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Centre d'Economie de la Sorbonne

CES Working Papers

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2022.06R

Version révisée

Maison des Sciences Économiques, 106-112 boulevard de L'Hôpital, 75647 Paris Cedex 13
https://centredeconomiesorbonne.cnrs.fr/
ISSN : 1955-611X – eISSN : 2968-6687

A Dynamic Analysis of Criminal Networks

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June 21, 2023

Abstract

We take a novel approach based on differential games to the study of criminal networks. We extend the static crime network game (Ballester et al., 2006, 2010) to a dynamic setting where criminal activities negatively impact the accumulation of total wealth in the economy. We derive a Markov Perfect Equilibrium (MPE), which is unique within the class of strategies considered, and show that, unlike in the static crime network game, the vector of equilibrium crime efforts is not necessarily proportional to the vector of Bonacich centralities. Next, we conduct a comparative dynamic analysis with respect to the network size, the network density, and the marginal expected punishment, finding results in contrast with those arising in the static crime network game. We also shed light on a novel issue in the network theory literature, i.e., the existence of a *voracity effect*. Finally, we study the problem of identifying the optimal target in the population of criminals when the planner's objective is to minimize aggregate crime at each point in time. Our analysis shows that the key player in the dynamic and the static setting may differ, and that the key player in the dynamic setting may change over time.

JEL Classification: C73, D85, K42.

Keywords: differential games; Markov Perfect Equilibrium; criminal networks; Bonacich centrality; key player.

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Documents de travail du Centre d'Economie de la Sorbonne 2022.06R (Version révisée)

1 Introduction

It is natural to think about criminal and delinquent activities in terms of networks, and more specifically, social networks. Indeed, as argued in Lindquist and Zenou (2019), social network analysis can be quite useful for understanding more about the root causes of crime and delinquency, and for designing crime prevention policies. Not surprisingly, there exists a vast literature devoted to crime and networks (see, e.g., Lindquist and Zenou, 2019, for an overview). However, dynamic considerations in the context of criminal networks have not received enough attention so far. The present paper contributes to the networks and crime literature, by proposing a novel (dynamic) approach to the study of criminal networks, i.e., a differential game approach.

The natural fit between crime and social networks comes particularly from the fact that crime is primarily considered as a group activity and that social interactions heavily affect criminal behavior.[1] Indeed, the importance of social networks and peer influences in criminal activities has been acknowledged for a long time in the criminology and sociology literature (e.g., Sutherland, 1947; Haynie, 2001; Sarnecki, 2001; Warr, 2002). Also the economic literature is very active in the study of peer and network effects in crime. Sah (1991) and Glaeser et al. (1996) were the first to develop economic models of social interactions and crime, and were followed by others that proposed various theoretical foundations on peer and network effects in criminal activities (e.g., Calvó-Armengol and Zenou, 2004; Ballester et al., 2006, 2010; Cortés et al., 2019). In parallel to theoretical investigations, there is also strong empirical evidence of peer effects in crime (e.g., Ludwig et al., 2001; Kling et al., 2005; Patacchini and Zenou, 2012; Bayer et al., 2009; Damm and Dustmann, 2014).

There exists a sizeable literature on applications of differential games in the field of crime and crime control (e.g., Feichtinger, 1983; Dawid and Feichtinger, 1996; Dubovik and Parakhonyak, 2014; Faria et al., 2019), government corruption (e.g., Kemp and Long, 2009; Ngendakuriyo and Zaccour, 2013, 2017), counterfeiting (e.g., Crettez et al., 2020) and terrorism (e.g., Nova et al., 2010; Wrzaczek et al., 2017). This literature has been able to shed light on a number of important issues related to the dynamics of illegal activities carried out by individuals, firms, and governments. However, it has abstracted from the widely recognized fact that criminals are embedded in social networks (see Ballester et al., 2010). In this paper, we aim to fill this gap in the literature by merging two so far disjoint strands of research, namely, the research on the dynamics of crime without social networks, and the research on social networks without dynamics. Indeed, to the

[1] A classical example of crime as a group/family activity is that of Italian Mafias (e.g., Calderoni, 2012; Allum et al., 2019).

best of our knowledge, ours is the first analysis of criminal networks in a full-fledged dynamic game.

Identifying the optimal targets and key players in social networks is a fundamental problem in various social, economic and political situations. In parallel to information diffusion, technology adoption, marketing or political campaigning, this problem is of particular importance in crime, delinquency and terrorism; for overviews on targeting and pricing, and key players in different contexts, see Bloch (2016) and Zenou (2016), respectively. Numerous studies in the economics literature characterize optimal targets by well established or new centrality measures (e.g., Ballester et al., 2006; Galeotti and Goyal, 2009; Candogan et al., 2012; Bloch and Querou, 2013; Banerjee et al., 2013, 2019; Bimpikis et al., 2016; Demange, 2017; Grabisch et al., 2018; Galeotti et al., 2020). Some works consider network formation in the setting of Ballester et al. (2006). Liu et al. (2012) develop a network formation model to determine key criminals, i.e., those who once removed generate the highest possible reduction in aggregate crime level in a network. At each period of time, a criminal is chosen at random and decides with whom she/he wants to form a link, anticipating the criminal effort game played by all criminals after a new link has been added. König et al. (2014) develop a two-stage game, where agents play the game of Ballester et al. (2006) in the first stage, which is followed by a linking-formation process in the second stage. Network formation is also considered by Lee et al. (2021), who empirically identify the key player defined in Ballester et al. (2006).

Our analysis takes a different (dynamic) road and is conducted in terms of a differential game (see Başar and Olsder, 1995, Dockner et al., 2000, Haurie et al., 2012, and Long, 2010 for concepts and applications). As is well known, differential games are particularly useful for modeling economic problems which involve both dynamics and strategic behavior. We propose and analyze an infinite-horizon linear quadratic differential game based on the seminal papers by Ballester et al. (2006, 2010).[2] In our differential game, the state variable is the stock of total wealth legally produced in the economy. At each point in time, criminals embedded in a social network decide how much effort to make, taking as given the efforts of the other criminals. The sum of efforts by all criminals negatively affects the evolution of the state variable. As such, part of total wealth in the economy is transferred from the legal to the illegal sector.[3] We assume that players use Markovian strategies,

[2] On the class of linear-quadratic differential games see Dockner et al. (2000, Chapter 7). Some prominent examples of applications of linear-quadratic differential games in economics include Fershtman and Kamien (1987), Tsutsui and Mino (1990), Dockner and Long (1993), Benchekroun (2003, 2008), and Jun and Vives (2004).

[3] In a similar vein, there exists a (static) literature studying situations where power and coercion govern the exchange of resources, and stronger agents are able to take resources from weaker agents (see, e.g., Piccione and

i.e., they condition their crime efforts on the current state variable, and derive a Markov Perfect Equilibrium (MPE), which is unique within the class of (stationary) linear feedback strategies considered. Next, we perturb the equilibrium by changing the network size, the network density, the marginal expected punishment, and the implicit growth rate. Finally, we study the problem of identifying the key player, i.e., the player who, if removed, leads to the largest drop in aggregate crime.

Our main results can be summarized as follows. First, in the static game by Ballester et al. (2006), the Nash equilibrium is proportional to the Bonacich centrality (Bonacich, 1987). In our dynamic setting, instead, this proportionality does not hold in general. However, we do recover the result by Ballester et al. (2006) as a particular case, when the shadow price of total wealth in the economy is the same for all players (e.g., in a regular network). Moreover, while in the static setup aggregate crime is always strictly positive, in our dynamic setting corner solutions may prevail (although the focus of our analysis is on interior solutions). Second, we show that a social multiplier effect, which occurs when an increase in the number of criminals, or links, or both, leads to an increase in aggregate crime, does not necessarily arise. In general, the results of our comparative dynamic analysis with respect to the network size and density and with respect to the expected marginal punishment suggest that some of the conclusions reached in the static literature on criminal networks do not necessarily carry over to a Markovian environment. Conditions exist such that more criminals or more connected criminals induce lower crime in the economy, and conditions exist such that the impact on aggregate crime of an increase in the marginal expected punishment in the static and the dynamic setting differ. This holds true both in the short run and at the steady-state equilibrium. Third, we show that a faster growing economy (in the absence of crime) may cause an increase in aggregate crime, which, in the end, may dampen economic growth. This is related to the so called voracity effect (see Tornell and Lane, 1999), which, to the best of our knowledge, has not been studied in the network literature so far. Forth, we extend the analysis of the key player in Ballester et al. (2006, 2010), and show that the identification of the key player is more nuanced than in the static setting. In our dynamic game, the key player does not necessarily correspond to the player with the highest intercentrality measure. Moreover, in our dynamic game, it is possible that the key player changes over time. Beyond theoretical interest, this finding has clear policy implications: under certain circumstances, it is optimal (from an aggregate crime minimization perspective) to imprison a specific criminal only for a finite time, after which the same criminal should be reintegrated into society, and "substituted in jail" with a different

Rubinstein, 2004; Jordan, 2006; Piccione and Rubinstein, 2007).

criminal, either temporarily or ad infinitum.

The rest of the paper is organized as follows. In Section 2, we present the model, first by recalling the static setting of Ballester et al. (2010) in Section 2.1 and then by introducing our dynamic framework in Section 2.2. Section 3 is devoted to the MPE and Bonacich centrality. In Section 4, we conduct a comparative dynamic analysis with respect to the network size, the network density, the marginal expected punishment, and the implicit growth rate. In Section 5, we address the issue of identifying the key player in the network. Section 6 concludes. All the proofs are presented in the Appendices A till D.

2 Model

2.1 Static Setting

The static crime network game Our point of departure is the framework of Ballester et al. (2010) as recalled below. We consider a criminal network game with a set $N = \{1, \ldots, n\}$ of players (criminals[4]) embedded in a network g of social connections. Let $\mathbf{G} = [g_{ij}]$ denote the n-square adjacency matrix of network g, keeping track of the (direct) connections in the network. Criminals i and j are connected in g if and only if $g_{ij} = 1$, and $g_{ij} = 0$ otherwise. By convention, $g_{ii} = 0$. The criminals decide how much crime effort to exert. Let $\mathbf{x} = (x_1, \ldots, x_n)$ denote the population crime profile, with $x_i \geq 0$ being crime effort exerted by criminal $i \in N$.

