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Abstract

In a market economy, the aggregate production level depends not only on the aggregate variables but also on the distribution of individual characteristics (e.g., productivity, credit limit, ...). We point out that, due to financial frictions the equilibrium aggregate production may be non-monotonic in both individual productivity and credit limit. By consequence, the emergence of some firms (for example, improving productivity or relaxing credit limit) may not necessarily be beneficial to economic development.

JEL Classifications: D2, D5, E44, G10, O4.
Keywords: Productivity shock, financial shock, credit constraint, heterogeneity, productivity dispersion, distributional effects, efficiency, general equilibrium.

1 Introduction

The paper aims to investigate two basic questions in economics: what are the impacts of (individual and aggregate) productivity and financial shocks on the aggregate output?

Let us start by a simple framework: there are m agents and a singe good (which can be consumed or used to produce). Each agent i is endowed S_i units of good and has a production function $A_i f_i(k)$ where $A_i > 0$ is the productivity, and f_i is the autonomous production function. In an economy without market imperfections, the output must be

$$Y^{perfect} \equiv \max_{(k_i)\geq 0} \sum_i A_i f_i(k_i) \quad \text{subject to}: \quad \sum_i k_i \leq S \equiv \sum_i S_i. \tag{1}$$

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Of course, the output $Y^{perfect}$ is increasing in each productivity A_i. In other words, the productivity has a positive effect on the output. This insight is well-known in economics textbooks or classical papers (Solow, 1957; Romer, 1986, 1990).

Now, consider a competitive economy with financial market imperfections: each agent has a credit limit γ_i. The equilibrium aggregate production is

$$Y = \sum_i A_i f_i(k_i), \text{ where } (k_i) \text{ is the allocation of inputs in general equilibrium.} \quad (2)$$

Since the equilibrium allocation of capital (k_i) depends on the economy structure, the equilibrium production Y depends on productivities $\mathcal{A} \equiv (A_i)_{i=1}^m$ and credit limits $\gamma \equiv (\gamma_i)_{i=1}^m$ of agents. So, we write $Y = Y(A_1, \ldots, A_m, \gamma_1, \ldots, \gamma_m)$. We aim to investigate conditions under which the aggregate production increases or decreases in productivity A_i and credit limit γ_i. Our novel point is that the aggregate output (and hence aggregate TPF) may decrease even the productivity and credit limit of all agents increase, i.e.,

$$\begin{aligned} &Y(A'_1, \ldots, A'_m, \gamma'_1, \ldots, \gamma'_m) < Y(A_1, \ldots, A_m, \gamma_1, \ldots, \gamma_m) \\ &\qquad \text{for some } (A'_i), (\gamma'_i) \text{ such that } A'_i > A_i, \gamma'_i > \gamma_i, \forall i = 1, \ldots, m. \end{aligned} \quad (3)$$

First, we explain the role of productivity. The productivity is widely viewed as one of the most important determinants of economic growth. Our contribution is to argue that whether a positive productivity shock generates a positive (or negative) effect on the aggregate output depends on the distribution of productivity, the size of productivity shock, and the level of financial imperfection.

If the productivity of all agents increases (resp., decreases) at the same rate, then this change has a positive (resp., negative) effect on the aggregate production. However, when the productivity of agents increases at different rates, the aggregate production may decrease. This may happen if the TFP of less productive agents increases faster than that of more productive agents. Indeed, in such a case, less productive agents absorb more capital and produces more. Since the aggregate capital is limited, other producers (who are more productive) get less capital (because of market imperfections) and so they produce less. By consequence, the net effect may be negative. This happens if (1) the TFP of less productive agents is far from that of more productive producers, i.e., the productivity dispersion is high,[1] (2) the positive shock is quite small, (3) the credit constraint is tight.

Second, we focus on the effects of a change in credit limit (γ_i) on the aggregate production Y. One can expect that relaxing credit limits would have positive impact on the aggregate output as shown in Khan and Thomas (2013) (section VI. C), Midrigan and Xu (2014) (section II.B), Moll (2014) (Proposition 1), and Catherine, Chaney,

[1] Andrews, Criscuolo and Gal (2015) use a harmonised cross-country dataset, based on underlying data from the OECD-ORBIS database (Gal, 2013), to analyze the characteristics of firms that operate at the global productivity frontier and their relationship with other firms in the economy. Andrews, Criscuolo and Gal (2015) document growing productivity dispersion for several developed countries over the 2000s. Bouche, Cette, and Lecat (2021) present empirical evidence showing an increase in productivity dispersion between French firms during the period 1991-2016, with a growing productivity gap between frontier and laggard firms. See Goldin, Koutroumpis, Lafond, and Winkler (2021) for an excellent review on the slowdown in productivity growth.

Huang, Sraer, and Thesmar (2017). Our novel point is that, while a homogeneous positive credit shock improves the aggregate output, an asymmetric positive credit shock can reduce the output.

The intuition is similar to that in the case of productivity shocks we have mentioned above: If credit limits of less productive agents increase faster that those of more productive ones, less productive agents get more capital and more productive agents get less capital, hence the aggregate output may decrease. It should be noticed that although the aggregate output is not necessarily monotonic in credit limits of producers, it does not exceed that in the frictionless economy.

After having worked in a static model, we investigate our above questions in infinite-horizon models à la Ramsey. We prove that the non-monotonic effect of productivity and credit limit on the aggregate output cannot appear at the steady state in infinite-horizon models à la Ramsey. The reason is that the steady state interest rate only depends on the rate of time preferences of agents.

Therefore, we focus on the global dynamics of intertemporal equilibrium. First, our findings suggest that a permanent increase of productivity of less productive agents improves the aggregate output in the long run. However, when this productivity shock is quite small and credit constraints are tight, the aggregate output may decrease in the short-run and then increase from some period on.

Second, we look at the effects of credit limits. In the static model, an increase in the most productive agent's credit limit is always beneficial for the aggregate output. However, along intertemporal equilibrium, we show that an increase of the most productive producer's credit may reduce the output at every period. The intuition behind is that when her(his) credit limit goes up, the equilibrium interest rate increases, and hence, her(his) repayment also increases. This in turn reduces her(his) net worth in the next period. By consequence, her(his) saving and hence the production decrease. The mechanism can be summarized by the following schema:

$$\begin{aligned}&\text{Credit limit} \uparrow \;\Rightarrow\; \text{Interest rate} \uparrow \;\Rightarrow\; \text{Agent's net worth} \downarrow \;\Rightarrow\\ &\Rightarrow\; \text{Saving} \downarrow \;\Rightarrow\; \text{Production} \downarrow \;\Rightarrow \cdots\end{aligned} \tag{4}$$

As in the static model, this mechanism can happen because the credit limit of the most productive agent remains low and the productivity dispersion is high.

Our article is related to a growing literature on general equilibrium models with heterogeneous producers and financial frictions.[2] Let us mention some of them.[3] Midrigan and Xu (2014) consider a two-sector model with a collateral constraint that requires the debt of producer does not exceed a fraction of its capital stock. They focus on balanced growth equilibrium to study the role of collateral constraint in determining TFP. Their parameterizations consistent with the data imply fairly small losses from misallocation, but potentially sizable losses from inefficiently low levels of entry and technology adoption. Khan and Thomas (2013) develop a dynamic stochastic general

[2]The reader is referred to Matsuyama (2007), Quadrini (2011), Brunnermeier, Eisenbach, and Sannikov (2013) for more complete reviews on the macroeconomic effects of financial frictions and to Buera, Kaboski, and Shin (2015) for the relationship between entrepreneurship and financial frictions.

[3]While we focus on firm heterogeneity, there is a growing literature studying the roles of household heterogeneity in macroeconomics (the reader is refereed to Kaplan and Violante (2018) for an excellent review on this topic).

equilibrium with a representative household and heterogeneous firms facing a borrowing constraint (slightly different from ours) and focus on recursive equilibrium. They find that a negative shock to borrowing conditions can generate a large and persistent recession through disruptions to the distribution of capital. Buera and Shin (2013) develop a model with individual-specific technologies and collateral constraints to investigate the role of the misallocation and reallocation of resources in macroeconomic transitions. Buera and Shin (2013) find that collateral constraints have a large impact along the transition to the steady state. Moll (2014) studies the effect of collateral constraints on capital misallocation and aggregate productivity in a general equilibrium with a continuum of heterogeneous firms and financial frictions (modeled by a collateral constraint). Proposition 1 in Moll (2014) shows that the aggregate TFP is increasing in the leverage ratio which is the common across firms.[4]

Our paper differs from this literature in two points. First, the credit limit is individualized in our model while all credit parameters in the above studies are common across firms. Second, we argue that this credit heterogeneity plays an important role in the distribution of capital and of income as well as in the aggregate output. Indeed, we prove that the aggregate output and the aggregate TFP in our model may not be monotonic functions of the credit limits which are different across agents; they may display an inverted-U form.[5] However, we show that, if agents have the same credit limit, the aggregate output and the aggregate TFP are increasing functions of this common credit limit; this finding is consistent with the above literature.

Our paper is related to Baqaee and Farhi (2020) who build a general equilibrium model where productivity and wedge are exogenous parameters to study how the impact of (productivity and wedge) shocks can be decomposed into a pure technology effect and an allocative efficiency effect. There are some differences between Baqaee and Farhi (2020) and the present paper. First, Baqaee and Farhi (2020) model frictions by wedge while we model frictions by a credit constraint and the credit limit is our exogenous parameter. Second, Baqaee and Farhi (2020) provide a quantitative analysis by applying their approach to the firm-level markups in the U.S. but they do not provide conditions (based on exogenous parameters) under which the aggregate output is increasing or decreasing in productivity and friction level (wedge in their framework). Although we do not provide quantitative applications of our results, we show several conditions (based on exogenous parameters) under which the aggregate output is increasing or decreasing in productivity and friction level (credit limit in our framework). We also run some simulations and extend our analyses in infinite-horizon models while Baqaee and Farhi (2020) do not do this exercise.

Our paper also concerns the literature on the welfare effects of financial constraints. Jappelli and Pagano (1994, 1999) consider overlapping generations models with liquidity constraints and households living for three periods and argue that liquidity con-

[4]In both Buera and Shin (2013), Moll (2014), the collateral constraint, which is slightly different from ours, states that the capital of a firm does not exceed a leverage ratio of its financial wealth.

[5]Our finding is related to Aghion, Bergeaud, and Maghin (2018). They consider a model of firm dynamics and innovation with entry, exit, and credit constraints, based on Klette and Kortum (2004), Aghion, Akcigit, and Howitt (2015). They assume that intermediate firms (monopolist) cannot invest more than μ times their current market value in innovation. They argue that the credit access may harm productivity growth because it allows less efficient incumbent firms to remain longer on the market, which discourages entry of new and potentially more efficient innovators.

straints may increase or decrease welfares. The central point in Jappelli and Pagano (1994, 1999) is that liquidity constraints have two opposite effects on welfare: "they force the consumption of young below the unconstrained level but raise their permanent income by fostering capital accumulation". Obiols-Homs (2011) considers a general equilibrium with heterogeneous households (whose borrowings are bounded by an exogenous limit) and a representative firm. He argues that the borrowing limit has a negative on the welfare of borrower if its *quantity effect* dominates its *price effect*. As in Jappelli and Pagano (1994, 1999), the mechanism of Obiols-Homs (2011) relies on the role of supply of credit to households who need to smooth their consumption. By contrast, our mechanism focuses on credit to firms who need credit to finance their productive investment. Moreover, Obiols-Homs (2011) considers exogenous borrowing limits while we focus on credit constraints and our model has endogenous borrowing limits.

Catherine, Chaney, Huang, Sraer, and Thesmar (2017) build a dynamic general equilibrium model with heterogeneous firms and collateral constraints. They focus on the steady state and provide estimates suggesting that lifting financial frictions (modeled by collateral constraints) would increase aggregate welfare by 9.4% and aggregate output by 11%. Our paper differs from Catherine, Chaney, Huang, Sraer, and Thesmar (2017) in two aspects. First, although we also find that the aggregate output in the frictionless economy is higher than that in the economy with financial frictions, it is not a monotonic function of the degree of financial friction. Second, both individual and social welfares may not be monotonic in the degree of financial friction. Interestingly, lifting credit constraint may decrease the welfare of some agents.

Last but not least, our paper contributes to the debate concerning the slowdown in aggregate productivity growth that has been documented by several studies such as Andrews, Criscuolo and Gal (2015), Bouche, Cette, and Lecat (2021), Goldin, Koutroumpis, Lafond, and Winkler (2021); see Footnote 1. Our above analyses suggest that the interplay between credit constraints, high heterogeneity of productivity, asymmetry of productivity and financial shocks may generate a slowdown in aggregate productivity growth. The aggregate productivity growth rate may be far from that of most productive firms. It may be even lower than the smallest productivity growth rate of firms. Our approach, which is different from those in the literature, is based on the general equilibrium theory with financial frictions.

The rest of our article is organized as follows. Section 2 presents a two-period general equilibrium framework. Section 3.1 studies the effects of productivity shocks while Section 3.2 investigates the role of credit shocks. Section 4 extends our analyses in Sections 3.1 and 3.2 to infinite-horizon general equilibrium models à la Ramsey. Section 5 concludes. Formal proofs are gathered in the appendices.

2 A static framework

We consider a deterministic two-period economy with a finite number (m) of heterogeneous agents. There is a single good (numéraire) which can be consumed or used to produce. Each agent i has exogenous initial wealth (S_i units of good) at the initial date. To keep the model as simple as possible, we assume that agents just maximize their consumption in the second period and we focus on the output in this period.

Agents have two ways for investing. On the one hand, agent i can buy k_i units of physical capital at the initial date to produce $F_i(k_i)$ units of good at the second date, where F_i is the production function.[6] Since we are interested in the effect of individual productivity, we write

$$F_i(x) = A_i f_i(x), \tag{5}$$

where $A_i > 0$ represents the productivity of agent i while f_i is the original production function.

On the other hand, she can invest in a financial asset with real return r which is endogenous. Denote b_i the asset holding of agent i. She can also borrow and then pay back Rb_i in the next period. However, there is a borrowing constraint. The maximization problem of agent i can be described as follows:

$$(P_i): \quad \pi_i = \max_{k_i, b_i} \left[F_i(k_i) - Rb_i\right] \tag{6a}$$

$$\text{subject to: } 0 \leq k_i \leq S_i + b_i \text{ (budget constraint)} \tag{6b}$$

$$Rb_i \leq \gamma_i F_i(k_i) \text{ (borrowing constraint)} \tag{6c}$$

where $\gamma_i \in (0,1)$ is an exogenous parameter. Borrowing constraint (6c) means that the repayment does not exceed the market value of the borrower's project.[7][8] This is similar to the collateral constraint (4) in Kiyotaki (1998) or the so-called **earnings-based constraint** in Lian and Ma (2021). The better the commitment, the higher value of γ_i, the larger the set of feasible allocations of the agent i. Kiyotaki (1998) interprets γ_i as the collateral value of investment. In our paper, we call γ_i the *credit limit* of agent i.

The following table from the Enterprise Surveys (2018)'s panel datasets suggests that borrowing and collateral constraints matter for the development of firms.

An economy $\mathcal{E}$ with credit constraints is characterized by a list of fundamentals

$$\mathcal{E} \equiv (A_i, f_i, \gamma_i, S_i)_{i=1,\dots,m}.$$

Definition 1. *A list $(R, (k_i, b_i)_i)$ is an equilibrium if (1) for each i, given R, the allocation (b_i, k_i) is a solution of the problem (P_i), and (2) financial market clears $\sum_i b_i = 0$.*

[6] We can interpret the one-factor production function F_i as a reduced form for a setting with other factors of production. Indeed, suppose that the firm has a two-factor production function, say capital and labor, $G_i(k, N)$. For a given level of capital k_i, the firm chooses labor quantity N_i to maximize its profit $\max_{N_i \geq 0} [G_i(k_i, N_i) - wN_i]$. The first order condition writes $\frac{\partial G_i}{\partial N}(k_i, N_i) = w$. This implies that $N_i = N_i(k_i, w)$. So, $G_i(k_i, N_i) = G_i(k_i, N_i(k_i, w))$. We now define $F_i(k_i) \equiv G_i(k_i, N_i(k_i, w))$.

[7] Here, we follow Kiyotaki (1998) by assuming that the debtor is required to put her project as collateral in order to borrow: If she does not repay, the creditor can seize the collateral. Due to the lack of commitment (or just because the debtor is not willing to help the creditor take the whole value of the debtor's project), the creditor can only obtain a fraction γ_i of the total value of the project. Anticipating the possibility of default, the creditor limits the amount of credit so that the debt repayment will not exceed a fraction γ_i of the debtor's project value.

[8] Matsuyama (2007) (Section 2) considers a model with heterogeneous agents, which corresponds to our model with $k_i = 1$, $S_i = w, b_i = 1 - w$. However, different from our setup, investment projects in Matsuyama (2007) are non-divisible.

Economy	Proportion of loans requiring collateral (%)	Value of collateral needed for a loan (% of the loan amount)	Percent of firms not needing a loan	Percent of firms whose recent loan application was rejected	Proportion of investments financed internally (%)
All Countries	79.1	205.8	46.4	11.0	71.0
East Asia & Pacific	82.6	238.4	50.7	6.4	77.8
Europe & Central Asia	78.7	191.9	54.3	10.9	72.4
Latin America & Caribbean	71.3	198.5	45.0	3.1	62.7
Middle East & North Africa	77.4	183.0	51.8	10.2	71.1
South Asia	81.1	236.0	44.7	14.4	73.9
Sub-Saharan Africa	85.3	214.8	37.4	15.3	73.9

We require standard assumptions on the production function.

Assumption 1. *The production function f_i is concave, strictly increasing, $f_i(0) = 0$. The credit limit γ_i belongs the interval $(0, 1)$ for any i.*

Lemma 1. *Under Assumption 1, there exists an equilibrium.*

We can prove the equilibrium existence by using the standard argument (see, for instance, Bosi, Le Van, and Pham (2018))

Remark 1. Some authors (Buera and Shin, 2013; Moll, 2014) set $k_i \leq \theta w_i$, where $w_i \geq 0$ is the agent i's wealth and interpret that θ measures the degree of credit frictions (credit markets are perfect if $\theta = \infty$ while $\theta = 1$ corresponds to financial autarky, where all capital must be self-financed by entrepreneurs). In our framework, S_i plays a similar role of wealth w_i in Buera and Shin (2013), Moll (2014). Another way to introduce credit constraint is to set that $b_i \leq \theta k_i$. This corresponds to constraint (3) in Midrigan and Xu (2014). Other authors (Kocherlakota, 1992; Obiols-Homs, 2011) consider exogenous borrowing limits by imposing a short sales constraint: $b_i \leq B$ for any i. Under these three settings, the asset holding b_i is bounded from above by an upper bound which does not depend on the interest rate R.[9]

2.1 Equilibrium aggregate production and aggregate TFP

Given an equilibrium $(R, (k_i, b_i)_i)$, the aggregate output is $Y = \sum_i A_i f_i(k_i)$. Since Y depends on fundamentals, we write

$$Y = Y\big((A_i), (f_i), (S_i), (\gamma_i)\big). \tag{7}$$

The production in the market economy with financial imperfections depends on the distributions of productivity, initial wealth and credit limits.

In an economy with perfect financial market, the aggregate production is simply determined by

$$Y^{perfect} \equiv \max_{(k_i)\geq 0} \sum_i F_i(k_i) \quad \text{subject to :} \quad \sum_i k_i \leq S \equiv \sum_i S_i. \tag{8}$$

[9] Carosi, Gori, and Villanacci (2009) present a two-period general equilibrium model with uncertainty, numéraie assets, and participation constraints described by functions of agent's choices and prices. Carosi, Gori, and Villanacci (2009) prove the existence of equilibrium and study indeterminacy but do not provide comparative statics.

$Y^{perfect}$ is increasing in A_i, $\forall i$, and in S.

In equilibrium, we have $\sum_i k = S$. So, we have that $Y \leq Y^{perfect}$. This is consistent with a number of studies on the macroeconomic effects of financial constraints (Buera and Shin, 2013; Karaivanov and Townsend, 2014; Midrigan and Xu, 2014; Moll, 2014; Catherine, Chaney, Huang, Sraer, and Thesmar, 2017).

If we assume that $F_i(k) = A_i f(k)$ where A_i represents the individual productivity of agent i and f is a production function, then we can define the aggregate production function G and the aggregate TFP A by

$$\text{the aggregate TFP: } A \equiv \frac{Y}{F(S)} \tag{9}$$

$$\text{the aggregate production function: } G(S) \equiv Y = Af(S). \tag{10}$$

Our main goal is to explore conditions under which the aggregate output Y given by (7) and TFP A given by (9) are increasing or decreasing in individual productivity A_i and credit limit γ_i.

2.2 Individual choice and characterization of equilibrium

2.2.1 Individual optimal choice

We firstly study the individual optimal allocation of agents. When technologies are linear (where Inada condition does not hold), it is easy to obtain the following result

Lemma 2 (individual choice - linear production function)**.** *Assume that $F_i(K) = A_i K$. Let $R > 0$ be given. The solution for agent i's maximization problem is described as follows.*

1. *If $R \leq \gamma_i A_i$, then there is no solution ($k_i = \infty$).*
2. *If $A_i > R > \gamma_i A_i$, then agent i borrows from the financial market and the borrowing constraint is binding. We have $k_i = \frac{R}{R-\gamma_i A_i} S_i, a_i = \frac{\gamma_i A_i}{R-\gamma_i A_i} S_i, \pi_i = A_i k_i - R b_i = \frac{R(1-\gamma_i)}{R-\gamma_i A_i} A_i S_i$.*
3. *If $A_i = R$, then the solutions for the agent's problem include all pairs (k_i, b_i) such that $-S_i \leq b_i \leq \frac{\gamma_i}{1-\gamma_i} S_i$ and $k_i = b_i + S_i$.*
4. *If $A_i < R$, then agent i does not produce goods and invest all her initial wealth in the financial market: $k_i = 0, b_i = -S_i$.*

Next, we consider the case of strictly concave technology.

Assumption 2. *For any i, the function f_i is strictly increasing, strictly concave, twice continuously differentiable, $f_i(0) = 0$, $f_i(\infty) = \infty$, $f_i'(0) = \infty$, $f_i'(\infty) = 0$.*

Before present the properties of individual optimal choice, we introduce some notations:

Definition 2. *Given R, γ_i, A_i, S_i, denote $k_i^n = k_i^n(R/A_i)$ the unique solution to the equation $A_i f_i'(k) = R$ and $k_i^b = k_i^b(\frac{R}{\gamma_i A_i}, S_i)$ the unique solution to $R(k - S_i) = \gamma_i A_i f_i(k)$.*

k_i^b (k_i^n, respectively) represents the capital level of agent i when her borrowing constraint is binding (not binding, respectively). Under assumptions in Lemma 2, we can verify that: (1) k_i^n is strictly decreasing in R/A_i. Moreover, $\lim_{R/A_1 \to 0} k_i^n = +\infty$, and $\lim_{R/A_1 \to \infty} k_i^n = 0$. (2) k_i^b is strictly increasing in S_i but strictly decreasing in $\frac{R}{\gamma_i A_i}$. Moreover, $\lim_{R/A_i \to 0} k_i^b = +\infty$, and $\lim_{R/A_i \to \infty} k_i^b = S_i$.

The following result characterizes the solution of the problem (P_i).

Lemma 3 (individual choice - strictly concave production function)**.** *Under Assumption 2, there exists a unique solution to the problem (P_i). The optimal capital k_i is increasing in TFP A_i, credit limit γ_i but decreasing in the interest rate R.*

1. *If $R\frac{k_i^n(R/A_i)-S_i}{A_i f_i(k_i^n(R/A_i))} \geq \gamma_i$, then credit constraint is binding and the capital level is $k_i = k_i^b$. Moreover, $k_i = k_i^b \leq k_i^n$.*

2. *If $R\frac{k_i^n(R/A_i)-S_i}{A_i f_i(k_i^n(R/A_i))} < \gamma_i$, then credit constraint is not binding and $k = k_i^n$. In this case, we have $k_i = k_i^n < k_i^b$.*

Proof. See Appendix A. □

Lemma 3 leads us to introduce the following assumption.

Assumption 3. *The function $\frac{kf_i'(k)}{f_i(k)}$ is increasing in k.*

Under Assumptions 2 and 3, the function $\frac{(k-S_i)f_i'(k)}{f_i(k)}$ is strictly increasing in k. Therefore, the function $G_i(x) \equiv \frac{(k_i^n(x)-S_i)x}{f_i(k_i^n(x))}$ is strictly decreasing in x. Moreover, we can check that $\lim_{x \to +\infty} G_i(x) = -\infty$, $\lim_{x \to 0} G_i(x) = \lim_{k \to \infty} \frac{kf_i'(k)}{f_i(k)}$. By consequence, we obtain the following result.

Corollary 1. *Let Assumptions 2 and 3 be satisfied. So, if agent i's borrowing constraint is binding, we must have $\gamma_i \leq \lim_{x \to \infty} \frac{xF_i'(x)}{F_i(x)}$. By consequence, when $F_i(k) = A_i k^{\alpha_i}$ and $\gamma_i > \alpha_i$, then agent i's borrowing constraint is not binding.*

The following assumption and result show the interaction between interest rate, credit limit γ_i and borrowing constraint.

Assumption 4. $\gamma_i < \lim_{k \to \infty} \frac{kf_i'(k)}{f_i(k)}$.

Corollary 2. *Let Assumptions 2, 3 and 4 be satisfied. We can define R_i the unique value satisfying*

$$H_i(R_i) \equiv R_i \frac{k_i^n(R_i/A_i) - S_i}{A_i f_i(k_i^n(R_i/A_i))} = \gamma_i. \tag{11}$$

Then, we have that:

1. *Agent i's borrowing constraint is binding if and only if $H_i(R) \geq \gamma_i$ which is equivalent to $R \leq R_i \equiv H_i^{-1}(\gamma_i)$.*

2. *R_i/A_i does not depend on A_i, and $\lim_{A_i \to \infty} R_i = \infty$, $\lim_{A_i \to 0} R_i = 0$. R_i is increasing in productivity A_i but decreasing in γ_i and in S_i.*

3. *We also have* $k_i^n(R_i/A_i) = k_i^b(R_i/A_i)$.

The threshold R_i is exogenous. It represents the subjective interest rate of agent below which agent borrows so that her(his) borrowing constraint is binding. Point 2 of Lemma 2 indicates that the credit constraint of agent i is more likely to bind if the interest rate, her initial wealth and credit limit are low, and/or her productivity is high.

Remark 2. *Under Cobb-Douglas technology, i.e.,* $F_i(k) = A_i k^\alpha$*, we can compute that* $H_i(R) = \alpha\big(1 - \big(\frac{R}{\alpha A_i S_i^{\alpha-1}}\big)^{\frac{1}{1-\alpha}}\big)$ *is decreasing in* R *and* $H_i(0) = \alpha$*. So, if* $\alpha_i < \gamma_i$*, then borrowing constraint is not binding, whatever the level of interest rate* R*. When* $H_i(0) > \gamma_i$*, i.e.,* $\alpha > \gamma_i$*, we have* $R_i = \alpha A_i S_i^{\alpha-1}\big(1 - \frac{\gamma_i}{\alpha}\big)^{1-\alpha}$.

2.2.2 Characterization of general equilibrium

Thanks to above analyses, we can fully characterize the general equilibrium in two cases: (1) linear technology and (2) Assumptions 2, 3 and 4 are satisfied. We only state here our result for the case of strictly concave technologies. See Theorem 2 in Appendix A for the case of linear technology

Theorem 1 (characterization of general equilibrium: strictly concave technologies)**.** *Under Assumption 2, there exists a unique equilibrium. Assume, in addition, that Assumption 3 and 4 hold and* $R_1 < R_2 < \ldots < R_m$*, then the unique equilibrium is determined as follows:*

1. *In the regime* $\mathcal{R}_m$*, i.e., when*

$$S < \sum_{i=1}^{m} k_i^n(R_m/A_i), \tag{12}$$

credit constraint of any agent is not binding. In this case, the equilibrium coincides to that of the economy without credit constraints, and the interest rate is $R = R^* > R_m$*. Agent* i *borrows* ($k_i \geq S_i$) *if and only if* $F_i'(S_i) \geq R^*$.

