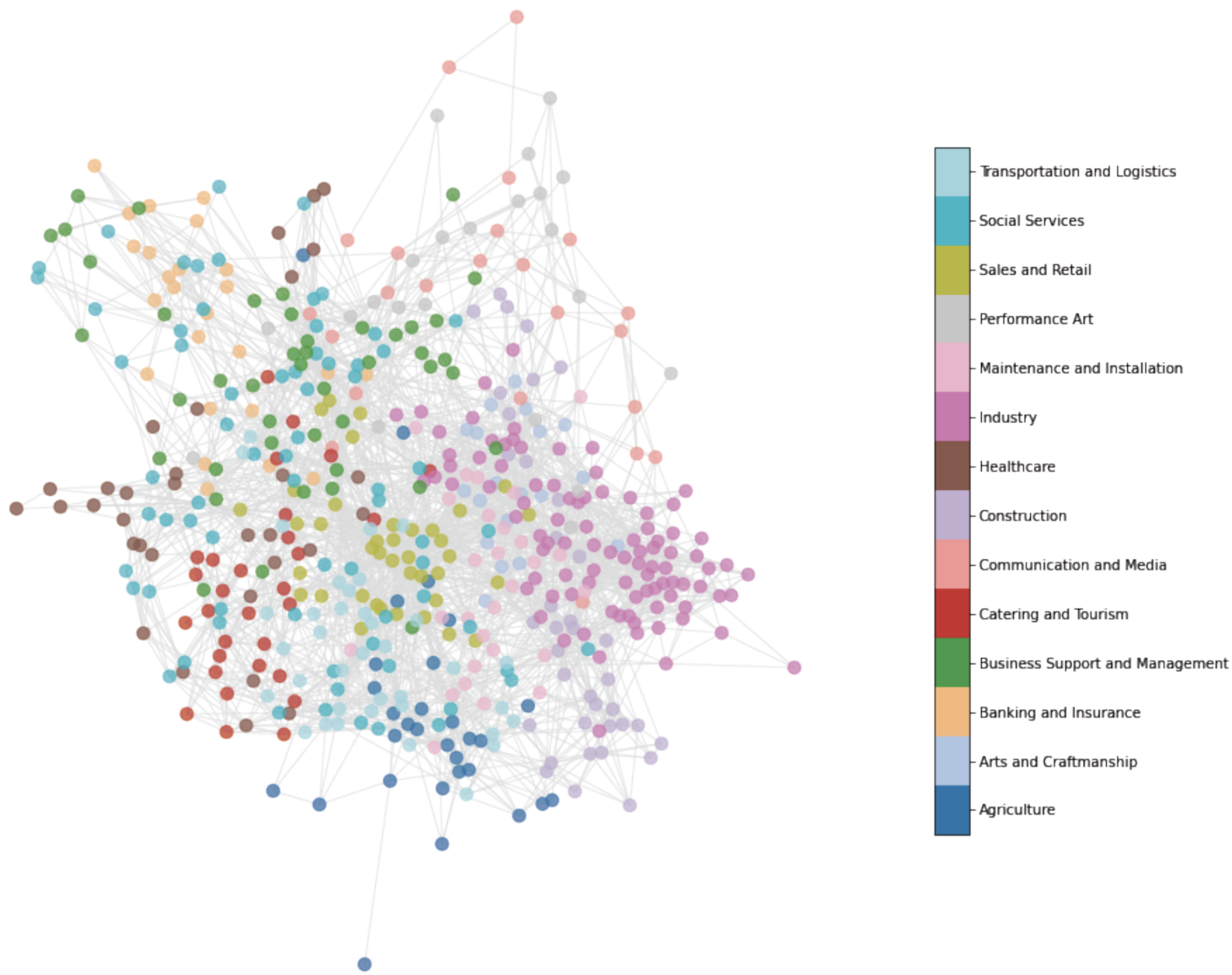

Figure 1: The occupational network

different families of occupations (1-digit level of aggregation) are represented with different colors. The arrangement of families of occupations across the network seems to make sense. The "Industry" occupations are clustered together in the east of the network, while "Social Services" and "Banking and Insurance" occupations are clustered together in the north-west ; in the center of the network are occupations from "Sales and Retail" or "Business Support and Management".[9] Note that the self-loops are not plotted for clarity reasons.

The occupational network also admits an alternative representation, through its associated adjacency matrix. Let $\boldsymbol{G} = (g_{ij})_{(i,j)\in\mathcal{N}^2}$ denotes the adjacency matrix of the occupational

[9]Let me note that many groups of occupations seem to overlap importantly with sectors, e.g. industry. In fact, a lot of occupations are specific to a certain industry but others, like accounting or sales, exist across all sectors.

network $\mathcal{G}$. The general coefficient of the adjacency matrix is defined as follows $g_{ij} = 1$ if occupations i and j are connected, and $g_{ij} = 0$ otherwise. Thanks to the simplifying assumptions, the adjacency matrix features two properties: (i) it is unweighted $g_{ij} \in \{0, 1\}$ and (ii) it is symmetric $g_{ij} = g_{ij}$ for any pair $(i, j) \in \mathcal{N}^2$. In what follows, I will also refer to the occupational network

2.3 Stylized facts

In this subsection, I document three stylized facts on the architecture of the occupational network.

Fact 1. *The occupational network is very sparse.*

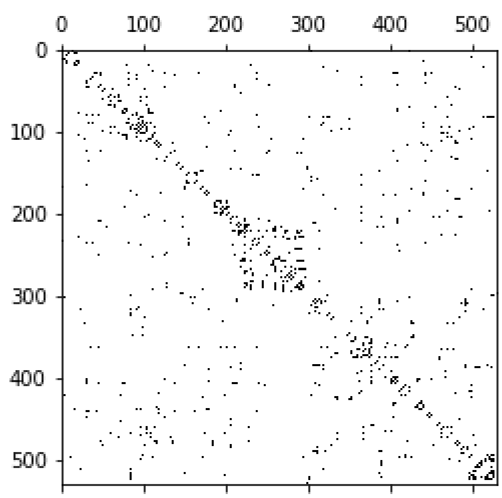

Figure 2: Plot of the adjacency matrix

A network is said sparse if it has few edges compared to the size of the network. In practice, it is measured by the edge density ratio, which is the ratio of the number of edges E to the maximum possible number of edges $E_{\max}$. The maximum possible number of edges (including self-loops) is given by the formula $E_{\max} = \frac{N(N-1)}{2} + N$, where N is the number of nodes. The number of edges in the occupational network is $E \approx 3500$ and the maximum possible number of edges in the occupational network is $E_{\max} \approx 140\,000$. This yields an edge density ratio of

$$\frac{E}{E_{\max}} \approx 2.5\%$$

This means that approximately only 2.5% of the maximum possible number of edges in the occupational network actually exist. This is very small. In economic terms, this entails that only a very small number of occupational transitions are possible. This suggests high frictions to occupational mobility.

The sparsity of the occupational network can also be readily observed by looking at the Figure 2, which plots the coefficients of the adjacency matrix (excluding self-loops). The unitary coefficients are represented as black dots, and nodes are clustered by groups of 1-digit occupations. The sparsity of black dots indicates that only a small fraction of coefficients of the adjacency matrix are non-zero.

Fact 2. *Occupations have heterogeneous numbers of connections.*

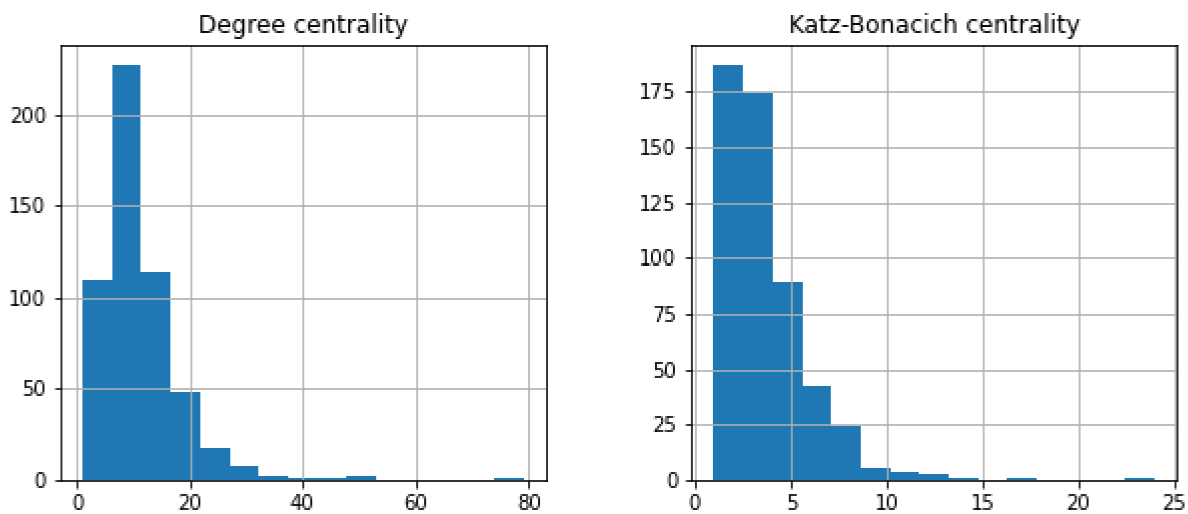

Figure 3: The distribution of degree and Katz-Bonacich centrality indices

The left hand side of Figure 3 plots the histogram of the the number of direct connections across occupations - also referred to as the occupations' degree. It shows there is large heterogeneity across occupations in terms of their degree. The degree distribution peaks around 10, meaning that the typical occupation has around 10 connections. Nevertheless, some occupations deviate importantly from the typical degree. The highest-degree occupation (sales representative) has 80 connections, which is almost ten times the typical degree. This points to the fact that there is important heterogeneity in the intensity of mobility frictions across occupations. Most occupations have few possible reallocation opportunities, but a few have many reallocation opportunities.

What about the number of indirect connections, i.e. the number of occupations at distance two or higher? The Katz-Bonacich centrality counts both the number of direct and indirect connections, where the number of connections at distance two or higher from the occupation are discounted proportionally to their distance. Let K_i the index of Katz-Bonacich centrality of occupation i in the occupational network. It is defined as follows

$$K_i = \alpha \sum_{k=0}^{+\infty} \beta^k (\boldsymbol{G}^k \mathbb{1})_i$$

where α and β are, respectively, a normalization and a discounting parameter. The term $(\boldsymbol{G}^k \mathbb{1})_i$ gives the number of neighbors at distance k from node i.[10] There exists an alternative, recursive formulation of Katz-Bonacich centrality, which I will use later : $K_i = \alpha + \beta \sum_{j \in \mathcal{N}} g_{ij} K_j$.

The right-hand side of Figure 3 plots the histogram of the occupations' Katz-Bonacich centrality index in the occupational network, with parameter values $\alpha = 1$ and $\beta = 0.05$. It shows there exists substantial heterogeneity across occupations in terms of Katz-Bonacich centrality too. This suggests too that occupations are very heterogeneously affected by occupational mobility frictions.