Following Becker (1968), Ballester et al. (2010) assume that criminals trade off the costs and benefits of crime activities when deciding about their crime efforts. The expected gains to criminal i are given by

$$u_i(\mathbf{x}, g) = \underbrace{z_i(\mathbf{x})}_{\text{proceeds}} - \underbrace{p_i(\mathbf{x}, g)}_{\text{apprehension}} \underbrace{f}_{\text{fine}}. \tag{1}$$

The proceeds $z_i(\mathbf{x})$ correspond to the gross crime payoffs of criminal i. Ballester et al. (2010) assume that the higher the criminal connections to a criminal and/or the higher the involvement in criminal activities of these connections, the lower i's probability to be caught $p_i(\mathbf{x}, g)$. Furthermore, for the sake of tractability, they restrict attention to the following expressions:

$$z_i(\mathbf{x}) = x_i \max \left\{ 1 - \delta \sum_{j=1}^{n} x_j, 0 \right\}, \tag{2}$$

[4] Ballester et al. (2010) focus on petty crimes and therefore consider delinquents rather than criminals. In the present paper, we consider criminal networks.

$$p_i(\mathbf{x}, g) = p_0 x_i \max \left\{ 1 - \phi \sum_{j=1}^{n} g_{ij} x_j, 0 \right\}, \tag{3}$$

where $\delta > 0$ is the global substitutability parameter, $\phi > 0$ is the local complementarity parameter, and p_0 is the marginal probability of being caught for an isolated criminal. It is assumed that, at an equilibrium $\mathbf{x}^*$,

$$1 - \delta \sum_{j=1}^{n} x_j^* \geq 0 \quad \text{and} \quad 1 - \phi \sum_{j=1}^{n} g_{ij} x_j^* \geq 0. \tag{4}$$

Then, by substituting $z_i(\mathbf{x})$ and $p_i(\mathbf{x}, g)$ given in (2) and (3) into $u_i(\mathbf{x}, g)$ given in (1), we get the following utility function[5] of criminal i:

$$u_i(\mathbf{x}, g) = (1 - \pi) x_i - \delta \sum_{j=1}^{n} x_i x_j + \pi \phi \sum_{j=1}^{n} g_{ij} x_i x_j, \tag{5}$$

where $\pi = p_0 f$ is the marginal expected punishment cost for an isolated criminal. We assume that $\pi < 1$.

The Bonacich centrality and Nash equilibrium Let $\mathbf{G}^k = [g_{ij}^{[k]}]$ denote the kth power of $\mathbf{G}$, where $k \in N$, keeping track of the indirect connections in the network. In particular, $\mathbf{G}^0 = \mathbf{I}$. Every coefficient $g_{ij}^{[k]} \geq 0$ measures the number of walks of length $k \geq 1$ in g between i and j, where a walk of length $k \geq 1$ between i and j is a sequence $(i_0, \ldots, i_k)$ of players such that $i_0 = i$, $i_k = j$, $i_p \neq i_{p+1}$ and $g_{i_p i_{p+1}} = 1$ for all $0 \leq p \leq k - 1$.

Definition 1 *Consider a network g with adjacency n-square matrix $\mathbf{G}$ and a scalar $a \geq 0$ such that the matrix*

$$\mathbf{M}(g, a) = [m_{ij}(g, a)] = [\mathbf{I} - a\mathbf{G}]^{-1} = \sum_{k=0}^{+\infty} a^k \mathbf{G}^k$$

is well defined and nonnegative. Hence, the coefficients $m_{ij}(g, a) = \sum_{k=0}^{+\infty} a^k g_{ij}^{[k]}$ count the number of walks in g that start at i and end in j, where walks of length k are weighted by a^k.

(i) The vector of Bonacich centralities of parameter a in g is

$$\mathbf{b}(g, a) = [\mathbf{I} - a\mathbf{G}]^{-1} \cdot \mathbf{1},$$

[5] The crime network game of Ballester et al. (2010) is developed by using the network model of Ballester et al. (2006) to the case of criminal networks. Ballester et al. (2006) consider the utility function $u_i(x_1, ..., x_n) = \alpha x_i - \frac{1}{2}(\beta - \gamma) x_i^2 - \gamma \sum_{j=1}^{n} x_i x_j + \lambda \sum_{j=1}^{n} g_{ij} x_i x_j$. Hence, we have the following parameterization: $\alpha = 1 - \pi$, $\lambda = \pi\phi$, $\gamma = \beta = \delta$.

where $\mathbf{1}$ *is the* n*-dimensional vector of ones*[6]*. Hence, the Bonacich centrality of node* i *given by*

$$b_i(g, a) = \sum_{j=1}^{n} m_{ij}(g, a)$$

counts the total number of walks in g *that start at* i*, where walks of length* k *are weighted by* a^k*. Note that* $b_i(g, a) \geq 1$*, with equality when* $a = 0$*.*

(ii) The vector of weighted Bonacich centralities of parameter a *in* g *is*

$$\mathbf{b}_{\mathbf{w}}(g, a) = [\mathbf{I} - a\mathbf{G}]^{-1} \cdot \mathbf{w}$$

with $\mathbf{w} = (\mathsf{w}_1, ..., \mathsf{w}_n)^T$*.*

Let $b\,(g, a)$ denote the sum of the Bonacich centralities of all criminals, i.e.,

$$b\,(g, a) = \sum_{i=1}^{n} b_i\,(g, a)\,.$$

Denote $\theta = \pi\phi/\delta$ and let $\rho(g)$ be the spectral radius of the adjacency matrix $\mathbf{G}$.

Ballester et al. (2006) show that if $\theta\rho(g) < 1$, then there exists a unique Nash equilibrium $\mathbf{x}_S^* = (x_{S,1}^*, \ldots, x_{S,n}^*)^T$, which is interior, and given by

$$\mathbf{x}_S^* = \frac{(1-\pi)\,\mathbf{b}\,(g, \theta)}{\delta\,[1 + b\,(g, \theta)]}.$$

Hence, the aggregate crime level $x_S^* = \sum_{i=1}^{n} x_{S,i}^*$ is equal to

$$x_S^* = \frac{(1-\pi)\,b\,(g, \theta)}{\delta\,[1 + b\,(g, \theta)]}.$$

2.2 Dynamic Setting

We extend the static game previously described to a dynamic setting. Time is continuous and denoted by $t \in [0, \infty)$. Let $y(t) \geq 0$ denote the aggregate stock of wealth which is legally produced and $x(t) = \sum_{i=1}^{n} x_i(t)$ the aggregate crime rate in the economy at t. The intertemporal relationship between $y(t)$ and $x(t)$ is captured by the following differential equation:

$$\dot{y}(t) = \mu y(t) - x(t), \quad y(0) = y_0 \geq 0, \tag{6}$$

with $\mu > 0$ denoting the *implicit* growth rate of total wealth and y_0 the initial level of total wealth in the economy. Crime is assumed to be wealth-reducing. The idea behind (6) is that criminal

[6] More precisely, $\mathbf{b}(g, a)$ is obtained from Bonacich centrality (Bonacich, 1987) by an affine transformation and $\mathbf{b}(g, a) = \mathbf{1} + \mathbf{k}(g, a)$ with $\mathbf{k}(g, a)$ being Katz prestige measure (Katz, 1953). In the literature, Bonacich centrality is also called Katz-Bonacich centrality, as the measure is due to both authors.

activities such as robberies and tax evasion have a negative impact on the accumulation of total wealth in the economy. Clearly, in the absence of crime, the growth rate of total wealth in the economy is strictly positive; otherwise, it can be negative (or nil).

Criminal i's objective functional is given by

$$J_i = \int_0^\infty e^{-rt} u_i(x_1, ..., x_n, g) dt,$$

with $u_i(x_1, ..., x_n, g)$ defined in (5) and $r > 0$ being the discount rate. Criminal i seeks to maximize J_i w.r.t. x_i subject to (6).

At each t, criminals, after observing $y(t)$, decide how much effort to make, taking as given the efforts of the other criminals. Our equilibrium concept is Markov Perfect Equilibrium (MPE). Specifically, we adopt the closed-loop (feedback) Nash equilibrium: criminals condition their crime effort only on the current state variable, which summarizes the entire history of the game. This restriction captures the notion that bygones are bygones (see Başar and Olsder, 1995; Dockner et al., 2000; Maskin and Tirole, 2001).[7] Note that, *a priori*, equilibrium crime efforts can be either increasing or decreasing in y. Formally, strategies are of the form $x_i(t) = \psi_i(t, y(t))$, where ψ_i is a decision rule specifying a level of crime effort for criminal i for any t and observed y.

The restrictions imposed on closed-loop strategies are given in the following definition.[8]

Definition 2 *A n-tuple of closed-loop strategies $(\psi_1, ..., \psi_n)$ is said to be admissible if $(i = 1, ..., n)$*

(i) $x_i(t) = \psi_i(t, y(t))$ is well defined for all $t \geq 0$

(ii) the function $t \rightarrow x_i(t) = \psi_i(t, y(t))$ is measurable

(iii) $\psi_i(t, 0) = 0$

(iv) the initial value problem $\dot{y}(t) = \mu y(t) - \sum_{i=1}^{n} \psi_i(t, y(t))$, $y(0) = y_0 > 0$, has a unique solution.

Property (iii), in particular, requires that criminal i makes zero effort if total wealth in the economy is nil. Indeed, a necessary condition for crime to exist is that total wealth in the economy is strictly positive.

Let $\boldsymbol{\psi}^*$ be a n-tuple of admissible closed-loop strategies, and $\boldsymbol{\psi}^*_{-i}$ be the $(n-1)$-tuple of admissible closed-loop strategies ψ^*_j, with $j = 1, ..., n$, $j \neq i$.

[7] By definition, history dependent strategies, such as trigger strategies, are ruled out.

[8] Similar restrictions are common in the differential game literature (e.g., Dockner and Sorger, 1996; Benchekroun, 2003, 2008; Colombo and Labrecciosa, 2015).

Definition 3 *The n-tuple $\boldsymbol{\psi}^*$ constitutes a Markov Perfect Equilibrium (MPE) if, for every possible initial condition (y_0, t_0):*

$$J_i(\boldsymbol{\psi}^*) \geq J_i(\psi_i, \boldsymbol{\psi}^*_{-i}) \text{ for all } i = 1, ..., n$$

for any closed-loop strategy ψ_i such that $(\psi_i, \boldsymbol{\psi}^*_{-i})$ is an admissible n-tuple of closed-loop strategies.