2. *In the regime* $\mathcal{R}_n$ *(with* $1 \leq n \leq m-1$*), i.e., when*

$$\sum_{i=1}^{n} k_i^n(\frac{R_n}{A_i}) + \sum_{i=n+1}^{m} k_i^b(\frac{R_n}{\gamma_i A_i}, S_i) > S \geq \sum_{i=1}^{n+1} k_i^n(\frac{R_{n+1}}{A_i}) + \sum_{i=n+2}^{m} k_i^b(\frac{R_{n+1}}{\gamma_i A_i}, S_i), \tag{13}$$

the equilibrium interest rate and agents' capital are determined by

$$\sum_{i=1}^{n} k_i^n(\frac{R}{A_i}) + \sum_{i=n+1}^{m} k_i^b(\frac{R}{\gamma_i A_i}, S_i) = S \equiv \sum_i S_i \tag{14a}$$

$$k_i = \begin{cases} k_i^n(\frac{R}{A_i}) & \text{if } i \leq n \\ k_i^b(\frac{R}{\gamma_i A_i}, S_i) & \text{if } i \geq n+1. \end{cases} \tag{14b}$$

Notice that $R_n < R \leq R_{n+1}$ *in this case. Any agent* i $(i \geq n+1)$ *borrows and her credit constraint is binding. The credit constraint of any agent* $i \leq n$ *is not binding. Moreover, agent* i $(i \leq n)$ *borrows if and only if* $F_i'(S_i) \geq R$.

Proof. See Appendix A.1. □

3 Impacts of productivity and credit shocks

This section aims to address our main question: How the aggregate output changes when productivities (A_i) and credit limits (γ_i) vary. First, given the interest rate R, the optimal quantity of physical capital k_i of agent i depends on R, her productivity A_i and credit limit γ_i. So, we write $k_i = k_i(A_i, \gamma_i, R)$. Second, in equilibrium, the equilibrium interest rate depends on agents' characteristics, including (A_i) and (γ_i). So, we can write $R = R(A_1, A_2, \dots, A_m, \gamma_1, \dots, \gamma_m)$. By consequence, the physical capital of agents and the aggregate output are

$$k_i = k_i\big(A_i, \gamma_i, R(A_1, A_2, \dots, A_m, \gamma_1, \dots, \gamma_m)\big), \quad Y = \sum_i A_i f_i(k_i)$$

3.1 Impacts of productivity shocks

In this section, we investigate conditions under which the aggregate production is increasing or decreasing when productivities (A_i) change.

3.1.1 A motivating example

Let us start by considering a simple economy having two agents with linear production functions $F_i(k) = A_i k$ with $\gamma_2 A_2 < A_1 < A_2$. In this case, we can explicitly compute the equilibrium interest rate and aggregate output (see Theorem 2 in Appendix A.2):

$$Y = \begin{cases} A_2(S_1 + S_2) & \text{if } A_1 < \gamma_2 A_2 \frac{S_1+S_2}{S_1} \\ A_1 S_1 + A_2 S_2 \dfrac{A_1(1-\gamma_2)}{A_1 - \gamma_2 A_2} & \text{if } A_1 \geq \gamma_2 A_2 \frac{S_1+S_2}{S_1} \end{cases} \tag{15}$$

This allows us to fully investigate the effects of productivity shocks. First, the aggregate output is always increasing in A_2 - the productivity of the most productive agent. Second, looking at the effect of A_1, we see that when $\gamma_2 A_2 \frac{S_1+S_2}{S_1} < A_1$, we have $\frac{\partial Y}{\partial A_1} = S_1 - \frac{(1-\gamma_2)\gamma_2 A_2^2 S_2}{(A_1 - \gamma_2 A_2)^2}$, and

$$\frac{\partial Y}{\partial A_1} \geq 0 \Leftrightarrow \frac{S_1}{S_2}\big(\frac{A_1}{A_2} - \gamma_2\big)^2 \geq (1-\gamma_2)\gamma_2 \tag{16}$$

So, the aggregate output displays an U-shape as a function of the least productive agent's credit limit. It is increasing in A_1 if the productivity ratio A_1/A_2 is higher than a threshold (or, equivalently, the productivity gap A_2/A_1 is lower than a threshold). Figure 1 illustrates an example. In this numerical simulation, we set $S_1 = 1, S_2 = 0.7$, $A_2 = 1, \gamma_2 = 0.2$, and let A_1 vary from $\gamma_2 A_2 \frac{S_1+S_2}{S_1} = 0.34$ to $A_2 = 2$. Then the output, as a function of A_1, is decreasing on the interval $(0.34, 0.54]$ and then increasing in the interval $(0.54, 1)$

We now consider the case where both productivities A_1 and A_2 change.

Proposition 1 (productivity shocks)**.** *Consider a two-agent economy having linear technologies $F_i(k) = A_i k$ $\forall i = 1, 2$ with $A_1 < A_2$, and borrowing constraints: $Rb_i \leq \gamma_i A_i k_i$.*

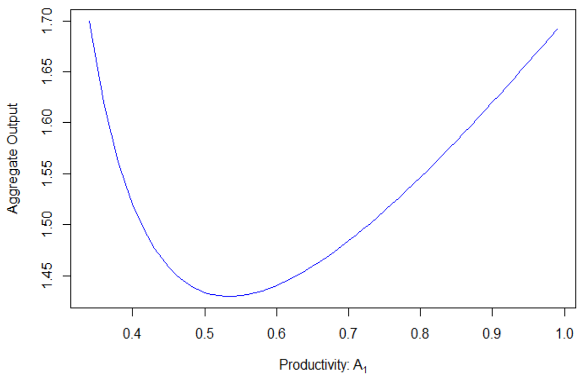


Figure 1: Non-monotonic effect of the agent 1's productivity.

Assume that there is a productivity shock that changes the productivity of agents from (A_1, A_2) *to* (A'_1, A'_2). *Assume that* $A'_2 > A'_1$. *Assume that the credit constraint of agent 2 is low so that* $\gamma_2 < \frac{A_1}{A_2}\frac{S_1}{S_1+S_2}$ *and* $\gamma_2 < \frac{A'_1}{A'_2}\frac{S_1}{S_1+S_2}$. *Then, the output change is*

$$Y(A'_1, A'_2) - Y(A_1, A_2) = (A'_1 - A_1)S_1 + A_2S_2(1-\gamma_2)\frac{A_1A'_2 - A'_1A_2}{(A_1 - \gamma_2A_2)(A'_1 - \gamma_2A'_2)} \quad (17)$$

(1) We have that:

$$\textit{If } \frac{A'_2}{A_2} \geq \frac{A'_1}{A_1} \geq 1, \textit{ then } Y(A'_1, A'_2) \geq Y(A_1, A_2) \quad (18)$$

(2) Assume that

$$S_2A_2(1-\gamma_2)\frac{\gamma_2A_2}{(A_1-\gamma_2A_2)^2} - S_1 > 0, \; \textit{i.e.}, \; \frac{S_1}{S_2}\Big(\frac{A_1}{A_2} - \gamma_2\Big)^2 < (1-\gamma_2)\gamma_2 \quad (19)$$

Then, there is a neighborhood $\mathcal{B}$ *of* (A_1, A_2) *such that*

$$\frac{Y(A'_1, A'_2) - Y(A_1, A_2)}{A'_1 - A_1} < 0 \quad (20a)$$

$$\forall (A'_1, A'_2) \in \mathcal{B} \textit{ satisfying } \frac{\frac{A'_2}{A_2} - 1}{\frac{A'_1}{A_1} - 1} < \frac{\gamma_2A_2}{A_1} - \frac{S_1(A_1-\gamma_2A_2)^2}{S_2A_1A_2(1-\gamma_2)} \textit{ and } A'_1 \neq A_1. \quad (20b)$$

Proof. See Appendix B. □

Condition (18) says that the aggregate output increases if the productivity of both firms increases and the productivity of the most productive agent increases faster than that of the less productive one.

Let us now focus on point 2 of Proposition 1. Here, condition (19) plays a very important role. It is satisfied if the ratio $\frac{A_1}{A_2}$ is low in the sense that $\frac{A_1}{A_2} < \gamma_2 + \big(\frac{\gamma_2(1-\gamma_2)S_2}{S_1}\big)^{0.5}$. This can be interpreted as a high productivity dispersion. Under this condition, we see that $\frac{\gamma_2 A_2}{A_1} - \frac{S_1(A_1-\gamma_2 A_2)^2}{S_2 A_1 A_2(1-\gamma_2)} \in (0,1)$. According to conditions (19) and (20a), under a positive shock that improves the TFP of all agents, the aggregate output may decrease:

$$Y(A_1', A_2') < Y(A_1, A_2), \forall A_1' > A_1, A_2' > A_2, (A_1', A_2') \in \mathcal{B} \text{ satisfying (20b)}.$$

Let us explain the economic intuition behind this result. Assume that the productivity dispersion is high and let us consider a small positive shock (both the TFP of both agents increases). If the productivity of the less productive agent increases faster than that of the most productive agent (i.e., $\frac{A_2'}{A_2}$ is low with respect to $\frac{A_1'}{A_1}$, see condition (20b)), the first agent absorbs more physical capital and the most productive agent gets less capital (i.e., $k_2(A_1', A_2') < k_2(A_1, A_2)$). By consequence, the aggregate output may decrease.

3.1.2 Individual productivity shocks

We now work with more general models. Consider an equilibrium and agent j whose capital k_j is strictly positive. Assume that we have the differentiability. In this case, condition $\sum_i k_i = S$ implies that $\frac{\partial k_j}{\partial A_j} + \frac{\partial R}{\partial A_j}\sum_{i=1}^m \frac{\partial k_i}{\partial R} = 0$. Thus, we have $\frac{\partial R}{\partial A_j} > 0$ and

$$\frac{\partial k_j}{\partial R}\frac{\partial R}{\partial A_j} + \frac{\partial k_j}{\partial A_j} = -\sum_{i\neq j}\frac{\partial k_i}{\partial R}\frac{\partial R}{\partial A_j} > 0 \tag{21}$$

where note that $\frac{\partial k_j}{\partial A_j} > 0$, $\frac{\partial k_i}{\partial R} < 0\ \forall i \neq j$.

We see that an increase in the TFP A_j of agent j would increase the equilibrium interest rate and the capital of agent j but decrease the capital of other agents. We can now explore the effect of the individual productivity A_j on the aggregate production.

Proposition 2 (effects of productivity shocks - general decompositions)**.** *Consider an equilibrium with $k_j > 0$. Let A_j vary and assume that the equilibrium outcomes are differentiable functions. We have*

$$\frac{\partial Y}{\partial A_j} = \underbrace{f_j(k_j) + A_j f_j'(k_j)\underbrace{\left(\frac{\partial k_j}{\partial R}\frac{\partial R}{\partial A_j} + \frac{\partial k_j}{\partial A_j}\right)}_{\geq 0}}_{\textit{Added production of agent } j} + \underbrace{\sum_{i\neq j} A_i f_i'(k_i)\underbrace{\frac{\partial k_i}{\partial R}}_{\leq 0}\underbrace{\frac{\partial R}{\partial A_j}}_{\geq 0}}_{\textit{Production losses of other agents}} \tag{22}$$

$$\frac{\partial Y}{\partial A_j} = \underbrace{f_j(k_j) + A_j f_j'(k_j)\underbrace{\frac{\partial k_j}{\partial A_j}}_{\geq 0}}_{\textit{Quantity effect}} + \underbrace{\sum_{i} A_i f_i'(k_i)\underbrace{\frac{\partial k_i}{\partial R}}_{\leq 0}\underbrace{\frac{\partial R}{\partial A_j}}_{\geq 0}}_{\textit{Price effect}} \tag{23}$$

$$\frac{\partial Y}{\partial A_j} = \underbrace{f_j(k_j)}_{\textit{Productivity effect}} + \underbrace{\sum_{i\neq j}\left(A_j f_j'(k_j) - A_i f_i'(k_i)\right)\underbrace{\frac{-\partial k_i}{\partial R}}_{\geq 0}\underbrace{\frac{\partial R}{\partial A_j}}_{\geq 0}}_{\textit{Allocation effect}}. \tag{24}$$

By consequence, $\partial Y/\partial A_j \geq 0$, $\forall j \in \mathcal{I}$, where $\mathcal{I} = \arg\max_{i>n}\{A_i f_i'(k_i)\}$. The aggregate output increases in A_i if the firm i has the highest total marginal factor productivity:

While we directly get (22) and (23) by taking the derivative of Y with respect to A_j, (24) is a consequence of (21).

Proposition 2 provides different interpretations of the effects of individual productivity shocks. For instance, when productivity A_j increases, the production of agent j increases but that of other agents decrease. This mechanism is represented by (22).

However, conditions in Proposition 2 are based on endogenous variables. The following result, based on exogenous parameters, shows the role of credit limit on the effect of productivity change.

Corollary 3. *1. Consider the case of strictly concave technology and let Assumptions 2 and 3 be satisfied. If one of the following conditions*

a The credit limit of agent any is high, in the sense that $\gamma_i > \lim_{x\to\infty} \frac{xf_i'(x)}{f_i(x)}, \forall i$.

b $\gamma_i < \lim_{x\to\infty} \frac{xf_i'(x)}{f_i(x)}, \forall i$, $R_1 < R_2 < \cdots < R_m$, and $S < \sum_{i=1}^m k_i^n(R_m/A_i)$.

is satisfied, the equilibrium coincides to that in the economy without frictions, and hence, the equilibrium aggregate output is increasing in each individual productivity A_i

2. Consider the linear technology: $F_i(k) = A_i k$ with $A_1 < \cdots < A_m$. We have $Y \leq Y^{perfect} \equiv A_m \sum_i S_i$. Moreover, $Y = Y^*$ if and only if $f_m A_m \geq A_{m-1}(1 - \frac{S_m}{S})$.

The intuition of point 1.a is that condition $\gamma_i > \lim_{x\to\infty} \frac{xf_i'(x)}{f_i(x)}$ ensures that agent i's borrowing constraint is not binding (see Corollary 1). By consequence, the equilibrium coincides to that in the economy without frictions. Therefore, the output is increasing in each productivity.

Under conditions in point 1.b of Corollary 3, Theorem 1 implies that the equilibrium coincides to that in the economy without frictions. Observe that $\sum_{i=1}^m k_i^n(R_m/A_i)$ is increasing in γ_m because R_i/A_i does not depend on A_i and $k_i^n(R/A_i)$ is decreasing in R/A_i. So, condition $S < \sum_{i=1}^m k_i^n(R_m/A_i)$ is more likely to be satisfied if the credit limit γ_m of the agent m who needs the credit the most (in the sense that $R_m > R_i$, $\forall i$) is quite high, then the credit constraints of this agent and all other ones are not binding. This is consistent with point 2 of Corollary 3.

To better understand 1.b, we look at the case where $F_i(k) = A_i k^\alpha$, $\forall i, \forall k$, with $\alpha > \gamma_i$ $\forall i$. According to Remark 2, we have $R_m = \alpha A_m S_m^{\alpha-1}(1 - \frac{\gamma_m}{\alpha})^{1-\alpha}$, and hence

$$\sum_{i=1}^m k_i^n(\frac{R_m}{A_i}) = \sum_{i=2}^m (\frac{\alpha A_i}{R_m})^{\frac{1}{1-\alpha}} = \sum_{i=1}^m (\frac{A_i}{A_m S_m^{\alpha-1}(1-\frac{\gamma_m}{\alpha})^{1-\alpha}})^{\frac{1}{1-\alpha}} = \sum_{i=1}^m (\frac{A_i}{A_m})^{\frac{1}{1-\alpha}} \frac{S_m}{1-\frac{\gamma_m}{\alpha}}.$$

So, condition $S < \sum_{i=1}^m k_i^n(R_m/A_i)$ becomes $\sum_{i=1}^m S_i < \sum_{i=1}^m (\frac{A_i}{A_m})^{\frac{1}{1-\alpha}} \frac{S_m}{1-\frac{\gamma_m}{\alpha}}$. This can be satisfied if γ_m is high in the sense that it is closed to α.[10]

It would be important to explore condition under which $\partial Y/\partial A_j$ may be negative. First, we focus on the case of strictly concave production functions.

[10]For instance, we can take $\gamma_i = \gamma < \alpha$, $S_i = s$, $\forall i$, and $A_1 < \cdots < A_m$. Then $R_1 < \cdots < R_m$. Moreover, $S < \sum_{i=1}^m k_i^n(R_m/A_i)$ becomes $m(1 - \frac{\gamma_m}{\alpha}) < \sum_{i=1}^m (\frac{A_i}{A_m})^{\frac{1}{1-\alpha}}$, which is satisfied if γ is closed to α.

Proposition 3 (effects of productivity shocks - strictly concave technology)**.** *Consider the case of strictly concave technology and let Assumptions 2 and 3 be satisfied. Assume also that* $R_2 < R_3 < \cdots < R_m$.

1. *There exists* $\bar{A}_1 > 0$ *such that the equilibrium output* Y *is increasing in* A_1 *on the interval* $(\bar{A}_1, \infty)$.

2. *Consider the case when* A_1 *is small. Denote*

$$D_2 = k_2^n(\frac{R_2}{A_2}) + \sum_{i=3}^{m} k_i^b(\frac{R_2}{\gamma_i A_i}, S_i), \quad D_3 = \sum_{i=2}^{3} k_i^n(\frac{R_3}{A_i}) + \sum_{i=3}^{m} k_i^b(\frac{R_3}{\gamma_i A_i}, S_i), \quad \dots$$
$$D_m = \sum_{i=2}^{m} k_i^n(\frac{R_m}{A_i})$$

Since $R_2 < R_3 < \cdots < R_m$, *we have* $D_2 > D_3 > \cdots > D_m > 0$.

   (a) *If* $S < D_m$, *then the output is increasing in* A_1 *when* A_1 *is small enough.*

   (b) *Assume that*

$$D_n > S > D_{n+1} \tag{25}$$
$$\gamma_i \frac{f_i(k)}{k f_i'(k)} < \frac{S_1}{S_1 + \sum_{t \geq n+1} S_t}, \forall i = n+1, \dots, m, \forall k \in (0, S) \tag{26}$$
$$\lim_{x \to +\infty} \frac{x}{f_1''(x)} < 0 \tag{27}$$

Then, for any A_1 *small enough, we have that* $\frac{\partial Y}{\partial A_1} < 0$.

Proof. See Appendix B. □

Proposition 3 shows the role of two important factors: credit limits (γ_i) and productivity A_1. According to point 1, when the productivity A_1 is high, a positive productivity shock is good for the aggregate output. The intuition behind is that when A_1 is high enough, the marginal productivity $A_1 f_1'(k_1)$ is the highest total marginal factor productivity, and hence, decomposition (24) ensures that $\frac{\partial Y}{\partial A_1} > 0$.

Regarding part 2.a of Proposition 3, condition $S < D_m$ is non-empty and can be satisfied with a large class of parameter.[11] Observe that D_m is increasing in $A_2, \dots, A_{m-1}$ but decreasing in A_m because R_i/A_i does not depend on A_i and $k_i^n(R/A_i)$ is decreasing in R/A_i. Moreover, D_m is increasing in agent m's credit limit γ_m. In other words, condition $S < D_m$ is likely to be satisfied if γ_m is quite high. In such a case, point 2.a ensures that, when A_1 is small enough, the credit constraints of all agents are not binding and hence the aggregate output is increasing in A_i, $\forall i \geq 1$.

[11] Indeed, let $F_i(k) = A_i k^\alpha$, $\forall i, \forall k$, with $\alpha > \gamma_i$. We have $R_m = \alpha A_m S_m^{\alpha-1}(1 - \frac{\gamma_m}{\alpha})^{1-\alpha}$, and hence $D_m = \sum_{i=2}^m (\frac{A_i}{A_m})^{\frac{1}{1-\alpha}} \frac{S_m}{1-\frac{\gamma_m}{\alpha}}$. When $S_i = s, \gamma_i = \gamma$, $\forall i$, and $A_2 < \cdots < A_m$, then we have $R_2 < \cdots < R_m$. Condition $S < D_m$ is equivalent to $m(1 - \frac{\gamma}{\alpha}) < \sum_{i=2}^m (\frac{A_i}{A_m})^{\frac{1}{1-\alpha}}$ which can be satisfied.

Let us now look at point 2.b. Condition $D_n > S > D_{n+1}$ ensures that when A_1 is low enough the credit constraint of any agent $i \geq n+1$ is binding while that of any agent $i \leq n$ is not. Condition (26) means that agents whose credit constraints are binding have a very low credit limit. In such a case, the aggregate output may be decreasing in productivity A_1 when A_1 is small enough. The fact that A_1 is very small ensures that the productivity dispersion is high. This is consistent with condition (16) in a two-agent model with linear production functions.

In the case of a two-agent model, we have the following result with more details and intuitive conditions.

Proposition 4. *Consider a two-agent model.*

1. *Let Assumptions 2, 3 and 4 be satisfied. Assume also that*

$$k_2^n(\frac{R_2}{A_2}) < S, \quad \gamma_2 < \frac{S_1}{S_1+S_2}\frac{Sf_2'(S)}{f_2(S)}, \quad \lim_{x\to+\infty}\frac{x}{f_1''(x)} < 0$$

   Then, for any A_1 small enough, we have that $\frac{\partial Y}{\partial A_1} < 0$.

2. *By consequence, in a two-agent economy with Cobb-Douglas production functions ($F_i(k) = A_i k^\alpha$) and $\gamma_2 < \alpha\frac{S_1}{S_1+S_2}$, we have $\frac{\partial Y}{\partial A_1} < 0$ for A_1 small enough.*

Proof. See Appendix B. □

With linear production functions, we have the following result which is a consequence of Theorem 2 in Appendix A.

Proposition 5 (effects of productivity shocks - linear technology)**.** *Assume that $F_i(k) = A_i k$ $\forall i, \forall k$ with $A_1 < \cdots < A_m$. Assume that $A_n > \max_i(\gamma_i A_i)$ and $\sum_{i=n+1}^m \frac{A_n S_i}{A_n-\gamma_i A_i} \leq S \leq \sum_{i=n}^m \frac{A_n S_i}{A_n-\gamma_i A_i}$. Then, the equilibrium interest rate $R^* = A_n$ and the aggregate output is*

$$Y = A_n\sum_{i=1}^n S_i + \sum_{i=n+1}^m \frac{A_n(1-\gamma_i)}{A_n-\gamma_i A_i}A_i S_i. \tag{28}$$

We also have that $\frac{\partial Y}{\partial A_j} > 0$, $\forall j > n$, and

$$\frac{\partial Y}{\partial A_n} = \sum_{i=1}^n S_i - \sum_{i=n+1}^m \frac{(1-\gamma_i)\gamma_i A_i^2 S_i}{(A_n-\gamma_i A_i)^2} \tag{29}$$

We can see clearly that $\frac{\partial Y}{\partial A_n}$ may have any sign. Since $\frac{\partial Y}{\partial A_n}$ is increasing in A_n, it can be negative when A_n is low and positive when A_n is high. This is consistent with our insights mentioned above.

3.1.3 General productivity shocks

Assume now that the TFP of all agents depends on an exogenous variable $x \in \mathbb{R}$ in the sense that $A_i = A_i(x)$ where A_i is a differentiable function of x and $A_i'(x) > 0$, $\forall i$. We wonder how the aggregate output changes when x varies. First, we have that

$$\begin{aligned} Y(x) &= \sum_i A_i(x) f_i\big(k_i(x)\big) = \sum_i A_i(x) f_i\Big(k_i\big(A_i(x), R(A_1(x), \ldots, A_m(x))\big)\Big) \\ \frac{\partial Y}{\partial x} &= \sum_i A_i'(x) f_i\big(k_i(x)\big) + \sum_i A_i(x) f_i'(k_i(x)) k_i'(x). \end{aligned} \tag{30}$$

Since the physical capital k_i depends on x, we simply write $k_i = k_i(x)$. We have

$$k_i'(x) = \frac{\partial k_i}{\partial R}\frac{\partial R}{\partial x} + \frac{\partial k_i}{\partial A_i}\frac{\partial A_i}{\partial x}, \quad \frac{\partial R}{\partial x} = \sum_j \frac{\partial R}{\partial A_j}\frac{\partial A_j}{\partial x}$$

Notice that $\frac{\partial k_i}{\partial A_i} \geq 0$, $\frac{\partial k_i}{\partial R} \leq 0$, $\frac{\partial A_i}{\partial x} \geq 0$, $\frac{\partial R}{\partial x} \geq 0$. So, we can expect that $k_i'(x)$ may have any sign. However, we have $\sum_i k_i'(x) = 0$ because $\sum_i k_i = S$ in equilibrium.

We have two decompositions which help us to understand why the aggregate output may be increasing or decreasing in the exogenous productivity shock x.

$$\begin{aligned} \frac{\partial Y}{\partial x} &= \underbrace{\sum_i A_i'(x) f_i\big(k_i(x)\big) + \sum_{i: k_i'(x) \geq 0} A_i(x) f_i'(k_i(x)) k_i'(x)}_{\text{Added production of agent } j} \\ &\quad + \underbrace{\sum_{i: k_i'(x) < 0} A_i(x) f_i'(k_i(x)) k_i'(x)}_{\text{Production losses of other agents}} \end{aligned} \tag{31}$$

$$\frac{\partial Y}{\partial x} = \sum_i A_i'(x) f_i\big(k_i(x)\big) + \underbrace{\sum_i A_i(x) f_i'(k_i(x)) \underbrace{\frac{\partial k_i}{\partial A_i}}_{>0} \underbrace{\frac{\partial A_i}{\partial x}}_{>0}}_{\text{Quantity effect}} + \underbrace{\sum_i A_i(x) f_i'(k_i(x)) \underbrace{\frac{\partial k_i}{\partial R}}_{<0} \underbrace{\frac{\partial R}{\partial x}}_{>0}}_{\text{Price effect}} \tag{32}$$

When the TFP of all agents changes at the same rate, we have the following result.

Proposition 6 (homogeneous productivity shocks)**.** *Consider an equilibrium. Assume that an exogenous shock makes the individual TFP vary from A_i to $A_i(x) = xA_i$, $\forall i$, where $x > 0$. Then, for this new economy, there is an equilibrium where $Y(x) = xY$, i.e., the aggregate output changes at the same rate.*

Proof. Denote $(R, (k_i, b_i))$ an equilibrium for the economy $\mathcal{E} \equiv (A_i, f_i, \gamma_i, S_i)_{i=1,\ldots,m}$ with borrowing constraints: $Rb_i \leq \gamma_i A_i f_i(k_i)$. We can check that $(R(x), (k_i, b_i))$, where $R(x) \equiv xR$, is an equilibrium for the new economy $\mathcal{E}(x) \equiv (A_i(x), f_i, \gamma_i, S_i)_{i=1,\ldots,m}$. In equilibrium, the new aggregate output is $Y(x) = \sum_i A_i(x) f_i(k_i) = xY$. □

Next, we consider the case where productivity changes are not proportional. In such a case, we argue that positive productivity shocks may reduce the aggregate output. Indeed, by using Taylor's theorem and Proposition 3, we obtain the following result.

Proposition 7 (asymmetric productivity shocks). *Consider an economy which satisfies conditions in case 2.(b) in Proposition 3, and $A_1 > 0$ small enough. Then, there exist $g \in (0,1)$ and a neighborhood $\mathcal{G}$ of $(A_1, \dots, A_m)$ such that*

$$\frac{Y(A'_1, \dots, A'_m) - Y(A_1, \dots, A_m)}{A'_1 - A_1} < 0, \tag{33}$$
$$\forall (A'_1, \dots, A'_m) \in \mathcal{G} \text{ satisfying } |\frac{A'_i - A_i}{A'_1 - A_1}| < g, \forall j.$$

Proof. Denote $A \equiv (A_1, \dots, A_m)$ and $A' \equiv (A'_1, \dots, A'_m)$. By Taylor's theorem, we have

$$Y(A') - Y(A) = \frac{\partial Y(A)}{\partial A_1}(A'_1 - A_1) + \sum_{i \geq 2} \frac{\partial Y(A)}{\partial A_i}(A'_i - A_i) + \sum_i h_i(A, A')(A'_i - A_i)$$

where $\lim_{A' \to A} h_i(A, A') = 0$.