Fact 3. *There exist bottlenecks in the occupational network*

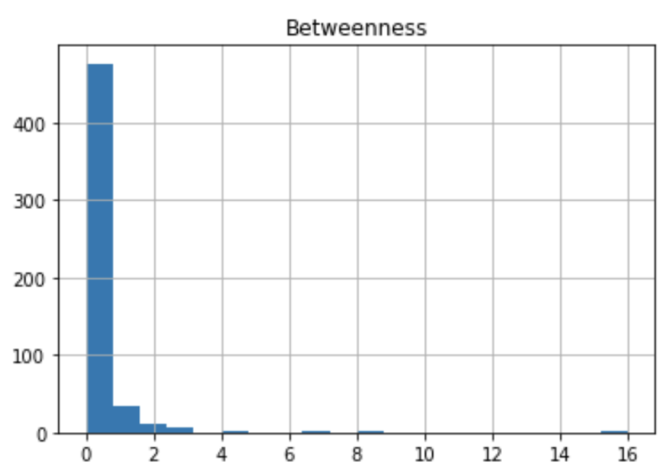

Figure 4: The distribution of betweenness centrality indices

A network has bottlenecks when a large fraction of paths between nodes must pass through a small subset of central nodes. These nodes are called bottleneck nodes. Intuitively, they

[10] To see why, first note that the vector $\boldsymbol{G}\mathbb{1}$ gives the number of direct connections, i.e. the number of neighbors at distance 1. Second, note that matrix $\boldsymbol{G}^2 = \boldsymbol{G}\boldsymbol{G}$ gives the number of connections at distance 2 between pairs of occupations, since $(\boldsymbol{G}^2)_{ij} = \sum_l g_{il} g_{lj}$ and $g_{il} g_{lj} = 1$ if and only if both the edges (i,l) and (l,j) exist. This can be easily generalized to the number of connections at distances higher than 2, using the relation $\boldsymbol{G}^k = \boldsymbol{G}^{k-1}\boldsymbol{G}$.

are the "weak points" of the network, where jamming can occur. In practice, the bottleneck nodes are identified as the nodes with high indices of betweenness centrality. Intuitively, betweenness centrality aims at capturing how much a node is "in-between" others.

Let B_i denote the betweenness centrality index of a node i, and let $N(k,l)$ and $N(k,i,l)$ denote, respectively, the number of shortest-path walks from k to l and the number of shortest-path walks from k to l passing through i. The betweenness centrality index of node i is defined as

$$B_i = \binom{N}{2}^{-1} \sum_{(k,l)\in\mathcal{N}^2} \frac{N(k,i,l)}{N(k,l)}$$

The intuition is the following. Consider an origin node k and a destination node l. The general term of the sum represents the fraction of shortest-path walks from k and to l which must pass through i. Then, the measure of betweenness centrality averages over all possible pairs of origin and destination nodes (k,l), where $\binom{N}{2}$ is the number of all possible pairs. Therefore, the betweenness centrality is the expected fraction of time that a node is between others, picking origin and destination nodes at random.[11]

Figure 4 plots the histogram of betweenness centrality indices in the occupational network. Most of the occupations have a very low index of betweenness centrality, less than 1%. This means that these occupations are on at most 1% of all reallocation paths between random origin and destination occupations. However, a small number of occupations have a very high level of betweenness centrality. The most betweenness central occupation (sales representative) has a betweenness centrality index of around 15%, i.e. more than ten times the typical value. This means that around 15% of reallocation paths between random origin and destination occupations must flow through this occupation. In the appendix, I provide the list of the 10 occupations with the highest index of betweenness centrality.

Figure 5 plots the occupational network with the size of nodes proportional to their index of betweenness centrality. Two lessons can be drawn from this figure: (i) bottleneck occupations mostly belong to the families of occupations "Business Support and Management" and "Sales and Retail", i.e. occupations in the middle of the skill distribution. (ii) bottleneck occupations are located in the center of the network, and in particular between the families of occupations "Industry" and "Social Services", i.e. routine and non-routine low-skill occupations. This means that reallocation dynamics from routine to non-routine low-skill occupations must pass through these middle-skill bottleneck occupations. However, the former have been hit quite bad by the job polarization process, and they might not let enough workers transit through them. In other words, job polarization has reinforced bottlenecks in

[11] Variants of the betweenness centrality measure also exist. For example, one could also pick origin and destination nodes in a non-random manner, or one could consider non-shortest path walks.

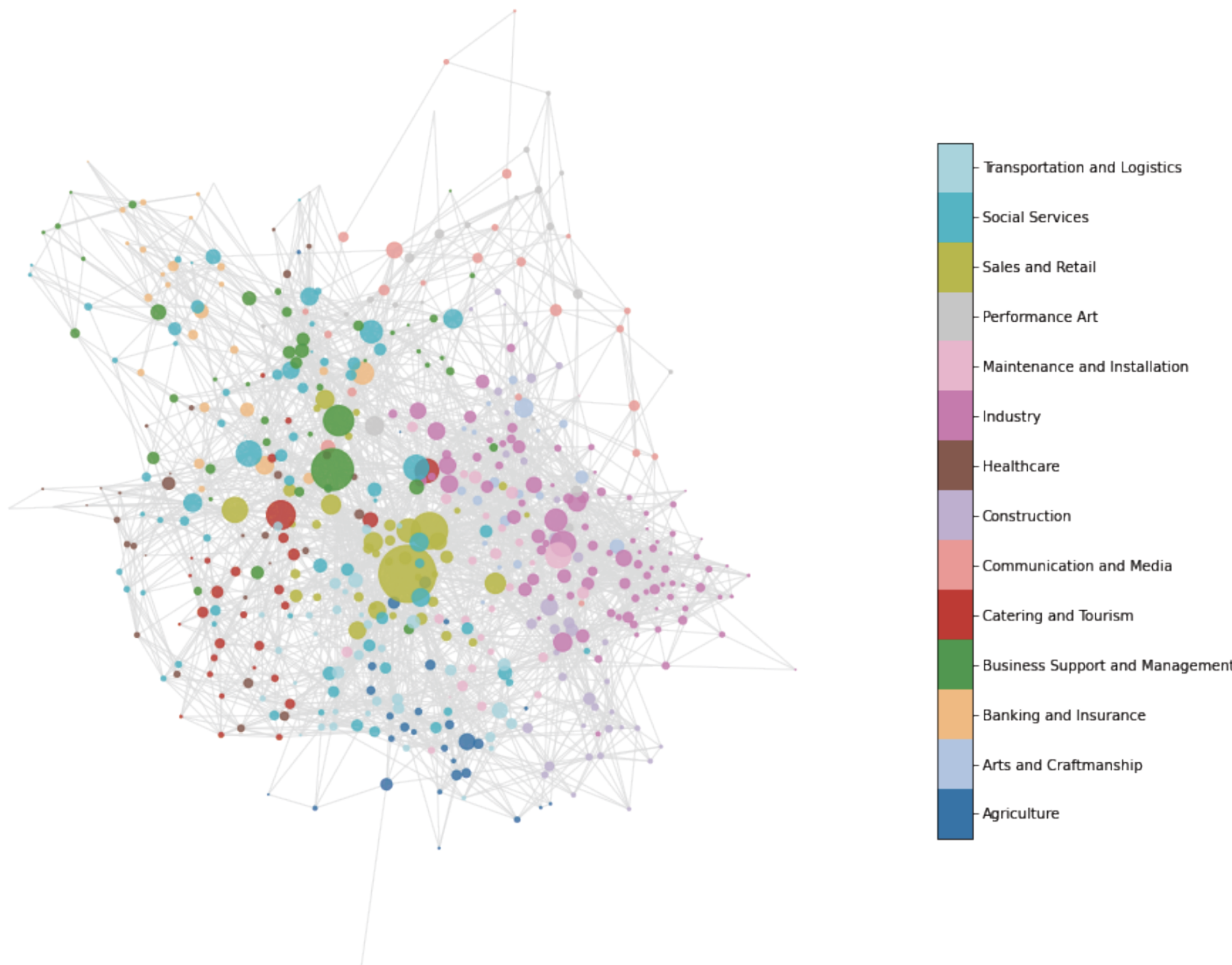

Figure 5: The bottleneck occupations in the occupational network

the occupational network, and therefore might have contributed to slow down the reallocation dynamics of workers after structural shocks.

To sum up, the bottleneck occupations play a key role for the reallocation dynamics. Intuitively, if it were difficult - for some exogenous reasons - for workers to transit through them, then the speed at which workers can reallocate between different parts of the network would be substantially affected. The theoretical model formalizes this intuition.

3 The model

In this section, I present a tractable model of frictional worker reallocation between occupations. The model builds on the textbook search & matching model of the labor market, in the spirit of Diamond (1982), Mortensen (1982) and Pissarides (1985). The key new assumption is that unemployed workers can search for jobs in adjacent occupations in the occupational network. This creates a process of worker reallocation, as workers move across occupations upon matching with a firm in an adjacent occupation.

3.1 Set up

Time is continuous, starts at zero and runs until infinity.

Occupational structure. There are N distinct occupations, indexed by $i \in \mathcal{N}$. Occupations are connected to each other by an occupational network, which controls the degree of occupational mobility frictions between pairs of occupations. The structure of the occupational network is given by the adjacency matrix $\boldsymbol{G} = (g_{ij})_{(i,j)\in\mathcal{N}^2}$ which is defined as follows: $g_{ij} = 1$ if professional transitions between occupations i and j are feasible, and $g_{ij} = 0$ otherwise. This means that the occupational network is assumed to be both unweighted $g_{i,j} \in \{0,1\}$ and undirected $g_{ij} = g_{ji}$. As is natural, occupations are connected to themselves: $g_{ii} = 1$. Moreover, I assume without loss of generality that the graph is connected, i.e. there exists a path from any occupation i to any other occupation j.[12]

Workers. There is a unit mass of infinitely-lived workers who are risk-neutral, consume all their income and discount time at the rate r. Workers can be in two states: either employed or unemployed. Workers receive a wage while employed, and unemployment benefits while unemployed. They can only search for jobs when they are unemployed.

Moreover, workers are spread across the economy's different occupations. Let $e_i(t)$ denote the number of workers employed in occupation i at date t and $u_i(t)$ the number of unemployed workers whose last employment was in occupation i at date t. The distribution of employed and unemployed workers across occupations at date t is denoted $\boldsymbol{x}(t) = ((e_i(t))_{i\in\mathcal{N}}, (u_i(t))_{i\in\mathcal{N}})'$. The initial distribution of workers is taken as given $x(0) = x_0$ and can be anything. The employed and unemployed fractions of the population are respectively given by $e(t) \equiv \sum_{i\in\mathcal{N}} e_i(t)$ and $u(t) \equiv \sum_{i\in\mathcal{N}} u_i(t)$ and verify $e(t) + u(t) = 1$ at all dates.

Firms. There exists an unbounded number of potential firms in each occupation. Firms pay a flow cost c_i to post a job vacancy. When a firm matches with a worker, the vacancy is filled and a firm-worker pair starts producing with productivity y_i. The output of the match

[12]This assumption is made without loss of generality because if the graph is disconnected, then it can be partitioned into a number of distinct connected sub-components and I can focus on each of them separately.

is homogeneous across occupations and its price is normalized to one.[13] The firm pays the worker a wage $w_i(t)$ and collects profits $y_i - w_i(t)$. The firm-worker match gets randomly destroyed at the exogenous Poisson rate δ_i.