Within the set of closed-loop strategies, in line with the bulk of the literature, we restrict attention to strategies of the symmetric and stationary type. The game is symmetric as the discount rate and the time horizon are common to all criminals, their ability to affect the evolution of the state is identical, and criminals' instantaneous payoffs and feasible sets take the same form. Strategies are of the stationary type due to the structure of the game: the equation of motion is autonomous, and the instantaneous payoffs as well as the feasible sets do not explicitly depend on time.[9]

3 MPE and Bonacich Centrality

In this section, we derive the MPE (unique in the class of strategies considered) and, in the same spirit as Ballester et al. (2006), study the relationship between the MPE and the vector of Bonacich centralities for general networks. In order to derive MPE strategies, we adopt the value function approach. Let $V_i(y)$ denote criminal i's value function, representing the discounted value of the stream of utilities (5) for a game that starts at y. By standard arguments (see Starr and Ho, 1969), MPE strategies must satisfy the following Hamilton-Jacobi-Bellman (HJB) equations ($i = 1, ...n$):

$$rV_i(y) = \max_{x_i \geq 0} \left\{ u_i(x_i, \boldsymbol{\psi}^*_{-i}, g) + V_i'(y) \left[\mu y - x_i - \sum_{j=1, j \neq i}^{n} \psi_j^*(y) \right] \right\}, \tag{7}$$

where $V_i'(y) = \partial V_i(y)/\partial y$ denotes the nonnegative shadow price of total wealth for criminal i. Maximization of the RHS of (7) yields the following necessary and sufficient (given the concavity of u_i) condition:

$$\frac{\partial u_i(x_i, \boldsymbol{\psi}^*_{-i}, g)}{\partial x_i} - V_i'(y) = 0.$$

Let $\underline{y}_{,i}$ be the value of y such that $\psi_i(y) = 0$, and $\overline{y}_i$ the value of y such that $V_i'(y) = 0$. The nonnegativity constraint on $\psi_i(y(t))$ implies that $\psi_i(y) = 0$ for all $y \leq \underline{y}_{,i}$, and the nonnegativity constraint on $V_i'(y)$ implies that $V_i'(y) = 0$ for all $y \geq \overline{y}_i$, with $i = 1, ..., n$. We label all players $i = 1, .., n$ according to their values $\underline{y}_{,i}$ such that $\underline{y}_{(k)}$ of player (k) does not exceed $\underline{y}_{(k+1)}$ of player

[9] Note that stationarity alone is not sufficient to rule out equilibria involving non stationary strategies. However, as pointed out in Dockner et al. (2000), non stationary equilibria are of less interest and therefore they are generally not considered in economic applications.

$(k+1)$ for all $k = 1, .., n-1$, i.e., we have $\underline{y}_{(1)} \leq \underline{y}_{(2)} \leq \cdots \leq \underline{y}_{(n)}$. In order for strategies to be admissible (as defined in Definition 2) and induce a trajectory of aggregate crime that converges to a (locally) stable steady state, we make the following assumptions, which we will maintain throughout.

Assumption 1 $\underline{y}_{,i} > 0$ for all $i = 1, ..., n$.

Assumption 2 The solution to the initial value problem $\dot{y}(t) = \mu y(t) - \sum\limits_{i=1}^{n} \psi_i\left(y\left(t\right)\right)$, $y(0) = y_0 \in (\underline{y}_{(n)}, \overline{y}_{i_{\min}})$, is bounded from below by $\underline{y}_{(n)}$ and from above by $\overline{y}_{i_{\min}}$, where $i_{\min} = \arg\min_{i \in N} \overline{y}_i$.

Assumptions 1 and 2 are required for conditions (iii) and (i) in Definition 2, respectively, to be satisfied. Moreover, Assumption 1 is required for the existence of a (locally) stable steady state, while Assumption 2 is required for that steady state to belong to the interval $(\underline{y}_{(n)}, \overline{y}_{i_{\min}})$.

We focus on the interval $\underline{y}_{(n)} < y < \overline{y}_{i_{\min}}$ where all criminals are active and play nondegenerate Markovian strategies (i.e., strategies that depend on the current state variable). Given the linear-quadratic structure of the game, we guess a value function of the form

$$V_i(y) = A_i \frac{y^2}{2} + B_i y + C_i \Rightarrow \frac{\partial V_i(y)}{\partial y} = A_i y + B_i \geq 0,$$

and consider (stationary) linear feedback strategies

$$\psi_i(y(t)) = \alpha_i y(t) + \beta_i \geq 0,$$

where α_i and β_i are constants the depend on the parameters of the model. The linearity of $\psi_i(y(t))$ ensures that the initial value problem $\dot{y}(t) = \mu y(t) - \sum\limits_{i=1}^{n} \psi_i\left(y\left(t\right)\right) = (\mu - \sum_{i=1}^{n} \alpha_i) y(t) - \sum_{i=1}^{n} \beta_i$, $y(0) = y_0 > 0$ has a unique solution, thus satisfying condition (iv) in Definition 2.

Theorem 1 *The matrix* $\mathbf{M}(g, \theta) = [\mathbf{I} - \theta \mathbf{G}]^{-1}$ *is well-defined and nonnegative if and only if* $\theta \rho(g) < 1$. *Assume that this condition holds. Within the class of (stationary) linear feedback strategies, there exists a unique MPE given by the n-tuple* $\boldsymbol{\psi}^*$, *with*

$$\boldsymbol{\psi}^* = \mathbf{x}_S^* - \frac{\mathbf{b}_{\mathbf{V}'}\left(g, \theta\right)}{\delta\left[1 + b\left(g, \theta\right)\right]} \quad \textit{for } \underline{y}_{(n)} < y < \overline{y}_{i_{\min}}$$

where

$$\mathbf{x}_S^* = \frac{\left(1 - \pi\right) \mathbf{b}\left(g, \theta\right)}{\delta\left[1 + b\left(g, \theta\right)\right]},$$

and $\mathbf{b}\left(g, \theta\right)$ *and* $\mathbf{b}_{\mathbf{V}'}\left(g, \theta\right)$ *are the vector of Bonacich centralities of parameter* θ *in* g *and the vector of weighted Bonacich centralities of parameter* θ *in* g, *with weights* $\mathbf{V}' = (V_1', ..., V_n')^T = (A_1 y + B_1, ..., A_n y + B_n)^T$, *respectively.*

Theorem 1 establishes that for $\underline{y}_{(n)} < y < \overline{y}_{i_{\min}}$ there exists a MPE characterized by crime efforts which are strictly positive and lower than their static counterparts for all criminals. This implies that the assumptions specified in (4), which have to hold in the static game, also hold in our dynamic game.

The reason why Theorem 1 focusses on the interval $(\underline{y}_{(n)}, \overline{y}_{i_{\min}})$ is twofold. Firstly, for $y \leq \underline{y}_{(n)}$, the nonnegativity constraint on the control variable of all criminals i such that $y \leq \underline{y}_i$ becomes binding and, therefore, these criminals abstain from committing crime (for these criminals, the shadow price of total wealth in the economy turns out to be higher than the marginal benefit from committing crime, $1-\pi$). This is in contrast with the static game studied in Ballester et al. (2006, 2010), in which aggregate effort is always strictly positive. An equilibrium involving active and inactive criminals would need to be constructed sequentially by solving a series of differential equations with constants of integration such that the value function of each criminal is continuously differentiable. In general, in the presence of inactive players, the equilibrium strategies of active players are complicated nonlinear functions. In order to keep the analysis analytically tractable, we then restrict our attention to levels of y in excess of $\underline{y}_{(n)}$. Secondly, for $y \geq \overline{y}_{i_{\min}}$, the nonnegativity constraint on the shadow price of total wealth in the economy for all criminals i such that $y \geq \overline{y}_i$ becomes binding (for these criminals, total wealth in the economy is so high that the dynamic constraint plays no role). Constructing a MPE for $y \geq \overline{y}_{i_{\min}}$ would require checking that no criminal has an incentive to deviate from their linear strategy (recall that in the interval $(\underline{y}_{(n)}, \overline{y}_{i_{\min}})$ criminals play linear strategies). Consider, for instance, a network structure in which there are only two types of criminals, Type 1 and Type 2, with $\underline{y}_{,1} > \underline{y}_{,2}$. In the interval $(\underline{y}_{,2}, \underline{y}_{,1})$, by definition, $V_2'(y) = 0 < V_1'(y)$. If y belongs to this interval, it has to be checked that Type 1 criminal has no incentive to move away from the linear strategy ψ_1^* and plays instead an impulse control that instantaneously brings y back to $\underline{y}_{,2}$. If Type 1 criminal does have an incentive to move away from the linear strategy ψ_1^*, levels of y in excess of $\underline{y}_{,2}$ are not sustainable.[10] In order to avoid possible discontinuities in the value functions, we then restrict our attention to levels of y below $\overline{y}_{i_{\min}}$.

The next two remarks are about the relationship between MPE and Bonacich centrality.

Remark 1 *Unlike in the static game studied in Ballester et al. (2006, 2010), the vector of equilibrium crime efforts, $\boldsymbol{\psi}^*$, is not necessarily proportional to the the vector of Bonacich centralities, $\mathbf{b}(g,\theta)$.*

[10] A similar result can be found in Benchekroun et al. (2014) in the context of a linear quadratic differential game of oligopoly exploitation of a common-pool renewable resource assumed to grow according to a linearized logistic function with firms having different marginal costs of extraction.

Remark 2 *When $V_i' = V'$ for all $i = 1, ..., n$, since $\mathbf{b}_{\mathbf{V}'}(g,\theta) = V'\mathbf{b}(g,\theta)$, $\boldsymbol{\psi}^*$ given in Theorem 1 becomes*

$$\boldsymbol{\psi}^* = \frac{(1-\pi-V')\,\mathbf{b}(g,\theta)}{\delta\,[1+b(g,\theta)]}.$$

In this case, as in the static game studied in Ballester et al. (2006, 2010), the vector of equilibrium crime efforts is proportional to the vector of Bonacich centralities; if $V_i' = V' = 0$ for all $i = 1, ..., n$ then $\boldsymbol{\psi}^*$ corresponds to $\mathbf{x}_S^*$ given in Theorem 1.

Corollary 1 below characterizes aggregate crime as a function of total wealth in the economy.