We can choose $\epsilon < 0, g < 1$ and (A'_i) such that $\frac{A'_i - A_i}{A'_1 - A_1}| < g$ and $\frac{\partial Y(A)}{\partial A_1} + \sum_{i \geq 2} \frac{\partial Y(A)}{\partial A_i} \frac{A'_i - A_i}{A'_1 - A_1} < \epsilon < 0$. In this case, we get (33). □

There are two key points that ensure (33). The first condition is $\frac{\partial Y(A)}{\partial A_1} < 0$, i.e., the output is decreasing in A_1 in a neighborhood of $(A_1, \dots, A_m)$; notice that this may happen only if A_1 is small enough. Of course, we have $\frac{Y(A'_1, \dots, A'_m) - Y(A_1, \dots, A_m)}{A'_1 - A_1} > 0$ if $\frac{\partial Y(A)}{\partial A_i} > 0$, $\forall i$. The second condition is $|\frac{A'_i - A_i}{A'_1 - A_1}| < g$, i.e., the productivity does not change at the same rate and that the productivity of the less productive agent (agent 1) increases faster than that of the most productive agents. This implies that agent 1 absorbs more capital than other ones.

3.2 Impacts of credit shocks

In this section, we investigate the effects of credit limits (γ_t) on the aggregate production, which help us to understand better the relationship between finance and economic growth. A meaningful question is whether financial development has positive effects on the economic growth. In our model, relaxing credit limit (i.e., increasing γ_i) can be interpreted as reduction of financial friction or improvement of the financial sector.

Consider an equilibrium and an agent j whose borrowing constraint is binding. Then, the equilibrium interest rate R is increasing in γ_j. Assume that we have the differentiability. Then, $\left(\sum_{i=1}^m \frac{\partial k_i}{\partial R}\right) \frac{\partial R}{\partial \gamma_j} + \frac{\partial k_j}{\partial \gamma_j} = 0$, which imply that

$$\frac{\partial k_j}{\partial R} \frac{\partial R}{\partial \gamma_j} + \frac{\partial k_j}{\partial \gamma_j} = -\sum_{i \neq j} \frac{\partial k_i}{\partial R} \frac{\partial R}{\partial \gamma_j} \geq 0 \tag{34}$$

where $\frac{\partial k_j}{\partial \gamma_j} \geq 0$, $\frac{\partial k_i}{\partial R} \leq 0$.

3.2.1 Impacts of individual credit shocks

We firstly look at the effects of credit limits $(\gamma_i)_i$.

Proposition 8 (effects of credit shocks)**.** *Consider an equilibrium. The equilibrium outcomes do not depend on credit limits γ_i of agents whose borrowing constraints are not binding. For any agent j whose borrowing constraint is binding, let γ_j vary and assume that the equilibrium outcomes are differentiable functions. Then, we have decompositions:*

$$\frac{\partial Y}{\partial \gamma_j} = \underbrace{F_j'(k_j)\underbrace{\Big(\frac{\partial k_j}{\partial R}\frac{\partial R}{\partial \gamma_j} + \frac{\partial k_j}{\partial \gamma_j}\Big)}_{\geq 0}}_{\textit{Added production of agent } j} + \underbrace{\sum_{i\neq j} F_i'(k_i)\underbrace{\frac{\partial k_i}{\partial R}}_{\leq 0}\underbrace{\frac{\partial R}{\partial \gamma_j}}_{\geq 0}}_{\textit{Production losses of other agents}} \tag{35}$$

$$\frac{\partial Y}{\partial \gamma_j} = \underbrace{F_j'(k_j)\underbrace{\frac{\partial k_j}{\partial \gamma_j}}_{\geq 0}}_{\textit{Quantity effect}} + \underbrace{\sum_{i} F_i'(k_i)\underbrace{\frac{\partial k_i}{\partial R}}_{\leq 0}\underbrace{\frac{\partial R}{\partial \gamma_j}}_{\geq 0}}_{\textit{Price effect}} \tag{36}$$

$$\frac{\partial Y}{\partial \gamma_j} = \underbrace{\frac{\partial R}{\partial \gamma_j}}_{\geq 0}\sum_{i\neq j}\big(F_j'(k_j) - F_i'(k_i)\big)\underbrace{\frac{-\partial k_i}{\partial R}}_{\geq 0} \tag{37}$$

While we directly get (35) and (36) by taking the derivative of Y with respect to γ_j, condition (37) is a consequence of (35) and (34).

In (35), the first term represents the marginal added production of agent j when her credit limit γ_j is relaxed while the two last term represent the marginal production loss of other agents. The aggregate output increases in γ_j if the marginal added production exceeds the marginal production loss.

Condition (37) represents the redistributive effect of a change in the credit limit γ_j. According to (37), we obtain the following result.

Corollary 4. *Denote $\mathcal{I}_n = \arg\max_i\{F_i'(k_i)\}$. Thus, we have that $\partial Y/\partial \gamma_j \geq 0\ \forall j \in \mathcal{I}_n$, i.e., the aggregate output is increasing in the credit limit of agents having the highest marginal productivity.*

We now provide conditions under which the aggregate output may be decreasing in credit limits.

Proposition 9 (individual credit shock)**.** *Assume that $F_i(k) = A_i k\ \forall i, k$. Assume that $\max_i(\gamma_i A_i) < A_1 < \cdots < A_m$. Consider the case where the equilibrium interest rate is belong to the interval (A_n, A_{n+1}). Then, we have that:*

1. $\frac{\partial Y}{\partial \gamma_{n+1}} < 0 < \frac{\partial Y}{\partial \gamma_m}$ *if $n+1 < m$.*[12]

[12]Moreover, if $n + 1 = m$ (i.e., only agent m produces), we have $\frac{\partial Y}{\partial \gamma_m} = 0$.

2. *Consider an entrepreneur i with $n+1 < i < m$, we have that:*

$$\frac{\partial Y}{\partial \gamma_i} > 0 \text{ if } A_i \text{ is high enough, i.e., } \frac{A_i - A_{i-1}}{A_m - A_i} > \frac{\sum_{t=i+1}^{m} \frac{\gamma_t A_t S_t}{(A_n - \gamma_t A_t)^2}}{\sum_{t=n+1}^{i-1} \frac{\gamma_t A_t S_t}{(A_{n+1} - \gamma_t A_t)^2}} \tag{38a}$$

$$\frac{\partial Y}{\partial \gamma_i} < 0 \text{ if } A_i \text{ is low enough, i.e., } \frac{A_i - A_{n+1}}{A_{i+1} - A_i} < \frac{\sum_{t=i+1}^{m} \frac{\gamma_t A_t S_t}{(A_{n+1} - \gamma_t A_t)^2}}{\sum_{t=n+1}^{i-1} \frac{\gamma_t A_t S_t}{(A_n - \gamma_t A_t)^2}}. \tag{38b}$$

Proof. See Appendix C. □

Condition $\frac{\partial Y}{\partial \gamma_{n+1}} < 0$ indicates that an increasing of the credit limit of the least productive producer harms the aggregate output while condition $\frac{\partial Y}{\partial \gamma_m} > 0)$ has a similar interpretation as in Corollary 4.

According to (38a) and (38b), the aggregate output is more likely to be increasing (resp., decreasing) in the credit limit of an agent if the TFP of this producer is quite close to those of more productive entrepreneurs (resp., that of the least productive entrepreneur) or/and credit limits and initial wealths of more productive agents $(\gamma_t)_{t>i}$ are low.

We complement our above points by a numerical example.

Example 1. Consider a three-agent economy with linear production functions $F_i(k) = A_i k, \forall i, \forall k$, and borrowing constraints are $Rb_i \leq \gamma_i A_i k_i$. In Appendix C, we completely compute the equilibrium. Assume now that fundamentals are given by $S_1 = 4$, $S_2 = 4$, $S_3 = 3$, $A_1 = 1$, $A_2 = 1.2$, $A_3 = 1.5$, $\gamma_1 = 0.2$.

First, we set $\gamma_3 = 0.3$ and we let γ_2 vary. Figure 2 shows the effects of the agent 2's credit limit γ_2 on the equilibrium interest rate and the aggregate output. When γ_2 varies from 0.15 to 0.45, the interest rate varies from $A_1 = 1$ to $A_2 = 1.2$. The aggregate output is not monotonic functions of γ_2: It is increasing in γ_2 in the regime $\mathcal{A}_1$ where the interest rate $R = A_1$, but decreasing in γ_2 in the regime $\mathcal{R}_1$ where the interest rate $R = R_1$ (consistent with Proposition 9), and then constant in the regime $\mathcal{A}_1$ where $R = A_2$.

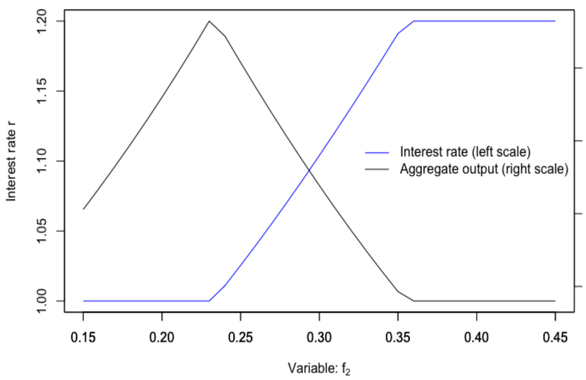


Figure 2: Non-monotonic effects of credit limit γ_2.

Second, we set $\gamma_2 = 0.3$ and let γ_3 vary. Figure 3 shows the effects of the most productive agent's credit limit γ_3 on the equilibrium interest rate and the aggregate output. The output is increasing in γ_3 (this is consistent with point 1 of Proposition 9).

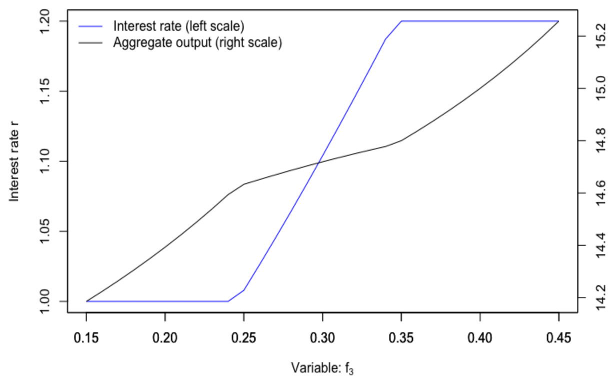


Figure 3: Monotonic effects of credit limit γ_3.

3.2.2 Impacts of general financial shocks

Assume that the credit limit of all agents depends on an exogenous variable $x \in \mathbb{R}$ in the sense that $\gamma_i = \gamma_i(x)$ where γ_i is a differentiable function of x and $\gamma_i'(x) > 0$.

We wonder how the aggregate output changes when x varies. We have that

$$Y(x) = \sum_i A_i f_i\big(k_i(x)\big) = \sum_i A_i f_i\Big(k_i\big(\gamma_i(x), R(\gamma_1(x), \dots, \gamma_m(x))\big)\Big). \tag{39}$$

Since the physical capital k_i depends on x, we have

$$k_i'(x) = \frac{\partial k_i}{\partial \gamma_i}\frac{\partial \gamma_i}{\partial x} + \frac{\partial k_i}{\partial R}\frac{\partial R}{\partial x}, \quad \frac{\partial R}{\partial x} = \sum_j \frac{\partial R}{\partial \gamma_j}\frac{\partial \gamma_j}{\partial x} \tag{40}$$

Recall that $\frac{\partial k_i}{\partial \gamma_i} \geq 0$, $\frac{\partial k_i}{\partial R} \leq 0$, $\frac{\partial \gamma_i}{\partial x} \geq 0$, $\frac{\partial R}{\partial x} \geq 0$ because $\frac{\partial R}{\partial x} = \sum_j \frac{\partial R}{\partial \gamma_j}\frac{\partial \gamma_j}{\partial x}$ and $\frac{\partial R}{\partial \gamma_j} \geq 0$, $\forall j$. So, we see that $k_i'(x)$ may have any sign. However, we know $\sum_i k_i'(x) = 0$ because $\sum_i k_i = S$ in equilibrium.

We have two decompositions which help us to understand why the aggregate output may be increasing or decreasing in the exogenous shock x.

$$\frac{\partial Y}{\partial x} = \underbrace{\sum_{i:k_i'(x)\geq 0} A_i f_i'(k_i(x))k_i'(x)}_{\text{Added production of agent } j} + \underbrace{\sum_{i:k_i'(x)<0} A_i f_i'(k_i(x))k_i'(x)}_{\text{Production losses of other agents}} \tag{41}$$

$$\frac{\partial Y}{\partial x} = \underbrace{\sum_i A_i f_i'(k_i(x)) \underbrace{\frac{\partial k_i}{\partial \gamma_i}}_{>0} \underbrace{\frac{\partial \gamma_i}{\partial x}}_{>0}}_{\text{Quantity effect}} + \underbrace{\sum_i A_i f_i'(k_i(x)) \underbrace{\frac{\partial k_i}{\partial R}}_{<0} \underbrace{\frac{\partial R}{\partial x}}_{>0}}_{\text{Price effect}} \tag{42}$$

We consider the case of homogeneous credit shock.

Proposition 10 (homogeneous credit shock)**.** *Assume either $F_i(k) = A_i k$, $\forall i, \forall k$ or Assumption 2 is satisfied. Assume also that $\gamma_i = \gamma \in (0,1)$, $\forall i$. Then the equilibrium aggregate output is an increasing function of the credit limit γ.*

Proof. See Appendix C. □

The intuition of the result is simple: all credit-constrained firms, who have higher marginal productivity, can borrow more from other agents who have lower marginal productivity, and hence produce more. This point is consistent with those in in Khan and Thomas (2013) (section VI. C), Midrigan and Xu (2014) (section II.B), Moll (2014) (Proposition 1), and Catherine, Chaney, Huang, Sraer, and Thesmar (2017).

We now assume that there is an aggregate shock on credit limits under which the new credit limits are $(\gamma'_i)_i$. Our novel point is that, even $\gamma'_i > \gamma_i \; \forall i$, the new aggregate output $Y' = Y(\gamma'_1, \dots, \gamma'_m)$ may be lower than $Y = Y(\gamma_1, \dots, \gamma_m)$. Formally, we have the following result.

Proposition 11 (general credit shocks)**.** *Assume that $F_i(k) = A_i k \; \forall i, k$, and $\max_i(\gamma_i A_i) < A_1 < \cdots < A_m$. Consider the regime $\mathcal{R}_n$ in which the equilibrium interest rate is in the interval $[A_n, A_{n+1})$. Consider an agent i such that $n+1 < i < m$ and assume that condition (38b) holds. Then there exist $g \in (0,1)$ and a neighborhood $\mathcal{G}$ of $(\gamma_1, \dots, \gamma_m)$ such that*

$$\frac{Y(\gamma'_1, \dots, \gamma'_m) - Y(\gamma_1, \dots, \gamma_m)}{\gamma'_i - \gamma_i} < 0, \tag{43}$$

$$\forall (\gamma'_1, \dots, \gamma'_m) \in \mathcal{G} \textit{ satisfying } \left|\frac{\gamma'_j - \gamma_j}{\gamma'_i - \gamma_i}\right| < g, \forall j \neq i.$$

We can apply the same argument used in Proposition 7 to prove Proposition 11.

Proposition 11 shows that the aggregate output may be reduced even the credit limits of all agents increase (i.e., $\gamma'_i > \gamma_i, \forall i$). It complements Proposition 10, Proposition 9, and those in Buera and Shin (2013), Khan and Thomas (2013), Midrigan and Xu (2014), Moll (2014), Catherine, Chaney, Huang, Sraer, and Thesmar (2017). Recall that these studies provide conditions under which relaxing credit limits has positive impact on the aggregate output.

3.3 Impacts of both productivity and financial shocks

Assume that the TFP and credit limit of all agents depend on an exogenous variable $x \in \mathbb{R}^+$ in the sense that $A_i = A_i(x)$, and $\gamma_i = \gamma_i(x)$ where A_i, γ_i is a differentiable function of x. The equilibrium aggregate production is

$$Y(x) = \sum_i A_i(x) f_i\Big(k_i\Big(A_i(x), \gamma_i(x), R\big(A_1(x), \dots, A_m(x), \gamma_1(x), \dots, \gamma_m(x)\big)\Big)\Big). \tag{44}$$

We have

$$\frac{\partial k_i}{\partial x} = \frac{\partial k_i}{\partial A_i}\frac{\partial A_i}{\partial x} + \frac{\partial k_i}{\partial \gamma_i}\frac{\partial \gamma_i}{\partial x} + \frac{\partial k_i}{\partial R}\frac{\partial R}{\partial x} \text{ and } \frac{\partial R}{\partial x} = \sum_i \frac{\partial R}{\partial A_i}\frac{\partial A_i}{\partial x} + \sum_i \frac{\partial R}{\partial \gamma_i}\frac{\partial \gamma_i}{\partial x}$$

If $A_i'(x) \geq 0$ and $\gamma_i'(x) \geq 0$, then we have $\frac{\partial R}{\partial x} \geq 0$. Since $\frac{\partial k_i}{\partial R} \leq 0$ (price effect), the capital stock k_i can be decreasing or increasing in the exogenous shock x. However, we have $\sum_i k_i'(x) = 0$ since $\sum_i k_i = \sum_i S_i$ is exogenous. By consequence, we have the following decomposition of the effect of the exogenous shock x:

$$\begin{aligned}\frac{\partial Y}{\partial x} &= \underbrace{\sum_i A_i'(x) f_i\big(k_i(x)\big) + \sum_{i: k_i'(x) \geq 0} A_i(x) f_i'(k_i(x)) k_i'(x)}_{\text{Added production of agent } j} \\ &\quad + \underbrace{\sum_{i: k_i'(x) < 0} A_i(x) f_i'(k_i(x)) k_i'(x)}_{\text{Production losses of other agents}}. \end{aligned} \tag{45}$$

We can also decompose $\frac{\partial Y}{\partial x}$ into quantity and price effects:

$$\begin{aligned}\frac{\partial Y}{\partial x} &= \underbrace{A_i'(x) f_i\big(k_i(x)\big) + \sum_i A_i(x) f_i'(k_i(x)) \Big(\underbrace{\frac{\partial k_i}{\partial A_i}}_{\geq 0} \underbrace{\frac{\partial A_i}{\partial x}}_{\geq 0} + \underbrace{\frac{\partial k_i}{\partial \gamma_i}}_{\geq 0} \underbrace{\frac{\partial \gamma_i}{\partial x}}_{\geq 0} \Big)}_{\text{Quantity effect}} \\ &\quad + \underbrace{\sum_i A_i(x) f_i'(k_i(x)) \underbrace{\frac{\partial k_i}{\partial R}}_{\leq 0} \underbrace{\frac{\partial R}{\partial x}}_{\geq 0}}_{\text{Price effect}} \end{aligned} \tag{46}$$

From these decompositions and our analyeses above, we can see that the aggregate output Y and the aggregate productivity defined by (9) may be increasing or decreasing in the shock x, depending on the nature of shocks and the distributions of productivity, credit limit and endowments.

3.4 Productivity growth, productivity dispersion and credit constraint

Consider the case $F_i(k) = A_i f(k), \forall i, \forall k$. The aggregate productivity TFP is defined by $TFP = Y/f(S)$. Assume that there is a shock that changes productivity from A_i to A_i' and credit limit from γ_i to γ_i'. The new TFP of the economy is $TFP' = Y'/f(S)$. We have

$$\frac{TFP'}{TFP} = \frac{\frac{Y(A_1', \ldots, A_m')}{f(S)}}{\frac{Y(A_1, \ldots, A_2)}{f(S)}} = \frac{Y(A_1', \ldots, A_m')}{Y(A_1, \ldots, A_2)}$$

We aim to understand the relationship between the aggregate productivity growth $\frac{TFP'}{TFP}$ and individual ones $\frac{A_1'}{A_1}, \ldots, \frac{A_m'}{A_m}$.

In the economy without frictions, by using the definition (8) we have that

$$\frac{TFP'}{TFP} = \frac{\max\{\sum_i A_i' f(k_i) : k_i \geq 0, \sum_i k_i \leq S\}}{\max\{\sum_i A_i f(k_i) : k_i \geq 0, \sum_i k_i \leq S\}}$$

Observe that $min_i\{\frac{A'_i}{A_i}\}A_if(k_i) \leq A'_if(k_i) \leq max_i\{\frac{A'_i}{A_i}\}A_if(k_i)$. So, obtain that $min_i\{\frac{A'_i}{A_i}\} \leq \frac{TFP'}{TFP} \leq max_i\{\frac{A'_i}{A_i}\}$.

However, when we consider economies with credit constraints, our above analyses (see Propositions 1, 3, 7, 11) show that the aggregate productivity growth $\frac{TFP'}{TFP}$ may be less than $min_i\{\frac{A'_i}{A_i}\}$. Indeed, for instance, we can choose (A_i) and (A'_i) so that all conditions in Proposition 7 are satisfied and $min_i\{\frac{A'_i}{A_i}\} > 1$. In this case, we have $Y(A'_1, \ldots, A'_m) - Y(A_1, \ldots, A_m) < 0$, or, equivalently, $\frac{TFP'}{TFP} < 1$. We summarize our points in the following result.

Proposition 12 (productivity growth, productivity dispersion and credit constraint). *Consider the case $F_i(k) = A_if(k)$, $\forall i, \forall k$. Assume that there is a shock that changes productivity from A_i to A'_i and credit limit from γ_i to γ'_i.*

1. *In the economy without frictions, we always have*

$$min_i\{\frac{A'_i}{A_i}\} \leq \frac{TFP'}{TFP} \leq max_i\{\frac{A'_i}{A_i}\} \tag{47}$$

2. *Consider economies with credit constraints $\mathcal{E} \equiv (A_i, f_i, \gamma_i, S_i)_{i=1,\ldots,m}$.*

   (a) *If $\frac{A'_i}{A_i} = g > 0$, $\forall i$, then Proposition 6 implies that $\frac{TFP'}{TFP} = g$.*

   (b) *However, under some situations as in Propositions 1, 3, 7, 11, we may have that*

$$\frac{TFP'}{TFP} < min_i\{\frac{A'_i}{A_i}\}. \tag{48}$$

By consequence, the aggregate productivity growth rate may be far from that of most productive firms. It may be even lower than the smallest productivity growth rate of firms. Our points contribute to the debate concerning the slowdown in aggregate productivity growth. For instance, by using data in 23 OECD countries over the 2000s, Andrews, Criscuolo and Gal (2015) document a slowdown in aggregate productivity growth, a rising productivity gap between the global frontier and other firms, and that productivity growth at the global frontier remained robust.

As recognized by Goldin, Koutroumpis, Lafond, and Winkler (2021), there is no single reason for the slowdown in aggregate productivity growth. We provides a supply-side point of view by using a general equilibrium model with credit constraint. Our above analyses suggest that the interplay between credit constraints, high heterogeneity of productivity, asymmetry of productivity and financial shocks may generate a slowdown in aggregate productivity growth.

4 Extension: Models à la Ramsey

We now consider an infinite-horizon model with credit constraints. Agent i maximizes her intertemporal utility subject to budget and borrowing constraints

$$\max_{(c_i,k_i,b_i)} \sum_{t=0}^{\infty} \beta_i^t u_i(c_{i,t}) \tag{49a}$$

$$\text{subject to: } c_{i,t} + k_{i,t} + R_t b_{i,t-1} \leq F_{i,t}(k_{i,t-1}) + b_{i,t} \tag{49b}$$

$$R_{t+1} b_{i,t} - \gamma_i F_{i,t}(k_{i,t}) \leq 0 \tag{49c}$$

where we assume that $b_{i,-1} = 0 \ \forall i$ and denote $w_{i,0} = F_{i,0}(k_{i,-1})$.

We make standard assumptions.

Assumption 5. *u_i is concave, continuously differentiable, $u_i'(0) = +\infty$.*

Definition 3. *An equilibrium is a list $((c_{i,t}, k_{i,t}, b_{i,t})_i, R_t)_{t\geq 0}$ such that (1) given (R_t), the allocation $(c_{i,t}, k_{i,t}, b_{i,t})$ is a solution of the above maximization problem, and (2) markets clear: $\sum_i b_{i,t} = 0$, $\sum_i (c_{i,t} + k_{i,t}) = \sum_i F_{i,t}(k_{t-1})$, $\forall t$.*[13]

Remark 3 (steady state analysis)**.** *Let Assumptions 1, 5 be satisfied. Assume that $F_{i,t} = F_i$, i.e., does not depend on time. Consider a steady state equilibrium with $k_i > 0$, $\forall i$.*

1. *The steady state interest rate is $R = 1/\max_i\{\beta_i\}$.*

2. *Assume, in addition, that $\beta_1 > \beta_i$, $\forall i \geq 2$. Then $A_1 f_1'(k_1) = R = 1/\beta_1$, agent 1's borrowing constraint is not binding, and for any $i \geq 2$,*

$$\frac{R - \gamma_i A_i f_i'(k_i)}{R} = A_i f_i'(k_i)(1-\gamma_i)$$

Hence, k_i is increasing in A_i. Since $R\beta_i \leq 1$, the value k_i is increasing in credit limit γ_i. By consequence, the steady state output $Y = \sum_i A_i f_i(k_i)$ is increasing in TFP A_i and credit limit γ_i for any i.

Proof. See Appendix D. □

According to this observation, the non-monotonicity of the aggregate output only appears in the transitional dynamics of the economy. Therefore, we will focus on transitional dynamics.

4.1 Impact of productivity along transitional dynamics

In general, it is difficult to comparing intertemporal equilibrium in infinite-horizon models. For the sake of tractability, we assume that there are two agents with utility function $u_i(c) = ln(c)$ and the linear production function $F_{i,t}(k) = A_{i,t}k$ with $A_{1,t} < A_{2,t}$, $\forall t$. In some cases, we can explicitly compute the equilibrium interest rate and aggregate output.

[13] Here, we do not focus the equilibrium existence; see, for instance, Bosi, Le Van, and Pham (2018).

Lemma 4. *Consider an infinite-horizon two-agent model with utility function $u_i(c) = ln(c)$ $\forall c$, $\forall t$, $\forall i = 1, 2$ and production functions $F_{i,t}(k) = A_{i,t}k$, $\forall k$, $\forall t$, $\forall i = 1, 2$. Assume that $\gamma_2 A_{2,t} < A_{1,t} < A_{2,t}$ $\forall t$, and*

$$\beta_1^t s_{1,0} - \beta_2^t \frac{\gamma_2 A_{2,t+1}}{A_{1,t+1} - \gamma_2 A_{2,t+1}} \frac{(1-\gamma_2)A_{2,t}}{A_{1,t} - \gamma_2 A_{2,t}} \cdots \frac{(1-\gamma_2)A_{2,1}}{A_{1,1} - \gamma_2 A_{2,1}} s_{2,0} > 0 \forall t \geq 0.$$

where $s_{i,0} = \beta_i w_{i,0}$. Then there exists an equilibrium with $R_t = A_{1,t}$ $\forall t$. At equilibrium, the aggregate output at date t, $(t \geq 1)$, is

$$\begin{aligned} Y_t =& \beta_1^{t-1} A_{1,t} A_{1,t-1} \cdots A_{1,1} s_{1,0} \\ &+ \beta_2^{t-1} (1-\gamma_2)^t \frac{A_{2,t} A_{1,t}}{A_{1,t} - \gamma_2 A_{2,t}} \frac{A_{2,t-1} A_{1,t-1}}{A_{1,t-1} - \gamma_2 A_{2,t-1}} \cdots \frac{A_{2,1} A_{1,1}}{A_{1,1} - \gamma_2 A_{2,1}} s_{2,0} \end{aligned} \tag{50}$$

Proof. See Appendix D. □

Lemma 4 allows us to investigate the impacts of productivity shocks. First, we look at the effects of individual productivity growth.

Proposition 13 (impact of individual productivity)**.** *Assume that there are two agents having production functions $F_i(k) = A_i k$ $\forall i$ with $A_1 < A_2$, and utility function $u_i(c) = ln(c)$ $\forall i$. Assume that*

$$A_1 - \gamma_2 A_2 > 0, \quad \beta_1 > \frac{(1-\gamma_2)A_2}{A_1 - \gamma_2 A_2} \beta_2, \quad s_{1,0} - \frac{\gamma_2 A_2}{A_1 - \gamma_2 A_2} s_{2,0} > 0. \tag{51}$$

Then, there is an equilibrium with the interest rate $R_t = A_1$ $\forall t$. In this equilibrium, we have that:

1. *If $\frac{s_{1,0}}{s_{2,0}} \frac{(A_1 - \gamma_2 A_2)^2}{(1-\gamma_2)\gamma_2 A_2^2} > 1$, then $\frac{\partial Y_t}{\partial A_1} > 0$ $\forall t$.*
2. *If $\frac{s_{1,0}}{s_{2,0}} \frac{(A_1 - \gamma_2 A_2)^2}{(1-\gamma_2)\gamma_2 A_2^2} < 1$, then there exists a time t_0 such that $\frac{\partial Y_t}{\partial A_1} < 0$ $\forall t \leq t_0$ and $\frac{\partial Y_t}{\partial A_1} \geq 0$ $\forall t > t_0$. Moreover, the time t_0 is decreasing in the productivity gap A_1/A_2.*

Moreover, we have that the output ratio $\frac{Y_{1,t}}{Y_{2,t}} \to \infty$, the share of net worth of the productive agent $\frac{W_{2,t}}{Y_t} \to 0$, and the growth rate $G_t \equiv \frac{Y_{t+1}}{Y_t} \to \beta_1 A_1$.