Labor markets. Each occupation corresponds to a different labor market. Search frictions prevent unemployed workers and recruiting firms to match immediately. Firms only search for workers in the occupation where the vacancy is posted. Let $v_i(t)$ denote the number of vacancies posted in occupation i at date t. Unemployed workers search for recruiting firms in all neighboring occupations. Let define $s_i(t)$ the number of unemployed workers searching in occupation i at date t

$$s_i(t) \equiv \sum_{j \in \mathcal{N}} g_{ij} u_j(t)$$

which is the sum of unemployed workers searching from adjacent occupations. Importantly, I assume that unemployed workers' total search effort is not divided across adjacent occupations. The interpretation is that unemployed workers can costlessly send CVs around, but only jobs in occupations with similar enough skills will consider the application. This assumption aims at capturing the fact that workers in occupations with more connections find, everything else being equal, jobs faster.[14]

The matching technology $\mathcal{M}(\cdot)$ is the same across occupations. The number of matches in occupation i is given by a Cobb-Douglas matching function $\mathcal{M}(s_i, v_i) = \zeta s_i^{\alpha} v_i^{1-\alpha}$. The tightness ratio in occupation i is defined as $\theta_i(t) = v_i(t)/s_i(t)$. Firms and workers meet randomly. Hence, each unemployment worker searching in occupation i faces a flow probability $p_i(t) = \mathcal{M}(s_i(t), v_i(t))/s_i(t) = \zeta \theta_i(t)^{1-\alpha}$ of matching with a firm in occupation i. Note that, thanks to the assumption of random search, the job finding rate in occupation i $p_i(t)$ is the same for all unemployed workers searching from neighboring occupations.[15] Similarly, each vacancy in occupation i faces a flow probability $q_i(t) = \mathcal{M}(s_i(t), v_i(t))/v_i(t) = \zeta \theta_i(t)^{-\alpha}$ of matching with a worker.

[13] Another interpretation is that each occupation produces one unit of occupation-specific output, which is sold on international markets at the exogenous price y_i. The assumption that the price is exogenous is key to ensure tractability of the worker distribution dynamics, and without it there are general equilibrium feedback effects between the price and the distribution of workers across occupation.

[14] An alternative assumption is that each worker is endowed with one unit of search, that she can split between all adjacent occupations. I consider this extension in the appendix. The reality is probably in-between these two assumptions, such that the workers' search search effort scales sublinearly with the number of neighbors.

[15] This is a strong assumption, which likely does not hold in the data: job finding rates probably vary across different origin occupations. Here, it stems partially from the assumption that the occupational network is non-weighted, which implies all adjacent occupations face the *same* level of occupational mobility frictions, and that unemployed workers cannot adjust choose their level search effort across different occupations.

Flow value of unemployment. Unemployed workers in occupation i receive unemployment benefits $b_i = by_i$, with $b \in (0,1)$. This assumption captures the fact that unemployment benefits are proportional to the wage in occupation i, which scales with productivity y_i in equilibrium.[16]

Wage determination. Workers and firms set wage according to a generalized Nash bargaining rule, with worker bargaining strength ϕ_i. The worker bargaining strength controls exogenously how the surplus is split between the worker and the firm. It is not to be confused with the agents' outside options, which is endogenous, and corresponds to the minimal outcome agents are willing to accept.

3.2 Value functions

In what follows, the economy is not in steady-state and all endogenous variable are a function of time t. I adopt the convention that $\dot{x} = dx/dt$.

Unemployment and employment. Let $E_i(t)$ be the value of being employed in occupation i at date t and $U_i(t)$ be the value of being unemployed with last employment in occupation i at date t. The value of being employed in occupation i $E_i(t)$ satisfies the Bellman equation

$$rE_i(t) = w_i(t) + \delta_i[U_i(t) - E_i(t)] + \dot{E}_i(t) \tag{1}$$

The interpretation is as follows. The flow value of being unemployed in occupation i (on the left-hand side) must be equal to a sum of three terms. The first term is the instantaneous flow payoff, here the utility that workers derive from consuming their wage w_i. The second term reflects the stochastic change in expected value coming from job destruction: at rate δ_i, workers in occupation i lose their job and suffer the loss $U_i - E_i < 0$.[17] The third term captures deterministic time variations along the transition towards the steady-state $\dot{E}_i(t)$.

The value of being unemployed from occupation i $U_i(t)$ satisfies the Bellman equation

$$rU_i(t) = b_i + \underbrace{\sum_{j \in \mathcal{N}} g_{ij} p_j(t) \max\{E_j(t) - U_i(t), 0\} + \dot{U}_i(t)}_{\text{Search in adjacent occupations}} \tag{2}$$

[16]Bilal (2021) makes a similar assumption, for tractability reasons.

[17]For simplicity, I assume that this condition always holds. If $E_i < U_i$ then employed worker would always prefer being unemployed rather than employed in occupation i. In the long run, the occupation would disappear.

The unemployed workers in occupation i consume their unemployment benefits b_i. They search for jobs in all adjacent occupations and at rate $p_j(t)$, they match with a firm in occupation j. They accept the job if the net expected gain from becoming employed is positive $E_j - U_i > 0$, otherwise they refuse and keep searching for alternative job offers. For the sake of simplicity, I assume unemployed workers always prefer to be employed, that is $E_j(t) - U_i(t) > 0$ for all i, j. The value function also varies over time along the transition towards the steady-state $\dot{U}_i(t)$.

Recruiting and producing firms. Let $J_i(t)$ be the value of a job filled for a firm in occupation i at date t and $V_i(t)$ be the value of an unfilled vacancy in occupation i at date t. The value of a job filled in occupation i $J_i(t)$ satisfies the Bellman equation

$$rJ_i(t) = y_i - w_i(t) + \delta_i[V_i(t) - J_i(t)] + \dot{J}_i(t) \tag{3}$$

The firm collects the flow profits $y_i - w_i(t)$. At rate δ_i, the firm-worker match gets exogenously destroyed and the firm suffers the loss $V_i(t) - J_i(t) < 0$. The value function can also vary over time along the transition dynamics towards the steady-state.

The value of an unfilled vacancy in occupation i $V_i(t)$ satisfies the Bellman equation

$$rV_i(t) = -c_i + q_i(t)[J_i(t) - V_i(t)] + \dot{V}_i(t) \tag{4}$$

The recruiting firm pays a flow cost c_i to keep its job vacancy posted. At rate q_i, the vacancy gets filled and the recruiting firm experiences the gain $J_i(t) - V_i(t) > 0$. The value function can also vary over time along the transition dynamics towards the steady-state.

Nash bargaining. When a firm and a worker meet, they bargain over the wage. Then, the wage is constantly renegotiated over the duration of the match. At any date t, the negotiated wage in occupation i $w_i(t)$ maximizes the generalized Nash product

$$\max_{w_i(t)} (E_i(t) - U_i(t))^{\phi_i} (J_i(t) - V_i(t))^{1-\phi_i}$$

Importantly, I make a simplifying assumption regarding the workers' outside option. I assume that the workers' outside option is the value to keep searching for jobs from occupation i onward, independently of the worker's last occupation. This way, all workers applying for vacancies in occupation i share the same outside options and earn the same wage. This assumption helps with analytical tractability.[18]

[18] Otherwise, wages would be a function of current and last occupations w_{ij}. As a consequence, the value of being employed for the worker and the value of a filled vacancy for the firm would depend on the worker's

The interior solution to the maximization program above writes

$$(1-\phi_i)(E_i(t)-U_i(t)) = \phi_i(J_i(t)-V_i(t)) \tag{5}$$

Free entry. Potential entrants can freely post vacancies in each occupation. This drives down the value of posting a vacancy to zero in each occupation and at all times

$$V_i(t) = 0 \tag{6}$$

3.3 Laws of motion

Variations in employment. The mass of employed workers in occupation i at date t $e_i(t)$ evolves over time according to

$$\dot{e}_i(t) = p_i(t)\sum_{j\in\mathcal{N}} g_{ij}u_j(t) - \delta_i e_i(t) \tag{7}$$

The variation in employment in occupation i is given by the difference between new hires and job destructions in occupation i. The number of new hires is the product of the job finding rate in occupation i $p_i(t)$ and the mass of unemployed workers searching in occupation i from neighboring occupations $\sum_{j\in\mathcal{N}} g_{ij}u_j(t)$. The number of the job destructions in occupation i is the product the job destruction rate δ_i and the mass of employed workers in occupation i $e_i(t)$.

Variations in unemployment. The mass of unemployed workers in occupation i at date t $u_i(t)$ evolves over time according to

$$\dot{u}_i(t) = \delta_i e_i(t) - \underbrace{\left(\sum_{j\in\mathcal{N}} g_{ij}p_j(t)\right)}_{\equiv\sigma_i(t)} u_i(t) \tag{8}$$

The variation in unemployment in occupation i is given by the difference between job destructions in occupation i and hires in adjacent occupations by unemployed workers searching from occupation i. The former is the product of unemployment outflow rate in occupation i, i.e. the sum of job finding rates in the adjacent occupations, and the mass of unemployed workers searching from occupation i.

current and last occupation, that is E_{ij} and J_{ij}. If the recruiting firm is able to discriminate between job seekers based on their last occupations, this means that tightness ratios also depend on the worker's last occupations, that is θ_{ij}. While arguably a realistic feature of the economy, as wages do indeed differ within occupations, I abstract from this complexity for now.

In what follows, let $\sigma_i(t) \equiv \sum_{j \in \mathcal{N}} g_{ij} p_j(t)$ denote the unemployment outflow rate in occupation i. Importantly, the unemployment outflow rate in i is not to be confused with the job finding rate in occupation i. The latter measures the probability of finding a job *in* occupation i, while the former measures the probability of finding a job *from* occupation i, by searching in adjacent occupations too. In standard search & matching models, both variables coincide, since there exists only a single homogeneous labor market, but this is not the case here.

Matrix form. The system of laws of motion can also be written in matrix form

$$\dot{\boldsymbol{x}}(t) = -\boldsymbol{Q}(t)\boldsymbol{x}(t) \quad \text{with} \quad \boldsymbol{Q}(t) = \begin{pmatrix} \text{diag}(\boldsymbol{\delta}) & -\text{diag}(\boldsymbol{p})(t)\boldsymbol{G} \\ -\text{diag}(\boldsymbol{\delta}) & \text{diag}(\boldsymbol{G}\boldsymbol{p}(t)) \end{pmatrix} \tag{9}$$

where $\boldsymbol{x}(t) = ((e_i(t))_{i \in \mathcal{N}}, (u_i(t))_{i \in \mathcal{N}})'$ is the distribution of employed and unemployed workers across occupations and $\text{diag}(\boldsymbol{p})(t)$ and $\text{diag}(\boldsymbol{\delta})$ are the diagonal matrices formed, respectively, with the vectors $\boldsymbol{p}(t) = (p_i(t))_{i \in \mathcal{N}}$ and $\boldsymbol{\delta} = (\delta_i)_{i \in \mathcal{N}}$.

In a slight abuse of notation, I refer to the matrix $\boldsymbol{Q}(t)$ as the transition rate matrix.[19] Note that the transition rate matrix is asymmetric, despite of the symmetry of the occupational network. The reason is that unemployment to employment transition rates (job finding rates) differ from employment to unemployment transition rates (destruction rates). Moreover, there is additional asymmetry coming from the heterogeneity of job finding rates across occupations.

Illustrative example. To fix ideas, let me present an illustrative example. Consider a stylized economy with three occupations connected by a line network (see figure 5 below). This network structure is the simplest network structure featuring heterogeneity in nodes' network centrality .[20] Indeed, the mid-point occupation 2 is clearly more central than the end-point occupations 1 and 3. This allows me to investigate in a simple context the role played by central bottleneck occupations in shaping worker reallocation speed.

For concreteness, let the nodes 1, 2 and 3 correspond to, respectively, textile worker, retail worker, and care worker occupations. A direct occupational mobility from textile worker to care worker is not possible, because the textile worker would lack the social skills required. However, a two-step occupational mobility is possible, since the textile worker could acquire these skills by working temporarily in retail before moving to the care worker occupation.

[19] This is true up to a sign transformation. The negative sign is just to ensure that the eigenvalues of the transition rate matrix are always positive.

[20] A two-node line network does not feature heterogeneity in network centrality, since both nodes have the same number of connections.