Corollary 1 *The MPE aggregate crime level is given by*

$$x^* = x_S^*\left[1 - \frac{1}{1-\pi}\frac{b_{\mathbf{V}'}(g,\theta)}{b(g,\theta)}\right] \quad \text{for } \underline{y}_{(n)} < y < \overline{y}_{i_{\min}}$$

where

$$x_S^* = \frac{(1-\pi)\,b(g,\theta)}{\delta\,[1+b(g,\theta)]}, \tag{8}$$

and $b_{\mathbf{V}'}(g,\theta)$ and $b(g,\theta)$ are the sum of the coordinates of the vector of weighted Bonacich centralities of parameter θ in g, with weights $\mathbf{V}' = (V_1', ..., V_n')^T = (A_1 y + B_1, ..., A_n y + B_n)^T$, and the sum of the coordinates of the vector of Bonacich centralities of parameter θ in g, respectively.

Corollary 2 below characterizes aggregate crime as a function of time.

Corollary 2 *Take $y_0 \in (\underline{y}_{(n)}, \overline{y}_{i_{\min}})$. The MPE trajectory of aggregate crime is given by*

$$x^*(t) = x_S^*\left[1 - \frac{b_{\mathbf{A}}(g,\theta)\,y^*(t) + b_{\mathbf{B}}(g,\theta)}{(1-\pi)\,b(g,\theta)}\right],$$

where

$$y^*(t) = \widehat{y} + (y_0 - \widehat{y})\exp\left[t\left(\mu + \frac{b_{\mathbf{A}}(g,\theta)}{\delta\,[1+b(g,\theta)]}\right)\right],$$

and

$$\widehat{y} = \frac{(1-\pi)\,b(g,\theta) - b_{\mathbf{B}}(g,\theta)}{\mu\delta\,[1+b(g,\theta)] + b_{\mathbf{A}}(g,\theta)} \in (\underline{y}_{(n)}, \overline{y}_{i_{\min}}),$$

and $b_{\mathbf{A}}(g,\theta)$ and $b_{\mathbf{B}}(g,\theta)$ are the sum of the coordinates of the vector of weighted Bonacich centralities of parameter θ in g with weights $\mathbf{A} = (A_1, ..., A_n)^T$ and the sum of the coordinates of the vector of weighted Bonacich centralities of parameter θ in g with weights $\mathbf{B} = (B_1, ..., B_n)^T$, respectively. The MPE trajectory of aggregate crime converges to $\widehat{x} = \mu\widehat{y}$ as $t \to \infty$ provided that

$$\mu + \frac{b_{\mathbf{A}}(g,\theta)}{\delta\,[1+b(g,\theta)]} < 0.$$

A necessary condition for the steady-state equilibrium to be stable is that $b_{\mathbf{A}}(g,\theta) < 0$, which implies that $b_{\mathbf{V}'}(g,\theta) = b_{\mathbf{A}}(g,\theta)\, y + b_{\mathbf{B}}(g,\theta)$ is decreasing in y. Hence, recalling from Corollary 1 that x^* is decreasing in $b_{\mathbf{V}'}(g,\theta)$, stability of the steady-state equilibrium implies that aggregate crime is increasing in y. Our theory then predicts that, *ceteris paribus*, higher (resp. lower) levels of aggregate crime should be observed in richer (resp. poorer) economies.

Aggregate crime as a function of time is depicted in Figure 1.

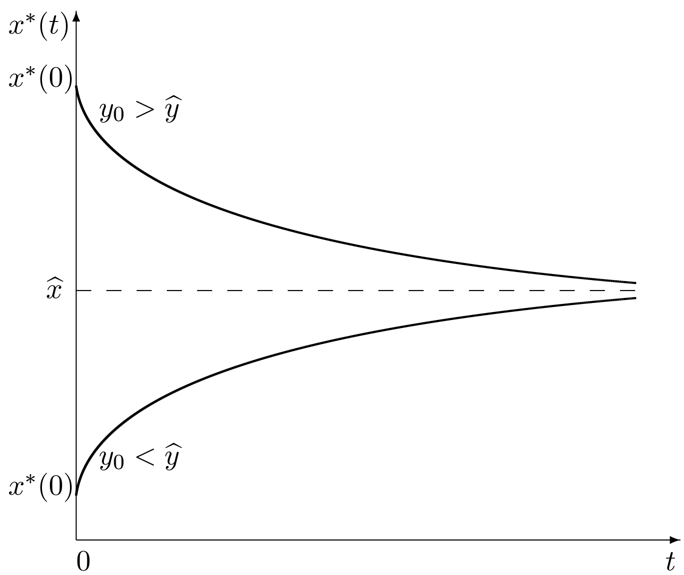

Figure 1: MPE trajectory of aggregate crime

In the remainder of this section, we provide two illustrative examples of Corollaries 1 and 2.

Example 1. Consider a regular network[11] with $n \geq 2$ criminals, each having the same degree d.[12] Routine calculations lead to

$$b_i(g,\theta) = \frac{1}{1-d\theta},$$

which implies that

$$b(g,\theta) = \frac{n}{1-d\theta}.$$

From (8) and given $\theta = \pi\phi/\delta$, it follows that

$$x_S^* = \frac{n(1-\pi)}{\delta(1+n) - d\pi\phi}.$$

[11] A network is regular if each of its nodes has the same degree, i.e., the same number of connections.

[12] It can be easily checked that, for regular networks, the condition $\theta\rho(g) < 1$ is satisfied when $\theta < 1/d$.

Next, we verify that, for regular networks, $b_{\mathbf{V}'}(g,\theta) = V'b(g,\theta)$ and therefore (from Corollary 1)

$$x^* = x_S^* \left(1 - \frac{V'}{1-\pi}\right).$$

By definition, $b_{\mathbf{V}',i}(g,\theta) = \sum_{j=1}^{n} m_{ij}(g,a)V_j'$, where $[m_{ij}(g,a)] = [\mathbf{I} - a\mathbf{G}]^{-1}$. Regular networks imply that $V_j' = V'$ for all $j = 1, ..., n$. Hence, $b_{\mathbf{V}',i}(g,\theta) = V' \sum_{j=1}^{n} m_{ij}(g,a)$. Given that $b_i(g,\theta) = \sum_{j=1}^{n} m_{ij}(g,a)$, it follows that $b_{\mathbf{V}',i}(g,\theta) = V'b_i(g,\theta)$ and that $b_{\mathbf{V}'}(g,\theta) = V'b(g,\theta)$.

Maximization of the RHS of (7) yields

$$\psi^* = \frac{1-\pi-V'}{\delta(1+n) - d\pi\phi}. \tag{9}$$

For each criminal, we guess a value function of the form $V = Ay^2/2 + By + C$. Using (7) and (9), the coefficients of V (for interior solutions) can be obtained by identification:

$$A = \frac{(r-2\mu)\left[\delta(1+n) - d\pi\phi\right]^2}{2\left[\delta n^2 - (n-1)\,d\pi\phi\right]}, \tag{10}$$

$$B = \frac{(\pi-1)(r-2\mu)\left[\delta\left(n^2+1\right) - (n-1)\,d\pi\phi\right]}{2\mu\left[\delta n^2 - (n-1)\,d\pi\phi\right]}, \tag{11}$$

and

$$C = \frac{(B+\pi-1)\left[\delta\left(Bn^2+\pi-1\right) - B(n-1)\,d\pi\phi\right]}{r\left[\delta(1+n) - d\pi\phi\right]^2}.$$

We have $A < 0$ since $r - 2\mu < 0$ and $\delta n^2 - (n-1)\,d\pi\phi > 0$ by Assumptions 1 and 2, and $B > 0$, since $\pi - 1 < 0$. Note that $A < 0$ implies that ψ_i^* is increasing in y.

The MPE aggregate crime can be written as

$$x^* = x_S^* \left(1 - \frac{Ay+B}{1-\pi}\right) \text{ for } \underline{y}_{(n)} < y < \overline{y}_{i_{\min}},$$

with A and B given in (10) and (11), respectively. By using the definitions of $\underline{y}_i$ and $\overline{y}_i$, it can be easily checked that $\underline{y}_i = \underline{y} = (1-\pi-B)/A > 0$ and $\overline{y}_i = \overline{y} = -B/A > 0$ for all $i = 1, ..., n$. It can also be easily checked that $\overline{y} - \underline{y} = (\pi - 1)/A > 0$.

The MPE trajectory of aggregate crime is given by (for $y_0 \in (\underline{y}, \overline{y})$)

$$x^*(t) = x_S^* \left[1 - \frac{Ay^*(t)+B}{1-\pi}\right],$$

where

$$y^*(t) = \widehat{y} + (y_0 - \widehat{y}) \exp\left[t\left(\mu + \frac{An}{\delta(1+n) - d\pi\phi}\right)\right], \tag{12}$$

with $\widehat{y}$ being the value of y that solves

$$\mu y = x_S^* \left(1 - \frac{Ay+B}{1-\pi}\right),$$

that is

$$\widehat{y} = \frac{n\left(1-\pi-B\right)}{An+\mu\left[\delta\left(1+n\right)-d\pi\phi\right]},$$

with A and B previously defined. Aggregate crime increases (resp. decreases) over time if $y_0 <$(resp. $>$)$\widehat{y}$. (12) corresponds to $y^*(t)$ given in Corollary 2. Indeed,

$$\frac{b_{\mathbf{A}}\left(g,\theta\right)}{\delta\left[1+b\left(g,\theta\right)\right]} = \frac{An}{\delta\left(1+n\right)-d\pi\phi},$$

since $b_{\mathbf{A}}\left(g,\theta\right) = An/(1-d\theta)$, with $\theta = \pi\phi/\delta$. Moreover, we have

$$\widehat{x} = x_S^*\left(1-\frac{A\widehat{y}+B}{1-\pi}\right),$$

which corresponds to $\widehat{x}$ given in Corollary 2 since, for regular networks, $b_{\mathbf{A}}\left(g,\theta\right) = Ab\left(g,\theta\right)$ and $b_{\mathbf{B}}\left(g,\theta\right) = Bb\left(g,\theta\right)$.

Example 2. As a second illustrative example, consider the star network g in Figure 2.