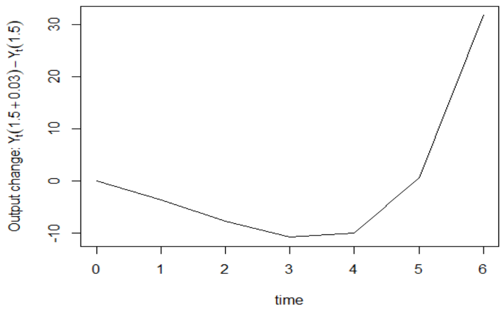
Proof. See Appendix D. □

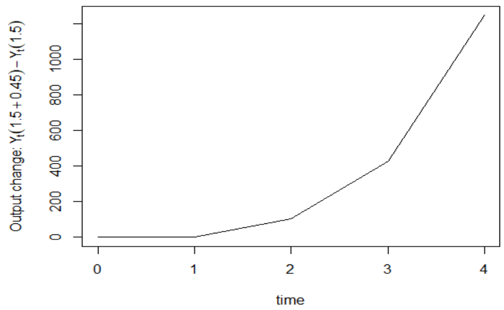
Condition (51) ensures that the less productive agent still produces, i.e., $k_{1,t} > 0$ $\forall t$. This happens if its TFP A_1 is not too low and the rate of time preference β_1 is high enough. Notice that these conditions imply that $\beta_1 A_1 > \beta_2 A_2$. This ensures that agent 1 still produces and the growth rate $\frac{Y_{t+1}}{Y_t}$ converges to $\beta_1 A_1$.

Proposition 13 allows us to understand the impact of a shock on the TFP of the less productive agent. Observe that, if A_1 increases, then the output will increase in the long run. However, point 2 of Proposition 13 indicates that, if A_1 increases but it is still low, the output may decrease in the short run and then increase in the long run.

Example 2. To complement our theoretical finding in Proposition 13, we run a simulation in a two-agent model with linear production function $F_i(k) = A_i k$, and $s_{1,0} = 200, s_{2,0} = 100, \beta_1 = 0.99, \beta_2 = 0.4, A_1 = 1.5, A_2 = 2.25$. The credit limit of agent 2 is $\gamma_2 = 0.4$. Let us denote $Y_t(A_1)$ the equilibrium aggregate output of the economy when the productivity of agent 1 is A_1. The following graphics show how the difference between $Y_t(A_1 + h) - Y_t(A_1)$ changes over time, where h is a productivity change.

First, when the productivity of agent 1 increases from 1.5 to 1.53 (a small productivity shock), the output goes down and then goes up. Precisely, $Y_t(1.5+0.03) - Y_t(1.5) < 0$ for $t = 1, 2, 3, 4$ and then $Y_t(1.5 + 0.03) - Y_t(1.5) > 0$, $\forall t \geq 5$.





Second, when there is a big productivity shock so that the productivity of agent 1 increases from 1.5 to 1.95, the output goes up at any period: $Y_t(1.5+0.45) - Y_t(1.5) > 0$, $\forall t \geq 1$. This is consistent with Proposition 1.

Impact of temporary productivity shocks

Let us look at the impact of temporary productivity shock. Assume that there is a productivity shock only at date 1, which affects the TFP of agent 1: $A_{1,1}$. We would like to understand how the aggregate output changes when $A_{1,1}$ varies. According to Lemma 4, we can compute

$$\frac{\partial Y_t}{\partial A_{1,1}} = \beta_1^{t-1} A_{1,t} \cdots A_{1,2} s_{1,0} - \beta_2^{t-1}(1-\gamma_2)^t \frac{A_{2,t}A_{1,t}}{A_{1,t} - \gamma_2 A_{2,t}} \cdots \frac{A_{2,2}A_{1,2}}{A_{1,2} - \gamma_2 A_{2,2}} \frac{\gamma_2 A_{2,1}^2}{(A_{1,1} - \gamma_2 A_{2,1})^2} s_{2,0}$$

$$(A_{1,t} \cdots A_{1,2})^{-1} \frac{\partial Y_t}{\partial A_{1,1}} = \beta_1^{t-1} s_{1,0} - \beta_2^{t-1} \frac{(1-\gamma_2)A_{2,t}}{A_{1,t} - \gamma_2 A_{2,t}} \cdots \frac{(1-\gamma_2)A_{2,1}}{A_{1,1} - \gamma_2 A_{2,1}} \Big(\frac{\gamma_2 A_{2,1}}{A_{1,1} - \gamma_2 A_{2,1}}\Big) s_{2,0}$$

Corollary 5 (impact of temporary productivity shocks)**.** *Under Assumptions in Lemma 4, there exists an equilibrium with $R_t = A_{1,t}$ $\forall t$. At equilibrium, we have that*

$$\frac{\partial Y_t}{\partial A_{1,1}} > 0 \Leftrightarrow \beta_1^{t-1} s_{1,0} - \beta_2^{t-1} \frac{(1-\gamma_2)A_{2,t}}{A_{1,t} - \gamma_2 A_{2,t}} \cdots \frac{(1-\gamma_2)A_{2,1}}{A_{1,1} - \gamma_2 A_{2,1}} \frac{\gamma_2 A_{2,1}}{A_{1,1} - \gamma_2 A_{2,1}} s_{2,0} > 0. \quad (52)$$

To illustrate our finding, let us focus on a particular case where $A_{1,t} = A_1$ $\forall t \geq 2$ and $A_{2,t} = A_2$ $\forall t \geq 1$. In this case, we have that, for $t \geq 2$,

$$\frac{\partial Y_t}{\partial A_{1,1}} > 0 \Leftrightarrow \beta_1^{t-1} s_{1,0} - \beta_2^{t-1} \Big(\frac{(1-\gamma_2)A_2}{A_1 - \gamma_2 A_2}\Big)^{t-1} \frac{(1-\gamma_2)\gamma_2 A_2^2}{(A_{1,1} - \gamma_2 A_2)^2} s_{2,0} > 0. \qquad (53)$$

By consequence, $\frac{\partial Y_t}{\partial A_{1,1}} > 0$ if β_1/β_2, $s_{1,0}/s_{2,0}$, A_{11} are high enough, A_2/A_1 is low.

4.2 Impact of credit limits along transitional dynamics

In this section, we aim to show the effects of credit limits on the aggregate output in intertemporal equilibrium. The following result provides a characterization of intertemporal equilibrium in a tractable case.

Lemma 5. *Consider a two-agent economy with linear production functions $F_i(k) = A_i k$, $\forall i = 1, 2$, where $\gamma_2 A_2 < A_1$. Assume that*

$$\beta_2 > \beta_1, \quad A_1 < \gamma_2 A_2 (1 + \frac{s_{2,0}}{s_{1,0}}) < A_2 \tag{54}$$

where $s_{i,0} \equiv \beta_i w_{i,0}$ and $w_{i,0}$ is the endowment of agent i at the initial date.

Then there exists an equilibrium with the interest rates

$$R_1 = \gamma_2 A_2 (1 + \frac{s_{2,0}}{s_{1,0}}) \in (A_1, A_2) \tag{55a}$$

$$R_t = A_2, \forall t \geq 2. \tag{55b}$$

Denote $s_{i,t} \equiv k_{i,t} - b_{i,t}$ the individual saving of agent i at date t. In such an equilibrium, we have

$$\begin{aligned}
&\text{Agent 1: } s_{1,0} = \beta_1 w_{1,0}, \quad s_{1,1} = \beta_1 \gamma_2 A_2 (s_{1,0} + s_{1,0}), \quad s_{1,t} = (\beta_1 A_2)^{t-1} s_{1,1}, \forall t \geq 1 \\
&\qquad k_{1,t} = 0, \forall t \geq 0 \\
&\text{Agent 2: } s_{2,0} = \beta_2 w_{2,0}, \quad s_{2,1} = \beta_2 (1 - \gamma_2) A_2 (s_{1,0} + s_{1,0}), \quad s_{2,t} = (\beta_2 A_2)^{t-1} s_{2,1}, \forall t \geq 1 \\
&\qquad k_{2,0} = s_{2,0} + s_{1,0} \\
&\qquad k_{2,t} = s_{2,t} + s_{1,t} = A_2^t \Big(\beta_1^t \gamma_2 + \beta_2^t (1 - \gamma_2)\Big)(s_{1,0} + s_{2,0}) \forall t \geq 1.
\end{aligned}$$

Proof. See Appendix D. □

In such an equilibrium, only the most productive agent produces. Notice that her borrowing constraint at date 1 is binding but her borrowing constraints from date 2 on are not. From date 2 on, the equilibrium interest rate equals the productivity of the most productive agent: $R_t = A_2$, $\forall t \geq 1$. However, interest rate between the initial date and date 1 is different from the productivity of agent 2 because the credit limit γ_2 is not so high (in the sens that $\gamma_2 < \frac{s_{1,0}}{s_{1,0}+s_{2,0}}$) and the productivity gap is high (in the sense that $\frac{A_1}{A_2} < \gamma_2(1 + \frac{s_{2,0}}{s_{1,0}})$; see condition (54)). Notice that Kiyotaki (1998)'s Section 2 only focuses on the case where the equilibrium interest rate equals the rate of return on investment of unproductive agents, i.e., $R_t = A_1$, $\forall t$.

Thanks to Lemma 5, we can observe the effect of the credit limit.

Proposition 14. *Under assumptions in Lemma 5, the aggregate output is given by*

$$\begin{aligned}
Y_1 &= A_2 k_{2,0} = \frac{A_2 R_1}{R_1 - \gamma_2 A_2} s_{2,0} = A_2 (s_{1,0} + s_{2,0}) \\
\forall t \geq 2, \quad Y_t &= A_2 k_{2,t-1} = A_2^t \Big(\beta_1^{t-1} \gamma_2 + \beta_2^{t-1} (1 - \gamma_2)\Big)(s_{1,0} + s_{2,0})
\end{aligned}$$

The aggregate output at date 1 does not depend on the credit limit γ_2 of the most productive agent and

$$\frac{\partial Y_t}{\partial \gamma_2} < 0, \forall t \geq 2. \tag{56}$$

It means that the aggregate output from date 2 on decreases when the most productive agent's credit limit increases. This is different from the standard view on the effects of financial constraints as shown in Buera and Shin (2013), Khan and Thomas (2013), Midrigan and Xu (2014), Moll (2014), Catherine, Chaney, Huang, Sraer, and Thesmar (2017).

Let us explain the insight of this finding. Denote $W_{i,t}$ the net worth of agent i at date t. We have $W_{i,t} = F_i(k_{i,t-1}) - R_t b_{i,t-1}$. In equilibrium in Proposition 14, the net worth of agent 2 at date 1 is

$$W_{2,1} = A_2 k_{2,0} - R_1 b_{2,0} = (1-\gamma_2)(s_{1,0} + s_{2,0}).$$
$$W_{2,t} = A_2 k_{2,t-1} - R_1 b_{2,t-1} = (1-\gamma_2) A_2 k_{2,t-1} = (1-\gamma_2) A_2^t \Big(\beta_1^{t-1}\gamma_2 + \beta_2^{t-1}(1-\gamma_2)\Big)(s_{1,0} + s_{2,0})$$

These are decreasing in the credit limit γ_2. The reason behind is that when γ_2 goes up, the interest rate R_1 increases which makes the repayment $R_1 b_{2,0}$ increases. However, the capital $k_{2,0}$ of agent 2 is already the sum of aggregate saving $s_{1,0} + s_{2,0}$ which can no longer increase. By consequence, the net worth $W_{2,1} = A_2 k_{2,0} - R_1 b_{2,0}$ decreases.[14] This makes the saving of agent 2 go down, and, hence, the output decreases. The mechanism can be summarized by the following schema:

$$\begin{aligned} &\textbf{Credit limit } \gamma_2 \uparrow \Rightarrow \textbf{Interest rate} \uparrow \Rightarrow \textbf{Agent 2's net worth} \downarrow \Rightarrow \\ &\Rightarrow \textbf{Saving} \downarrow \Rightarrow \textbf{Production} \downarrow \Rightarrow \cdots \end{aligned} \tag{57}$$

However, this mechanism does not happen when the credit limit γ_2 of agent 2 is high enough.

Remark 4. *In equilibrium without borrowing constraints, we have $R_t = A_2$, $\forall t$, and*

$$\begin{aligned} s_{i,0} &= \beta_i w_{i,0}, \forall i \\ s_{i,t} &= \beta_i A_2 s_{i,t-1} = (\beta_i A_2)^t s_{i,0}, \forall t \geq 1, \forall i = 1,2 \\ k_{1,t} &= 0, \quad b_{1,t} = -s_{1,t} \\ k_{2,t} &= s_{1,t} + s_{2,t} = A_2^t(\beta_1^t s_{1,0} + \beta_2^t s_{2,0}) \\ Y_t^{perfect} &= A_2 k_{2,t-1} = A_2^t(\beta_1^{t-1} s_{1,0} + \beta_2^{t-1} s_{2,0}) \end{aligned}$$

Since $\gamma_2(1 + \frac{s_{2,0}}{s_{1,0}}) < 1$, we can verify that, for all $t \geq 2$,

$$Y_t = A_2^t \Big(\beta_1^{t-1}\gamma_2 + \beta_2^{t-1}(1-\gamma_2)\Big)(s_{1,0} + s_{2,0}) < A_2^t(\beta_1^{t-1} s_{1,0} + \beta_2^{t-1} s_{2,0}) = Y_t^{perfect}.$$

[14] This insight is consistent with that in a two-period model in Pham and Pham (2021).

4.2.1 A three-agent model

Lemma 6. *Consider an economy with three agents having linear production functions $F_i(k) = A_i k$ and collateral constraint $R_t b_{i,t-1} \leq \gamma_i A_i k_{i,t-1}$.*

Assume that, $\beta_3 \geq \max(\beta_1, \beta_2)$, $\gamma_2 A_2 < \gamma_3 A_3 < A_1 < A_2 < A_3$, and

$$\frac{\gamma_2}{1-\gamma_2} s_{2,0} + \frac{\gamma_3 A_3}{A_2 - \gamma_3 A_3} s_{3,0} < s_{1,0} < \frac{\gamma_2 A_2}{A_1 - \gamma_2 A_2} s_{2,0} + \frac{\gamma_3 A_3}{A_1 - \gamma_3 A_3} s_{3,0}$$
$$(\beta_1 \gamma_2 + \beta_2(1-\gamma_2)) A_2 s_{2,0} \leq (\beta_3 - \beta_1)\gamma_3 A_3 s_{3,0}$$

where $s_{i,0} \equiv \beta_i w_{i,0}$ and $w_{i,0}$ is the endowment of agent i at the initial date.

Then there exists an equilibrium with the interest rates

$$R_t = A_3, \forall t \geq 2, \tag{58a}$$

$$R_1 \in (A_1, A_2) \text{ is determined by } s_{1,0} = \frac{\gamma_2 A_2}{R_1 - \gamma_2 A_2} s_{2,0} + \frac{\gamma_3 A_3}{R_1 - \gamma_3 A_3} s_{3,0}. \tag{58b}$$

In such an equilibrium, the individual saving $s_{i,t} \equiv k_{i,t} - b_{i,t}$ is given by

$$\text{Agent 1: } s_{1,0} = \beta_1 w_{1,0}, \quad s_{1,t} = \beta_1 R_t s_{1,t-1} \ \forall t \geq 1, \quad s_{1,t} = \beta_1^t R_t \cdots R_1 s_{1,0}$$
$$\text{Agents 2 and 3: } s_{i,0} = \beta_i w_{i,0}, \forall i; \quad s_{i,1} = \beta_i \frac{(1-\gamma_i) A_i R_1}{R_1 - \gamma_i A_i} s_{i,0}, \forall i = 2, 3$$
$$s_{i,t} = \beta_i A_3 s_{i,t-1} = (\beta_i A_3)^{t-1} s_{i,1}, \forall t \geq 2, \forall i = 1, 2, 3.$$

The output is given by

$$Y_1 = \sum_i A_i k_{i,0} = \frac{A_2 R_1}{R_1 - \gamma_2 A_2} s_{2,0} + \frac{A_3 R_1}{R_1 - \gamma_3 A_3} s_{3,0}$$
$$Y_2 = A_3^{t-1}\Big(\beta_1^{t-1} R_1 s_{1,0} + \beta_2^{t-1}(1-\gamma_2)\frac{A_2 R_1}{R_1 - \gamma_2 A_2} s_{2,0} + \beta_3^{t-1}(1-\gamma_3)\frac{A_3 R_1}{R_1 - \gamma_3 A_3} s_{3,0}\Big).$$

Proof. See Appendix D. □

In such an equilibrium, the most productive agent (agent 2) produces at any period while the least productive one never produces. At the period 1, there are two producers (agent 2 and agent 3) thanks to condition (58a). However, from period 2 on, only agent 3 is producer.

Let us focus on the effects of the credit limits on the aggregate production. According to the formula of the output and equation (58b), we can compute the derivative of the output and obtain the following result.

Proposition 15. *Let conditions in Lemma 6 be satisfied. Then, we have that*

$$\frac{\partial Y_1}{\partial \gamma_3} = \frac{\partial R_1}{\partial \gamma_3}(A_3 - A_2)\frac{\gamma_2 A_2}{(R_1 - \gamma_2 A_2)^2} s_{3,0} > 0 \tag{59a}$$

$$\frac{\partial Y_1}{\partial \gamma_2} = \frac{\partial R_1}{\partial \gamma_2}(A_2 - A_3)\frac{\gamma_3 A_3}{(R_1 - \gamma_3 A_3)^2} s_{3,0} < 0 \tag{59b}$$

For any period $t \geq 2$, we have that

$$\frac{\partial Y_t}{\partial \gamma_3} = \frac{\partial R_1}{\partial \gamma_3} A_3^{t-1}\Big(\beta_1^{t-1}s_{1,0} + \big(\beta_3^{t-1}(1-\gamma_3)A_3 - \beta_2^{t-1}(1-\gamma_2)A_2\big)\frac{\gamma_2 A_2 s_{2,0}}{(R_1-\gamma_2 A_2)^2}\Big)$$
$$- A_3^{t-1}\beta_3^{t-1}\frac{A_3 R_1}{R_1 - \gamma_3 A_3}s_{3,0} \tag{60a}$$
$$\frac{\partial Y_t}{\partial \gamma_2} = \frac{\partial R_1}{\partial \gamma_2} A_3^{t-1}\Big(\beta_1^{t-1}s_{1,0} + \big(\beta_2^{t-1}(1-\gamma_2)A_2 - \beta_3^{t-1}(1-\gamma_3)A_3\big)\frac{\gamma_3 A_3 s_{3,0}}{(R_1-\gamma_3 A_3)^2}\Big)$$
$$- A_3^{t-1}\beta_2^{t-1}\frac{A_2 R_1}{R_1 - \gamma_2 A_2}s_{2,0} \tag{60b}$$

Since $\gamma_2 < \gamma_3$, the output at period 1 is increasing in γ_3 (the credit limit of the most productive producer) but decreasing in γ_2 (the credit limit γ_2 of the least productive producer). This is consistent with Corollary 4 or point 1 in Proposition 9.

In the long run (when t is high enough), we have $\beta_3^{t-1}(1-\gamma_3)A_3 - \beta_2^{t-1}(1-\gamma_2)A_2 > 0$ because $\beta_3 > \beta_2$. In such an equilibrium, when the credit limit of agent 2 increases, the aggregate output may decrease in any period.

Example 3. *We illustrate our insight by the following simulation. In this simulation, we set that $\beta_1 = 0.2, \beta_2 = 0.2, \beta_3 = 0.95$, $s_{1,0} = 4 = \beta_1 w_{1,0}, s_{2,0} = 4 = \beta_2 w_{2,0}, s_{3,0} = 3 = \beta_3 w_{3,0}$, $\gamma_1 = 0.2, \gamma_3 = 0.3$. Productivity: $A_1 = 1, A_2 = 1.2, A_3 = 1.5$. We draw the output path for two cases: $\gamma_2 = 0.3$ and $\gamma_2 = 0.35$. We observe that*

$$Y_t(\gamma_2 = 0.35) < Y_t(\gamma_2 = 0.30), \forall t \geq 1.$$

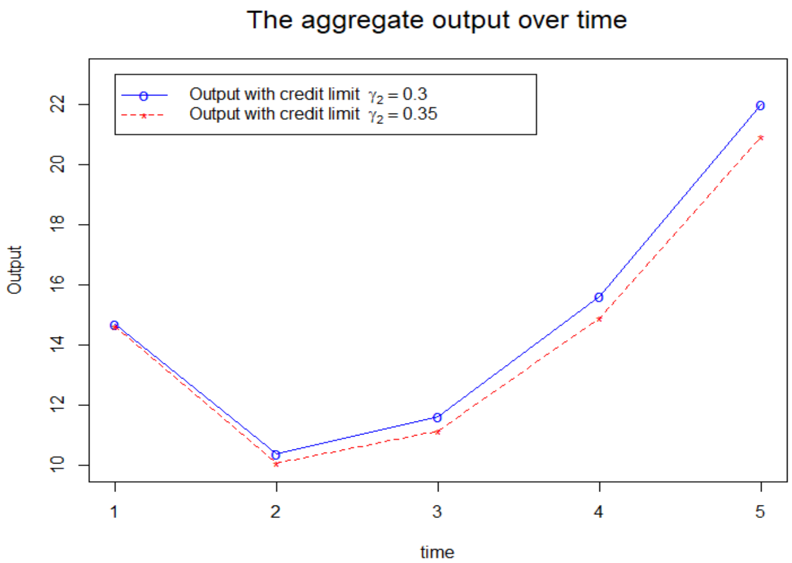


Figure 4: Effects of credit limits γ_2 on the aggregate output.

5 Conclusion

We have build general equilibrium models with borrowing constraints to explain why the aggregate output may be decreasing (increasing, respectively) when the productivity or credit limit of producers increases (decreases, respectively). A positive homogeneous (productivity or financial) shock has a positive impact on the aggregate output. However, positive, asymmetric shocks may reduce the aggregate production. We have pointed out that both financial frictions and the productivity gap (or dispersion of productivity distribution) matter for this phenomenon.

A Proofs for Sections 2

Proof of Lemma 3. Since $F_i'(0) = \infty$, we have $k_i > 0$ at optimum. The Lagrange function is

$$L = F_i(k_i) - Rb_i + \lambda_i(S_i + b_i - k_i) + \mu_i(\gamma_i F_i(k_i) - Rb_i)$$

It is easy to see that (k_i, b_i) is a solution if and only if there exists (λ_i, μ_i) such that

$$\begin{aligned}&[k] : (1 + \mu_i\gamma_i)F_i'(k_i) = \lambda_i\\ &[a] : (1 + \mu_i)R = \lambda_i, \quad \mu_i \geq 0, \text{ and } \mu_i(\gamma_i F_i(k_i) - R_i b_i) = 0.\end{aligned}$$

These equations imply that:

$$A_i f_i(k_i) = F_i'(k_i) = R\frac{1 + \mu_i}{1 + \gamma_i\mu_i} \geq R. \tag{A.1}$$

Since F_i' is decreasing, we have $k_i \leq k_i^n(R/A_i)$.

We consider two cases.

Case 1: The credit constraint is binding: $\gamma F_i(k_i) = Rb_i$. In this case, (k_i, b_i) is the solutions of the following equations:

$$b_i = k_i - S_i \tag{A.2}$$

$$\gamma F_i(k_i) = R(k_i - S_i), \quad i.e., \quad \frac{\gamma_i}{R} = \frac{k_i}{F_i(k_i)} - \frac{S_i}{F_i(k_i)}. \tag{A.3}$$

Consider the function $k/F_i(k)$. Its derivative equals $\frac{F_i(k) - kF_i'(k)}{(F_i(k))^2}$ which is non-negative because F is concave. So, the function $G_i(k) \equiv \frac{k - S_i}{F_i(k)}$ is strictly increasing in k. Moreover, $\lim_{k\to 0} G_i(k) < \gamma_i/R$ and $G_i(\infty) > \gamma_i/R$ (because $F_i'(\infty) < 1$). Therefore, there exists a unique solution k_i of equation (A.3), and this is positive. It is actually k_i^b.

We now investigate condition $k_i \leq k_i^n$. Since $G_i(k_i) = \gamma_i/R$, condition $k_i \leq k_i^n$ is equivalent to $G_i(k_i^n) \geq \gamma_i/R$ (because $G_i(k_i) = \gamma_i/R$) or, equivalently, $R\frac{k_i^n(R/A_i) - S_i}{F_i(k_i^n(R/A_i))} \geq \gamma_i$.

Conversely, assume that $R\frac{k_i^n(R/A_i) - S_i}{F_i(k_i^n(R/A_i))} \geq \gamma_i$. We choose $k_i = k_i^b$. Then, by definition of k_i^b, we have $k_i \in (S_i, \infty)$. Therefore, we have

$$R > R(1 - \frac{S_i}{k_i}) = \gamma_i\frac{F_i(k_i)}{k_i} \geq \gamma_i F_i'(k_i)$$

where the last inequality follows the fact that F_i is concave. It means that $R > \gamma_i F_i'(k_i)$. So, we can define μ_i, λ_i by

$$1 - \frac{F_i'(k_i)}{R} = \mu_i\Big(\frac{F_i'(k_i)}{R} - \gamma_i\Big), \quad \lambda_i = R(1+\mu_i).$$

Therefore, (λ_i, μ_i) and (k_i, b_i) satisfy conditions $[k]$ and $[b]$ above. It means that (k_i, b_i) is a solution.

Case 2: $\gamma_i F_i(k_i) > Rb_i$. In this case, we have $\mu_i = 0$, and hence $F_i'(k_i) = R$, i.e, $k_i = k_i^n$. It remains to check that this value of k_i satisfies the condition: $\gamma_i F_i(k_i) > Rb_i = R(S_i - k_i)$, i.e., $\gamma_i/R > G_i(k_i^n)$.

Observe that if $RG_i(k_i^n) < (\geq)\gamma_i$, then $G_i(k_i^n) < (\geq)\gamma_i/R = G_i(k_i^b)$, which implies that $k_i^n < (\geq) k_i^b$.

The converse is easy. Notice that, in this case, agent borrows (i.e., $b_i > 0$) if and only if $k_i > S$ or equivalently $k_i^n > S$. This means that her wealth is low and/or interest rate is low and/or her productivity is high.

$\square$

A.1 Proof of Theorem 1 (economy with strictly concave technologies)

To simplify notations, we write $k_i^n(R)$ and $k_i^b(R)$ instead of $k_i^n(\frac{R}{A_i})$ and $k_i^b(\frac{R}{\gamma_i A_i}, S_i)$ (see Definition 2). We also introduce the so-called aggregate capital demand function:

$$B_n(R) \equiv \begin{cases} \sum_{i=1}^{n} k_i^n(R) + \sum_{i=n+1}^{m} k_i^b(R) & \text{if } n \leq m-1 \\ \sum_{i=1}^{m} k_i^n(R) & \text{if } n = m. \end{cases}$$

Lemma 7. $B_n(R_n) > B_{n+1}(R_{n+1}) = B_n(R_{n+1})$.

Proof. Indeed, since $R_n < R_{n+1}$, we notice that

$$\begin{aligned} B_n(R_n) &\equiv \sum_{i=1}^{n} k_i^n(R_n) + \sum_{i=n+1}^{m} k_i^b(R_n) \\ &> B_n(R_{n+1}) = \sum_{i=1}^{n} k_i^n(R_{n+1}) + \sum_{i=n+1}^{m} k_i^b(R_{n+1}) = \sum_{i=1}^{n+1} k_i^n(R_{n+1}) + \sum_{i=n+2}^{m} k_i^b(R_{n+1}) \end{aligned}$$

where the last equality follows $k_{n+1}^b(R_{n+1}) = k_{n+1}^n(R_{n+1})$. Therefore, $B_n(R_n) > B_{n+1}(R_{n+1}) = B_n(R_{n+1})$ $\forall n$. $\square$

We state an intermediate step whose proof is based on Lemma 3 and Corollary 2.