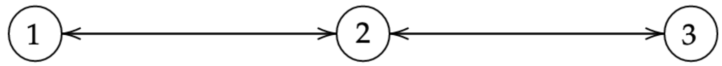

Figure 6: A Line Occupational Network with 3 Nodes

In this example, the transition rate matrix $\boldsymbol{Q}(t)$ writes

$$\boldsymbol{Q}(\boldsymbol{t}) = \begin{pmatrix} \delta_1 & 0 & 0 & -p_1(t) & -p_1(t) & 0 \\ 0 & \delta_2 & 0 & -p_2(t) & -p_2(t) & -p_2(t) \\ 0 & 0 & \delta_3 & 0 & -p_3(t) & -p_3(t) \\ -\delta_1 & 0 & 0 & p_1(t)+p_2(t) & 0 & 0 \\ 0 & -\delta_2 & 0 & 0 & p_1(t)+p_2(t)+p_3(t) & 0 \\ 0 & 0 & -\delta_3 & 0 & 0 & p_2(t)+p_3(t) \end{pmatrix}$$

3.4 Definition of the equilibrium

Definition of the equilibrium. *The equilibrium is a collection of time-varying value functions* $\{E_i(t)\}_{i\in\mathcal{N}}, \{U_i(t)\}_{i\in\mathcal{N}}$, $\{J_i(t)\}_{i\in\mathcal{N}}$, $\{V_i(t)\}_{i\in\mathcal{N}}$, *time-varying wages* $\{w_i(t)\}_{i\in\mathcal{N}}$, *time-varying labor market tightness ratios* $(\{\theta_i(t)\}_{i\in\mathcal{N}}$ *and a time-varying distribution of workers* $x(t)$, *such that:*

i) The value functions satisfy the Bellman equations (1)-(4)

ii) At any date t, the wages satisfy the Nash bargaining condition (5)

iii) At any date t, the free entry condition holds (6)

iv) The distribution of workers satisfies the matrix law of motion (9)

Block recursive equilibrium. The equilibrium satisfies the property of block recursivity, as defined in Menzio and Shi (2010) and Menzio and Shi (2011). An equilibrium is block recursive if the agents' value and policy functions (here, the wages and the tightness ratios) do not depend on the worker distribution. This is the case here, because the worker distribution does not show up in equations (1)-(6).

The property of block recursivity is key to ensure the tractability of models with heterogeneous agents. Here, it means that the equilibrium tightness ratios and wages can be solved independently of the distribution of workers across occupations. This gives me my solution strategy. First, I solve for the equilibrium wages and tightness ratios, for any distribution of workers across occupations and states. Second, I solve for the dynamics of the distribution of workers along the transition, given the equilibrium wages and tightness ratios.

Intuitively, the property of block recursivity is obtained thanks to the simplifying assumption regarding the workers' outside options: namely, unemployed workers coming from different occupations share the same outside option when bargaining with a firm. This way, all workers earn the same wage within an occupation, independently of what was their last occupations. Therefore the value of posting a vacancy - which is a function of the expected wage - does not depend on the composition of the pool of workers searching in the occupation.

4 Characterization of the equilibrium

In this section, I provide an analytical characterization of the equilibrium. In particular, I leverage the tractability of the search and matching framework to elicit how the structure of the occupational network shapes worker reallocation dynamics.

Towards solving for my main result on the transition dynamics, I characterize the equilibrium distributions of wages and tightness ratios. These intermediary results show that the scope of the network approach goes beyond the question of transition dynamics, and can address topics related to labor market power and wage inequality.

4.1 Characterization of the equilibrium wage and tightness ratios

Transition dynamics. In theory, the wages and the tightness ratios vary along the transition towards the steady-state. Indeed, the equations (1)-(6) define a system of differential equations, giving rise to potentially complex time dynamics. In practice, however, the equilibrium wages and tightness ratios are constant along the transition.

Proposition 1. (Steadiness of wages and tightness ratios)
The equilibrium wages and the tightness ratios have unstable dynamics. The only non-explosive solution is the trajectory with constant steady-state value. For all $t > 0$,

$$w_i(t) = w_{i\infty} \quad \text{and} \quad \theta_i(t) = \theta_{i\infty} \tag{10}$$

where $w_{i\infty}$ is the steady-state value of $w_i(t)$ and $\theta_{i\infty}$ is the steady-state value of $\theta_i(t)$.

Proof. See appendix.

This proposition generalizes a standard result in the canonical search and matching model.[21] It means that wages and tightness ratios are "jump variables": that is, they adjust immediately to their steady-state levels after a change in the parameters of the economy. The same holds for the value functions. The reason is that the value and policy functions are forward-looking, and therefore dynamically unstable, so that the only non-explosive solution is the steady-state value. By contrast, the distribution of workers across occupations and employment states is slow-moving and converges only gradually towards its steady-state value. In the rest of the paper, I drop the steady-state subscript on the equilibrium value and policy functions.

This result makes the study of transition dynamics very tractable. It implies that the distribution of transition probabilities $\boldsymbol{p}(t) = (p_i(t))_{i\in\mathcal{N}}$ is constant along the transition. Therefore, the transition rate matrix $\boldsymbol{Q}(t)$, which governs the reallocation of workers between occupations, is constant along the transition too. The system of law of motions (9) becomes a system of first-order ordinary differential equations with constant coefficients.

Equilibrium wages. The wages are determined by the Nash bargaining condition, which can be rewritten in terms of flow variables as follows

$$w_i = \phi_i y_i + (1-\phi_i) rU_i \tag{11}$$

It states that the wage is a weighted average of the firm productivity y_i and of the unemployed worker's flow value of being unemployed rU_i, where the weight is controlled by the worker bargaining strength ϕ_i. The firm and the workers strike a balance between the highest wage acceptable by the firm (the firm productivity) and lowest acceptable wage by the worker (the flow value of unemployment), i.e. the reservation wage. In what follows, I refer to the flow value of unemployment as the reservation wage and denote it $\bar{w}_i \equiv rU_i$.

The reservation wage is an endogenous object, which depends on the value of the alternative job opportunities existing on the labor markets. The novelty here is that the equilibrium reservation wage is shaped by the occupational network, since unemployed workers can find alternative jobs in adjacent occupations too. Intuitively, occupations with a high number of connections should have higher reservation wages, because they have access to more alternative job opportunities. The next proposition solves for the distribution of equilibrium reservation wages and clarify its relation with the structure of the occupational network.

[21] See for example Pissarides (2000) for the case with homogeneous workers.

Proposition 2. (Equilibrium reservation wage)
Let define the normalized matrix of mobility flows[22] $\boldsymbol{M}(\boldsymbol{\theta}) = \{\mu_{ij}(\boldsymbol{\theta})\}_{(ij)\in\mathcal{N}^2}$ *as follows*

$$\mu_{ij}(\boldsymbol{\theta}) = \frac{g_{ij}p(\theta_j)}{r + \sum_{l\in\mathcal{N}} g_{il}p(\theta_l)} \tag{12}$$

There exists a unique vector of equilibrium reservation wage $\bar{\boldsymbol{w}} = (\bar{w}_i)_{i\in\mathcal{N}}$*, and it solves*

$$\bar{w}_i = \Big[1 - \sum_{j\in\mathcal{N}} \mu_{ij}\Big] b_i + \sum_{j\in\mathcal{N}} \mu_{ij}(\boldsymbol{\theta})\Big[(1-\beta_j)y_j + \beta_j \bar{w}_j\Big] \tag{13}$$

where $\beta_j \equiv \frac{r(1-\phi_j)+\delta_j}{r+\delta_j}$ *is a damping parameter. In matrix form, the solution writes*

$$\bar{\boldsymbol{w}} = \Big[\boldsymbol{I} - \text{diag}(\boldsymbol{\beta})\boldsymbol{M}(\boldsymbol{\theta})\Big]^{-1}\Big[b(\boldsymbol{I} - \text{diag}(\boldsymbol{M}(\boldsymbol{\theta})\mathbb{1})) + (\boldsymbol{I} - \text{diag}(\boldsymbol{\beta}))\boldsymbol{M}(\boldsymbol{\theta})\Big]\boldsymbol{y} \tag{14}$$

Proof. See appendix.

Equation (13) defines a recursion between reservation wages in the network: reservation wages are a function of reservation wages in adjacent occupations. The reason is straightforward. The reservation wage in an occupation depends on the attainable wages in adjacent occupations, which are themselves are function of the reservation wage due to Nash bargaining. The strength of the relation between adjacent reservation wages is controlled by the distribution of job finding rates, through the coefficients of the normalized matrix of mobility flows, and the distribution of damping parameters.

This result clarifies how structure of the occupational network affects the distribution of reservation wages in the economy. The reservation wage in an occupation is equal to its index of Katz-Bonacich centrality in the network defined by the normalized matrix of mobility flows $M(\theta)$.[23] Indeed, Katz-Bonacich centrality defines a node's centrality in a similar recursive manner: a node is central if it is connected to other central nodes. Therefore, the indices of Katz-Bonacich centrality and the reservation wages solve the same set of equations and are equal. This is consistent with the intuition that central occupations have high bargaining power.

This result has implications for the distribution of equilibrium wages too. The equilibrium wages can be easily recovered from the equilibrium reservation wages, using the Nash bargaining condition (11). Therefore, the equilibrium wages inherit the properties of the reservation wages. In fact, up to a linear transformation, the equilibrium wage is also given

[22]This is a slight abuse of notation, since the rows of the normalized matrix of mobility flows do not sum to unity because of the discount rate.

[23]I consider here a generalized version of the Katz-Bonacich centrality, with heterogeneous damping parameters and heterogeneous exogenous constants.

by in the index of Katz-Bonacich in the network defined by the normalized matrix of mobility flows. This means that, everything else being equal, central occupations have higher wages. It sheds light on a new determinant of between-occupation wage inequality.

Finally, this result also matters for the dynamics of worker reallocation. Anticipating a little bit, high wages discourages job vacancy posting by firms in central occupations. As a consequence, this proposition predicts that central occupations have, everything else being equal, lower job finding rates. This contributes to slow down the worker reallocation dynamics. Indeed, it makes it more difficult for workers to transit through central occupations, which connects different parts of the occupational network.

Equilibrium tightness ratios. The equilibrium tightness ratio solves the free entry condition (6), which can be rewritten as follows

$$\frac{c_i}{q(\theta_i)} = J_i \tag{15}$$

This equation states that the expected cost of posting a vacancy must equalize the expected benefit of posting a vacancy. The expected cost of posting a vacancy amounts to paying c_i per unit of time for an expected time $1/q(\theta_i)$, while the expected benefit of posting a vacancy is the value of a filled vacancy J_i.

The benefit of posting a vacancy is endogenous, and equal to the present value of future expected profits. The novelty here is that wages - a key determinant of firm profits - are a function of the network structure, through the bargaining channel explained above. Therefore, firm vacancy posting will also depend of network structure.

Proposition 3. (Equilibrium tightness ratios)
The distribution of equilibrium tightness ratios θ solves the following multi-dimensional fixed-point equation

$$\boldsymbol{\theta} = \Gamma(\boldsymbol{\theta}) \quad \text{where} \quad \Gamma(\boldsymbol{\theta}) = \Big[\text{diag}(\boldsymbol{\eta})\big(\boldsymbol{y} - \bar{\boldsymbol{w}}(\boldsymbol{\theta})\big)\Big]^{\circ\frac{1}{\alpha}} \tag{16}$$

with diag($\boldsymbol{\eta}$) the diagonal matrix with coefficients $\eta_i \equiv \frac{\zeta(1-\phi_i)}{c_i(r+\delta_i)}$ and the power $\circ\frac{1}{\alpha}$ being applied element-wise.

Proof. See appendix.