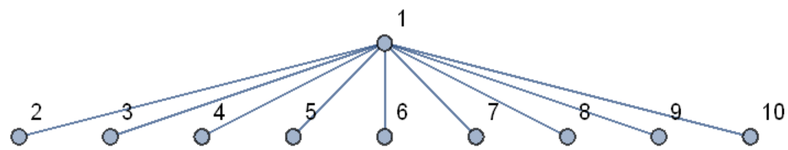

Figure 2: Star network

Let $\phi = \delta = 1$ and $\pi = 0.05$, implying that $\theta = 0.05$.[13] Moreover, let $\mu = 1$ and $r = 0.001$. Routine calculations lead to $b_1\left(g,\theta\right) = 1.4834$ and $b_i\left(g,\theta\right) = 1.0742$ with $i = 2,...,10$. It follows that $b\left(g,\theta\right) = 11.1509$ and that $x_S^* = 0.8718$.

In order to compute x^* we need $b_{\mathbf{V}'}\left(g,\theta\right)$. Performing the maximization indicated in (7) leads to the following value functions (for interior solutions):

$$V_1 = A_1y^2/2 + B_1y + C_1,$$

and

$$V_i = A_iy^2/2 + B_iy + C_i,$$

with $i = 2,...,10$, where $A_1 = -1.1603$, $B_1 = 0.9580$, $C_1 = 7.8871$, $A_i = -1.2037$, $B_i = 0.9593$, and $C_i = 7.1271$. Hence, $V_1' = -1.1603y + 0.9580$, $V_i' = -1.2037y + 0.9593$, and $b_{\mathbf{V}'}\left(g,\theta\right) = 10.6947 - 13.3582y$. From Corollary 1, we then get $x^* = -0.0083 + 1.0994y$.

[13] It can be easily checked that $\rho(g) = 1.15$, therefore the condition $\theta\rho(g) < 1$ is satisfied.

The MPE trajectory of aggregate crime is given by

$$x^*(t) = -0.0083 + 1.0994y^*(t),$$

where

$$y^*(t) = 0.084 + (y_0 - 0.084)\exp[-0.0994t], \tag{13}$$

for $y_0 \in (0.0083, 0.7969)$, which is the interval of y where all criminals are active and play non-degenerate Markovian strategies. It can be easily checked that (13) corresponds to $y^*(t)$ given in Corollary 2. If y_0 <(resp. >)0.084, aggregate crime increases (resp. decreases) over time; as $t \to \infty$, aggregate crime converges to 0.084.

4 Comparative Dynamics

In this section, we conduct a comparative dynamic analysis with respect to the network size, the network density,[14] and the marginal expected punishment. First, we evaluate how aggregate crime responds in the neighborhood of a given initial stock of total wealth (short-run impact). Second, we evaluate how aggregate crime responds at the steady-state equilibrium (long-run impact). We also study the impact of an increase in the implicit growth rate of total wealth in the economy on the rate of growth.

4.1 Network Size and Density

In the static game studied in Ballester et al. (2010), aggregate crime is increasing in either network size or density or both, a feature often referred to as *social multiplier effect*. In the absence of dynamic considerations, policies aimed at reducing aggregate crime should be designed so as to reduce the number of criminals or the number of links or both. In our dynamic game, instead, things are more involved, and a social multiplier effect does not necessarily exist.

Consider two networks, g with associated adjacency matrix $\mathbf{G}$ and g' with associated adjacency matrix $\mathbf{G}'$, with $g \subset g'$ (meaning that g' contains either more criminals n or more links or both). Formally: for all i, j, $g'_{ij} = 1$ if $g_{ij} = 1$. Recall that $b(g, \theta)$ counts the total number of weighted walks in g. Hence, $b(g, \theta)$ is an increasing function in g (for the inclusion ordering), as more links

[14] The density of a network is a relative fraction of possible links that are present in the network. In other words, it is the average degree of all n nodes in the network divided by $n - 1$.

imply more such walks. Call[15] $\Delta x_S^* = x_S^*(g') - x_S^*(g)$. We have

$$\begin{aligned}\Delta x_S^* &= \frac{(1-\pi)\, b\,(g',\theta)}{\delta\,[1+b\,(g',\theta)]} - \frac{(1-\pi)\, b\,(g,\theta)}{\delta\,[1+b\,(g,\theta)]} \\ &= \frac{(1-\pi)}{\delta}\left[\frac{[1+b\,(g)]\, b\,(g',\theta) - [1+b\,(g',\theta)]\, b\,(g,\theta)}{[1+b\,(g',\theta)]\,[1+b\,(g,\theta)]}\right],\end{aligned}$$

implying that

$$\Delta x_S^* \overset{s}{=} [1+b\,(g)]\, b\left(g'\right) - \left[1+b\left(g'\right)\right] b\,(g) = b\left(g'\right) - b\,(g) > 0,$$

where $\overset{s}{=}$ means same sign as.

4.1.1 Short-run impact

Call $\Delta x^* = x^*(g') - x^*(g)$ and define

$$\omega\,(g,\theta) = \frac{b_{\mathbf{V}'}\,(g,\theta)}{b\,(g,\theta)} \geq 0.$$

For $y_0 \in (\max\{\underline{y}_{(n)}(g), \underline{y}_{(n)}(g')\}, \min\{\overline{y}_{i_{\min}}(g), \overline{y}_{i_{\min}}(g')\})$, we have[16]

$$x^*(g) = x_S^*(g)\left[1 - \frac{\omega\,(g,\theta)}{1-\pi}\right].$$

It follows that the total effect of an increase in the number of criminals, or the number of links, or both, can be decomposed into the sum of two effects, a static effect, which is positive, and a dynamic effect, the sign of which is a priori ambiguous:

$$\Delta x^* = \underbrace{\Delta x_S^*}_{\text{static effect}} + \underbrace{\frac{\omega\,(g,\theta)\, x_S^*(g) - \omega\,(g',\theta)\, x_S^*(g')}{1-\pi}}_{\text{dynamic effect}}.$$

Δx^* can be rewritten as

$$\Delta x^* = x_S^*(g')\left[1 - \frac{\omega\,(g',\theta)}{1-\pi}\right] - x_S^*(g)\left[1 - \frac{\omega\,(g,\theta)}{1-\pi}\right].$$

Clearly, when ω is decreasing in g, $\Delta x^* \overset{s}{=} \Delta x_S^* > 0$ since $x_S^*(g') > x_S^*(g)$ and $1 - \omega\,(g',\theta)\,/(1-\pi) > 1 - \omega\,(g,\theta)\,/(1-\pi)$. When instead ω is increasing in g, it is possible that $\Delta x^* < 0$. Take, for instance, $\omega\,(g',\theta) \to 1-\pi$ and $\omega\,(g',\theta) > \omega\,(g,\theta)$. It follows that $\lim_{\omega(g',\theta)\to 1-\pi} \Delta x^* = -x_S^*(g)[1 - \omega\,(g,\theta)\,/(1-\pi)] < 0$.

The above discussion leads to the following proposition.

[15] In this subsection and Section 5, we add explicitly g and g' to the notation of aggregate crime, as we compare the aggregate crime in different networks.

[16] A priori, the order relationships between $\underline{y}_{(n)}(g)$ and $\underline{y}_{(n)}(g')$ and between $\overline{y}_{i_{\min}}(g)$ and $\overline{y}_{i_{\min}}(g')$ are ambiguous.

Proposition 1 *Take* $y_0 \in (\max\{\underline{y}_{(n)}(g), \underline{y}_{(n)}(g')\}, \min\{\overline{y}_{i_{\min}}(g), \overline{y}_{i_{\min}}(g')\})$. *If* $\omega(g', \theta) < \omega(g, \theta)$ *then* $\Delta x^* > 0$. *However, if* $\omega(g', \theta) > \omega(g, \theta)$ *then it is possible that* $\Delta x^* < 0$.

A necessary condition for $\omega(g', \theta) > \omega(g, \theta)$ is $b_{\mathbf{V}'}(g', \theta) > b_{\mathbf{V}'}(g, \theta)$, since $b(g', \theta) > b(g, \theta)$. Then, a social multiplier effect does not necessarily exist when the sum of shadow prices of total wealth in the economy is higher in network g' than in network g. In the next two examples, we show that, counterintuitively, an increase in the number of criminals, or the number of links, can lead to a decrease in aggregate crime (in the neighborhood of a given y_0).

Example 1 (continued). For regular networks, the function $\omega(g, \theta)$ reduces to $V' = Ay + B$, with A and B given in (10) and (11), respectively. Let $n = \delta = 5$, $d = 2$, $\phi = 1$, and $\pi = 0.3$. Moreover, let $\mu = 1$ and $r = 0.001$. It follows that $V' = -7.0467y_0 + 0.7282$ and that $x^*(g) = -0.0048 + 1.1984y_0$. Consider now a network g' resulting from an increase in d from 2 to 4, leading to $V' = -6.8971y_0 + 0.7288$ and $x^*(g') = -0.005 + 1.1974y_0$. Setting $y_0 = \widehat{y}(g) = 0.0241$, at $t = 0$, we get

$$x^*(g'') = \begin{cases} 0.0241 \text{ for } g'' = g \text{ with } d = 2 \\ 0.0239 \text{ for } g'' = g' \text{ with } d = 4 \end{cases}$$

In the static game, instead,

$$x^*_S(g'') = \begin{cases} 0.1191 \text{ for } g'' = g \text{ with } d = 2 \\ 0.1215 \text{ for } g'' = g' \text{ with } d = 4 \end{cases}$$

This example shows that, unlike in the static game, more connected criminals can be associated with lower aggregate crime.[17]

Example 2 (continued). Consider the network g' resulting from adding one periphery criminal to the network g depicted in Figure 2. The total number of periphery criminals increases from 9 to 10. Criminal 1 remains the centre of the star. The total number of criminals then increases from 10 to 11. Routine calculations lead to $b_1(g', \theta) = 1.5385$ and $b_i(g', \theta) = 1.0769$ with $i = 2, ..., 11$. It follows that $b(g', \theta) = 12.3077$ and $x^*_S(g') = 0.8786$. Performing the maximization indicated in (7) leads to the following value functions (for interior solutions):

$$V_1 = A_1 y^2/2 + B_1 y + C_1,$$

and

$$V_i = A_i y^2/2 + B_i y + C_i,$$

[17] Note that $V'|_{d=4,y_0=0.05} = 0.383\,90 > V'|_{d=2,y_0=0.05} = 0.375\,84$.