Lemma 8. *Let assumptions in Theorem 1 be satisfied. Consider an equilibrium $((k_i, b_i)_i, R)$ and an index $n \in \{1, \dots, m-1\}$.*

1. *If $R > R_m$, Lemma 3 implies that credit constraint of any agent is not binding. So, the equilibrium coincides to that of the economy without credit constraints. Therefore, we have $R = R^* > R_m$.*

2. *If $R > R_n$, then credit constraint of any agent $i \leq n$ is not binding. Hence $k_i = k_i^n(R) < k_i^n(R_n)$ $\forall i \leq n$. Condition $R > R_n$ also implies that $k_i^b(R) < k_i^b(R_n)$. Therefore, we have $\sum_i S_i < B_n(R_n)$.*

3. *If $R \leq R_{n+1}$, then credit constraint of any agent $i \geq n+1$ is binding, and hence $k_i = k_i^b(R) \geq k_i^b(R_{n+1})$ $\forall i \geq n+1$. Moreover, we have $k_i \geq k_i^n(R) \geq k_i^n(R_{n+1})$. Therefore, we have $\sum_i S_i \geq B_n(R_{n+1})$.*

We now prove Theorem 1. Let us consider an equilibrium. Since there is at least one agent whose credit constraint is not binding, we have $R > R_1$.

Step 1. Suppose that $R \in (R_n, R_{n+1}]$. So, credit constraint of any agent $i \geq n+1$ is binding and that of any agent $i \leq n$ is not binding. Hence, the capital demand is

$$\sum_i k_i = \sum_{i=1}^{n} k_i^n(R) + \sum_{i=n+1}^{m} k_i^b(R). \tag{A.4}$$

Therefore, the equilibrium interest rate is determined by

$$\sum_{i=1}^{n} k_i^n(R) + \sum_{i=n+1}^{m} k_i^b(R) = S \equiv \sum_i S_i. \tag{A.5}$$

The left-hand side is decreasing in r, and hence this equation has a unique solution.

Since $R \in (R_n, R_{n+1}]$, we have

$$\sum_{i=1}^{n} k_i^n(R_n) + \sum_{i=n+1}^{m} k_i^b(R_n) > \sum_i S_i \geq \sum_{i=1}^{n} k_i^n(R_{n+1}) + \sum_{i=n+1}^{m} k_i^b(R_{n+1}).$$

Conversely, if this condition holds, by using properties of functions k_i^b, k_i^n, we can easily prove that $R \in (R_n, R_{n+1}]$. Indeed, if $R > R_{n+1}$, then point 2 of Lemma 8 implies that $S < B_{n+1}(R_{n+1})$. This contradicts to $S \geq B_{n+1}(R_{n+1})$. If $R \leq R_n$, then point 3 of Lemma 8 implies that $S \geq B_{n-1}(R_n) = B_n(R_n)$. This contradicts to $S < B_n(R_n)$. Therefore, we obtain $R \in (R_n, R_{n+1}]$.

Step 2. We now suppose that $R^* > R_m$. We will prove that credit constraint of any agent is not binding. Suppose that

$$\mathcal{B} = \{i \in \{1, \dots, m\} : \text{ agent i's borrowing constraint is binding}\} \neq \emptyset.$$

Let $n : 1 \leq n \leq m-1$ be the highest element in $\mathcal{B}$, i.e., credit constraint of any agent $i \geq n+1$ is binding while that of any agent $i \leq n$ is not. We have $R \in (R_n, R_{n+1}]$. So, $k_i^b(R) \geq k_i^b(R_n + 1) > k(R_m)$ and $k_i^n(R) \geq k_i^n(R_{n+1}) \geq k_i^n(R_m)$. Hence, we get that

$$\sum_i S_i = \sum_{i=1}^{n} k_i^n(R) + \sum_{i=n+1}^{m} k_i^b(R) \geq \sum_{i=1}^{m} k_i^n(R_m). \tag{A.6}$$

However, by definition of R^*, we have

$$\sum_i S_i = \sum_{i=1}^{m} k_i^n(R^*) < \sum_{i=1}^{m} k_i^n(R_m). \tag{A.7}$$

This is a contradiction.

Step 3. We now prove that $R_n \leq R^* \ \forall n \leq m-1$. Indeed, in the regime $\mathcal{R}_n$, for any $i \geq n+1$, agent i's credit constraint is binding. Hence, Lemma 3 follows that $k_i^b(R_n) \leq k_i^n(R_n) \ \forall i \geq n+1$. Consequently, we get that

$$\sum_{i=1}^{m} k_i^n(R^*) = S = \sum_{i=1}^{n} k_i^n(R_n) + \sum_{i=n+1}^{m} k_i^b(R_n) \leq \sum_{i=1}^{m} k_i^n(R_n)$$

which implies that $R^* \geq R_n$.

A.2 Characterization of general equilibrium with linear technologies

Denote

$$\mathbb{D}_n \equiv \sum_{i=n}^{m} \frac{A_n S_i}{A_n - \gamma_i A_i} \ \forall n \geq 1, \quad \mathbb{B}_n \equiv \sum_{i=n+1}^{m} \frac{A_n S_i}{A_n - \gamma_i A_i} \ \forall n \geq 1. \tag{A.8}$$

where by convention, $\sum_{i=n}^{m} x_i = 0$ if $n > m$. By definition (A.8), we observe that

$$\frac{S_m}{1-\gamma_m} = \mathbb{D}_m < \cdots < \mathbb{D}_{n+1} < \mathbb{B}_n < \mathbb{D}_n < \mathbb{B}_{n-1} < \cdots < \mathbb{B}_1 = \sum_{i=2}^{m} \frac{A_1 S_i}{A_1 - \gamma_i A_i}. \tag{A.9}$$

Denote R_n^L the greatest solution of the following equation:[15]

$$\underbrace{\sum_{i=n+1}^{m} \frac{\gamma_i A_i}{R - \gamma_i A_i} S_i}_{\text{Asset demand}} = \underbrace{\sum_{i=1}^{n} S_i}_{\text{Asset supply}} \quad \text{or equivalently} \quad \underbrace{\sum_{i=n+1}^{m} \frac{R S_i}{R - \gamma_i A_i}}_{\text{Capital demand}} = \underbrace{S}_{\text{Capital supply}} \tag{A.10}$$

Theorem 2 (characterization of general equilibrium with linear technologies)**.** *Assume that $F_i(K) = A_i K \ \forall i$ and $A_1 < \cdots < A_m$. Then, there exists a unique equilibrium. The equilibrium interest rate is determined by the following:*

$$R = \begin{cases} A_i & \textit{in the regime } \mathcal{A}_i. \\ R_i^L & \textit{in the regime } \mathcal{R}_i. \end{cases} \tag{A.11}$$

where

1. *the regime $\mathcal{A}_n$ (with $n \in \{1, \ldots, m\}$) is the set of all economies satisfying $A_n > \max_i(\gamma_i A_i)$ and $\mathbb{B}_n \leq S \leq \mathbb{D}_n$*
2. *the regime $\mathcal{R}_n$ (with $n \in \{1, \ldots, m-1\}$) is the set of all economies satisfying*
   (a) *either $\max_i(\gamma_i A_i) < A_n < R_n^L < A_{n+1}$ (or equivalently $\max_i(\gamma_i A_i) < A_n$ and $\mathbb{D}_{n+1} < S < \mathbb{B}_n$)*

[15] It should be noticed that the function $f(x) \equiv \sum_{i=n+1}^{m} \frac{xS_i}{x - \gamma_i A_i}$ is not continuous at point $\gamma_i A_i$ with $i \geq n+1$. However, it is continuous and decreasing in the interval $(\max_{i \geq n+1}(\gamma_i A_i), \infty)$. Then, the equation $f(x) = S$ has a unique solution, r_n, in such interval.

(b) *or* $A_n \leq \max_i(\gamma_i A_i) < R_n^L < A_{n+1}$ *(or equivalently* $A_n \leq \max_i(\gamma_i A_i) < R_n^L$ *and* $\mathbb{D}_{n+1} < S$*).*

Remark 5. *In Theorem 2, we assume that $A_1 < \cdots < A_m$. However, we can characterize the set of equilibria in the general case where some agents have the same productivity. Indeed, without lost of generality, we can (1) rank that $A_i \leq A_{i+1}$, $\forall i$, and assume that (2) the set $\{A_i : i \in \{1, \dots, m\}\}$ has the cardinal p, $p \leq m$ and its distinct values are $(A_{i_t})_{t=1}^p$, where $A_1 = A_{i_1} < A_{i_2} < \cdots < A_{i_p} = A_m$. We can decompose that*

$$A_1, A_2, \dots, A_m = \underbrace{A_1, \dots, A_1}_{i_1 \text{ times}}, \underbrace{A_{i_1+1}, \dots, A_{i_1+i_2}}_{i_2 \text{ times}}, \dots, \underbrace{A_{i_1+\dots+i_{p-1}}, \dots, A_m}_{i_m \text{ times}}$$

Let us denote $\mathbb{A}_t \equiv A_{i_t}$, $\quad \mathbb{S}_t \equiv \sum_{i:A_i=A_{i_t}} S_i$. Then, we can use the same argument in Theorem 2 (but we replace m by p, A_i by $\mathbb{A}_i$, S_i by $\mathbb{S}_i$) to determine the unique equilibrium interest rate. However, there may be multiple equilibrium allocations when one of the sets $\{i : A_i = A_{i_1}\}, \dots, \{i : A_i = A_{i_p}\}$ has multiple elements.

A.2.1 Proof of Theorem 2

Theorem 2 is a direct consequence of the existence of equilibrium and Lemmas 9-13 below. First, the following result is a direct consequence of Lemma 2.

Lemma 9. *Assume that $A_1 < A_2 < \cdots < A_m$. If $\max_i(\gamma_i A_i) \geq A_n$ and there exists an equilibrium, then $R > A_n$.*

By comparing $\mathbb{B}_n, \mathbb{D}_n$ with the aggregate capital supply $S \equiv \sum_{i=1}^m S_i$, we obtain the following result.

Lemma 10. *Assume that $A_1 < A_2 < \cdots < A_m$. Denote $S \equiv \sum_{i=1}^m S_i$ the aggregate capital. Consider an equilibrium.*

1. *If $A_n > \max_i(\gamma_i A_i)$ and $R > A_n$, then $\mathbb{B}_n > S$. Consequently, if $A_n > \max_i(\gamma_i A_i)$ and $\mathbb{B}_n \leq S$, then $R \leq A_n$.*

2. *If $A_n > \max_i(\gamma_i A_i)$ and $R < A_n$, then $S > \mathbb{D}_n$. Consequently, if $A_n > \max_i(\gamma_i A_i)$ and $S \leq \mathbb{D}_n$, then $R \geq A_n$.*

Proof. 1. Since $R > A_i$ for any $i = 1, \dots, n$, Lemma 2 implies that $k_i = 0, a_i = -S_i$ $\forall i = 1, \dots, n$. Hence, we have, by using market clearing condition,

$$\sum_{i=1}^{n} S_i = -\sum_{i=1}^{n} a_i = \sum_{i=n+1}^{m} a_i \leq \sum_{i=n+1}^{m} \frac{\gamma_i A_i}{R - \gamma_i A_i} S_i < \sum_{i=n+1}^{m} \frac{\gamma_i A_i}{A_n - \gamma_i A_i} S_i \quad \text{(A.12)}$$

where the first inequality follows $b_i \leq \frac{\gamma_i A_i S_i}{R - \gamma_i A_i}$ while the last inequality follows $R > A_n > \max_i(\gamma_i A_i)$ and the fact that the function $Func(R) \sum_{i=n+1}^m \frac{\gamma_i A_i}{R - \gamma_i A_i} S_i$ is decreasing in $(\max_i(\gamma_i A_i), +\infty)$. Notice that this function is not decreasing in the interval $(0, \infty)$.

2. Since $R < A_n$, again Lemma 2 implies that $k_i = \frac{R}{R-\gamma_i A_i} S_i$ and $a_i = \frac{\gamma_i A_i}{R-\gamma_i A_i} S_i \forall i \geq n$. We have

$$\sum_{i=n}^{m} \frac{A_n S_i}{A_n - \gamma_i A_i} S < \sum_{i=n}^{m} \frac{R S_i}{R - \gamma_i A_i} = \sum_{i=n}^{m} k_i \leq \sum_{i=1}^{m} S_i = S \tag{A.13}$$

where the first inequality follows $A_n > R > \max_i(\gamma_i A_i)$.

□

Lemma 11. *$R = A_n$ if and only if $A_n > \max_i(\gamma_i A_i)$ and $\mathbb{B}_n \leq S \leq \mathbb{D}_n$.*[16]

Proof. If $R = A_n$, we have $k_i = 0 \ \forall i \leq n-1$ and $k_i = \frac{RS_i}{R-\gamma_i A_i} \ \forall i \geq n+1$. This implies that $A_n = R > \max_i(\gamma_i A_i)$. Since $0 \leq k_n \leq \frac{RS_i}{R-f_n A_n}$, we have

$$\sum_{i=n+1}^{m} \frac{RS_i}{R - \gamma_i A_i} \leq \sum_{i} k_i = \sum_{i=n}^{m} k_i \leq \sum_{i=n}^{m} \frac{RS_i}{R - \gamma_i A_i} = \sum_{i=n} \frac{A_n S_i}{A_n - \gamma_i A_i} \tag{A.14}$$

By converse, suppose that $A_n > \max_i(\gamma_i A_i)$ and $\sum_{i=n+1}^{m} \frac{A_n S_i}{A_n - \gamma_i A_i} \leq S \leq \sum_{i=n}^{m} \frac{A_n S_i}{A_n - \gamma_i A_i}$. Applying points 1 and 2 of Lemma 10, we have $R \geq A_n$ and $R \leq A_n$. Hence $R = A_n$. □

By combining Lemma 10 and the fact that $R > \max_i(\gamma_i A_i)$, we obtain the following result.

Lemma 12. *Assume that $A_1 < A_2 < \cdots < A_m$. Consider an equilibrium. If $R \in (A_n, A_{n+1})$, then $A_{n+1} > \max_i(\gamma_i A_i)$ and $R = R_n^L$ (hence $R_n^L \in (A_n, A_{n+1})$).*

We now identify the necessary and sufficient conditions under which $R = R_n^L$.

Lemma 13. *$R = R_n^L \neq A_n$ if and only if one of the following conditions is satisfied:*

1. *$\max_i(\gamma_i A_i) < A_n < r_n^L < A_{n+1}$, or equivalently $\max_i(\gamma_i A_i) < A_n$ and $\mathbb{D}_{n+1} < S < \mathbb{B}_n$*

2. *$A_n \leq \max_i(\gamma_i A_i) < R_n^L < A_{n+1}$, or equivalently $A_n \leq \max_i(\gamma_i A_i) < R_n^L$ and $\mathbb{D}_{n+1} < S$.*

In any case, we have that $R_n^L \in [A_n, A_{n+1})$.

Proof. **Part 1**. Assume that $R = R_n^L \neq A_n$. By definition of R and R_n^L, we have $\sum_{i=n+1}^{m} \frac{RS_i}{R-\gamma_i A_i} = S$, and $R_n^L > \max_i(\gamma_i A_i)$. We will prove that $R = R_n^L \in (A_n, A_{n+1})$.

If $R \leq A_n$, then $R < A_{n+1}$, and hence $k_i = \frac{RS_i}{R-\gamma_i A_i} \ \forall i \geq n+1$. Since $\sum_{i=n+1}^{m} \frac{RS_i}{R-\gamma_i A_i} = S = \sum_i k_i$. We have $k_i = 0 \ \forall i \leq n$, and hence $k_n = 0$. This implies that $R \geq A_n$. Therefore, we have $R = A_n$, a contradiction. Thus, we have $R > A_n$.

If $R \geq A_{n+1}$, we have $k_i = 0 \ \forall i \leq n$. Hence $S = \sum_i k_i \leq \sum_{i=n+1}^{m} \frac{RS_i}{R-\gamma_i A_i}$. Since $\sum_{i=n+1}^{m} \frac{RS_i}{R-\gamma_i A_i} = S$, we have $k_i = \frac{RS_i}{R-\gamma_i A_i} \ \forall i \geq n+1$. Hence $A_{n+1} \geq R$. So, $R = A_{n+1}$. We have just proved that $R \leq A_{n+1}$. By definition of R, we get that $A_{n+1} >$

[16] We need condition $A_n > M \equiv \max_i(\gamma_i A_i)$ because that $R > \max_i(\gamma_i A_i)$. Condition $\sum_{i=n+1}^{m} \frac{A_n S_i}{A_n - \gamma_i A_i} \leq S$ ensures that $R \leq A_n$ while condition $S \leq \sum_{i=n}^{m} \frac{A_n S_i}{A_n - \gamma_i A_i}$ ensures that $R \geq A_n$.

$\max_i(\gamma_i A_i)$. If $R_n^L = A_{n+1}$, then applying Lemma 11, we have $\sum_{i=n+2}^{m} \frac{A_{n+1}S_i}{A_{n+1}-\gamma_i A_i} = \mathbb{B}_{n+1} \leq S$. However, by definition of R_n^L, we have $\sum_{i=n+1}^{m} \frac{A_{n+1}S_i}{A_{n+1}-\gamma_i A_i} = S$, contradiction. Therefore, we obtain $R_n^L < A_{n+1}$.

We have just proved that $R_n^L \in (A_n, A_{n+1})$. Applying point 2 of Lemma 10, we have $S > \mathbb{D}_{n+1}$. There are two cases:

1. $\max_i(\gamma_i A_i) \geq A_n$. In this case, we have $A_n \leq \max_i(\gamma_i A_i) < R_n^L < A_{n+1}$.

2. $\max_i(\gamma_i A_i) < A_n$. We get $\max_i(\gamma_i A_i) < A_n < R_n^L < A_{n+1}$. Notice that, in this case, $R_n^L \in (A_n, A_{n+1})$ is equivalent to $\mathbb{D}_{n+1} < S < \mathbb{B}_n$.

Part 2. Conversely, assume that (i) $A_n \leq \max_i(\gamma_i A_i) < R_n^L < A_{n+1}$ or (ii) $\max_i(\gamma_i A_i) < A_n < R_n^L < A_{n+1}$.

1. If $A_n \leq \max_i(\gamma_i A_i) < R_n^L < A_{n+1}$. Condition $A_n \leq \max_i(\gamma_i A_i)$ implies that $R > A_n$. Then $k_i = 0\ \forall i \leq n$, and hence $S = \sum_{i=n+1}^{m} k_i \leq \sum_{i=n+1}^{m} \frac{RS_i}{R-\gamma_i A_i}$

   By definition R_n^L, we have $S = \sum_{i=n+1}^{m} \frac{R_n^L S_i}{R_n^L-\gamma_i A_i}$. Since the function $f(X) \equiv \sum_{i=n+1}^{m} \frac{XS_i}{X-\gamma_i A_i}$ is decreasing in the interval $(\max_{i\geq n+1}(\gamma_i A_i), \infty)$ and $R, R_n^L > \max_i(\gamma_i A_i)$, we have $R \leq R_n^L$. This implies that $R \in (A_n, A_{n+1})$. Therefore, Lemma 12 implies that $R = R_n^L$.

2. If $\max_i(\gamma_i A_i) < A_n$ and $\mathbb{D}_{n+1} < S < \mathbb{B}_n$. We have $S < \mathbb{D}_n$ because $\mathbb{D}_n > \mathbb{B}_n$. According to point 2 of Lemma 10, we have $R \geq A_n$.

   Condition $S > \mathbb{D}_{n+1}$ implies that $S > \mathbb{B}_{n+1}$ because $\mathbb{D}_{n+1} > \mathbb{B}_{n+1}$. According to point 1 of Lemma 10, we have $R \leq A_{n+1}$.

   If $R = A_{n+1}$, then Lemma 11 implies that $S \leq \mathbb{D}_{n+1}$. This is a contradiction because $S > \mathbb{D}_{d+1}$.

   If $R = A_n$, Lemma 11 implies that $S \in [\mathbb{B}_n, \mathbb{D}_n]$. However, $S \leq \mathbb{B}_n$. Thus, we have $S = \mathbb{B}_n = \sum_{i=n+1}^{m} \frac{A_n S_i}{A_n-\gamma_i A_i}$. Since $A_n > \max_i(\gamma_i A_i)$, then $A_n = R_n^L$, a contradiction.

   Summing up, we have $R \in (A_n, A_{n+1})$. By applying point 3 of Lemma 12, we have $R = R_n^L$.

□

Remark 6. We can check that the regimes in Theorem 2 are not overlap, and the union of these regimes is equal to the set of economies satisfying $A_1 < \cdots < A_m$, or, formally,

$$\mathbf{E} = \cup_{i=1}^{m} \mathcal{A}_i \cup \cup_{i=1}^{m-1} \mathcal{R}_i \tag{A.15a}$$

$$\mathcal{X} \cap \mathcal{Y} = \emptyset\ \forall X, Y \in \{\mathcal{A}_1, \ldots, \mathcal{A}_m, \mathcal{R}_1, \ldots, \mathcal{R}_{m-1}\} \text{ and } X \neq Y. \tag{A.15b}$$

Denote $M \equiv \max_i(\gamma_i A_i)$. By definition, we see that:

1. The economy $\mathcal{E} \equiv (F_i, \gamma_i, S_i)_{i=1,\ldots,m} \in \mathcal{A}_1$ if and only if $A_1 > \max_i(\gamma_i A_i)$ and $S > \mathbb{B}_1$.

2. $\mathcal{E} \in \mathcal{A}_m$ if and only if $S \leq \mathbb{D}_m$.

3. $\mathcal{E} \in \mathcal{A}_n$ with $n \in \{2, \ldots, m-1\}$ if and only if $A_n > \max_i(\gamma_i A_i)$ and $\mathbb{B}_n \leq S \leq \mathbb{D}_n$.

4. $\mathcal{R}_n \equiv \mathcal{R}_{n,1} \cup \mathcal{R}_{n,2}$ with $n \in \{1, \ldots, m-1\}$ where
   (a) $\mathcal{R}_{n,1}$ is the set of economies such that $A_n > \max_i(\gamma_i A_i)$ and $\mathbb{D}_{n+1} < S < \mathbb{B}_n$.
   (b) $\mathcal{R}_{n,2}$ is the set of economies such that $A_{n+1} > \max_i(\gamma_i A_i) \geq A_n$ and $\mathbb{D}_{n+1} < S$.

We now prove (A.15a) which implies the existence of equilibrium. It suffices to verify that $\mathbf{E} \subset \cup_{i=1}^m \mathcal{A}_i \cup \cup_{i=1}^{m-1} \mathcal{R}_i$. Let us consider an economy $\mathcal{E}$. There are only two cases.

1. $\max_i(\gamma_i A_i) < A_1$. In this case, we have $\max_i(\gamma_i A_i) < A_n \ \forall n$. Therefore, it is easy to see that $\mathcal{E} \in \cup_{i=1}^m \mathcal{A}_i \cup \cup_{i=1}^{m-1} \mathcal{R}_{i,1} \subset \cup_{i=1}^m \mathcal{A}_i \cup \cup_{i=1}^{m-1} \mathcal{R}_i$.

2. There exists $n \in \{1, \ldots, m-1\}$ such that $A_{n+1} > \max_i(\gamma_i A_i) \geq A_n$. There are two sub-cases.
   (a) $S > \mathbb{D}_{n+1}$. In this case, $\mathcal{E} \in \mathcal{R}_{n+1,2}$.
   (b) $S \leq \mathbb{D}_{n+1}$. Recall that $M < A_{n+1}$. In this case, we will prove that $\mathcal{E} \in \cup_{i=n+1}^m \mathcal{A}_i \cup \cup_{i=n+1}^{m-1} \mathcal{R}_i$. Indeed, since $S \leq \mathbb{D}_{n+1}$, there are $2(m-n)-1$ cases.
      i. If there exists $i \in \{n+1, m-1\}$ such that $\mathbb{B}_i \leq S \leq \mathbb{D}_i$. Then $\mathcal{E} \in \mathcal{A}_i$ because $A_i \geq A_{n+1} > \max_i(\gamma_i A_i)$.
      ii. If there exists $i \in \{n+1, m-1\}$ such that $\mathbb{D}_{i+1} \leq S \leq \mathbb{B}_i$. Then $\mathcal{E} \in \mathcal{R}_{i,1}$ because $A_i \geq A_{n+1} > \max_i(\gamma_i A_i)$.
      iii. Last, if $S \leq \mathbb{D}_m$, then $\mathcal{E} \in \mathcal{R}_m$.

Proof of (A.15b). Observe that the equilibrium interest rate is unique if (A.15b) holds. We have to prove that:

$$\mathcal{A}_n \cap \mathcal{A}_h = \emptyset \ \forall n \neq h \tag{A.16a}$$
$$\mathcal{A}_n \cap \mathcal{R}_{h,1} = \emptyset \ \forall n, h \tag{A.16b}$$
$$\mathcal{A}_n \cap \mathcal{R}_{h,2} = \emptyset \ \forall n, h \tag{A.16c}$$
$$\mathcal{R}_n \cap \mathcal{R}_h = \emptyset \ \forall n \neq h. \tag{A.16d}$$

Following (A.9), it is easy to see that the two first equalities hold.

We now prove that $\mathcal{A}_n \cap \mathcal{R}_{h,2} = \emptyset \ \forall n, h$. Suppose that there exists $\mathcal{E} \in \mathcal{A}_n \cap \mathcal{R}_{h,2}$. It means that (1) $A_n > \max_i(\gamma_i A_i)$ and $\mathbb{B}_n \leq S \leq \mathbb{D}_n$, and (ii) $A_{h+1} > \max_i(\gamma_i A_i) \geq A_h$ and $\mathbb{D}_{h+1} < S$. From these conditions we get $A_n > A_h$, and hence $n \geq h+1$. Thus, we obtain $S > \mathbb{D}_{h+1} \geq \mathbb{D}_n \geq S$, a contradiction. Therefore, we have $\mathcal{A}_n \cap \mathcal{R}_{h,2} = \emptyset \ \forall n, h$.

Last, we prove $\mathcal{R}_n \cap \mathcal{R}_h = \emptyset$, or equivalently $\mathcal{R}_{n,i} \cap \mathcal{R}_{h,j} = \emptyset \ \forall i, j \in \{1, 2\}$, $\forall n \neq h$. Without loss of generality, we can assume that $n < h$. It is easy to see that $\mathcal{R}_{n,1} \cap \mathcal{R}_{h,1} = \emptyset$ and $\mathcal{R}_{n,2} \cap \mathcal{R}_{h,2} = \emptyset$. We now prove that $\mathcal{R}_{n,1} \cap \mathcal{R}_{h,2} = \emptyset$ and $\mathcal{R}_{n,2} \cap \mathcal{R}_{h,1} = \emptyset$.

1. Suppose that there exists $\mathcal{E} \in \mathcal{R}_{n,1} \cap \mathcal{R}_{h,2}$. It means that $A_n > \max_i(\gamma_i A_i)$; $\mathbb{D}_{n+1} < S < \mathbb{B}_n$; $A_{h+1} > \max_i(\gamma_i A_i) \geq A_h$; $\mathbb{D}_{h+1} < S$. Since $h > n$, then $A_h > A_n > \max_i(\gamma_i A_i)$. This is a contradiction because $\max_i(\gamma_i A_i) \geq A_h$. So, we have $\mathcal{R}_{n,1} \cap \mathcal{R}_{h,2} = \emptyset$.

2. Suppose that there exists $\mathcal{E} \in \mathcal{R}_{n,2} \cap \mathcal{R}_{h,1}$. It means that $A_{n+1} > \max_i(\gamma_i A_i) \geq A_n$; $\mathbb{D}_{n+1} < S$;$A_h > \max_i(\gamma_i A_i)$; $\mathbb{D}_{h+1} < S < \mathbb{B}_h$.

   Since $h \geq n+1$, we have $\mathbb{B}_h \leq \mathbb{B}_{n+1} < \mathbb{D}_{n+1} < S < \mathbb{B}_h$, a contradiction.