The distribution of equilibrium tightness ratios solves a complex multidimensional fixed-point problem. The intuition is the following. The expected benefit of posting a vacancy depends on the wage, which is a function of the tightness ratios in adjacent occupations, through the bargaining channel emphasized above. Indeed, the tightness ratios in adjacent occupations affect the probability to find jobs in those occupations, and therefore the reservation wage of the workers. Thanks to the bargaining channel, the equilibrium tightness ratios are therefore interconnected in the occupational network.

Due to the strong non-linearity of the recursion, there does not exist a closed-form solution. Therefore, I cannot derive a clear analytical connection between the distribution of tightness ratios and the structure of the occupational network, as with the distribution of equilibrium wages. Thanks to the analytical characterization of the fixed-point problem, however, the equilibrium tightness ratios can be solved easily and quickly with standard software fixed-point routines.

The interconnectedness of tightness ratios implies that changes in tightness ratios in some occupations have spill-over effects on tightness ratios in other occupations. The spill-over effects can be either positive or negative. Comparative statics reveal there are two effects at play. On the one hand, a higher tightness ratio in an occupation raises the *level* of the unemployment outflow rates in the adjacent occupations, which has a negative effect on tightness ratios in adjacent occupations. Indeed, this increases the workers' bargaining power in these occupations, pushing up the negotiated wage, and hence discouraging vacancy posting by firms.

On the other hand, a higher tightness ratio in an occupation changes the *composition* of transition probabilities across adjacent occupations. This has either a positive or a negative effect on tightness ratios in adjacent occupations, depending on whether the occupation experiencing the increase in tightness ratio has high or low wage. If the increase in tightness ratio is in an occupation with high wage, then the future expected wage of unemployed workers goes up, and hence their bargaining power and negotiated wage go up too. This discourages vacancy posting by firms.

An increase in the tightness ratios of central occupations creates negative spillovers for adjacent occupations. Indeed, proposition 2 implies that central occupations have higher wages, everything else being equal. Therefore, both the *level* and *composition* effects are negative. This means that an increase in the tightness ratio of a central occupation has a negative effect on tightness ratios in adjacent occupations.

This result matters for the worker reallocation dynamics. It implies that, everything else being equal, occupations with low tightness ratios will tend to have occupations with high tightness ratios close to them. As a result, workers from depressed labor markets - with low tightness ratios - can easily reallocate to dynamics labor markets - with high tightness ratios, since they will tend to be located close to each other. This contributes to speed up the reallocation process.

4.2 Characterization of the worker distribution

In this subsection, I characterize how the structure of the occupational network shapes the worker distribution, both at the steady-state and during the transition dynamics.

Steady-state worker distribution Let $\boldsymbol{x}_\infty$ denote the steady-state worker distribution. It solves

$$0 = -\boldsymbol{Q}\boldsymbol{x}_\infty \tag{17}$$

In the appendix, I show that the worker distribution always admits a steady-state distribution, which is unique and stable. The key intuition is that the steady-state worker distribution is, up to a normalizing constant, the eigenvector associated to the zero eigenvalue. The unicity of the steady-state means that the zero eigenvalue has multiplicity one, while the stability means that all other eigenvalues have positive real parts.[24]

I go further and give a full analytical characterization of the steady-state worker distribution, in terms of the transition probabilities. My main result is a closed-form expressions for the steady-state aggregate unemployment rate.

Proposition 4. (Steady-state aggregate unemployment rate)
Let define the weighted average destruction rates $\overline{\delta}$ and the weighted average unemployment outflow rates $\overline{\sigma}$ as follows

$$\overline{\delta} \equiv \sum_{i\in\mathcal{N}} \delta_i \cdot \frac{e_{i\infty}}{e_\infty} \quad \text{and} \quad \overline{\sigma} \equiv \sum_{i\in\mathcal{N}} \sigma_i \cdot \frac{u_{i\infty}}{u_\infty} \tag{18}$$

with the steady-state employment and unemployment shares across occupations given by

$$\frac{e_{i\infty}}{e_\infty} = \frac{p_i\sigma_i/\delta_i}{\sum_{j\in\mathcal{N}} p_j\sigma_j/\delta_j} \quad \text{and} \quad \frac{u_{i\infty}}{u_\infty} = \frac{p_i}{\sum_{l\in\mathcal{N}} p_l} \tag{19}$$

Then, the steady-state aggregate unemployment rate is given by

$$u_\infty = \frac{\overline{\delta}}{\overline{\delta} + \overline{\sigma}} \tag{20}$$

Proof. See appendix.

The closed-form expression of the aggregate unemployment rate is very similar to its counterpart in the standard search and matching model. In the standard model, the steady-state unemployment rate is equal to $u_\infty = \frac{\delta}{\delta+\sigma}$ where δ is the homogeneous job destruction rate and σ is the homogeneous unemployment outflow rate. Here, the homogeneous transition

[24] In fact, numerical simulations show that the eigenvalues are always real.

rates are replaced with their economy-wide weighted averages counterparts.[25] At first-sight, it seems that despite the complexity of the network architecture, the steady-state aggregate unemployment rate only depends on the average transition probabilities.

However, this intuition abstracts from the role played by the network structure in shaping the average transition probabilities, through the employment and unemployment shares. In fact, the next corollary shows that two network statistics play a key role in shaping the level of the steady-state aggregate unemployment rate.

Corollary. (Network structure and the steady-state aggregate unemployment rate)
Let define the covariance between the degree and the job finding rate $Cov(d_i, p_i)$ and the covariance between job finding rates in adjacent occupations $Cov_{i\sim j}(p_i, p_j)$ as follows

$$\text{Cov}(d_i, p_i) \equiv \frac{1}{N}\sum_{i\in\mathcal{N}}(d_i - \overline{d})(p_i - \overline{p}) \quad \text{and} \quad \text{Cov}_{i\sim j}(p_i, p_j) \equiv \frac{1}{2E}\sum_{(i,j)\in\mathcal{N}^2} g_{ij}(p_i - \overline{p})(p_j - \overline{p}) \tag{21}$$

where $\overline{d} \equiv \frac{1}{N}\sum_{i\in\mathcal{N}} d_i$ and $\overline{p} \equiv \frac{1}{N}\sum_{i\in\mathcal{N}} p_i$ are respectively the unweighted average degree and job finding rate.

Assume that destruction rates are homogeneous: that is, $\delta_i = \delta$ for all i. Then the steady-state aggregate unemployment rate can be rewritten as follows:

$$u_\infty = \frac{\delta}{\delta + \overline{d}\,\overline{p} + 2\text{Cov}(d_i, p_i) + \frac{\overline{d}}{\overline{p}}\text{Cov}_{i\sim j}(p_i, p_j)} \tag{22}$$

Proof. See appendix.

The first moment is an inverse measure of network bottlenecks: low values indicate that high-degree, well-connected nodes have low job finding rates, hence implying that few workers can transit through these central occupations. The second moment is a measure of assortativity of labor market conditions in the network: high values indicate that labor market conditions are very correlated across adjacent occupations in the occupational network, hence implying that occupations with high (or low) job finding rates will be clustered together in the network.

This result has key policy implications. In order to decrease steady-state aggregate unemployment, policy-makers often distribute subsidies to boost the worker job finding rate. The corollary sheds light on how best to distribute these subsidies. It implies that policy

[25] Note that the weights used to compute the average job destruction rates and unemployment outflow rates differ. It uses employment shares for the former and unemployment shares for the latter. Since both distribution do not coincide in the general case, this matters. This is typically a situation where theory can inform the aggregation of micro-parameters.

makers should (i) concentrate subsidies in high-degree occupations, rather than equally subsidize every occupation, and (ii) concentrate subsidies in clusters of occupations, rather than disseminate it across very dissimilar occupations.

Finally, note that the entire steady-state worker distribution can be recovered thanks to this proposition. Indeed, the steady-state level of unemployment and employment in occupation i can be recovered using $u_{i\infty} = \frac{u_{i\infty}}{u_\infty} \cdot u_\infty$ and $e_{i\infty} = \frac{e_{i\infty}}{e_\infty} \cdot (1 - u_\infty)$.

Transition dynamics of the worker distribution. What are the effects of productivity shocks on the dynamics of worker reallocation across occupations?

Consider the following scenario. At $t = 0^-$, the economy is at steady-state, and the distribution of workers across occupations is equal to the steady-state distribution implied by the current transition rate matrix $\boldsymbol{x}_0 = \boldsymbol{x}_\infty(\boldsymbol{Q})$. At $t = 0$, the distribution of productivity is hit by a unanticipated permanent shock, and becomes $\tilde{\boldsymbol{y}}$. The tightness ratios are jump variables, and therefore they immediately adjust to their new steady-state level $\tilde{\boldsymbol{\theta}}$. As a result, the job finding rates and the transition rate matrix adjust immediately too, and their new steady-state levels are respectively $\tilde{\boldsymbol{p}}$ and $\tilde{\boldsymbol{Q}}$. The new steady-state worker distribution implied by the new transition rate matrix is $\tilde{\boldsymbol{x}}_\infty(\tilde{\boldsymbol{Q}})$. The worker distribution of workers at $t = 0$ deviates from its new steady-state distribution: that is, we have $\boldsymbol{x}_0 \neq \tilde{\boldsymbol{x}}_{\boldsymbol{\infty}}(\tilde{\boldsymbol{Q}})$. The worker distribution then gradually converges towards the new steady-state worker distribution along the transition: in other words, the workers reallocate across the different occupations. This paper studies the speed at which this process of reallocation is taking place.

Let me characterize a bit further the dynamics of worker reallocation implied by the productivity shock. Denote by $\boldsymbol{\delta x}(t) \equiv \boldsymbol{x}(t) - \tilde{\boldsymbol{x}}_\infty$ the vector of deviations from the steady-state distribution. Positive elements of $\boldsymbol{\delta x}(t)$ correspond to excesses of workers in some occupations (relative to the new steady-state), while negative elements correspond to shortages of workers in other occupations (relative to the new steady-state). The total deviation from steady-state is measured as $\|\boldsymbol{\delta x}(t)\| = \sum_i \mid \delta x_i(t) \mid$. During the transition, the vector of deviations from steady-state solves:

$$\dot{\boldsymbol{\delta x}}(t) = -\tilde{\boldsymbol{Q}} \boldsymbol{\delta x}(t) \quad \text{with} \quad \boldsymbol{\delta x}(0) = \boldsymbol{x}_0 - \tilde{\boldsymbol{x}}_\infty \tag{23}$$

Eigen-decomposition of the transition rate matrix. The law of motion of the vector of deviations is a first-order matrix differential equations with constant coefficients. Assume that the transition rate matrix is diagonalizable with real eigenvalues.[26] Then, the vector of

[26]This assumption is always verified in numerical simulations and seems to always hold, but I have not found a proof yet. If the matrix is not diagonalizable, there exist solutions too, in terms of Jordan forms.

deviation from the steady-state distribution can be expressed in terms of the the transition rate matrix's eigenvalues and eigenvectors, as follows:

$$\boldsymbol{\delta x}(t) = \sum_{k=2}^{2N} \gamma_k(\boldsymbol{\delta x}_0) e^{-\lambda_k t} \boldsymbol{v}_k \tag{24}$$

where $\gamma_k(\boldsymbol{\delta x}_0)$, λ_k and $\boldsymbol{v}_k$ are, respectively, the loadings and the eigenvalues and eigenvectors of the transition rate matrix. Without loss of generality, I arrange the eigenvalues in increasing order $0 = \lambda_1 < ... < \lambda_{2N}$ and normalize the eigenvectors such that $\|\boldsymbol{v}_k\| = 1$.