with $i=2,...,11$, where $A_1=-1.1402$, $B_1=0.9565$, $C_1=6.5605$, $A_i=-1.1844$, $B_i=0.9576$, and $C_i=5.9203$. Hence, $V_1'=-1.1402y_0+0.9565$, $V_i'=-1.1844y_0+0.9576$, and $b_{\mathbf{V}'}(g',\theta)=11.7837-14.5093y_0$. From Corollary 1, we then get $x^*(g')=-0.0069+1.0903y_0$.[18] Recall that $x^*(g)=-0.0083+1.0994y_0$. Let $y_0=0.4$. It follows that aggregate crime at $t=0$ is given by

$$x^*(g'')=\begin{cases}0.4315 \text{ for } g''=g \text{ with } n=10\\ 0.4292 \text{ for } g''=g' \text{ with } n=11\end{cases}$$

and

$$x_S^*(g'')=\begin{cases}0.8718 \text{ for } g''=g \text{ with } n=10\\ 0.8786 \text{ for } g''=g' \text{ with } n=11\end{cases}$$

This example shows that, unlike in the static game, more criminals can be associated with lower aggregate crime.[19]

4.1.2 Long-run impact

Denote $\Delta\widehat{y}=\widehat{y}(g')-\widehat{y}(g)$, where $\widehat{y}(g)$ is implicitly given by

$$\mu\widehat{y}(g)=x_S^*(g)\left[1-\frac{\omega(g,\theta)}{1-\pi}\right]. \tag{14}$$

The RHS of (14) is increasing in y (since $\omega(g,\theta)$ is decreasing in y) and intersects the LHS of (14) from below (for stability of the steady state). Take $y_0=\widehat{y}(g)$. Hence, $\Delta x^*>$(resp. $<$) 0 implies that $\Delta\widehat{y}<$(resp. $>$) 0. Clearly, if $\Delta x^*>0$, which occurs, for instance, when $\omega(g',\theta)<\omega(g,\theta)$, then $x^*(g')>\mu y$. Consequently, $x^*(g')$ intersects μy at a point to the left of $\widehat{y}(g)$.

From Corollary 2, we know that $\widehat{y}$ and $\widehat{x}$ are positively correlated. Hence, we can write the following proposition.

Proposition 2 *Take $y_0=\widehat{y}(g)$. If $\Delta x^*>$(resp. $<$) 0 then $\Delta\widehat{x}<$(resp. $>$) 0.*

[18] The interval of y where all criminals are active and play nondegenerate Markovian strategies is $(0.0069, 0.80847)$.

[19] Note that $b_{\mathbf{V}'}(g',\theta)|_{n=11,y_0=0.4}=5.980>b_{\mathbf{V}'}(g,\theta)|_{n=10,y_0=0.4}=5.351$. $b_{\mathbf{V}'}(g',\theta)>b_{\mathbf{V}'}(g,\theta)$ is a necessary condition for $\omega(g',\theta)>\omega(g,\theta)$.

The case $\Delta\widehat{x} < 0$ in Proposition 2 is illustrated in Figure 3.

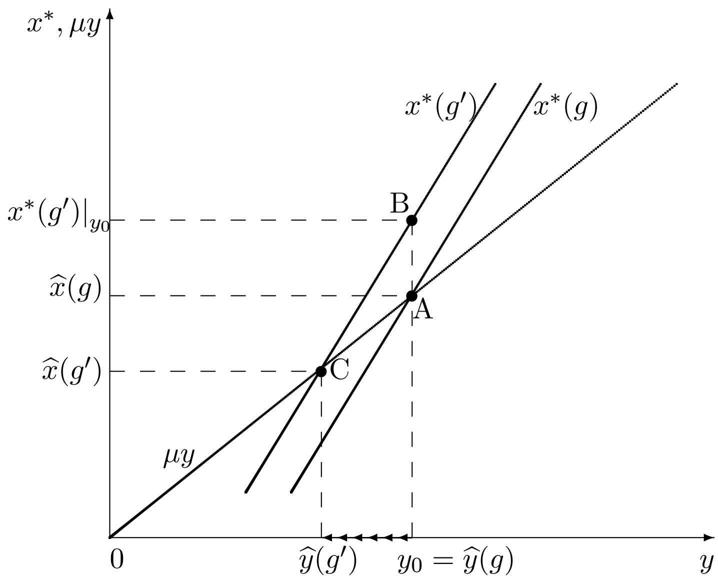

Figure 3: Comparative steady-state analysis with $y_0 = \widehat{y}(g)$

As can be seen in Figure 3, there exists a trade-off between short and long run: aggregate crime increases in the short run and decreases in the long run. Starting from point A, aggregate crime moves up to point B, then down to point C (moving along $x^*(g')$). Indeed, when $\Delta\widehat{x} < 0$ a social multiplier effect exists only in the short run.

A numerical example of the case $\Delta\widehat{x} > 0$ is provided below.

Example 1 (continued). For a regular network with $n = \delta = 5$, $\phi = 1$, $\pi = 0.3$, $\mu = 1$ and $r = 0.001$, the trajectories of aggregate crime are given by

$$x^*(g'', t) = \begin{cases} 0.0241 + (1.1984y_0 - 0.0289)\exp[-0.1984t] \text{ for } g'' = g \text{ with } d = 2 \\ 0.0253 + (1.1974y_0 - 0.0303)\exp[-0.1974t] \text{ for } g'' = g' \text{ with } d = 4 \end{cases}$$

Clearly, as $t \to \infty$, aggregate crime converges to 0.0241 with $d = 2$ and to 0.0253 with $d = 4$. Therefore, in line with the static game, long-run aggregate crime is lower with $d = 2$ than with $d = 4$. However, as previously shown in Example 1, for $t = 0$ and $y_0 = 0.0241$, in contrast with the static game, aggregate crime is higher with $d = 2$ than with $d = 4$. This implies that, for the given parameter values, the trajectories of aggregate crime with $d = 2$ and $d = 4$ intersect.

A final remark is in order. Proposition 2 establishes that if $y_0 = \widehat{y}(g)$ then the long-run response of an increase in the number of criminals, or the number of links, or both, is the opposite in sign to the short-run response. Nevertheless, if $y_0 \neq \widehat{y}(g)$ we can have either an increase or a decrease in aggregate crime, not only in the short run but also at the steady state. The case of a decrease in aggregate crime is illustrated in Figure 4 as well as in the numerical example below.

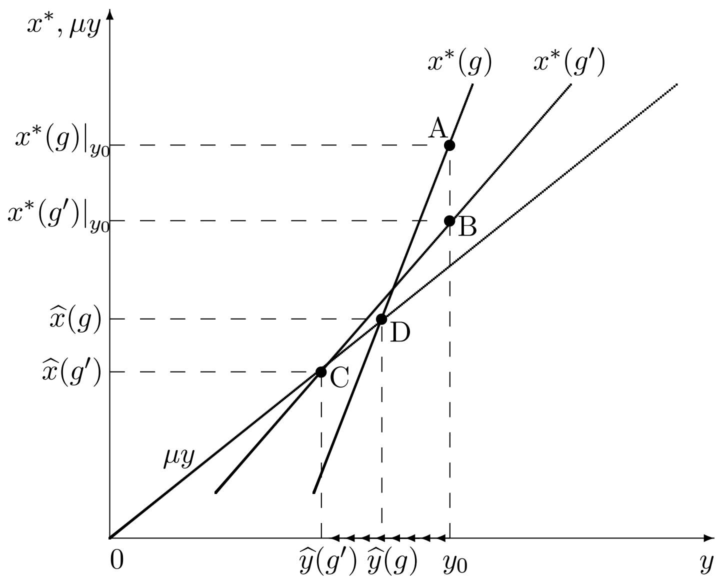

Figure 4: Comparative steady-state analysis with $y_0 \neq \widehat{y}(g)$

Starting from point A, aggregate crime moves down first to point B, then to point C (moving along $x^*(g')$).

Example 2 (continued). For a star network with $\phi = \delta = 1$, $\pi = 0.05$, $\mu = 1$ and $r = 0.001$, the trajectories of aggregate crime are given by

$$x^*(g'', t) = \begin{cases} 0.084 + (1.0994y_0 - 0.0923)\exp[-0.0994t] \text{ for } g'' = g \text{ with } n = 10 \\ 0.0761 + (1.0903y_0 - 0.0829)\exp[-0.0903t] \text{ for } g'' = g' \text{ with } n = 11 \end{cases}$$

Clearly, as $t \to \infty$, aggregate crime converges to 0.084 with $n = 10$ and to 0.0761 with $n = 11$. Therefore, in contrast with the static game, long-run aggregate crime is higher with $n = 10$ than with $n = 11$. Moreover, in contrast with the static game, as previously shown in Example 2 for $t = 0$ and $y_0 = 0.4$, aggregate crime is higher with $n = 10$ than with $n = 11$. This implies that more criminals are associated with a lower aggregate crime both in the short and the long run.

4.2 Marginal Expected Punishment

In the static game studied in Ballester et al. (2010), the impact of π on x_S^* can be decomposed into the sum of a direct and an indirect effect as follows:

$$\frac{\partial x_S^*}{\partial \pi} = \underbrace{\left(-\frac{x_S^*}{1-\pi}\right)}_{\text{direct effect}} + \underbrace{\left(\frac{\partial x_S^*}{\partial \theta}\frac{\phi}{\delta}\right)}_{\text{(static) indirect effect}}$$

where

$$\frac{\partial x_S^*}{\partial \theta} = \frac{(1-\pi)}{\delta\left[1+b\left(g,\theta\right)\right]^2}\frac{\partial b\left(g,\theta\right)}{\partial \theta} > 0.$$

The direct effect is negative, whereas the (static) indirect effect is positive, implying that the impact of π on x_S^* is ambiguous.