B Proofs for Section 3.1

Proof of Proposition 1. In the case $\gamma_2 < \frac{A_1}{\lambda A_2}\frac{S_1}{S_1+S_2} = \frac{A_1'}{A_2'}\frac{S_1}{S_1+S_2}$ and $\gamma_2 < \frac{A_1}{A_2}\frac{S_1}{S_1+S_2}$, we have that

$$\begin{aligned} Y(A_1', A_2') - Y(A_1, A_2) &= A_1'S_1 + A_2'S_2\frac{A_1'(1-\gamma_2)}{A_1' - \gamma_2 A_2'} - A_1S_1 - A_2S_2\frac{A_1(1-\gamma_2)}{A_1 - \gamma_2 A_2} \\ &= (A_1' - A_1)S_1 + A_2S_2(1-\gamma_2)\frac{A_1A_2' - A_1'A_2}{(A_1 - \gamma_2 A_2)(A_1' - \gamma_2 A_2')} \end{aligned}$$

Point 1. When $\frac{A_2'}{A_2} \geq \frac{A_1'}{A_1} \geq 1$, we have $(A_1' - A_1)S_1 > 0$ and $A_1A_2' - A_1'A_2 \geq 0$. By consequence, we get that $Y(A_1', A_2') - Y(A_1, A_2) > 0$.
Point 2. We can compute that

$$\begin{aligned} \frac{\partial Y}{\partial A_1} &= S_1 - A_2S_2(1-\gamma_2)\frac{\gamma_2 A_2}{(A_1 - \gamma_2 A_2)^2} \\ \frac{\partial Y}{\partial A_2} &= \frac{S_2A_1^2(1-\gamma_2)}{(A_1 - \gamma_2 A_2)^2} \end{aligned}$$

So, we have that

$$\begin{aligned} &\frac{\partial Y}{\partial A_1}(A_1, A_2)(A_1' - A_1) + \frac{\partial Y}{\partial A_1}(A_1, A_2)(A_2' - A_2) \\ =&\Big(S_1 - A_2S_2(1-\gamma_2)\frac{\gamma_2 A_2}{(A_1 - \gamma_2 A_2)^2}\Big)(A_1' - A_1) + \Big(\frac{S_2A_1^2(1-\gamma_2)}{(A_1 - \gamma_2 A_2)^2}\Big)(A_2' - A_2) \\ =&(A_1' - A_1)\Big(S_1 - A_2S_2(1-\gamma_2)\frac{\gamma_2 A_2}{(A_1 - \gamma_2 A_2)^2} + \frac{S_2A_1^2(1-\gamma_2)}{(A_1 - \gamma_2 A_2)^2}\frac{A_2' - A_2}{A_1' - A_1}\Big) \\ =&(A_1' - A_1)\Big(\frac{S_2A_1A_2(1-\gamma_2)}{(A_1 - \gamma_2 A_2)^2}\frac{\frac{A_2'}{A_2} - 1}{\frac{A_1'}{A_1} - 1} - \Big(S_2(1-\gamma_2)\frac{\gamma_2 A_2^2}{(A_1 - \gamma_2 A_2)^2} - S_1\Big)\Big). \end{aligned}$$

Therefore, we have

$$\frac{\frac{\partial Y}{\partial A_1}(A_1, A_2)(A_1' - A_1) + \frac{\partial Y}{\partial A_1}(A_1, A_2)(A_2' - A_2)}{A_1' - A_1} < 0$$

if $A_1' \neq A_1$, $S_2(1-\gamma_2)\frac{\gamma_2 A_2^2}{(A_1 - \gamma_2 A_2)^2} - S_1 > 0$ and

$$\frac{\frac{A_2'}{A_2} - 1}{\frac{A_1'}{A_1} - 1} < \frac{\gamma_2 A_2}{A_1} - \frac{S_1(A_1 - \gamma_2 A_2)^2}{S_2A_1A_2(1-\gamma_2)}.$$

By Taylor's theorem, we get point 2.

□

Proof of Corollary 3. Point 1.a is a direct consequence of Corollary 1. Point 1.b is a direct consequence of Theorem 1.

We now prove point 2. We make use of Theorem 2. We firstly consider the regime $\mathcal{R}_n$ with $n \leq m-1$. In this regime, we have

$$Y = Y_n = \sum_{i=n+1}^{m} \frac{rA_iS_i}{r - f_iA_i} \leq \sum_{i=n+1}^{m} A_m \frac{rS_i}{r - f_iA_i} = A_mS. \tag{A.17}$$

Notice that $Y = A_mS$ if and only if $n+1 = m$.

We now consider the regime $\mathcal{A}_n$ with $n \leq m$. In this regime, we have

$$\begin{aligned} Y &= A_n \sum_{i=1}^{n} S_i + \sum_{i=n+1}^{m} \frac{A_n(1-f_i)A_iS_i}{A_n - f_iA_i} = A_n \sum_{i=1}^{n} S_i + A_n \sum_{i=n+1}^{m} \frac{A_n(A_i - A_n)}{A_n - f_iA_i} \\ &\leq A_nS + (A_m - A_n) \sum_{i=n+1}^{m} \frac{A_n}{A_n - f_iA_i} \leq A_nS + (A_m - A_n)S = A_mS. \end{aligned}$$

where the last inequality is from the condition $\sum_{i=n+1}^{m} \frac{A_nS_i}{A_n - f_iA_i}$ in the regime $\mathcal{A}_n$. It is easy to see that $Y = A_mS$ if and only if either (i) $n+1 > m$ or (ii) $n+1 = m$ and $\frac{A_{m-1}}{A_{m-1} - f_mA_m}S_m = S$. Combining these two cases, we obtain our result.

□

Proof of Proposition 3. Since the production function has Cobb-Douglas form, all agents produce in equilibrium. According to (24), we have

$$\frac{\partial Y}{\partial A_1} = \underbrace{f_1(k_1)}_{\text{Productivity effect}} + \underbrace{\sum_{i \neq 1} \big(A_1 f_1'(k_1) - A_i f_i'(k_i)\big) \underbrace{\frac{-\partial k_i}{\partial R}}_{\geq 0} \underbrace{\frac{\partial R}{\partial A_1}}_{\geq 0}}_{\text{Allocation effect}}. \tag{A.18}$$

According to FOCs, we have

$$\begin{aligned} &[k] : (1 + \mu_i\gamma_i)F_i'(k) = \lambda_i \\ &[a] : (1 + \mu_i)R = \lambda_i, \quad \mu_i \geq 0, \text{ and } \mu_i(\gamma_iF_i(k_i) - R_ib_i) = 0. \end{aligned}$$

These equations imply that:

$$\gamma_iA_if_i'(k_i) \leq R = A_if_i'(k_i)\frac{1 + \gamma_i\mu_i}{1 + \mu_i} \leq A_if_i'(k_i), \forall i. \tag{A.19}$$

This implies that $R \geq \max_j \gamma_jF_j'(k_j) \geq \max_j \gamma_jF_j'(S))$. Thus, $R \geq \max_j \gamma_jF_j'(S)) > 0$, $\forall A_1$.

1. When A_1 is high enough. Note that $\lim_{A_1 \to \infty} R_1 = \infty$. Hence, for A_1 high enough, we have that $R_1 > S$. We prove that the equilibrium interest rate goes to infinity when A_1 goes to infinity. Indeed, if agent 1's borrowing constraint is

not binding, we have $R = A_1 f_1'(k_1) > A_1 f_1'(S)$. If agent 1's borrowing constraint is binding, we have $R(k_1 - S_1) = \gamma_1 A_1 f_1(k_1)$ which implies that

$$R = \frac{\gamma_1 A_1 f_1(k_1)}{k_1 - S_1} \geq \frac{\gamma_1 A_1 f_1(S_1)}{S - S_1}$$

Hence, $R \geq \min\left(A_1 f_1'(S), \frac{\gamma_1 A_1 f_1(S_1)}{S - S_1}\right)$. From this, we obtain that $\lim_{A_1 \to \infty} R = \infty$.

Now, condition $\lim_{A_1 \to \infty} R = \infty$ implies that borrowing constraint of any agent $i \geq 2$ is not binding for A_1 high enough. So, $A_1 f_1'(k_1) \geq R = A_i f_i'(k_i)$, $\forall i \geq 1$. By combining this and condition (A.18), we get that $\frac{\partial Y}{\partial A_1} > 0$ for A_1 high enough.

2. We will prove that when A_1 is small enough, the productivity effect is smaller than the allocation effect. To show $\frac{\partial Y}{\partial A_1} < 0$ for A_1 small enough, we will prove that $\lim_{A_1 \to 0} k_1 = 0$, $\lim_{A_1 \to 0} A_1 f_1'(k_1) - A_2 f_2'(k_2) < 0$, $\lim_{A_1 \to 0} \frac{-\partial k_2}{\partial R} > 0$, and $\lim_{A_1 \to 0} \frac{\partial R}{\partial A_1} > 0$.

   Since $A_1 f_1'(k_1) \geq R \geq \max_j \gamma_j F_j'(S)) > 0$, we have $\lim_{A_1 \to 0} f_1'(k_1) = \infty$. Therefore, we have

$$\lim_{A_1 \to 0} k_1 = 0, \text{ and } \lim_{A_1 \to 0} \sum_{i \neq 1} k_i = S. \tag{A.20}$$

   Since $\lim_{A_1 \to 0} k_1 = 0$, we get that $\gamma_1 A_1 f_1(k_1) - R b_1 = \gamma_1 A_1 f_1(k_1) - R k_1 + R S_1 > 0$ for A_1 small enough. It means that the borrowing constraint of agent 1 is not binding. To sum up, we have

$$R = A_1 f_1'(k_1) \geq \max_j \gamma_j F_j'(S)) > 0, \text{ for } A_1 \text{ small enough.}$$

   Denote

$$B_1 = B_1(R_1) \equiv k_1^n(R_1) + \sum_{i=2}^{m} k_i^b(R_1), \quad B_2 = B_2(R_2) = \sum_{i=1}^{2} k_i^n(R_2) + \sum_{i=3}^{m} k_i^b(R_2)$$
$$B_m = B_m(R_m) = \sum_{i=1}^{m} k_i^n(R_m)$$

   where, to simplify notations, we write $k_i^n(R)$ and $k_i^b(R)$ instead of $k_i^n(\frac{R}{A_i})$ and $k_i^b(\frac{R}{\gamma_i A_i}, S_i)$ (see Definition 2). We see that $D_i \equiv B_i - k_i^n(R_i)$, $\forall i$. Notice also that $B_1, \ldots, B_m$ depend on A_1 but $D_2, D_3, \ldots, D_m$ do not. Moreover, $\lim_{A_1 \to 0}(B_i - D_i) = 0$, $\forall i \geq 2$ because $\lim_{A_1 \to 0} k_1^n(R_i) = 0$, $\forall i \geq 2$.

   Condition $R_2 < R_3 < \cdots < R_m$ implies that $D_2 > \cdots > D_m$. Since $\lim_{A_1 \to 0} B_1 = +\infty$ and $\lim_{A_1 \to 0}(B_i - D_i) = 0$, $\forall i \geq 2$, we have $B_1 > B_2 > \cdots > B_m$ for A_1 small enough.

   (a) $S < D_m$. Then we have $S < B_m$. According to Theorem 1, the equilibrium coincides to that of the economy without frictions. Therefore, the output is increasing in A_1.

(b) Let $D_n > S > D_{n+1}$. In this case, we have $B_n > S > B_{n+1}$ for any A_1 small enough. According to Theorem 1, the equilibrium interest rate R is in the interval $(R_n, R_{n+1}]$ and determined by

$$\sum_{i=1}^{n} k_i^n(R) + \sum_{i=n+1}^{m} k_i^b(R) = S \equiv \sum_i S_i \tag{A.21}$$

Denote $Z_2(R) = \sum_{i=2}^{n} k_i^n(R) + \sum_{i=n+1}^{m} k_i^b(R)$. When A_1 tends to zero, we have $\lim_{A_1 \to 0} k_1^n(R) = 0$ and $\lim_{A_1 \to 0} R = R(0)$ where $R(0) > 0$ is uniquely determined by $Z_2(R(0)) = S$.

For $i \geq n+1$, agent i's borrowing constraint is binding: $R(k_i - S_i) = \gamma_i A_i f_i(k_i)$ for any A_1 small enough. Let A_1 tend to zero, we have k_i tends to $k_i(0)$, R tends to $R(0)$, and

$$\gamma_i A_i f_i(k_i(0)) = R(0)(k_i(0) - S_i).$$

Let σ be such that

$$\gamma_i \frac{f_i(k)}{k f_i'(k)} < \sigma < \frac{S_1}{S_1 + S_{n+1} + \cdots + S_m}, \forall i \geq n+1, \forall k \in (0, S). \tag{A.22}$$

According to condition (26), we have

$$\begin{aligned} R(0) - A_i f_i'(k_i(0)) &= \frac{\gamma_i A_i f_i(k_i(0))}{k_i(0) - S_i} - A_i f_i'(k_i(0)) && \text{(A.23)} \\ &\leq \frac{A_i f_i'(k_i(0))}{k_i(0) - S_i}\big(\sigma k_i(0) - (k_i(0) - S_i)\big) && \text{(A.24)} \end{aligned}$$

By market clearing condition, we have

$$\sum_{i=n+1}^{m} k_i = \sum_{i=2}^{m}(S_i - k_i) + S_1 - k_1 + \sum_{i=n+1}^{m} S_i \geq S_1 - k_1 + \sum_{i=n+1}^{m} S_i$$

Let A_1 tend to zero, we get that $\sum_{i=n+1}^{m} k_i(0) \geq S_1 + \sum_{i=n+1}^{m} S_i$. Thus,

$$\begin{aligned} &\sum_{i=n+1}^{m} \Big(\sigma k_i(0) - (k_i(0) - S_i)\Big) = \sum_{i=n+1}^{m} \Big(S_i - (1-\sigma) k_i(0)\Big) \\ &\leq \sum_{i=n+1}^{m} S_i - (1-\sigma)\big(S_1 + \sum_{i=n+1}^{m} S_i\big) < 0 \end{aligned}$$

Therefore, there exists $j \in \{n+1, \ldots, m\}$ such that $\sigma k_j(0) - (k_j(0) - S_j) < 0$, and hence

$$R(0) - A_j f_j'(k_j(0)) \leq \frac{A_j f_j'(k_j(0))}{k_j(0) - S_j}\big(\sigma k_j(0) - (k_j(0) - S_j)\big) < 0. \tag{A.25}$$

Now, by noting that $A_1 f'(k_1) = R$, we have

$$\frac{\partial Y}{\partial A_1} \leq f_1(k_1) + \big(R - A_j f_j'(k_j)\big)\frac{-\partial k_j}{\partial R}\frac{\partial R}{\partial A_1} \tag{A.26}$$

Again, by the market clearing condition

$$k_1^n(\frac{R}{A_1}) + \sum_{i\neq 2} k_i(R) = S \tag{A.27}$$

we have that

$$(k_1^n)'(\frac{R}{A_1})\frac{R'(A_1)A_1 - R}{A_1^2} + \sum_{i\neq 2}\frac{\partial k_i}{\partial R}R'(A_1) = 0 \tag{A.28}$$

$$\Leftrightarrow R'(A_1)\Big(\frac{1}{A_1}(k_1^n)'(\frac{R}{A_1}) + \sum_{i\neq 2}\frac{\partial k_i}{\partial R}\Big) = (k_1^n)'(\frac{R}{A_1})\frac{R}{A_1^2}$$

$$\Leftrightarrow R'(A_1)A_1\Big(\frac{1}{R} + \frac{A_1}{R}\frac{\sum_{i\neq 2}\frac{\partial k_i}{\partial R}}{(k_1^n)'\big(\frac{R}{A_1}\big)}\Big) = 1 \tag{A.29}$$

Since $\frac{\partial k_i}{\partial R} < 0, \forall i \neq 1$, and $(k_1^n)'\big(\frac{R}{A_1}\big) < 0$, we have $R'(A_1) > 0$.
By definition of k_1^n, we have $f_1'(k_1^n(x)) = x$. So, $(k_1^n)'(x)f_1''(k_1^n(x)) = 1$, and hence,

$$\lim_{A_1\to 0}\frac{R}{A_1}(k_1^n)'(\frac{R}{A_1}) = \lim_{A_1\to 0}\frac{\frac{R}{A_1}}{f_1''(\frac{R}{A_1})} = \lim_{x\to\infty}\frac{x}{f_1''(x)} < 0.$$

By combining this with (A.29), $\lim_{R\to R(0)}\frac{\partial k_i}{\partial R} < 0$, $\forall i$, and $\lim_{A_1\to 0} R = R(0) > 0$, we get that

$$\lim_{A_1\to 0} R'(A_1) = +\infty. \tag{A.30}$$

By combining (A.26), (A.25), (A.30), and $\lim_{R\to R(0)}\frac{\partial k_j}{\partial R} < 0$, we get that $\frac{\partial Y}{\partial A_1} < 0$ for any $A_1 > 0$ small enough.

□

Proof of Proposition 4. First, we state a corollary of Theorem 1.

Corollary 6. *Let Assumptions 2, 3 and 4 be satisfied. Consider a two-agent model and assume that $R_1 < R_2$.*

1. *In the regime $\mathcal{R}_2$, i.e., when $S < k_1^n(\frac{R_2}{A_1}) + k_2^n(\frac{R_2}{A_2})$, credit constraint of any agent is not binding.*

2. *In the regime $\mathcal{R}_1$, i.e., when $S \geq k_1^n(\frac{R_2}{A_1}) + k_2^n(\frac{R_2}{A_2})$,[17] the equilibrium interest rate R is determined by*

$$k_1^n(\frac{R}{A_1}) + k_2^b(\frac{R}{\gamma_2 A_2}, S_2) = S \equiv \sum_i S_i. \tag{A.31}$$

[17]Notice that we always have that $k_1^n(R_1) = k_1^b(R_1)$, $k_2^n(R_2) = k_2^b(R_2)$, and $k_1^n(R_1) + k_2^b(R_1) = k_1^b(R_1) + k_2^b(R_1) > S$.

In this regime, $R_1 < R \leq R_2$, agent 2 borrows and her credit constraint is binding while agent 1 is lender.

Now, we prove part 1 of Proof of Proposition 4. Since Inada condition holds, all agents produce in equilibrium. According to (24), we have

$$\frac{\partial Y}{\partial A_1} = f_1(k_1) + \big(A_1 f_1'(k_1) - A_2 f_2'(k_2)\big) \underbrace{\frac{-\partial k_2}{\partial R}}_{>0} \underbrace{\frac{\partial R}{\partial A_1}}_{>0}. \tag{A.32}$$

To show $\frac{\partial Y}{\partial A_1} < 0$ for A_1 small enough, we will prove that $\lim_{A_1 \to 0} k_1 = 0$, $\lim_{A_1 \to 0} A_1 f_1'(k_1) - A_2 f_2'(k_2) < 0$, $\lim_{A_1 \to 0} \frac{-\partial k_2}{\partial R} > 0$, and $\lim_{A_1 \to 0} \frac{\partial R}{\partial A_1} > 0$.

According to FOCs, we have

$$\begin{aligned}
&[k] : (1 + \mu_i \gamma_i) F_i'(k) = \lambda_i \\
&[a] : (1 + \mu_i) R = \lambda_i, \quad \mu_i \geq 0, \text{ and } \mu_i(\gamma_i F_i(k_i) - R_i b_i) = 0.
\end{aligned}$$

These equations imply that:

$$\gamma_i A_i f_i'(k_i) \leq R = A_i f_i'(k_i) \frac{1 + \gamma_i \mu_i}{1 + \mu_i} \leq A_i f_i'(k_i), \forall i. \tag{A.33}$$

This implies that $R \geq \gamma_2 F_2'(k_2) \geq \gamma_2 F_2'(S)$. Thus, $R \geq \gamma_2 F_2'(S)$, $\forall A_1$. Since $R \leq A_1 f_1'(k_1)$. So, we have $\lim_{A_1 \to 0} f_1'(k_1) = \infty$. Therefore, we have

$$\lim_{A_1 \to 0} k_1 = 0, \text{ and } \lim_{A_1 \to 0} k_2 = S. \tag{A.34}$$

By consequence, we get that $\gamma_1 A_1 f_1(k_1) - R b_1 = \gamma_1 A_1 f_1(k_1) - R k_1 + R S_1 > 0$ for A_1 small enough. It means that the borrowing constraint of agent 1 is not binding. To sum up, we have $R = A_1 f_1'(k_1) \geq \gamma_2 F_2'(S)$ for A_1 small enough.

Since R_2 does not depend on A_1, we observe that $\lim_{A_1 \to 0} k_1^n(R_2/A_1) = 0$. So, by combining with the assumption $k_2^n(\frac{R_2}{A_2}) < S$, we have $k_1^n(\frac{R}{A_1}) + k_2^n(\frac{R}{A_2}) < S$ for A_1 small enough. According to point 3 of Lemma 2, we have $R_1 < R_2$ for A_1 small enough. Hence, we can apply Corollary 6 to obtain that the borrowing constraint of agent 2 is binding in equilibrium. It means that $\gamma_2 A_2 f_2(k_2) - R k_2 + R S_2 = 0$.

Look at the market clearing condition: $k_1^n(\frac{R}{A_1}) + k_2^b(\frac{R}{\gamma_2 A_2}, S_2) = S \equiv \sum_i S_i$. When A_1 converges to 0, we have $k_1^n(\frac{R}{A_1})$ converges to 0. So, R converges to $R(0)$ satisfying $k_2^b(\frac{R(0)}{\gamma_2 A_2}, S_2) = S$. So, we have

$$\lim_{A_1 \to \infty} \big(A_1 f_1'(k_1) - A_2 f_2'(k_2)\big) = \lim_{A_1 \to \infty} \big(R - A_2 f_2'(k_2)\big) = R(0) - A_2 f_2'(S). \tag{A.35}$$

Since agent 2's borrowing constraint is binding: $R(k_2 - S_2) = \gamma_2 A_2 f_2(k_2)$ for any A_1 small enough. Let A_1 tend to zero, we have $\gamma_2 A_2 f_2(S) = R(0)(S - S_2) = R(0) S_1$. So, we have

$$R(0) - A_2 f_2'(S) = \frac{\gamma_2 A_2 f_2(S)}{S_1} - A_2 f_2'(S) < 0 \tag{A.36}$$

because we assume that $\gamma_2 < \frac{S_1}{S_1+S_2}\frac{Sf_2'(S)}{f_2(S)}$.

Again, by the market clearing condition $k_1^n(\frac{R}{A_1}) + k_2^b(\frac{R}{\gamma_2 A_2}, S_2) = S$, we have

$$\begin{aligned}
(k_1^n)'(\frac{R}{A_1})\frac{R'(A_1)A_1 - R}{A_1^2} + \frac{\partial k_2^b}{\partial x_1}(\frac{R}{\gamma_2 A_2}, S_2)\frac{R'(A_1)}{(\gamma_2 A_2)^2} = 0 \\
\Leftrightarrow R'(A_1)\Big(\frac{1}{A_1}(k_1^n)'(\frac{R}{A_1}) + \frac{1}{(\gamma_2 A_2)^2}\frac{\partial k_2^b}{\partial x_1}(\frac{R}{\gamma_2 A_2}, S_2)\Big) = (k_1^n)'(\frac{R}{A_1})\frac{R}{A_1^2} \\
\Leftrightarrow R'(A_1)A_1\Big(\frac{1}{R} + \frac{A_1}{(\gamma_2 A_2)^2 R}\frac{\frac{\partial k_2^b}{\partial x_1}\big(\frac{R}{\gamma_2 A_2}, S_2\big)}{(k_1^n)'\big(\frac{R}{A_1}\big)}\Big) = 1
\end{aligned} \tag{A.37}$$

First, since $\frac{\partial k_2^b}{\partial x_1} < 0$ and $(k_1^n)'\big(\frac{R}{A_1}\big) < 0$, we have $R'(A_1) > 0$.

Recall that $f_1'(k_1^n(x)) = x$. So, we have $(k_1^n)'(x)f_1''(k_1^n(x)) = 1$, and hence,

$$\lim_{A_1 \to 0}\frac{R}{A_1}(k_1^n)'(\frac{R}{A_1}) = \lim_{A_1 \to 0}\frac{\frac{R}{A_1}}{f_1''(\frac{R}{A_1})} = \lim_{x \to +\infty}\frac{x}{f_1''(x)} < 0.$$

By combining this with (A.37) and $\lim_{A_1 \to 0} R = R(0) > 0$, we get that

$$\lim_{A_1 \to 0} R'(A_1) = +\infty. \tag{A.38}$$

It is easy to see that, when A_1 is small enough, the $\frac{\partial k_2}{\partial R} = \frac{\partial k_2^b}{\partial R}\big(\frac{R}{\gamma_2 A_2}, S_2\big)$. Thus,

$$\lim_{A_1 \to 0}\frac{\partial k_2}{\partial R} = \lim_{R \to R(0)}\frac{\partial k_2^b}{\partial R}(\frac{R}{\gamma_2 A_2}, S_2) < 0. \tag{A.39}$$

By combining (A.34), (A.36), (A.38), (A.39) and (A.32), we conclude that $\frac{\partial Y}{\partial A_1} < 0$ for any $A_1 > 0$ small enough.

We now consider the Cobb-Douglas production functions. In such a case, condition $k_2^n(\frac{R_2}{A_2}) < S$ becomes $\gamma_2 < \alpha\frac{S_1}{S_1+S_2}$. For the sake of simplicity, we write k_2^n instead of $k_2^n(\frac{R_2}{A_2})$. Recall that $R_2 = A_2 f_2'(k_2^n) = A_2\alpha(k_2^n)^{\alpha-1}$. Hence,

$$\begin{aligned}
&\big(k_2^n - S_2\big)R_2 = \gamma_2 A_2 f_2\big(k_2^n\big) \Leftrightarrow (k_2^n - S_2)R_2 = \gamma_2 A_2(k_2^n)^\alpha \\
&\Leftrightarrow (k_2^n - S_2)A_2\alpha(k_2^n)^{\alpha-1} = \gamma_2 A_2(k_2^n)^\alpha \\
&\Leftrightarrow (\alpha - \gamma_2)A_2(k_2^n)^\alpha = S_2 A_2\alpha(k_2^n)^{\alpha-1} \Leftrightarrow (\alpha - \gamma_2)k_2^n = \alpha S_2.
\end{aligned}$$

Therefore, condition $S_1 + S_2 > k_2^n(R_2/A_2)$ becomes $(S_1 + S_2)(\alpha - \gamma_2) > \alpha S_2$, or, equivalently, $\alpha\frac{S_1}{S_1+S_2} > \gamma_2$. $\square$

C Proofs for Section 3.2

Proof of Proposition 9. Under assumptions in Proposition 9, we can prove that the equilibrium interest rate is in (A_n, A_{n+1}) if and only if

$$\sum_{i=n+1}^{m}\frac{A_{n+1}S_i}{A_{n+1} - \gamma_i A_i} < \sum_i S_i < \sum_{i=n+1}^{m}\frac{A_n S_i}{A_n - \gamma_i A_i} \tag{A.40}$$

Then, when $R \in (A_n, A_{n+1})$, it is determined by

$$\underbrace{\sum_{i=n+1}^{m} \frac{\gamma_i A_i}{R - \gamma_i A_i} S_i}_{\text{Asset demand}} = \underbrace{\sum_{i=1}^{n} S_i}_{\text{Asset supply}} \quad \text{or equivalently} \quad \underbrace{\sum_{i=n+1}^{m} \frac{RS_i}{R - \gamma_i A_i}}_{\text{Capital demand}} = \underbrace{S}_{\text{Capital supply}} \tag{A.41}$$

Agents $1, \dots, n$ are lenders while agents $n+1, \dots, m$ are borrowers. It is easy to see that $\frac{\partial Y}{\partial \gamma_i} = 0$, $\forall i \leq n$. For $i \geq n+1$, by using condition $\sum_{i=n+1}^{m} \frac{RS_i}{R - \gamma_i A_i} = \sum_i S$, we can compute that

$$\frac{\partial Y}{\partial \gamma_i} = \sum_{j=n+1}^{m} A_j S_j \frac{\partial \Big(\frac{R}{R - \gamma_j A_j} \Big)}{\partial \gamma_i} = \frac{\partial R}{\partial \gamma_i} \Big(A_i \sum_{j=n+1}^{m} \frac{\gamma_j A_j S_j}{(R - \gamma_j A_j)^2} - \sum_{j=n+1}^{m} \frac{\gamma_j S_j A_j^2}{(R - \gamma_j A_j)^2} \Big).$$

The first point is a direct consequence of this expression and the fact that $A_m > \dots > A_{n+1}$. Let us prove the second point. We have, by noticing that $R \in (A_n, A_{n+1})$ and $A_{t+1} > A_t$ $\forall t$,

$$A_i \sum_{t=n+1}^{m} \frac{\gamma_t A_t S_t}{(R - \gamma_t A_t)^2} - \sum_{t=n+1}^{m} \frac{\gamma_t S_t A_t^2}{(R - \gamma_t A_t)^2} \tag{A.42}$$

$$= \sum_{t=n+1}^{i-1} \frac{\gamma_t A_t S_t}{(R - \gamma_t A_t)^2} (A_i - A_t) - \sum_{t=i+1}^{m} \frac{\gamma_t A_t S_t}{(R - \gamma_t A_t)^2} (A_t - A_i) \tag{A.43}$$

$$\geq \sum_{t=n+1}^{i-1} \frac{\gamma_t A_t S_t}{(A_{n+1} - \gamma_t A_t)^2} (A_i - A_{i-1}) - \sum_{t=i+1}^{m} \frac{\gamma_t A_t S_t}{(A_n - \gamma_t A_t)^2} (A_m - A_i). \tag{A.44}$$

Combining this with the expression of $\frac{\partial Y}{\partial \gamma_i}$, we obtain (38a).