Let me comment on each term. The eigenvectors of the transition matrix correspond to groups of occupations co-moving together along the transition. The mathematical intuition is that, along the transition, the eigenvectors are simply scaled down by a constant factor every period. This means that the deviations in each occupation of this group will shrink at the same pace along the transition: $\boldsymbol{\delta x}(t) = \gamma_k(\boldsymbol{x}_0)\boldsymbol{v}_k$ implies $\boldsymbol{\delta x}(t+1) = e^{-\lambda_k}\boldsymbol{\delta x}(t)$. This interpretation of the eigenvectors of the transition matrix is consistent with its use in machine learning as a clustering method (called spectral clustering) to detect groups of observations with similar characteristics.

To give a concrete example of groups of co-moving occupations, consider the the carpenter and woodcutter occupations: their skills are similar and therefore employment in both occupation might display a tight positive co-movement along the transition, because workers from one occupation will likely make transition to the other, and vice versa. Importantly, note that the co-movements between occupations are allowed to take a negative sign: that is, an increase in employment in one occupation might be contemporary with a decrease in employment in another occupation. This is in particular the case for reallocation episodes, where there is an excess number of workers in certain occupations and a shortage of workers in others.

The eigenvalues of the transition matrix control the speed of at which the groups of co-moving occupations, i.e. the eigenvectors, converge to the steady-state. The lower the value of the eigenvalues, the slower the convergence of the group of co-moving occupations towards the steady-state. Moreover, since λ_2 is the lowest non-zero eigenvalue, v_2 corresponds to the group of occupations converging at the slowest pace towards steady-state. Similarly, λ_3 is the second lowest non-zero eigenvalue, and therefore v_3 corresponds to the group of occupations with the second slowest speed of convergence towards steady-state, etc.

The inverse of the eigenvalues of the transition rate matrix gives a measure of transition *time* for groups of co-moving occupations. To understand why, let me perform a simple dimensional analysis. Speed is defined in units of distance per unit of time, so the inverse of speed should give units of time per unit of distance. Normalizing units of distance to one gives the desired result. In fact, this measure of transition time is related to another very popular measure of transition time, called the half-life. The half-life of the convergence for a group of co-moving occupations is the time it takes to halve the initial deviation. It is proportional to the inverse of the eigenvalue by a factor of log(2), such that $\tau_k \equiv \frac{\log 2}{\lambda_k}$.

The loadings of the spectral decomposition control the quantitative importance of each group of co-moving occupations in shaping the transition dynamics. Indeed, one same occupation can belong to different groups of co-moving occupations, and the loadings reflect how important is each group for the transition dynamics. To make an analogy, a same person can belong to different groups of friends, but put different weight on how important each group is to her. Importantly, the loadings are a function of the vector of initial deviation from steady-state, and therefore capture the dependency of the transition dynamics on the initial deviation.[27] In particular, asymmetric shocks will have different loadings distributions than symmetric shocks, etc.

The distribution of loadings plays a key role in determining the aggregate worker reallocation speed, by shifting weight on slow or fast groups of co-moving occupations. To capture this intuition, let me define the spectral transition time as

$$\tau \equiv \sum_{k=2}^{2N} \omega_k \cdot \frac{\log 2}{\lambda_k} \quad \text{where} \quad \omega_k \equiv \frac{|\,\gamma_k(\boldsymbol{\delta x}_0)\,|}{\sum_{m=2}^{2N} |\,\gamma_m(\boldsymbol{\delta x}_0)\,|}$$

The spectral transition time is a weighted average of the half-life of the different groups of co-moving occupations (eigenvectors), where the weights are proportional to the absolute value of the loadings. The larger the weights on slow groups of co-moving occupations, the higher the aggregate reallocation time.

How does the spectral measure of transition time I propose compare to the rest of the literature? The growing literature on the time evolution of distribution has proposed another measure of transition time for distributions, called the cumulative impulse response (CIR) (Alvarez et al. (2016), Baley and Blanco (2021), Alvarez and Lippi (2021)). The CIR is defined as the area under the curve of the impulse response function: that is, $\text{CIR} = \int_0^\infty \frac{\|\boldsymbol{\delta x}(t)\|}{\|\boldsymbol{\delta x}(0)\|}\text{dt}$. The spectral measure of transition time has two advantages over the CIR: (i) it does not require to simulate future transition dynamics, which can be very costly numerically if the dimension of the system is large, and (ii) it is more explicit on the role played by each eigenvector in shaping the transition time. In fact, numerical simulations with random initial conditions in the illustrative example developed later show that both measures of transition times are highly correlated, with a coefficient of correlation of around $\rho = 0.98$.[28]

[27] They can be recovered by projecting the initial condition on the eigenspace $\boldsymbol{\delta x}_0 = \sum_{k=2}^{\infty} \gamma_k(\boldsymbol{\delta x}_0)\boldsymbol{v}_k$. Or equivalently by taking the dot product between the left eigenvectors $\boldsymbol{w}_k$ and the initial deviation: $\gamma_k(\boldsymbol{\delta x}_0) = \boldsymbol{w}_k' \boldsymbol{\delta x}_0$

[28] This is not too surprising as both measures of transition time are the same in the one-dimensional case. Indeed, $\text{CIR} = \int_0^\infty \frac{|\boldsymbol{\delta x}(t)|}{|\boldsymbol{\delta x}(0)|} dt = \int_0^\infty \frac{|\delta x_0| e^{-\lambda t}}{|\delta x_0|} dt = \frac{1}{\lambda}$

4.3 Illustrative example

In this subsection, I study the effect of productivity shocks on worker reallocation in the stylized economy presented above. There are three occupations - textile, retail and care - which are connected by a line network. Retail workers can move to both textile and care worker occupations, but the textile and care workers can only move to the retail occupation.

This network structure is simple enough so that the results are easily interpretable, but at the same time sufficiently rich to investigate the role played by the central bottleneck occupation in shaping worker reallocation dynamics.

Distribution of productivity shocks. At time $t = 0^-$, I assume that the distribution of productivity across occupations is $\boldsymbol{y} = \bar{y} \cdot (1, \chi, 1)$, where $\bar{y}$ is a scaling parameter and χ controls the difference in productivity between central and non-central occupations. If $\chi < 1$, the central occupation (retail) is less productive than the end-point occupations (textile and care). The parameter χ is an inverse measure of bottleneck intensity in the occupational network: low productivity in the central occupation yields a low job finding rate in this occupation, and therefore low flows of worker transiting through this occupation.

At time $t = 0$, the productivity distribution is hit by an unanticipated permanent shock and becomes $\tilde{\boldsymbol{y}}$. I distinguish between three types of shocks: (i) a symmetric productivity shock affecting every occupation with the same intensity $\tilde{\boldsymbol{y}} = \bar{y} \cdot (1 - \Delta, \chi - \Delta, 1 - \Delta)$, where Δ is the size of the shock; (ii) an asymmetric productivity shock affecting the end-point occupations (textile and care) with the same intensity and the same sign $\tilde{\boldsymbol{y}} = \bar{y} \cdot (1 - \Delta, \chi, 1 - \Delta)$; (iii) an asymmetric shock affecting the end-point occupations (textile and care) with the same intensity but opposite sign $\tilde{\boldsymbol{y}} = \bar{y} \cdot (1 + \Delta, \chi, 1 - \Delta)$. For brevity, I refer to the same-sign asymmetric productivity shock as Type I asymmetric shock, and to the opposite-sign asymmetric shock as Type II asymmetric shock.

In what follows, I set the parameter values as follows $\bar{y} = 5$, $\chi = 1$ and $\Delta = 0.05$, $\alpha = 0.5$, $\zeta = 0.5$, $\phi = 0.5$, $c = 30$, and $\delta = 1\%$. Solving for the equilibrium yields unemployment outflow rates in the range of $40 - 60\%$. The theoretical unemployment outflow rates and the job separation rates are quite close from their actual monthly values in the US. This means that one period approximately corresponds to one month in the simulation. I turn to study the worker reallocation dynamics generated by the different types of productivity shocks. I have four results.

Transition speeds. First, the different productivity shocks have very heterogeneous transition speed. Figure 7 plots the total deviation from steady-state $\|\boldsymbol{\delta x}(t)\|$ over time, for each type of shock. After a symmetric shock, the total deviation from steady-state converges very

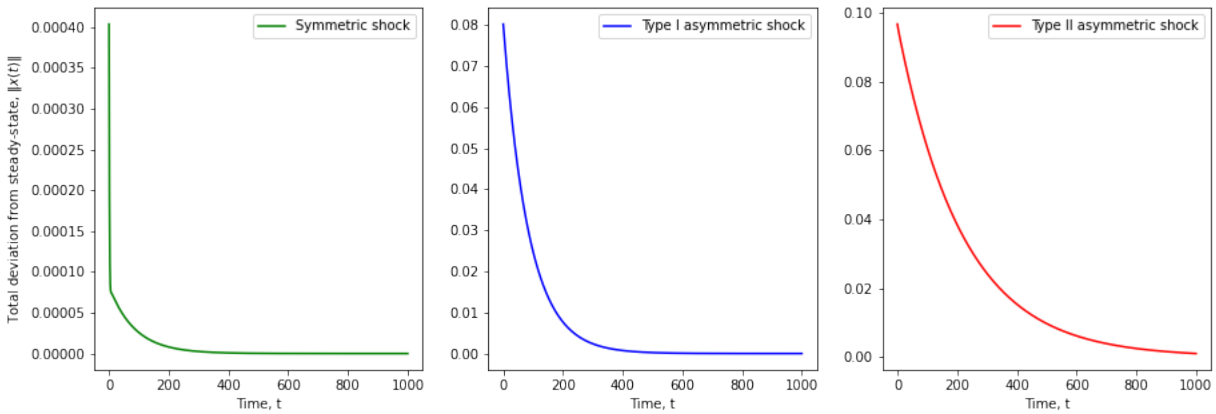

Figure 7: The total deviation from steady-state over time, after different productivity shocks

fast to zero. By contrast, the total deviation from steady-state converges much more slowly after asymmetric shocks, and Type II asymmetric shocks have slower adjustment dynamics than Type I asymmetric shocks.

This is reflected in the measure of transition time. The transition time is approximately 10 months for the symmetric shock, 60 months for the type I asymmetric shock, and 150 months for the Type II asymmetric shock. In other words, the transition time of the Type II asymmetric shock is around one order of magnitude higher than the transition time of the symmetric shock.

How does it compare with the transition time predicted by the standard search and matching model? The transition time predicted by the standard search and matching model is $\log(2)/(\bar{\sigma}+\bar{\delta})$, where $\bar{\sigma}$ is the average unemployment outflow rate and $\bar{\delta}$ is the average separation rate.[29] For realistic levels of transition rates, this yields a transition time of approximately 1.5 months. In other words, the transition time after Type II asymmetric shock is around two orders of magnitude slower than the transition time in the standard model.