4.2.1 Short-run impact

The short-run impact of π on x^* is given by (for interior solutions)

$$\begin{aligned}
\frac{\partial x^*}{\partial \pi} &= \underbrace{\left(-\frac{x_S^*}{1-\pi}\right)}_{\text{direct effect}} + \underbrace{\left(\frac{\partial x_S^*}{\partial \theta}\frac{\phi}{\delta}\right)}_{\text{static indirect effect}} + \underbrace{\left(\frac{\partial x_S^*}{\partial \theta}\frac{\phi}{\delta}\frac{\omega}{\pi-1} + \frac{x_S^*}{\pi-1}\frac{\partial \omega}{\partial \pi}\right)}_{\text{dynamic indirect effect}} \\
&= \underbrace{\left(-\frac{x_S^*}{1-\pi}\right)}_{\text{direct effect}} + \underbrace{\frac{\partial x_S^*}{\partial \theta}\frac{\phi}{\delta}\left(1-\frac{\omega}{1-\pi}\right) + \frac{x_S^*}{\pi-1}\frac{\partial \omega}{\partial \pi}}_{\text{total indirect effect}}
\end{aligned}$$

While the static indirect effect is always positive, the total indirect effect can be negative. Indeed, it is easy to verify that the total indirect effect is negative if

$$\frac{\partial x_S^*}{\partial \theta}\frac{\phi}{\delta}\left(1-\pi-\omega\right) < x_S^*\frac{\partial \omega}{\partial \pi}. \tag{15}$$

Letting $E_k f(z,k) = [k/f(z,k)]\partial f(z,k)/\partial k$ denote the elasticity of $f(z,k)$ with respect to k, (15) can be rewritten as

$$\frac{E_\pi \omega}{E_\theta x_S^*} > \frac{1-\pi-\omega}{\omega}. \tag{16}$$

Clearly, a necessary condition for (16) to hold is that ω be increasing in π (since the RHS of (16) and $E_\theta x_S^*$ are both positive). When ω is decreasing in π, the total indirect effect has the same sign as the static indirect effect, i.e., it is positive. In this case, as in the static game analyzed in Ballester et al. (2010), the impact of π on aggregate crime is ambiguous. When instead ω is increasing in π, the total indirect effect is negative. In this case, an increase in π unambiguously leads to a decrease in aggregate crime.

The above discussion leads to the following proposition.

Proposition 3 *If* $E_\pi \omega / E_\theta x_S^* > (1-\pi-\omega)/\omega$ *then* $\partial x^*/\partial\pi < 0$.

When the dynamic indirect effect is positive (resp. negative), we have $\partial x^*/\partial\pi >$(resp. $<$)$\partial x_S^*/\partial\pi$. Assume that $\partial x^*/\partial\pi > \partial x_S^*/\partial\pi$. There are two cases in which the sign of one derivative implies the sign of the other. If $\partial x^*/\partial\pi < 0$ then $\partial x_S^*/\partial\pi < 0$; if $\partial x_S^*/\partial\pi > 0$ then $\partial x^*/\partial\pi > 0$. Assume now that $\partial x^*/\partial\pi < \partial x_S^*/\partial\pi$. There are two cases in which the sign of one derivative implies the sign of the other. If $\partial x_S^*/\partial\pi < 0$ then $\partial x^*/\partial\pi < 0$; if $\partial x^*/\partial\pi > 0$ then $\partial x_S^*/\partial\pi > 0$. Interestingly, the qualitative impact of π on aggregate crime in the static and the dynamic model may differ: when $\partial x^*/\partial\pi > \partial x_S^*/\partial\pi$ we can have $\partial x^*/\partial\pi > 0$ together with $\partial x_S^*/\partial\pi < 0$; when, instead, $\partial x^*/\partial\pi < \partial x_S^*/\partial\pi$ we can have $\partial x^*/\partial\pi < 0$ together with $\partial x_S^*/\partial\pi > 0$.

In what follows, we provide two numerical examples illustrating the possible divergence between the static and the dynamic model in terms of impact of π on aggregate crime.

Example 1 (continued). Consider an increase in parameter π from 0.01 to 0.02 within a regular network. Let $n = 5$, $d = 4$, $\delta = 0.1$, $\phi = 0.8$, $\mu = 1$, $r = 0.001$, and $y_0 = 1.8447$. At $t = 0$, aggregate crime is given by

$$x^* = \begin{cases} 1.8447 \text{ for } \pi = 0.01 \\ 1.7994 \text{ for } \pi = 0.02 \end{cases}$$

In the static game, instead,

$$x_S^* = \begin{cases} 8.7148 \text{ for } \pi = 0.01 \\ 9.1418 \text{ for } \pi = 0.02 \end{cases}$$

In this example, an increase in the expected marginal punishment leads to a decrease in aggregate crime in the dynamic game and to an increase in aggregate crime in the static game.

Example 2 (continued). Consider an increase in parameter π from 0.01 to 0.02 within a star network. Let $n = 10$, $\delta = \phi = 0.1$, $\mu = 1$, $r = 0.001$, and $y_0 = 0.8623$. At $t = 0$, aggregate crime is given by

$$x^* = \begin{cases} 0.8623 \text{ for } \pi = 0.01 \\ 0.8629 \text{ for } \pi = 0.02 \end{cases}$$

In the static game, instead,

$$x_S^* = \begin{cases} 9.0152 \text{ for } \pi = 0.01 \\ 8.9402 \text{ for } \pi = 0.02 \end{cases}$$

In this example, an increase in the expected marginal punishment leads to an increase in aggregate crime in the dynamic game and to a decrease in aggregate crime in the static game.

4.2.2 Long-run impact

Denote

$$F = \mu y - x_S^* \left[1 - \frac{\omega(g,\theta)}{1-\pi}\right].$$

By implicit differentiation, we have

$$\frac{d\widehat{y}}{d\pi} = -\frac{\partial F/\partial \pi}{\partial F/\partial y} = \frac{-\frac{x_S^*}{1-\pi}\left(1+\frac{\partial \omega}{\partial \pi}\right) + \frac{\partial x_S^*}{\partial \theta}\frac{\phi}{\delta}\left(1-\frac{\omega}{1-\pi}\right)}{\mu + \frac{x_S^*}{1-\pi}\frac{\partial \omega}{\partial y}}.$$

Since

$$\mu + \frac{x_S^*}{1-\pi}\frac{\partial \omega}{\partial y} < 0$$

is required for the stability of the steady state, it follows that

$$\frac{d\widehat{y}}{d\pi} \overset{s}{=} -\frac{\partial x^*}{\partial \pi}.$$

From Corollary 2, we know that $\widehat{y}$ and $\widehat{x}$ are positively correlated. Hence, we can write the following proposition.

Proposition 4 *Take $y_0 = \widehat{y}(g)$. If $\partial x^*/\partial \pi >$(resp. $<$) 0 then $d\widehat{x}/d\pi <$(resp. $>$) 0.*

Proposition 4 is illustrated in the two examples below.

Example 1 (continued). For a regular network with $n = 5$, $d = 4$, $\delta = 0.1$, $\phi = 0.8$, $\mu = 1$, and $r = 0.001$, the trajectories of aggregate crime are given by

$$x^*(t) = \begin{cases} 1.8447 + (1.1967 y_0 - 2.2076)\exp[-0.1967t] \text{ for } \pi = 0.01 \\ 2.0786 + (1.1937 y_0 - 2.4812)\exp[-0.1937t] \text{ for } \pi = 0.02 \end{cases}$$

Clearly, as $t \to \infty$, aggregate crime converges to 1.8447 with $\pi = 0.01$ and to 2.0786 with $\pi = 0.02$. Therefore, in line with the static game, long-run aggregate crime is higher with $\pi = 0.02$ than with $\pi = 0.01$. However, as previously shown in Example 1, for $t = 0$ and $y_0 = 1.8447$, in contrast with the static game, aggregate crime is lower with $\pi = 0.02$ than with $\pi = 0.01$. This implies that, for the given parameter values, the trajectories of aggregate crime with $\pi = 0.01$ and $\pi = 0.02$ intersect.

Example 2 (continued). For a star network with $n = 10$, $\delta = \phi = 0.1$, $\mu = 1$, and $r = 0.001$, the trajectories of aggregate crime with $\pi = 0.01$ and $\pi = 0.02$ are given by

$$x^*(t) = \begin{cases} 0.8623 + (1.0994 y_0 - 0.9481)\exp[-0.0994t] \text{ for } \pi = 0.01 \\ 0.8567 + (1.0994 y_0 - 0.9419)\exp[-0.0994t] \text{ for } \pi = 0.02 \end{cases}$$

Clearly, as $t \to \infty$, aggregate crime converges to 0.8623 with $\pi = 0.01$ and to 0.8567 with $\pi = 0.02$. It is immediate to check that, for $y_0 = 0.8623$, which was previously considered in Example 2, short run aggregate crime is lower with $\pi = 0.01$ than with $\pi = 0.02$. This implies that, for the given parameter values, the trajectories of aggregate crime intersect.

4.3 Implicit Growth Rate and Voracity Effect

In this subsection, we investigate the possibility that an increase in the implicit growth rate of total wealth in the economy (e.g. a productivity gain) lowers economic growth, i.e., whether a *voracity effect* (see Tornell and Lane, 1999) arises. Formally, a voracity effect exists when $\partial(\dot{y}^*(t)/y^*(t))/\partial\mu < 0$. Note that, in the absence of crime, $\dot{y}^*(t)/y^*(t) = \mu$, therefore a voracity effect never arises. The question addressed in this subsection is new in the network theory literature. In this respect, the focus of this subsection is different from that of the previous subsections, which was on the comparison between the static and the dynamic impact of an increase in the number of criminals or links (or both), or an increase in the marginal expected punishment.

Let $\eta^*(t) = \dot{y}^*(t)/y^*(t)$. Take $y_0 \in (\underline{y}_{(n)}, \overline{y}_{i_{\min}})$. From Corollary 2, we obtain:

$$\dot{y}^*(t) = (y_0 - \widehat{y}) \left(\mu + \frac{b_{\mathbf{A}}(g,\theta)}{\delta[1 + b(g,\theta)]} \right) \exp\left[t \left(\mu + \frac{b_{\mathbf{A}}(g,\theta)}{\delta[1 + b(g,\theta)]} \right) \right].$$

It follows that the rate of growth of $y^*(t)$ can be written as

$$\eta^*(t) = \frac{(y_0 - \widehat{y}) \left(\mu + \frac{b_{\mathbf{A}}(g,\theta)}{\delta[1 + b(g,\theta)]} \right) \exp\left[t \left(\mu + \frac{b_{\mathbf{A}}(g,\theta)}{\delta[1 + b(g,\theta)]} \right) \right]}{\widehat{y} + (y_0 - \widehat{y}) \exp\left[t \left(\mu + \frac{b_{\mathbf{A}}(g,\theta)}{\delta[1 + b(g,\theta)]} \right) \right]}.$$

Differentiating $\eta^*(t)$ with respect to μ and evaluating the derivative at $t = 0$ gives

$$\begin{aligned} \left.\frac{\partial \eta^*(t)}{\partial \mu}\right|_{t=0} &= \delta y_0 [1 + b(g,\theta)] (y_0 - \widehat{y}) \left\{ \delta[1 + b(g,\theta)] + \frac{\partial b_{\mathbf{A}}(g,\theta)}{\partial \mu} \right\} \\ &\quad - \{ \delta\mu[1 + b(g,\theta)] + b_{\mathbf{A}}(g,\theta) \} \frac{\partial \widehat{y}}{\partial \mu}, \end{aligned}$$

which can be simplified to

$$\left.\frac{\partial \eta^*(t)}{\partial \mu}\right|_{t=0, y_0=\widehat{y}} = -\{ \delta\mu[1 + b(g,\theta)] + b_{\mathbf{A}}(g,\theta) \} \frac{\partial \widehat{y}}{\partial \mu}, \tag{17}$$

after setting $y_0 = \widehat{y}$. Since the expression in curly brackets in (17) is negative (for stability of the steady state) we have

$$\left.\frac{\partial \eta^*(t)}{\partial \mu}\right|_{t=0, y_0=\widehat{y}} \overset{s}{=} \frac{\partial \widehat{y}}{\partial \mu}.$$

We can then state the following proposition.