We also have

$$A_i \sum_{t=n+1}^{m} \frac{\gamma_t A_t S_t}{(R - \gamma_t A_t)^2} - \sum_{t=n+1}^{m} \frac{\gamma_t S_t A_t^2}{(R - \gamma_t A_t)^2} \tag{A.45}$$

$$= \sum_{t=n+1}^{i-1} \frac{\gamma_t A_t S_t}{(R - \gamma_t A_t)^2} (A_i - A_t) - \sum_{t=i+1}^{m} \frac{\gamma_t A_t S_t}{(R - \gamma_t A_t)^2} (A_t - A_i) \tag{A.46}$$

$$< \sum_{t=n+1}^{i-1} \frac{\gamma_t A_t S_t}{(A_n - \gamma_t A_t)^2} (A_i - A_{n+1}) - \sum_{t=i+1}^{m} \frac{\gamma_t A_t S_t}{(A_{n+1} - \gamma_t A_t)^2} (A_{i+1} - A_i). \tag{A.47}$$

Combining this with the expression of $\frac{\partial Y}{\partial \gamma_i}$, we obtain (38b). □

Proof of Example 1. We focus here on the case $max(\gamma_2 A_2, \gamma_3 A_3) < A_1$ (in this case the interest rate R may take any value in $[A_1, A_m]$). Applying Theorem 2, we can check that the interest rate is uniquely determined by

$$R = \begin{cases} A_1 \text{ if } S_1 \geq \frac{\gamma_3 A_3}{A_1-\gamma_3 A_3} S_3 + \frac{\gamma_2 A_2}{A_1-\gamma_2 A_2} S_2 \\ R_1 \text{ if } \frac{\gamma_3 A_3}{A_2-\gamma_3 A_3} S_3 + \frac{\gamma_2}{1-\gamma_2} S_2 < S_1 < \frac{\gamma_3 A_3}{A_1-\gamma_3 A_3} S_3 + \frac{\gamma_2 A_2}{A_1-\gamma_2 A_2} S_2 \\ A_2 \text{ if } \frac{\gamma_3 A_3}{A_2-\gamma_3 A_3} S_3 - S_2 \leq S_1 \leq \frac{\gamma_3 A_3}{A_2-\gamma_3 A_3} S_3 + \frac{\gamma_2}{1-\gamma_2} S_2 \\ R_2 \text{ if } \frac{\gamma_3}{1-\gamma_3} S_3 - S_2 < S_1 < \frac{\gamma_3 A_3}{A_2-\gamma_3 A_3} - S_2 \\ A_3 \text{ if } S_1 \leq \frac{\gamma_3}{1-\gamma_3} S_3 - S_2 \end{cases} \tag{A.48}$$

where $R_2 = \gamma_3 A_3 \Big(1 + \frac{S_3}{S_1+S_2}\Big)$ and R_1 is the highest solution of the equation:

$$\frac{\gamma_2 A_2}{R-\gamma_2 A_2} S_2 + \frac{\gamma_3 A_3}{R-\gamma_3 A_3} S_3 = S_1. \tag{A.49}$$

This equation implies that $R(S_2(R-\gamma_3 A_3)+S_3(R-\gamma_2 A_2)) = S(R-\gamma_2 A_2)(R-\gamma_3 A_3)$, or equivalently

$$S_1 R^2 - R\Big((S_1+S_2)\gamma_2 A_2 + (S_1+S_3)\gamma_3 A_3\Big) + S\gamma_2 A_2 \gamma_3 A_3 = 0. \tag{A.50a}$$

So, the rate R_1 is computed by

$$R = \frac{(S_1+S_2)\gamma_2 A_2 + (S_1+S_3)\gamma_3 A_3 + \sqrt{\Delta}}{2S_1} \tag{A.50b}$$

$$\text{where } \Delta \equiv \big((S_1+S_2)\gamma_2 A_2 + (S_1+S_3)\gamma_3 A_3\big)^2 - 4S_1 S\gamma_2 A_2 \gamma_3 A_3 \tag{A.50c}$$

There are 5 different cases. In each case, we can explicitly compute equilibrium outcomes thanks to Lemma 2.

□

Proof of Proposition 10 (homogeneous credit limit). Since $F_i'(k_t) \geq R$, there are two cases. (1) If $F_i'(k_i) = R$, then we have hence $\frac{\partial k_i}{\partial \gamma} < 0$. (2) If $F_i'(k_i) > R$, then borrowing constraint of this agent is binding.

The market clearing condition $\sum_i k_i = \sum_i S_i$ implies that

$$\sum_{i:F_i'(k_i)=R} \frac{\partial k_i}{\partial \gamma} + \sum_{i:F_i'(k_i)>R} \frac{\partial k_i}{\partial \gamma} = 0.$$

So, we have $\sum_{i:F_i'(k_i)>R} \frac{\partial k_i}{\partial \gamma} > 0$.

We now claim that $\frac{\partial k_i}{\partial \gamma} > 0$ for any agent with $F_i'(k_i) > R$. For such agents we have $\gamma F_i(k_i^n) - R(k_i^n - S_i)$. Taking the derivative with respect to γ of both sides of this equation, we have

$$F_i(k_i) + \gamma F_i'(k_i)\frac{\partial k_i}{\partial \gamma} = \frac{\partial R}{\partial \gamma}(k_i - S_i) + R\frac{\partial k_i}{\partial \gamma} \tag{A.51}$$

$$i.e., \quad \frac{\partial k_i}{\partial \gamma} = \Big(\frac{\partial R}{\partial \gamma}\frac{\gamma}{R} - 1\Big)\frac{F_i(k_i)}{R - \gamma F_i'(k_i)}. \tag{A.52}$$

By summing with respect to i such that $F_i'(k_i) > R$ and noticing that $\sum_{i:F_i'(k_i)>R} \frac{\partial k_i}{\partial \gamma} > 0$ and $R - \gamma F_i'(k_i) > 0$ $\forall i$, we get that $\frac{\partial R}{\partial \gamma}\frac{\gamma}{R} - 1 \geq 0$. From this and (A.52), we obtain $\frac{\partial k_i}{\partial \gamma} > 0$ $\forall i$ such that $F_i'(k_i) > R$.

We now observe that

$$\frac{\partial Y}{\partial \gamma} = \sum_{i:F_i'(k_i)=R} F_j'(k_i)\frac{\partial k_i}{\partial \gamma} + \sum_{i:F_i'(k_i)>R} F_i'(k_i)\frac{\partial k_i}{\partial \gamma} \geq R\Big(\sum_{i=1}^{m} \frac{\partial k_i}{\partial \gamma}\Big) = 0. \tag{A.53}$$

□

D Proofs of Section 4

Firstly, we provide a sufficient condition to check whether a sequence is an intertemporal equilibrium.

Lemma 14. *Let the borrowing constraint of agent i at date t is $B_i(R_t, b_{i,t-1}, k_{i,t-1}) \leq 0$, where*

- *The function B_i is continuously differentiable, increasing in b_i and R, but decreasing in k_i. Moreover, for R given, the function $B_i(R, \cdot, \cdot)$ is convex.*
- *$B_i(R, k_i, b_i) \leq 0$ implies that $F_i(k_i) - Rb_i \geq 0$.*
- *$\sum_i B_i(R, k_i, b_i) < 0, \forall R > 0, k_i \geq 0,$ such that $\sum_i k_i > 0, \sum_i b_i = 0$.*

If the sequences $(R_t, (c_{i,t}, k_{i,t}, b_{i,t})_i)_t$ and $(\lambda_{i,t}, \mu_{i,t}, \eta_{i,t})_{i,t}$ satisfy the following conditions:

1. *$c_{i,t}, l_{i,t}, \lambda_{i,t}, \eta_{i,t}, \mu_{i,t+1}$ are non-negative and $R_t > 0$ for any t.*
2. *$c_{i,t} + k_{i,t} + R_t b_{i,t-1} = F_{i,t}(k_{i,t-1}) + b_{i,t}$, and $B_i(R_t, b_{i,t-1}, k_{i,t-1}) \leq 0$, $\forall i, \forall t$.*
3. *$\sum_i b_{i,t} = 0$, $\forall t$.*
4. *$\sum_{t=0}^{\infty} \lambda_{i,t} c_{i,t} < \infty, \sum_{t=0}^{\infty} \beta_i^t u_i(c_{i,t}) < \infty$.*
5. *TVCs: $\lim_{T\to\infty} \beta_i^t u_i'(c_{i,t})(k_{i,t} - b_{i,t}) = 0$, $\forall i$.*
6. *FOCs: $\forall i, \forall t$,*

$$\begin{aligned}
\beta_i^t u_i'(c_{i,t}) &= \lambda_{i,t} \\
\lambda_{i,t} &= F_{i,t+1}'(k_{i,t})\lambda_{i,t+1} - \mu_{i,t+1}\frac{\partial B_i}{\partial k}(R_{t+1}, b_{i,t}, k_{i,t}) + \eta_{i,t} \\
\eta_{i,t} k_{i,t} &= 0 \\
\lambda_{i,t} &= R_{t+1}\lambda_{i,t+1} + \mu_{i,t+1}\frac{\partial B_i}{\partial b}(R_{t+1}, b_{i,t}, k_{i,t}) \\
\mu_{i,t+1} B_i(R_{t+1}, b_{i,t}, k_{i,t}) &= 0,
\end{aligned}$$

then the list $(R_t, (c_{i,t}, k_{i,t}, b_{i,t})_i)$ is an intertemporal equilibrium.

Proof of Lemma 14. Before proving this part, we should notice that this result requires neither $u_i(0) = 0$ nor $u_i'(0) = \infty$. Let us prove our result. It is sufficient to prove the optimality of (c_i, k_i, b_i) for all i. Let (c_i', k_i', b_i') be a plan satisfying all budget and borrowing constraints and $b_{i,-1}' - b_{i,-1} = 0 = k_{i,-1}' - k_{i,-1}$. We have

$$\sum_{t=0}^{T} \beta_i^t (u_i(c_{i,t}) - u_i(c_{i,t}')) \geq \sum_{t=0}^{T} \beta_i^t u_i'(c_{i,t})(c_{i,t} - c_{i,t}') = \sum_{t=0}^{T} \lambda_{i,t}(c_{i,t} - c_{i,t}')$$

$$c_{i,t} + k_{i,t} + R_t b_{i,t-1} \leq F_{i,t}(k_{i,t-1}) + b_{i,t} \tag{B.1}$$
$$B_i(R_t, b_{i,t-1}, k_{i,t-1}) \leq 0 \tag{B.2}$$

Budget constraints imply that

$$\begin{aligned}
& c_{i,t} = F_{i,t}(k_{i,t-1}) + b_{i,t} - k_{i,t} - R_t b_{i,t-1}, \quad c_{i,t}' \leq F_{i,t}(k_{i,t-1}') + b_{i,t}' - k_{i,t}' - R_t b_{i,t-1}' \\
& \lambda_{i,t}(c_{i,t} - c_{i,t}') \\
\geq & \lambda_{i,t}(F_{i,t}(k_{i,t-1}) + b_{i,t} - k_{i,t} - R_t b_{i,t-1} - F_{i,t}(k_{i,t-1}') - b_{i,t}' + k_{i,t}' + R_t b_{i,t-1}') \\
= & \lambda_{i,t}\big(F_{i,t}(k_{i,t-1}) - F_{i,t}(k_{i,t-1}')\big) - \lambda_{i,t}(k_{i,t} - k_{i,t}') + \lambda_{i,t}(b_{i,t} - b_{i,t}') - \lambda_{i,t} R_t (b_{i,t-1} - b_{i,t-1}')
\end{aligned}$$

According to FOCs, we have

$$\begin{aligned}
\lambda_{i,t} k_{i,t}' &= \lambda_{i,t+1} F_{i,t+1}'(k_{i,t}) k_{i,t}' - \mu_{i,t+1} \frac{\partial B_i}{\partial k}(R_{t+1}, b_{i,t}, k_{i,t}) k_{i,t}' + \eta_{i,t} k_{i,t}' \\
\lambda_{i,t} b_{i,t}' &= R_{t+1} \lambda_{i,t+1} b_{i,t}' + \mu_{i,t+1} \frac{\partial B_i}{\partial b}(R_{t+1}, b_{i,t}, k_{i,t}) b_{i,t}'
\end{aligned}$$

This implies that

$$\lambda_{i,t}(k_{i,t} - k_{i,t}') = \lambda_{i,t+1} F_{i,t+1}'(k_{i,t})(k_{i,t} - k_{i,t}') - \mu_{i,t+1} \frac{\partial B_i}{\partial k}(R_{t+1}, b_{i,t}, k_{i,t})(k_{i,t} - k_{i,t}') + \eta_{i,t}(k_{i,t} - k_{i,t}') \tag{B.3}$$

$$\lambda_{i,t}(b_{i,t} - b_{i,t}') = R_{t+1} \lambda_{i,t+1}(b_{i,t} - b_{i,t}') + \mu_{i,t+1} \frac{\partial B_i}{\partial b}(R_{t+1}, b_{i,t}, k_{i,t})(b_{i,t} - b_{i,t}') \tag{B.4}$$

Therefore, we have that

$$
\begin{aligned}
\sum_{t=0}^{T} \lambda_{i,t}(c_{i,t} - c'_{i,t}) \geq \sum_{t=0}^{T} \Big(\lambda_{i,t}\big(F_{i,t}(k_{i,t-1}) - F_{i,t}(k'_{i,t-1})\big) - \lambda_{i,t}(k_{i,t} - k'_{i,t})\Big) \\
+ \sum_{t=0}^{T} \Big(\lambda_{i,t}(b_{i,t} - b'_{i,t}) - \lambda_{i,t}R_t(b_{i,t-1} - b'_{i,t-1})\Big)
\end{aligned}
$$

$$
\begin{aligned}
\geq & \sum_{t=0}^{T-1} \big(\lambda_{i,t+1}F'_{i,t+1}(k_{i,t}) - \lambda_{i,t}\big)(k_{i,t} - k'_{i,t}) - \lambda_{i,T}(k_{i,T} - k'_{i,T}) \\
& + \sum_{t=0}^{T-1} \big(\lambda_{i,t} - \lambda_{i,t+1}R_{t+1}\big)(b_{i,t} - b'_{i,t}) + \lambda_{i,T}(b_{i,T} - b'_{i,T}) \\
= & \lambda_{i,T}\big(k'_{i,T} - b'_{i,T} - (k_{i,T} - b_{i,T})\big) + \sum_{t=0}^{T-1} \mu_{i,t+1}\frac{\partial B_i}{\partial k}(R_{t+1}, b_{i,t}, k_{i,t})(k_{i,t} - k'_{i,t}) - \eta_{i,t}(k_{i,t} - k'_{i,t}) \\
& + \sum_{t=0}^{T-1} \mu_{i,t+1}\frac{\partial B_i}{\partial b}(R_{t+1}, b_{i,t}, k_{i,t})(b_{i,t} - b'_{i,t}) \\
= & \lambda_{i,T}\big(k'_{i,T} - b'_{i,T} - (k_{i,T} - b_{i,T})\big) + \sum_{t=0}^{T-1} \eta_{i,t}(k'_{i,t} - k_{i,t}) \\
& + \sum_{t=0}^{T-1} \mu_{i,t+1}\Big(\frac{\partial B_i}{\partial k}(R_{t+1}, b_{i,t}, k_{i,t})(k_{i,t} - k'_{i,t}) + \frac{\partial B_i}{\partial b}(R_{t+1}, b_{i,t}, k_{i,t})(b_{i,t} - b'_{i,t})\Big)
\end{aligned}
$$

We consider $\mu_{i,t+1}\Big(\frac{\partial B_i}{\partial k}(R_{t+1}, b_{i,t}, k_{i,t})(k_{i,t} - k'_{i,t}) + \frac{\partial B_i}{\partial b}(R_{t+1}, b_{i,t}, k_{i,t})(b_{i,t} - b'_{i,t})\Big)$.

$$
\begin{aligned}
& \mu_{i,t+1}\Big(\frac{\partial B_i}{\partial k}(R_{t+1}, b_{i,t}, k_{i,t})(k_{i,t} - k'_{i,t}) + \frac{\partial B_i}{\partial b}(R_{t+1}, b_{i,t}, k_{i,t})(b_{i,t} - b'_{i,t})\Big) \\
= & \mu_{i,t+1}(B_i(R_{t+1}, b_{i,t}, k_{i,t}) - B_i(R_{t+1}, b'_{i,t}, k'_{i,t})) \\
& + \mu_{i,t+1}\Big(B_i(R_{t+1}, b'_{i,t}, k'_{i,t}) - B_i(R_{t+1}, b_{i,t}, k_{i,t}) \qquad \text{(B.5)} \\
& \qquad - \frac{\partial B_i}{\partial k}(R_{t+1}, b_{i,t}, k_{i,t})(k'_{i,t} - k_{i,t}) - \frac{\partial B_i}{\partial b}(R_{t+1}, b_{i,t}, k_{i,t})(b'_{i,t} - b_{i,t})\Big) \\
\geq & \mu_{i,t+1}(B_i(R_{t+1}, b_{i,t}, k_{i,t}) - B_i(R_{t+1}, b'_{i,t}, k'_{i,t})) = -\mu_{i,t+1}B_i(R_{t+1}, b'_{i,t}, k'_{i,t})) \geq 0, \qquad \text{(B.6)}
\end{aligned}
$$

because the function $B_i(R_{t+1}, \cdot, \cdot)$ is convex.

It remains to prove that $\liminf_{T\to\infty} \lambda_{i,T}\big(k'_{i,T} - b'_{i,T} - (k_{i,T} - b_{i,T})\big) \geq 0$.

According to (B.3) and (B.4), we have

$$\begin{aligned}
&\lambda_{i,t}\big(k'_{i,t}-b'_{i,t}-(k_{i,t}-b_{i,t})\big)\\
=&R_{t+1}\lambda_{i,t+1}(b_{i,t}-b'_{i,t})+\mu_{i,t+1}\frac{\partial B_i}{\partial b}(R_{t+1},b_{i,t},k_{i,t})(b_{i,t}-b'_{i,t})\\
&-\Big(\lambda_{i,t+1}F'_{i,t+1}(k_{i,t})(k_{i,t}-k'_{i,t})-\mu_{i,t+1}\frac{\partial B_i}{\partial k}(R_{t+1},b_{i,t},k_{i,t})(k_{i,t}-k'_{i,t})+\eta_{i,t}(k_{i,t}-k'_{i,t})\Big)\\
=&R_{t+1}\lambda_{i,t+1}(b_{i,t}-b'_{i,t})-\lambda_{i,t+1}F'_{i,t+1}(k_{i,t})(k_{i,t}-k'_{i,t})+\eta_{i,t}(k_{i,t}-k'_{i,t})\\
&+\mu_{i,t+1}\frac{\partial B_i}{\partial b}(R_{t+1},b_{i,t},k_{i,t})(b_{i,t}-b'_{i,t})+\mu_{i,t+1}\frac{\partial B_i}{\partial k}(R_{t+1},b_{i,t},k_{i,t})(k_{i,t}-k'_{i,t})\\
\geq&R_{t+1}\lambda_{i,t+1}(b_{i,t}-b'_{i,t})-\lambda_{i,t+1}F'_{i,t+1}(k_{i,t})(k_{i,t}-k'_{i,t}).
\end{aligned}$$

because of (B.6) and the fact that $\eta_{i,t}(k_{i,t}-k'_{i,t})=-\eta_{i,t}k'_{i,t})\leq 0$.

Since $F_{i,t+1}$ is concave, we have $F'_{i,t+1}(k_{i,t})(k_{i,t}-k'_{i,t})\leq F_{i,t+1}(k_{i,t})-F_{i,t+1}(k'_{i,t})$. So, we get that

$$\begin{aligned}
\lambda_{i,t}\big(k'_{i,t}-b'_{i,t}-(k_{i,t}-b_{i,t})\big)\geq&R_{t+1}\lambda_{i,t+1}(b_{i,t}-b'_{i,t})-\lambda_{i,t+1}\big(F_{i,t+1}(k_{i,t})-F_{i,t+1}(k'_{i,t})\big)\\
=&\lambda_{i,t+1}\big(R_{t+1}b_{i,t}-F_{i,t+1}(k_{i,t})\big)+\lambda_{i,t+1}\big(F_{i,t+1}(k'_{i,t})-R_{t+1}b'_{i,t}\big)
\end{aligned}$$

We have $F_{i,t+1}(k'_{i,t})-R_{t+1}b'_{i,t}\geq 0$ because $B_i(R_{t+1},k'_{i,t},b'_{i,t})\leq 0$.

The budget constraint at date t implies that $\lambda_{i,t}(c_{i,t}+k_{i,t}-b_{i,t})=\lambda_{i,t}\big(F_{i,t}(k_{i,t-1})-R_tb_{i,t-1}\big)$. Since $\lim_{t\to\infty}\lambda_{i,t}c_{i,t}=0=\lim_{t\to\infty}\lambda_{i,t}(k_{i,t}-b_{i,t})$, we get that $\lim_{t\to\infty}\lambda_{i,t}\big(F_{i,t}(k_{i,t-1})-R_tb_{i,t-1}\big)=0$. By consequence, we obtain that $\liminf_{T\to\infty}\lambda_{i,T}\big(k'_{i,T}-b'_{i,T}-(k_{i,T}-b_{i,T})\big)\geq 0$. □

Proof of Remark 3. Steady state analysis. Let us focus on an interior equilibrium (i.e., $k_{i,t}>0$, $\forall i,t$), we can write the FOCs

$$\begin{aligned}
\beta_i^t u'_i(c_{i,t})&=\lambda_{i,t}\\
\lambda_{i,t}&=F'_{i,t}(k_{i,t})(\lambda_{i,t+1}+\gamma_i\mu_{i,t+1})\\
\lambda_{i,t}&=R_{t+1}(\lambda_{i,t+1}+\mu_{i,t+1})\\
\mu_{i,t+1}(R_{t+1}b_{i,t}-\gamma_iF_i(k_{i,t}))&=0
\end{aligned}$$

where $\mu_{i,t}\geq 0$ is the multiplier with respect to the constraint $R_tb_{i,t-1}-\gamma_iF_{i,t}(k_{i,t-1})\leq 0$.

According to FOCs and the fact that B_i is increasing in b but decreasing in k, we have that $1\geq R_{t+1}\max_i\frac{\beta_iu'_i(c_{i,t+1})}{u'_i(c_{i,t})}$, $\forall i$. Since $k_{i,t}>0$, $\forall i,\forall t$, there exists an agent, say agent i, whose borrowing constraint at date $t+1$ is not binding. It means that $\mu_{i,t+1}=0$. By consequence, we have $1=R_{t+1}\frac{\beta_iu'_i(c_{i,t+1})}{u'_i(c_{i,t})}=R_{t+1}\max_j\frac{\beta_ju'_j(c_{j,t+1})}{u'_j(c_{j,t})}$. Therefore, we have $R=1/\max_i\{\beta_i\}$ at steady state.

The first-order conditions imply that $\lambda_{i,t}\frac{R_{t+1}-\gamma_iF'_{i,t}(k_{i,t})}{R_{t+1}}=F'_{i,t}(k_{i,t})\lambda_{i,t+1}(1-\gamma_i)$. By consequence, we obtain point 2.

□

Proof of Lemma 4. The maximization problem of agent i is

$$\max_{(c_i,k_i,b_i)} \sum_{t=0}^{\infty} \beta_i^t u_i(c_{i,t})$$
$$\text{subject to: } c_{i,t} + k_{i,t} + R_t b_{i,t-1} \leq A_{i,t} k_{i,t-1} + b_{i,t}$$
$$R_t b_{i,t-1} \leq \gamma_i A_{i,t}(k_{i,t-1})$$

Denote $s_{i,t} = k_{i,t} - b_{i,t}$ the net saving of agent i at date t.
Let $R_t = A_{1,t}\ \forall t$.
For agent 1, we have $c_{1,t} + (k_{1,t} - b_{1,t}) \leq A_{1,t}(k_{1,t-1} - b_{1,t-1})$. We can compute that

$$s_{1,0} = \beta_1 w_{1,0}, \quad s_{1,t} = \beta_1 A_{1,t} s_{1,t-1}\ \forall t \geq 1$$
$$s_{1,t} = \beta_1^t A_{1,t} \cdots A_{1,1} s_{1,0}$$

For agent 2, since $A_{2,t} > R_t = A_{1,t}\ \forall t$, her borrowing constraint is always binding: $R_t b_{2,t-1} = \gamma_2 A_{2,t} k_{2,t-1}$. Therefore, we have

$$s_{2,t} = k_{2,t}\Big(1 - \frac{\gamma_2 A_{2,t+1}}{R_{t+1}}\Big), \quad A_{2,t}k_{2,t-1} - R_t b_{2,t-1} = (1-\gamma_2)A_{2,t}k_{2,t-1}\ \forall t \geq 1.$$

From this, we can compute that

$$s_{2,t} = \beta_2 \frac{(1-\gamma_2)A_{2,t}R_t}{R_t - \gamma_2 A_{2,t}} s_{2,t-1} = \Big(\beta_2 \frac{(1-\gamma_2)A_{2,t}R_t}{R_t - \gamma_2 A_{2,t}}\Big) \cdots \Big(\beta_2 \frac{(1-\gamma_2)A_{2,1}R_1}{R_1 - \gamma_2 A_{2,1}}\Big) s_{2,0}$$
$$k_{2,t} = \frac{1}{1 - \frac{\gamma_2 A_{2,t+1}}{R_{t+1}}} s_{2,t} = \frac{R_{t+1}}{R_{t+1} - \gamma_2 A_{2,t+1}} s_{2,t}$$
$$b_{2,t} = \frac{\gamma_2 A_{2,t+1}}{R_{t+1}} k_{2,t} = \frac{\gamma_2 A_{2,t+1}}{R_{t+1} - \gamma_2 A_{2,t+1}} s_{2,t}$$

Therefore, we can find the capital of the first agent

$$\begin{aligned} k_{1,t} =& s_{1,t} + b_{1,t} = s_{1,t} - b_{2,t} \\ =& \beta_1^t A_{1,t} \cdots A_{1,1} s_{1,0} - \frac{\gamma_2 A_{2,t+1}}{R_{t+1} - \gamma_2 A_{2,t+1}} \Big(\beta_2 \frac{(1-\gamma_2)A_{2,t}R_t}{R_t - \gamma_2 A_{2,t}}\Big) \cdots \Big(\beta_2 \frac{(1-\gamma_2)A_{2,1}R_1}{R_1 - \gamma_2 A_{2,1}}\Big) s_{2,0} \end{aligned}$$

In order to keep $k_{1,t} > 0\ \forall t$, we need that

$$\beta_1^t A_{1,t} \cdots A_{1,1} s_{1,0} - \frac{\gamma_2 A_{2,t+1}}{R_{t+1} - \gamma_2 A_{2,t+1}} \Big(\beta_2 \frac{(1-\gamma_2)A_{2,t}R_t}{R_t - \gamma_2 A_{2,t}}\Big) \cdots \Big(\beta_2 \frac{(1-\gamma_2)A_{2,1}R_1}{R_1 - \gamma_2 A_{2,1}}\Big) s_{2,0} > 0$$
$$\Leftrightarrow \beta_1^t s_{1,0} - \beta_2^t \frac{\gamma_2 A_{2,t+1}}{A_{1,t+1} - \gamma_2 A_{2,t+1}} \frac{(1-\gamma_2)A_{2,t}}{A_{1,t} - \gamma_2 A_{2,t}} \cdots \frac{(1-\gamma_2)A_{2,1}}{A_{1,1} - \gamma_2 A_{2,1}} s_{2,0} > 0$$

or equivalently

$$\frac{s_{2,0}}{s_{1,0}} \frac{\beta_1}{\beta_2} \frac{\gamma_2}{1-\gamma_2} \Big(\frac{\beta_2}{\beta_1} \frac{(1-\gamma_2)A_{2,1}}{A_{1,1} - \gamma_2 A_{2,1}} \cdots \frac{\beta_2}{\beta_1} \frac{(1-\gamma_2)A_{2,t+1}}{A_{1,t+1} - \gamma_2 A_{2,t+1}}\Big) < 1 \forall t.$$

This happens if $\sup_t \dfrac{\beta_2}{\beta_1}\dfrac{1-\gamma_2}{\frac{A_{1,t}}{A_{2,t}}-\gamma_2} < 1$ and $\frac{s_{2,0}}{s_{1,0}}\frac{\beta_1}{\beta_2}\frac{\gamma_2}{1-\gamma_2} \leq 1$. By applying Lemma 14, we can check that the above list $(R_t, (c_{i,t}, k_{i,t}, b_{i,t})_i)$ is an equilibrium.