Loadings distributions. Second, the different productivity shocks have heterogeneous distribution of loadings. Figure 8 plots the distribution of the absolute value of the loadings - i.e. the weights in aggregate transition time - for each type of productivity shocks. It shows that the symmetric shock puts high weight on the loadings l_4 and l_6, which correspond to the fastest deviation profiles. By contrast, the asymmetric shocks put weight on loadings corresponding to the slowest fundamental deviation profiles. The Type I asymmetric shock puts almost only weight on the loading corresponding to the second-slowest deviation profile l_3,

[29] Indeed, the law of motion for aggregate unemployment in the standard search and matching model is: $\dot{u}(t)=\bar{\delta}(1-u(t))-\bar{\sigma}u(t)=\bar{\delta}-(\bar{\delta}+\bar{\sigma})u(t)$. The dynamics of unemployment is then given by: $u(t)=u_0e^{-(\bar{\delta}+\bar{\sigma})t}$

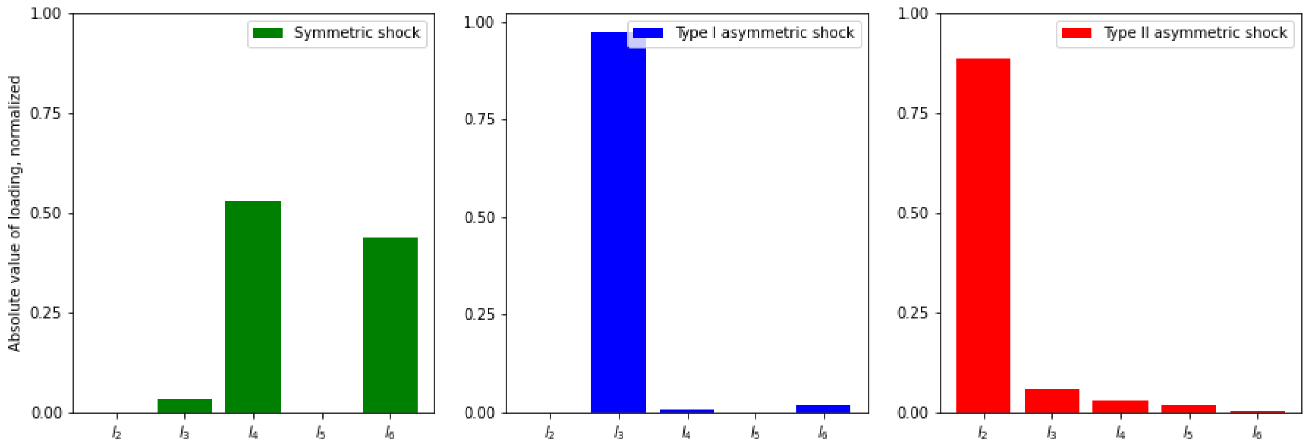

Figure 8: The distribution of loadings, after different productivity shocks

while the Type II asymmetric shock puts almost only weight on the loading corresponding to the slowest deviation profile l_2. Thus, the reason why productivity shocks have heterogeneous aggregate transition times is because they put weight on different fundamental deviation profiles. What do these shock profiles correspond to?

Deviation profiles. Third, the productivity shocks generate different patterns of deviations from steady-state. Figure 9 plots the vector of deviations from steady-state over time, after either Type I or Type II asymmetric shocks. I compute the deviation from steady-state in an occupation as the sum of the deviations from steady-state employment and unemployment in that occupation. The intensity of worker overabundance (shortages) relative to the steady-state is represented in red (blue). The nodes' colors become lighter as the overabundance or shortages disappear over time, because workers reallocate across occupations.

Type I and Type II asymmetric shocks generate very different worker reallocation dynamics across occupations. After a Type I asymmetric shock, the economy has too many workers in end-point occupations (textile and care), which it must reallocate towards the central occupation (retail). After a Type II asymmetric shock, the economy has too many workers in one end-point occupations (textile), which it must reallocate towards the other end-point occupation (care). This requires that workers from the textile occupation transit through the central retail occupation. The central bottleneck occupation (retail) plays very different role in the reallocation process, after Type I or Type II productivity shocks.

Bottleneck intensity. Fourth, the effect of bottleneck intensity on the speed of worker reallocation dynamics varies greatly across productivity shocks. Figure 10 plots the transition

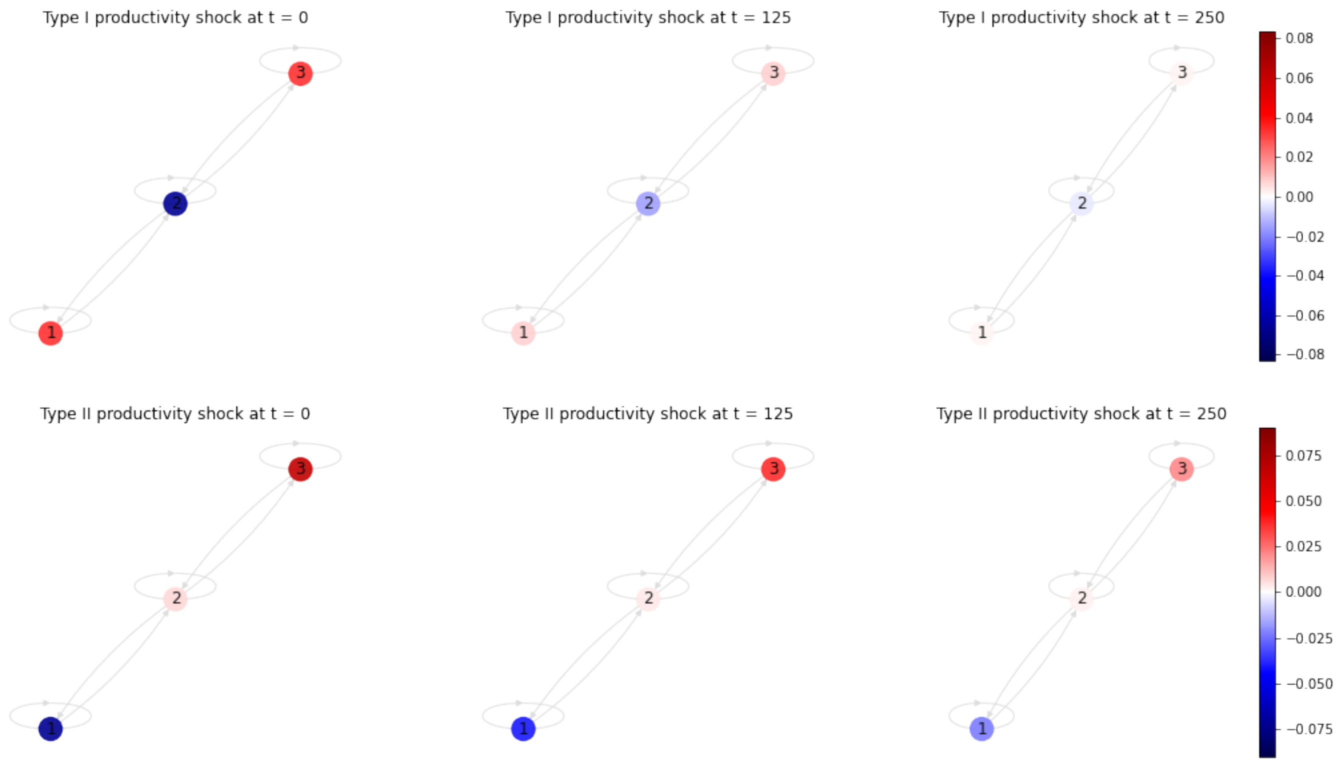

Figure 9: The deviation profiles over time, after different productivity shocks

time after Type I or Type II asymmetric productivity shocks, as a function of the inverse measure of bottleneck intensity χ. The transition time after Type I productivity shocks does not vary with the level of bottleneck intensity. By contrast, the transition time after Type II productivity shock reacts strongly to the the level of bottleneck intensity: a 10% decrease of productivity in the central bottleneck occupation increases transition time by 50%.

In other words, shocks to the central bottleneck occupation have a granular effect on reallocation dynamics, after Type II productivity shocks. This implies that the average transition rates are not informative about the speed of reallocation after Type II reallocation shocks, only the transition rate in the central bottleneck occupation is. For this reason, and contrary to common wisdom, large flows of workers across occupations are not necessarily indicative of fast worker reallocation dynamics.

The reason why transition times reacts differently to Type I and Type II asymmetric productivity shocks stems from the difference in reallocation dynamics emphasized above. Type I asymmetric productivity shocks generate reallocation *towards* the central occupation, while

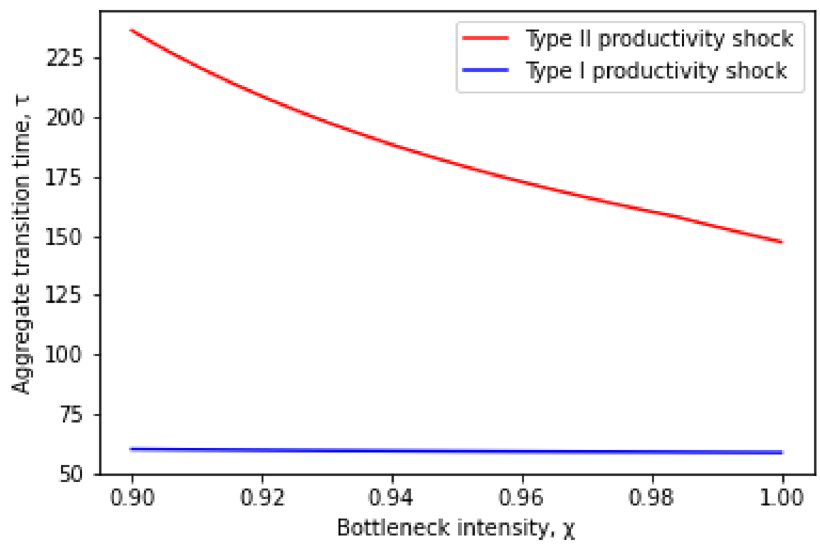

Figure 10: The transition time as a function

Type II asymmetric productivity shocks generate reallocation *through* the central occupation. To sum up, the interaction between Type II asymmetric shocks and high bottleneck intensity can lead to very sluggish worker reallocation dynamics.

5 Conclusion

In this paper, I study the dynamics of worker reallocation between occupations. The originality of my approach is to view the architecture of bilateral occupational mobility frictions between occupations as a network, which I call the occupational network. I argue that the structure of this network matters for worker reallocation dynamics.

I identify the structure of the occupational network using a French database on occupational characteristics, and I study its structure using tools from network theory. I find three stylized facts: (i) the occupational network is very sparse, (ii) occupations differ widely in their number of connections, and (iii) there exists bottlenecks in network.

Motivated by this evidence, I build tractable model of worker reallocation in order to understand how the structure of the network affects the dynamics of worker reallocation. I extend a standard search and matching model with heterogeneous occupations connected by an occupational network. I find that asymmetric shocks can generate transition dynamics which are two orders of magnitude slower than in the standard model; and that the higher the intensity of bottlenecks, the slower the reallocation dynamics after asymmetric shocks.

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6 Appendix

6.1 Proofs of the baseline model

6.1.1 Proof of Proposition 2

Network recursion. The Bellman equation for the value of being unemployed in occupation i writes

$$rU_i = b_i + \sum_{j \in \mathcal{N}} g_{ij} p_j (E_j - U_i)$$

Rearranging terms, this becomes

$$\Big[r + \sum_j g_{ij} p_j\Big] U_i = b_i + \sum_j g_{ij} p_j E_j$$

Rearranging terms in the Bellman equation for the value of being employed and plugging the Nash bargainin condition, one finds

$$E_j = \frac{y_j + \delta_j U_j}{r + \delta_j} = \frac{\phi_j}{r + \delta_j} y_j + \frac{r(1 - \phi_j) + \delta_j}{r + \delta_j} U_j$$

Plugging in the expression above yields

$$\Big[r + \sum_j g_{ij} p_j\Big] U_i = b_i + \sum_j g_{ij} p_j \Big[\frac{\phi}{r + \delta_j} y_j + \frac{r(1 - \phi) + \delta_j}{r + \delta_j} U_j\Big]$$

Rearranging terms and multipliying by r yields

$$rU_i = \frac{r}{r + \sum_j g_{ij} p_j} b_i + \sum_j \frac{g_{ij} p_j}{r + \sum_j g_{ij} p_j} \left(\frac{r\phi}{r + \delta_j} y_j + \frac{r(1 - \phi_j) + \delta_j}{r + \delta_j} rU_j\right)$$

Finally, note that

$$\frac{r}{r + \sum_j g_{ij} p_j} = 1 - \sum_j \frac{g_{ij} p_j}{r + \sum_j g_{ij} p_j}$$

By identifying the different terms with their respective definition one recovers the expression in equation (13).