We now compute the aggregate production

$$
\begin{aligned}
Y_t &= A_{1,t}k_{1,t-1} + A_{2,t}k_{2,t-1}\\
&= A_{1,t}\Big(\beta_1^{t-1}A_{1,t-1}\cdots A_{1,1}s_{1,0} - \frac{\gamma_2 A_{2,t}}{R_t-\gamma_2 A_{2,t}}\Big(\beta_2\frac{(1-\gamma_2)A_{2,t}R_{t-1}}{R_{t-1}-\gamma_2A_{2,t-1}}\Big)\cdots\Big(\beta_2\frac{(1-\gamma_2)A_{2,1}R_1}{R_1-\gamma_2A_{2,1}}\Big)s_{2,0}\Big)+\\
&+ A_{2,t}\frac{R_t}{R_t-\gamma_2A_{2,t}}\Big(\beta_2\frac{(1-\gamma_2)A_{2,t-1}R_{t-1}}{R_{t-1}-\gamma_2A_{2,t-1}}\Big)\cdots\Big(\beta_2\frac{(1-\gamma_2)A_{2,1}R_1}{R_1-\gamma_2A_{2,1}}\Big)s_{2,0}\\
&= \beta_1^{t-1}A_{1,t}A_{1,t-1}\cdots A_{1,1}s_{1,0}\\
&+ \beta_2^{t-1}(1-\gamma_2)^t\frac{A_{2,t}A_{1,t}}{A_{1,t}-\gamma_2A_{2,t}}\frac{A_{2,t-1}A_{1,t-1}}{A_{1,t-1}-\gamma_2A_{2,t-1}}\cdots\frac{A_{2,1}A_{1,1}}{A_{1,1}-\gamma_2A_{2,1}}s_{2,0}.
\end{aligned}
$$

□

Proof of Proposition 13. According to (50), we have

$$Y_t = \beta_1^{t-1}A_1^t s_{1,0} + \beta_2^{t-1}(1-\gamma_2)^tA_2^t\Big(\frac{A_1}{A_1-\gamma_2A_2}\Big)^t s_{2,0}.$$

So, we get that

$$\frac{1}{tA_1^{t-1}}\frac{\partial Y_t}{\partial A_1} = \beta_1^{t-1}s_{1,0} - (1-\gamma_2)\frac{\gamma_2A_2^2}{(A_1-\gamma_2A_2)^2}\Big(\frac{\beta_2(1-\gamma_2)A_2}{A_1-\gamma_2A_2}\Big)^{t-1}s_{2,0} \tag{B.7}$$

The output is more likely to be increasing in the TFP A_1 of the least productive agent (i.e., $\frac{\partial Y_t}{\partial A_1} > 0$) if (1) the productivity gap $\frac{A_2}{A_1}$ is low or (2) the initial income gap $\frac{s_{2,0}}{s_{1,0}}$ is low or (3) the time preference gap $\frac{\beta_2}{\beta_1}$ is low.

1. If $\frac{s_{1,0}}{s_{2,0}}\frac{(A_1-\gamma_2A_2)^2}{(1-\gamma_2)\gamma_2A_2^2} > 1$, then it is easy to see that $\frac{\partial Y_t}{\partial A_1} > 0\ \forall t$.
2. If $\frac{s_{1,0}}{s_{2,0}}\frac{(A_1-\gamma_2A_2)^2}{(1-\gamma_2)\gamma_2A_2^2} < 1$, we have immediately that $\frac{\partial Y_1}{\partial A_1} < 0$. According to equation (B.7) and the fact that $\lim_{t\to\infty}\left(\frac{\beta_1(A_1-\gamma_2A_2)}{(1-\gamma_2)A_2\beta_2}\right)^t = \infty$, we get point 2 of Proposition 13.

Notice that the capital of the first agent is

$$k_{1,t} = s_{1,t} + b_{1,t} = s_{1,t} - b_{2,t} = (\beta_1R)^t s_{1,0} - \frac{\gamma_2A_2}{R-\gamma_2A_2}\Big(\frac{\beta_2(1-\gamma_2)A_2R}{R-\gamma_2A_2}\Big)^t s_{2,0}$$

Condition $k_{1,t} > 0\ \forall t$ is equivalent to

$$\beta_1^t s_{1,0} - \frac{\gamma_2A_2}{A_1-\gamma_2A_2}\Big(\frac{\beta_2(1-\gamma_2)A_2}{A_1-\gamma_2A_2}\Big)^t s_{2,0} > 0\ \forall t \geq 0.$$

This happens if and only if $\beta_1 \geq \frac{(1-\gamma_2)A_2}{A_1-\gamma_2A_2}\beta_2$ and $s_{1,0} - \frac{\gamma_2A_2}{A_1-\gamma_2A_2}s_{2,0} > 0$. Here, we only focus on the case $\beta_1 > \frac{(1-\gamma_2)A_2}{A_1-\gamma_2A_2}\beta_2$. Since $A_2 > A_1$, this condition implies that $\beta_1 > \beta_2$.

We now prove the last part of Proposition 13. The net worth of agent 2: $W_{2,t} \equiv$ Output - Payment.

$$\begin{aligned}
W_{2,t} &\equiv \text{Output - Payment} = A_2 k_{2,t-1} - R_t b_{2,t-1} = (1-\gamma_2) A_2 k_{2,t-1} \\
&= (1-\gamma_2) A_{2,t} \frac{R_t}{R_t - \gamma_2 A_{2,t}} \Big(\beta_2 \frac{(1-\gamma_2) A_{2,t-1} R_{t-1}}{R_{t-1} - \gamma_2 A_{2,t-1}} \Big) \cdots \Big(\beta_2 \frac{(1-\gamma_2) A_{2,1} R_1}{R_1 - \gamma_2 A_{2,1}} \Big) s_{2,0} \\
&= (1-\gamma_2) A_2 \frac{A_1}{A_1 - \gamma_2 A_2} \Big(\beta_2 \frac{(1-\gamma_2) A_2 A_1}{A_1 - \gamma_2 A_2} \Big)^{t-1} s_{2,0}
\end{aligned}$$

First, we look at

$$\begin{aligned}
\frac{Y_{1,t} + Y_{2,t}}{Y_{2,t}} &= \frac{Y_t}{Y_{2,t}} = \frac{\beta_1^{t-1} A_1^t s_{1,0} + \beta_2^{t-1} (1-\gamma_2)^t \big(\frac{A_2 A_1}{A_1 - \gamma_2 A_2} \big)^t s_{2,0}}{A_2 \frac{A_1}{A_1 - \gamma_2 A_2} \Big(\beta_2 \frac{(1-\gamma_2) A_2 A_1}{A_1 - \gamma_2 A_2} \Big)^{t-1} s_{2,0}} \\
&= \frac{\beta_1^{t-1} A_1^t s_{1,0}}{A_2 \frac{A_1}{A_1 - \gamma_2 A_2} \Big(\beta_2 \frac{(1-\gamma_2) A_2 A_1}{A_1 - \gamma_2 A_2} \Big)^{t-1} s_{2,0}} + (1-\gamma_2) \\
&= \frac{\beta_1^{t-1} (A_1 - \gamma_2 A_2)^t s_{1,0}}{\beta_2^{t-1} A_2^t (1-\gamma_2)^{t-1} s_{2,0}} + (1-\gamma_2)
\end{aligned}$$

This is increasing in the ratio A_1/A_2. Since $\frac{\beta_1 (A_1 - \gamma_2 A_2)}{\beta_2 (1-\gamma_2) A_2} > 1$, we have $\frac{Y_t}{Y_{2,t}} \to \infty$, or equivalently $\frac{Y_{1,t}}{Y_{2,t}} \to \infty$. The reason is that the rate o time preference of the less productive agent β_1 is high enough. We can also check that the share of net worth of the productive agent $\frac{W_{2,t}}{Y_t} \to 0$.

We now look at the growth rate

$$\begin{aligned}
G_t = \frac{Y_{t+1}}{Y_t} &= \frac{\beta_1^t A_1^{t+1} s_{1,0} + \beta_2^t (1-\gamma_2)^{t+1} \big(\frac{A_2 A_1}{A_1 - \gamma_2 A_2} \big)^{t+1} s_{2,0}}{\beta_1^{t-1} A_1^t s_{1,0} + \beta_2^{t-1} (1-\gamma_2)^t \big(\frac{A_2 A_1}{A_1 - \gamma_2 A_2} \big)^t s_{2,0}} \\
&= \frac{\frac{\beta_2}{\beta_1} \big(\frac{\beta_1}{\beta_2} \frac{A_1 - \gamma_2 A_2}{A_2 (1-\gamma_2)} \big)^{t+1} + s_{2,0}}{\frac{\beta_2}{\beta_1} \big(\frac{\beta_1}{\beta_2} \frac{A_1 - \gamma_2 A_2}{A_2 (1-\gamma_2)} \big)^{t} + s_{2,0}} \frac{\beta_2^t (1-\gamma_2)^{t+1} \big(\frac{A_2 A_1}{A_1 - \gamma_2 A_2} \big)^{t+1}}{\beta_2^{t-1} (1-\gamma_2)^t \big(\frac{A_2 A_1}{A_1 - \gamma_2 A_2} \big)^t} \\
&= \frac{\frac{\beta_2}{\beta_1} \big(\frac{\beta_1}{\beta_2} \frac{A_1 - \gamma_2 A_2}{A_2 (1-\gamma_2)} \big)^{t+1} + s_{2,0}}{\frac{\beta_2}{\beta_1} \big(\frac{\beta_1}{\beta_2} \frac{A_1 - \gamma_2 A_2}{A_2 (1-\gamma_2)} \big)^{t} + s_{2,0}} \frac{\beta_2 (1-\gamma_2) A_2 A_1}{A_1 - \gamma_2 A_2}
\end{aligned}$$

Since $\frac{\beta_1}{\beta_2} \frac{A_1 - \gamma_2 A_2}{A_2 (1-\gamma_2)} > 1$, we have that $G_t \to \frac{\beta_1}{\beta_2} \frac{A_1 - \gamma_2 A_2}{A_2 (1-\gamma_2)} \frac{\beta_2 (1-\gamma_2) A_2 A_1}{A_1 - \gamma_2 A_2} = \beta_1 A_1$.

□

Proof of Lemma 5. Let us consider such an equilibrium. Since $A_1 < R_1 < A_2$, and $R_t = A_2 > A_1, \forall t$, agent 1 does not produce and only invests in the financial asset.

Denote $s_{i,t} \equiv k_{i,t} - b_{i,t}$. Agent 1 is lender, $k_{1,t} = 0$, $s_{1,t} = -b_{1,t}$, $\forall t$. We can compute that

$$\begin{aligned}
&s_{1,0} = \beta_1 w_{1,0}, \quad s_{1,t} = \beta_1 R_t s_{1,t-1} \ \forall t \geq 1 \\
&s_{1,t} = \beta_1^t R_t \cdots R_1 s_{1,0}
\end{aligned}$$

For agent 2, since $A_2 > R_1$, her borrowing constraints at date 0 is binding: $R_1 b_{2,0} = \gamma_2 A_2 k_{2,0}$. Therefore, we have

$$\begin{aligned} A_2 k_{2,0} - R_1 b_{2,0} &= (1-\gamma_2) A_2 k_{2,0} \\ s_{2,0} &= k_{2,0}\Big(1 - \frac{\gamma_2 A_2}{R_1}\Big) \\ k_{2,0} &= \frac{R_1}{R_1 - \gamma_2 A_2} s_{2,0}, \quad b_{2,0} = \frac{\gamma_2 A_2}{R_1 - \gamma_2 A_2} s_{2,0} \end{aligned}$$

The budget constraints of agent 2 write

$$\begin{aligned} c_{2,0} + s_{2,0} &= w_{2,0} \\ c_{2,1} + s_{2,1} &= (1-\gamma_2) A_2 k_{2,0} = \frac{(1-\gamma_2) A_2 R_1}{R_1 - \gamma_2 A_2} s_{2,0} \\ c_{2,t} + s_{2,t} &= A_2 s_{2,t}, \forall t \geq 2 \\ s_{2,t} &= k_{2,t} - b_{2,t}, \forall t \geq 2. \end{aligned}$$

From this and the FOCs, we can compute the individual saving

$$\begin{aligned} s_{i,0} &= \beta_i w_{i,0}, \forall i \\ s_{2,1} &= \beta_2 \frac{(1-\gamma_2) A_2 R_1}{R_1 - \gamma_2 A_2} s_{2,0} \\ s_{i,t} &= \beta_i A_2 s_{i,t-1}, \forall t \geq 2, \forall i = 1, 2. \end{aligned}$$

We now look at equilibrium. From the market clearing condition $\sum_i b_{i,t} = 0$, we have that

$$s_{1,0} = b_{2,0} = \frac{\gamma_2 A_2}{R_1 - \gamma_2 A_2} s_{2,0} \Leftrightarrow R_1 = \gamma_2 A_2 (1 + \frac{s_{2,0}}{s_{1,0}})$$

By consequence, we find the saving of all agents:

$$\begin{aligned} s_{i,0} &= \beta_i w_{i,0}, \forall i \\ s_{1,1} &= \beta_1 R_1 s_{1,0} = \beta_1 \gamma_2 A_2 (1 + \frac{s_{2,0}}{s_{1,0}}) s_{1,0} = \beta_1 \gamma_2 A_2 (s_{1,0} + s_{2,0}) \\ s_{2,1} &= \beta_2 \frac{(1-\gamma_2) A_2 R_1}{R_1 - \gamma_2 A_2} s_{2,0} = \beta_2 \frac{(1-\gamma_2) A_2 \gamma_2 A_2 (1 + \frac{s_{2,0}}{s_{1,0}})}{\gamma_2 A_2 (1 + \frac{s_{2,0}}{s_{1,0}}) - \gamma_2 A_2} s_{2,0} = \beta_2 (1-\gamma_2) A_2 (s_{1,0} + s_{2,0}) \\ s_{i,t} &= \beta_i A_2 s_{i,t-1} = (\beta_i A_2)^{t-1} s_{i,1}, \forall t \geq 2, \forall i = 1, 2. \end{aligned}$$

It remains to find the sequence of capital $(k_{i,t})$. We have, $\forall t \geq 1$

$$\begin{aligned}
k_{2,0} &= s_{1,0} + s_{2,0}\\
k_{2,t} &= s_{2,t} + b_{2,t} = s_{2,t} - b_{1,t} = s_{2,t} + s_{1,t}, \forall t \geq 1\\
k_{2,1} &= \beta_1 R_1 s_{1,0} + \beta_2 \frac{(1-\gamma_2)A_2 R_1}{R_1 - \gamma_2 A_2} s_{2,0}\\
&= \beta_1 s_{1,0}\gamma_2 A_2 (1 + \frac{s_{2,0}}{s_{1,0}}) + \beta_2 \frac{(1-\gamma_2)A_2\gamma_2 A_2(1+\frac{s_{2,0}}{s_{1,0}})}{\gamma_2 A_2(1+\frac{s_{2,0}}{s_{1,0}}) - \gamma_2 A_2} s_{2,0}\\
&= \beta_1 \gamma_2 A_2 (s_{1,0} + s_{2,0}) + \beta_2 \frac{(1-\gamma_2)A_2\gamma_2 A_2 \frac{s_{1,0}+s_{2,0}}{s_{1,0}}}{\gamma_2 A_2 \frac{s_{2,0}}{s_{1,0}}} s_{2,0}\\
&= \beta_1 \gamma_2 A_2 (s_{1,0} + s_{2,0}) + \beta_2 (1-\gamma_2) A_2 (s_{1,0} + s_{2,0})\\
k_{2,t} &= \sum_i s_{i,t} = \sum_i (\beta_i A_2)^{t-1} s_{i,1}, \forall t \geq 1\\
&= (\beta_1 A_2)^{t-1}\beta_1\gamma_2 A_2 (s_{1,0} + s_{2,0}) + (\beta_2 A_2)^{t-1}\beta_2(1-\gamma_2)A_2(s_{1,0} + s_{2,0})\\
&= A_2^t \Big(\beta_1^t \gamma_2 + \beta_2^t (1-\gamma_2)\Big)(s_{1,0} + s_{2,0})
\end{aligned}$$

We now check that the above list $((c_{i,t}, k_{i,t}, b_{i,t})_i, R_t)_t$ is an equilibrium. We use Lemma 14. It is easy to verify the market clearing conditions and the FOCs.

Condition $R_1 \in (A_1, A_2)$ is ensured by the assumption that

$$A_1 < \gamma_2 A_2 (1 + \frac{s_{2,0}}{s_{1,0}}) < A_2$$

- We verify borrowing constraints: $R_{t+1} b_{2,t} \leq \gamma_2 A_2 k_{2,t}$. This is satisfied for $t = 0$. Let us consider $t \geq 1$. This means that $k_{2,t} - s_{2,t} = b_{2,t} \leq \gamma_2 k_{2,t}$, or, equivalently, $(1-\gamma_2)k_{2,t} \leq s_{2,t}$. So, we must prove, for any $t \geq 1$,

$$s_{1,t} = k_{2,t} - s_{2,t} \leq \big(\frac{1}{1-\gamma_2} - 1\big) s_{2,t} = \frac{\gamma_2}{1-\gamma_2} s_{2,t}, \forall t \geq 1.$$

  These conditions are satisfied because $\beta_2 > \beta_1$.

- Tranversality conditions: $\lim_{T\to\infty} \beta_i^t u_i'(c_{i,t})(k_{i,T} - b_{i,T}) = 0$. It is easy to verify these conditions because $\beta_i \in (0,1)$ and $u'(c) = 1/c$.

□

Proof of Lemma 6. Let us consider such an equilibrium. Since $A_1 < R_1 < A_2 < A_3$, and $R_t = A_3, \forall t$, agent 1 does not produce and only invests in the financial asset.

Denote $s_{i,t} \equiv k_{i,t} - b_{i,t}$. Agent 1 is lender, $k_{1,t} = 0$, $s_{1,t} = -b_{1,t}$, $\forall t$. We can compute that

$$\begin{aligned}
s_{1,0} &= \beta_1 w_{1,0}, \quad s_{1,t} = \beta_1 R_t s_{1,t-1} \ \forall t \geq 1\\
s_{1,t} &= \beta_1^t R_t \cdots R_1 s_{1,0}
\end{aligned}$$

For agent i=$2,3$, since $A_3 > A_2 > R_1$, their borrowing constraints at date 0 are binding: $R_1 b_{i,0} = \gamma_i A_i k_{i,0}$. Therefore, we have

$$\begin{aligned}
A_i k_{i,0} - R_1 b_{i,0} &= (1-\gamma_i) A_i k_{i,0}, \ \forall i = 2,3 \\
s_{i,0} &= k_{i,0}\Big(1 - \frac{\gamma_i A_i}{R_1}\Big) \\
k_{i,0} &= \frac{R_1}{R_1 - \gamma_i A_i} s_{i,0}, \quad b_{i,0} = \frac{\gamma_i A_i}{R_1 - \gamma_i A_i} s_{i,0}
\end{aligned}$$

The budget constraint writes

$$\begin{aligned}
c_{i,0} + s_{i,0} &= w_{i,0} \\
c_{i,1} + s_{i,1} &= (1-\gamma_i) A_i k_{i,0} = \frac{(1-\gamma_i) A_i R_1}{R_1 - \gamma_i A_i} s_{i,0} \\
c_{i,t} + s_{i,t} &= A_3 s_{i,t}, \forall t \geq 2, \forall i = 2,3 \\
s_{2,t} &= k_{2,t} - b_{2,t} = -b_{2,t}, \quad k_{2,t} = 0, \forall t \geq 1 \\
s_{3,t} &= k_{3,t} - b_{3,t}, \forall t \geq 2.
\end{aligned}$$

From this and the FOCs, we can compute the individual saving

$$\begin{aligned}
s_{i,0} = \beta_i w_{i,0}, \forall i, \quad s_{i,1} &= \beta_i \frac{(1-\gamma_i) A_i R_1}{R_1 - \gamma_i A_i} s_{i,0}, \forall i = 2,3 \\
s_{i,t} &= \beta_i A_3 s_{i,t-1}, \forall t \geq 2, \forall i = 1,2,3.
\end{aligned}$$

We now look at equilibrium. From the market clearing condition $\sum_i b_{i,t} = 0$, we have that

$$s_{1,0} = b_{2,0} + b_{3,0} = \frac{\gamma_2 A_2}{R_1 - \gamma_2 A_2} s_{2,0} + \frac{\gamma_3 A_3}{R_1 - \gamma_3 A_3} s_{3,0}.$$

This is to determine the interest rate R_1. We have known $R_t = A_3, \forall t \geq 2$. It remains to find the sequence of capital $(k_{i,t})$. We have, $\forall t \geq 1$

$$\begin{aligned}
k_{3,t} &= s_{3,t} + b_{3,t} = s_{3,t} - b_{1,t} - b_{2,t} = s_{3,t} + s_{1,t} + s_{2,t}, \forall t \geq 1 \\
k_{3,1} &= \beta_1 R_1 s_{1,0} + \sum_{i=2,3} \beta_i \frac{(1-\gamma_i) A_i R_1}{R_1 - \gamma_i A_i} s_{i,0} \\
k_{3,t+1} &= \sum_i s_{i,t+1} = \sum_i \beta_i A_3 s_{i,t}, \forall t \geq 1
\end{aligned}$$

To sum up, we find that

$$\begin{aligned}
s_{i,0} = \beta_i w_{i,0}, \forall i, \quad k_{1,t} &= 0, \forall t \geq 0, k_{2,t} = 0, \forall t \geq 1 \\
k_{i,0} &= \frac{R_1}{R_1 - \gamma_i A_i} s_{i,0}, \quad b_{i,0} = \frac{\gamma_i A_i}{R_1 - \gamma_i A_i} s_{i,0}, \ \forall i = 2,3.
\end{aligned}$$

From this, we can compute the aggregate output:

$$Y_1 = \sum_i A_i k_{i,0} = \frac{A_2 R_1}{R_1 - \gamma_2 A_2} s_{2,0} + \frac{A_3 R_1}{R_1 - \gamma_3 A_3} s_{3,0}$$

$$\begin{aligned}
\forall t \geq 2, \quad Y_t &= A_3 \sum_i s_{i,t-1} = A_3 \sum_i (\beta_i A_3)^{t-2} s_{i,1} \\
&= A_3 \Big((\beta_1 A_3)^{t-2} \beta_1 R_1 s_{1,0} + \sum_{i=2,3} \beta_i (\beta_i A_3)^{t-2} \frac{(1-\gamma_i) A_i R_1}{R_1 - \gamma_i A_i} s_{i,0} \Big) \\
&= A_3^{t-1} \Big(\beta_1^{t-1} R_1 s_{1,0} + \sum_{i=2,3} \beta_i^{t-1} \frac{(1-\gamma_i) A_i R_1}{R_1 - \gamma_i A_i} s_{i,0} \Big) \\
&= A_3^{t-1} \Big(\beta_1^{t-1} R_1 s_{1,0} + \beta_2^{t-1} (1-\gamma_2) \frac{A_2 R_1}{R_1 - \gamma_2 A_2} s_{2,0} + \beta_3^{t-1} (1-\gamma_3) \frac{A_3 R_1}{R_1 - \gamma_3 A_3} s_{3,0} \Big)
\end{aligned}$$

We now check that the above list $((c_{i,t}, k_{i,t}, b_{i,t})_i, R_t)_t$ is an equilibrium. We use Lemma 14. It is easy to verify the market clearing conditions and the FOCs.

The financial market clearing condition at date 0 becomes $s_{1,0} = \frac{\gamma_2 A_2}{R_1 - \gamma_2 A_2} s_{2,0} + \frac{\gamma_3 A_3}{R_1 - \gamma_3 A_3} s_{3,0}$. So, condition $R_1 \in (A_1, A_2)$ is ensured by the assumption that

$$\frac{\gamma_3 A_3}{A_2 - \gamma_3 A_3} s_{3,0} + \frac{\gamma_2}{1-\gamma_2} s_{2,0} < s_{1,0} < \frac{\gamma_3 A_3}{A_1 - \gamma_3 A_3} s_{3,0} + \frac{\gamma_2 A_2}{A_1 - \gamma_2 A_2} s_{2,0}$$

- We verify borrowing constraints: $R_{t+1} b_{3,t} \leq \gamma_3 A_3 k_{3,t}$. This is satisfied for $t = 0$. Let us consider $t \geq 1$. since $R_{t+1} = A_3$, $\forall t \geq 1$, the borrowing constraint becomes $k_{3,t} - s_{3,t} = b_{3,t} \leq \gamma_3 k_{3,t}$, or, equivalently, $(1-\gamma_3) k_{3,t} \leq s_{3,t}$. So, we must prove, for any $t \geq 0$,

  $$s_{1,t} + s_{2,t} = k_{3,t} - s_{3,t} \leq \big(\frac{1}{1-\gamma_3} - 1 \big) s_{3,t} = \frac{\gamma_3}{1-\gamma_3} s_{3,t}.$$

  Since $s_{i,t} = \beta_i A_3 s_{i,t-1}, \forall t \geq 2, \forall i = 1, 2, 3$, and $\beta_3 \geq \max(\beta_1, \beta_2)$, it suffices to prove this condition at date 1. Condition $s_{1,1} + s_{2,1} \leq \frac{\gamma_3}{1-\gamma_3} s_{3,1}$ is equivalent to

  $$\beta_1 R_1 s_{1,0} + \beta_2 \frac{(1-\gamma_2) A_2 R_1}{R_1 - \gamma_2 A_2} s_{2,0} \leq \frac{\gamma_3}{1-\gamma_3} \beta_3 \frac{(1-\gamma_3) A_3 R_1}{R_1 - \gamma_3 A_3} s_{3,0}$$

  This is satisfied. Indeed, since $s_{1,0} = \frac{\gamma_3 A_3}{R_1 - \gamma_3 A_3} s_{3,0} + \frac{\gamma_2 A_2}{R_1 - \gamma_2 A_2} s_{2,0}$, we have

  $$\begin{aligned}
  &\beta_1 R_1 s_{1,0} + \beta_2 \frac{(1-\gamma_2) A_2 R_1}{R_1 - \gamma_2 A_2} s_{2,0} \\
  =& \beta_1 R_1 \Big(\frac{\gamma_3 A_3}{R_1 - \gamma_3 A_3} s_{3,0} + \frac{\gamma_2 A_2}{R_1 - \gamma_2 A_2} s_{2,0} \Big) + \beta_2 \frac{(1-\gamma_2) A_2 R_1}{R_1 - \gamma_2 A_2} s_{2,0} \\
  =& \beta_1 R_1 \frac{\gamma_3 A_3}{R_1 - \gamma_3 A_3} s_{3,0} + (\beta_1 \gamma_2 + \beta_2 (1-\gamma_2) A_2 s_{2,0} \frac{R_1}{R_1 - \gamma_2 A_2} \\
  \leq& \beta_1 R_1 \frac{\gamma_3 A_3}{R_1 - \gamma_3 A_3} s_{3,0} + (\beta_3 - \beta_1) \gamma_3 A_3 s_{3,0} \frac{R_1}{R_1 - \gamma_2 A_2} = \frac{\gamma_3}{1-\gamma_3} \beta_3 \frac{(1-\gamma_3) A_3 R_1}{R_1 - \gamma_3 A_3} s_{3,0}
  \end{aligned}$$

- Tranversality conditions: $\lim_{T\to\infty}\beta_i^t u_i'(c_{i,t})(k_{i,T}-b_{i,T})=0$. It is easy to verify these conditions because $\beta_i \in (0,1)$ and $u'(c)=1/c$.

□

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