Existence and unicity. The vector of equilibrium reservation wages $\bar{\boldsymbol{w}}$ can be written in matrix form as follows

$$\bar{\boldsymbol{w}} = \Big[\boldsymbol{I} - \text{diag}(\boldsymbol{\beta})\boldsymbol{M}(\boldsymbol{\theta})\Big]^{-1} \Big[b(\boldsymbol{I} - \text{diag}(\boldsymbol{M}(\boldsymbol{\theta})\mathbb{1})) + (\boldsymbol{I} - \text{diag}(\boldsymbol{\beta}))\boldsymbol{M}(\boldsymbol{\theta})\Big]\boldsymbol{y} \tag{25}$$

There exists a unique vector of equilibrium reservation wages iff the matrix $[I - \text{diag}(\beta)M(\theta)]$ is invertible. This matrix is invertible iff

$$\rho[\text{diag}(\beta)M(\theta)] < 1$$

where $\rho[\text{diag}(\beta)M(\theta)]$ is the modulus of the leading eigenvalue, i.e. the eigenvalue with the largest modulus, of the matrix $\text{diag}(\beta)M(\theta)$. To see why, note that:

$$[I - \text{diag}(\beta)M(\theta)]^{-1} = I + \text{diag}(\beta)M(\theta) + \frac{1}{2}(\text{diag}(\beta)M(\theta))^2 + \dots$$

This formula is the multi-dimensional equivalent of the formula for geometric series. Intuitively, the sum of infinite terms converges if and only if the matrices $(\text{diag}(\beta)M(\theta))^k$ converge sufficiently fast to zero when $k \to \infty$, which requires that the leading value of $\text{diag}(\beta)M(\theta)$ has modulus less than one.

Let me show that this sufficient condition is verified here, i.e. that the modulus of the leading eigenvalue is lower than unity. I use an important result on non-negative irreducible matrices namely the Perron-Frobenius theorem. According to the theorem, the modulus of the leading eigenvalue of a non-negative irreducible matrix is bounded from above by the largest matrix's row sums. The theorem can be applied here because the matrix $\text{diag}(\beta)M(\theta)$ is indeed non-negative and irreducible, since the graph G is connected. In addition, it can be easily checked that

$$\rho[\text{diag}(\beta)M(\theta)] \leq \max_i \sum_j \{\text{diag}(\beta)M(\theta)\}_{ij} = \max_i \beta_i \sum_j \mu_{ij} < 1$$

This proves that the matrix $[I - \text{diag}(\beta)M(\theta)]$ is invertible and hence that there exists a unique solution to the system of equations (13).

6.1.2 Proof of Proposition 3

Recursion. Let me explain the derivation of $\Gamma(\theta)$. Using the Bellman equation for the value of a job to the firm (3) and the Nash sharing rule, the value of a job to the firm in occupation i can be rewritten as

$$J_i(\theta) = \frac{1-\phi_i}{r+\delta_i}(y_i - \bar{w}_i(\boldsymbol{\theta}))$$

Plugging this expression in the free entry condition (14) yields

$$\frac{c_i}{q(\theta_i)} = \frac{1-\phi_i}{r+\delta_i}(y_i - \bar{w}_i(\boldsymbol{\theta}))$$

Using the fact that $q(\theta_i) = \zeta\theta_i^{-\alpha}$ and rearranging terms, one finally gets

$$\theta_i = \left[\frac{(1-\phi_i)}{\zeta c_i(r+\delta_i)}(y_i - \bar{w}_i(\boldsymbol{\theta}))\right]^{\frac{1}{\alpha}}$$

In matrix form, this writes

$$\boldsymbol{\theta} = \Gamma(\boldsymbol{\theta}) \quad \text{with} \quad \Gamma(\boldsymbol{\theta}) \equiv \Big[\text{diag}(\zeta)\big(\boldsymbol{y} - \bar{\boldsymbol{w}}(\boldsymbol{\theta})\big)\Big]^{\circ\frac{1}{\alpha}}$$

Existence and unicity. Proving there exists a unique solution to the multidimensional fixed-point problem (15) is not trivial. I am still figuring out a proof.

6.1.3 Proof of existence, unicity and stability of the steady-state worker distribution

First, the existence of the steady-state distribution derives from a key property of transition rate matrices, namely that their columns sum to zero. Let me remind that the steady-state distribution solves

$$\boldsymbol{Q}\boldsymbol{x}_\infty = 0$$

Therefore, a necessary and sufficient condition for the existence of the steady-state distribution is that the eigenvalue zero belongs to the spectrum of the transition rate matrix, that is

$$0 \in \mathrm{Sp}(\boldsymbol{Q})$$

Now, note that the columns of the transition rate matrix sum to zero. This implies that

$$\mathbb{1}\boldsymbol{Q} = \boldsymbol{0} \quad \implies \boldsymbol{Q}'\mathbb{1} = \boldsymbol{0}$$

where $\mathbb{1}$ is the vector of ones $\mathbb{1} = (1, \ldots, 1)'$. This means that $\mathbb{1}$ is an eigenvector associated to the eigenvalue value 0 of the transpose of the transition matrix $\boldsymbol{Q}'$ and hence $0 \in \mathrm{Sp}(\boldsymbol{Q}')$. Since matrices and their transposes share the same spectrum, we have $0 \in \mathrm{Sp}(\boldsymbol{Q})$ too. This shows that the process of the worker distribution admits a steady-state distribution.

Second, the unicity of the steady-state distribution derives from the assumption that the occupational network is connected. A sufficient and necessary condition for the unicity of the steady-state distribution is that the eigenvalue zero is unique, i.e. its multiplicity is one. The assumption that the occupational network is connected implies that the transition rate matrix is connected too. In linear algebra terms, this means that the transition matrix $\boldsymbol{Q}$ and its transpose $\boldsymbol{Q}'$ are irreducible, allowing me to apply the Perron-Frobenius theorem. The Perron-Frobenius theorem states that the leading eigenvalue is unique and the eigenvector associated to the leading eigenvalue is the only one with strictly positive elements. Since the eigenvector $\mathbb{1}$ has only strictly positive elements, the associated eigenvalue 0 is the leading eigenvalue of $\boldsymbol{Q}'$ and it is unique. Since the transition rate matrix and its transpose share the same spectrum, the transition matrix $\boldsymbol{Q}$ has a unique eigenvalue zero too. This proves there exists a unique steady-state distribution.

Third, the stability of the steady-state distribution derives also from the fact that the columns of a transition rate matrix sum to zero. A sufficient and necessary condition for the stability of the steady-state distribution is that the eigenvalues of $\boldsymbol{Q}$ have a positive real part, that is:

$$\mathrm{Re}(\lambda_k) \geq 0 \quad \text{for all } k$$

To show this, I use the Gershgorin circle theorem applied to the the transpose of the transition rate matrix $\boldsymbol{Q}'$. The theorem states that all its eigenvalues belong to at least one of Gershgorin circles, with center Q'_{ii} and radius $\sum_{j\neq i}|Q'_{ij}|$. Now, observe that for all i:

$$Q'_{ii} = Q_{ii} > 0 \quad \text{and} \quad \sum_{j\neq i}|Q'_{ij}| = \sum_{j\neq i}|Q_{ji}| = Q_{ii}$$

since the columns of $\boldsymbol{Q}$ sum to zero. This implies that all Gerschgorin circles of $\boldsymbol{Q}'$ belong to $\mathbb{R}_+ \times \mathbb{R}$ and hence that all eigenvalues of $\boldsymbol{Q}'$ have positive real parts. Using the fact that the transition matrix and its transpose $\boldsymbol{Q}'$ share the same spectrum, all eigenvalues of are positive too. This proves that the steady-state worker distribution is stable.

6.1.4 Proof of Proposition 4

At steady-state, the worker distribution solves the following system of variables

$$0 = \delta_i e_i - \Big(\sum_{j\in\mathcal{N}} g_{ij} p_j\Big) u_i \tag{26}$$

$$0 = p_i \sum_{j\in\mathcal{N}} g_{ij} u_j - \delta_i e_i \tag{27}$$

where the steady-state subscript is dropped temporarily for clarity reason.

Aggregation unemployment rate. Let rewrite the equation (29) as follows

$$\delta_i \frac{e_i}{e} e = \sigma_i \frac{u_i}{u} u$$

Summing for occupations i and using the fact that $e = 1 - u$, I find

$$\sum_{i\in\mathcal{N}} \delta_i \frac{e_i}{e} e = \sum_{i\in\mathcal{N}} \sigma_i \frac{u_i}{u} u \quad \Longleftrightarrow \quad u = \frac{\sum_{i\in\mathcal{N}} \delta_i \frac{e_i}{e}}{\sum_{i\in\mathcal{N}} \delta_i \frac{e_i}{e} + \sum_{i\in\mathcal{N}} \sigma_i \frac{u_i}{u}}$$

This is the desired closed-form expression for the aggregate unemployment rate. Note that it remains to solve for the steady-state employment and unemployment shares.

Unemployment and employment shares. I turn now the derivation of the steady-state unemployment and employment shares. Eliminating e_i from equation (29) and (20), one gets

$$\Big(\sum_{j\in\mathcal{N}} g_{ij} p_j\Big) u_i = p_i \sum_{j\in\mathcal{N}} g_{ij} u_j \tag{28}$$

Now, let me guess a functional form for the steady-state unemployment in occupation i: $u_i = p_i C$, where C is a multiplicative constant, and plug it in the equation above. I find

$$\Big(\sum_{j\in\mathcal{N}} g_{ij}p_j\Big)p_i C = p_i \sum_{j\in\mathcal{N}} g_{ij}p_j C \quad \iff \quad 1 = 1$$

The guess solves the equation, for any multiplicative constant C. Finally, the constant is found by using the normalizing relation $\sum_{i\in\mathcal{N}} u_i = u$. It is easy to check that one must have $C = \frac{u}{\sum_{l\in\mathcal{N}} p_l}$. Therefore, the steady-state unemployment share writes

$$\frac{u_i}{u} = \frac{p_i}{\sum_{l\in\mathcal{N}} p_l}$$

Now, I turn to the steady-state employment shares. Rearranging terms in equation (29) gives the relation $u_i = \frac{\delta_i e_i}{\sigma_i}$. By plugging this expression in equation (31), I find

$$\Big(\sum_{j\in\mathcal{N}} g_{ij}p_j\Big)\frac{\delta_i e_i}{\sigma_i} = p_i \sum_{j\in\mathcal{N}} g_{ij}\Big(\frac{\delta_j e_j}{\sigma_j}\Big)$$

Let me guess a functional form for the steady-state employment in occupation i $e_i = \frac{p_i\sigma_i}{\delta_i}C$, where C is a multiplicative constant, and plug it in the equation above. I find

$$\Big(\sum_{j\in\mathcal{N}} g_{ij}p_j\Big)\frac{\delta_i}{\sigma_i}\frac{p_i\sigma_i}{\delta_i}C = p_i \sum_{j\in\mathcal{N}} g_{ij}\Big(\frac{\delta_j}{\sigma_j}\frac{p_j\sigma_j}{\delta_j}C\Big) \quad \iff 1 = 1$$

The guess solves the equation, for any multiplicative constant C. Finally, the constant is found by using the normalizing relation $\sum_{i\in\mathcal{N}} e_i = e$. It is easy to check that one must have $C = \frac{e}{\sum_{l\in\mathcal{N}} p_l\sigma_l/\delta_l}$. Therefore, the steady-state unemployment share writes

$$\frac{e_i}{e} = \frac{p_i\sigma_i/\delta_i}{\sum_{l\in\mathcal{N}} p_l\sigma_l/\delta_l}$$