Proposition 5 *Take $y_0 \in (\underline{y}_{(n)}, \overline{y}_{i_{\min}})$. If t is sufficiently close to zero and y_0 is sufficiently close to $\widehat{y}$, $\partial\eta^*(t)/\partial\mu < 0$, i.e., there exists a voracity effect.*

The intuitive explanation for the occurrence of a voracity effect is that the indirect effect of an increase in total crime, which, given the wealth-reducing nature of crime, is negative, outweighs the direct positive effect of an increase in μ. Consequently, an increase in the implicit growth rate of total wealth in the economy depresses economic growth.[20] In the remainder of this subsection, we provide two numerical examples.

Example 1 (continued). We reconsider the same parameter values as those previously considered in Example 1: $n = \delta = 5$, $d = 2$, $\phi = 1$, $\pi = 0.3$, $\mu = 1$ and $r = 0.001$. For these parameter values, the steady state of total wealth in the economy is given by $\widehat{y} = 0.024$. We now increase parameter μ from 1 to 1.1. The steady state of total wealth in the economy becomes $\widehat{y} = 0.022$. Clearly, the rate of growth of total wealth in the economy with $y_0 = 0.024$ and $t = 0$ drops from zero to a negative value.

Example 2 (continued). We reconsider the same parameter values as those previously considered in Example 2: $n = 10$, $\phi = \delta = 1$, $\pi = 0.05$, $\mu = 1$ and $r = 0.001$. For these parameter values, the steady state of total wealth in the economy is given by $\widehat{y} = 0.084$. We now increase parameter μ from 1 to 1.1. The steady state of total wealth in the economy becomes $\widehat{y} = 0.0767$. Clearly, the rate of growth of total wealth in the economy with $y_0 = 0.084$ and $t = 0$ drops from zero to a negative value.

5 Key Player

In this section, building on Ballester et al. (2006, 2010), we study the problem of identifying the optimal target in the population of criminals when the planner's objective is to minimize aggregate crime at each point in time. Let g_{-i} denote the network resulting from removing criminal i from network g, and let $x^*(g_{-i}, t)$ denote the level of aggregate crime associated with network g_{-i} at time $t \in [0, \infty)$. Take y_0 such that all criminals in g and g_{-i} are active and play nondegenerate Markovian strategies for all $i = 1, .., n$ and all $t \in [0, \infty)$.

From Corollary 2, the trajectory of aggregate crime in g_{-i} for $t \in [0, \infty)$ is given by

$$x^*(g_{-i}, t) = x_S^*(g_{-i}) \left[1 - \frac{b_{\mathbf{A}}(g_{-i}, \theta)\, y^*(g_{-i}, t) + b_{\mathbf{B}}(g_{-i}, \theta)}{(1-\pi)\, b(g_{-i}, \theta)}\right],$$

[20] A similar result can be found in a number of related papers (e.g., Tornell and Lane, 1999; Long and Sorger, 2006; Van der Ploeg, 2011). However, to our knowledge, it has never been derived in the context of criminal networks.

where

$$x_S^*\left(g_{-i}\right)=\frac{(1-\pi)\,b\left(g_{-i},\theta\right)}{\delta\left[1+b\left(g_{-i},\theta\right)\right]},$$

and

$$y^*\left(g_{-i},t\right)=\widehat{y}\left(g_{-i}\right)+\left[y_0-\widehat{y}\left(g_{-i}\right)\right]\exp\left[t\left(\mu+\frac{b_{\mathbf{A}}\left(g_{-i},\theta\right)}{\delta\left[1+b\left(g_{-i},\theta\right)\right]}\right)\right],$$

with

$$\widehat{y}\left(g_{-i}\right)=\frac{(1-\pi)\,b\left(g_{-i},\theta\right)-b_{\mathbf{B}}\left(g_{-i},\theta\right)}{\mu\delta\left[1+b\left(g_{-i},\theta\right)\right]+b_{\mathbf{A}}\left(g_{-i},\theta\right)}$$

being the corresponding (locally stable) steady-state level of y.

The planner's problem is to remove the criminal who is associated with the largest drop in aggregate crime at each t. Formally:

$$\begin{aligned}&\max\left\{x^*(g,t)-x^*(g_{-i},t)\,|\,i=1,...,n\right\}=\min\left\{x^*(g_{-i},t)\,|\,i=1,...,n\right\}\\=\;&\min\left\{x_S^*\left(g_{-i}\right)\left[1-\frac{1}{1-\pi}\frac{b_{\mathbf{V}'}^*\left(g_{-i},\theta\right)}{b\left(g_{-i},\theta\right)}\right]\middle|\,i=1,...,n\right\},\ \forall t\in[0,\infty),\end{aligned}$$

with $b_{\mathbf{V}'}^*\left(g_{-i},\theta\right)=b_{\mathbf{A}}\left(g_{-i},\theta\right)y^*\left(g_{-i},t\right)+b_{\mathbf{B}}\left(g_{-i},\theta\right)$. We denote with i^* the solution to the above problem.

In the static game, where $\mathbf{A}=\mathbf{B}=\mathbf{0}$, the planner's problem becomes

$$\min\left\{x_S^*\left(g_{-i}\right)|\,i=1,...,n\right\},$$

which is equivalent to

$$\min\left\{b\left(g_{-i},\theta\right)|\,i=1,...,n\right\},$$

since $x_S^*\left(g_{-i}\right)$ is increasing in $b\left(g_{-i},\theta\right)$. We denote with i_S^* the solution to the static problem. From Ballester et al. (2006, 2010), we know that i_S^* is the criminal with the highest intercentrality of parameter θ in g, defined as

$$c_i\left(g,\theta\right)=b\left(g,\theta\right)-b\left(g_{-i},\theta\right)=\frac{b_i\left(g,\theta\right)^2}{m_{ii}\left(g,\theta\right)},$$

where $m_{ij}\left(g,\theta\right)$ are the coefficients of $\mathbf{M}(g,\theta)=[\mathbf{I}-\theta\mathbf{G}]^{-1}=\sum_{p=0}^{\infty}\theta^p\mathbf{G}^p$ counting the number of walks from i to j with walks of length p being discounted by θ^p. As pointed out in Ballester et al. (2006, 2010), the intercentrality measure $c_i\left(g,\theta\right)$ is equal to the sum of i's Bonacich centrality and i's contribution to every other player's Bonacich centrality. Keeping $b_i\left(g,\theta\right)$ fixed, $c_i\left(g,\theta\right)$ decreases with the proportion of i's Bonacich centrality due to self-loops, $m_{ii}\left(g,\theta\right)/b_i\left(g,\theta\right)$.

In our dynamic game, things are more involved, and maximizing $c_i\left(g,\theta\right)$ (or, equivalently, minimizing $b\left(g_{-i},\theta\right)$) does not necessarily lead to the largest drop in aggregate crime. Keeping $b_{\mathbf{V}'}^*\left(g_{-i},\theta\right)$ fixed, $x^*(g_{-i},t)$ is still increasing in $b\left(g_{-i},\theta\right)$. However, the removal of player i from g is not only captured by $b\left(g_{-i},\theta\right)$, but also by $b_{\mathbf{V}'}^*\left(g_{-i},\theta\right)$, which negatively impacts $x^*(g_{-i},t)$.

Theorem 2 *(i) A necessary (but not sufficient) condition for* $i^* \neq i^*_S$ *is*

$$\frac{b^*_{\mathbf{V}'}\left(g_{-i^*_S}, \theta\right)}{b\left(g_{-i^*_S}, \theta\right)} < \frac{b^*_{\mathbf{V}'}\left(g_{-i}, \theta\right)}{b\left(g_{-i}, \theta\right)},$$

for some $i \neq i^*_S$*. (ii) A sufficient (but not necessary) condition for* $i^* = i^*_S$ *is*

$$b^*_{\mathbf{V}'}\left(g_{-i^*_S}, \theta\right) > b^*_{\mathbf{V}'}\left(g_{-i}, \theta\right),$$

for all $i \neq i^*_S$*.*

Theorem 2 establishes that the key player in the static and the dynamic game may differ, either temporarily or *ad infinitum* (since $b^*_{\mathbf{V}'}$ changes over time). Moreover, in the dynamic game, the key player in the short run and the long run are not necessarily the same. In the remainder of this section, we provide an illustrative example of the possible divergence between the key player in the static and the dynamic game and between the key player in the short and the long run.

Example 3. Consider the network g in Figure 5 (see Ballester et al., 2006, 2010).

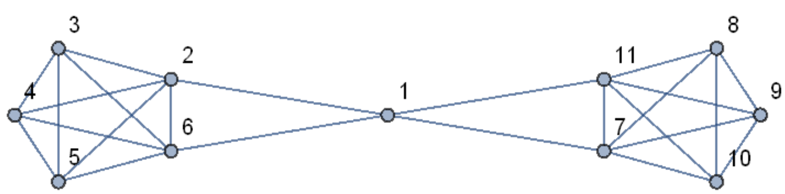

Figure 5: Bridge network with eleven criminals

As can be seen in Figure 5, there are three types of players, Type 1 (Player 1), Type 2 (Players 2,6,7,11) and Type 3 (Players 3,4,5,8,9,10). Let $\phi = \delta = 1$ and $\pi = 0.2$, implying that $\theta = 0.2$.[21] Moreover, let $\mu = 1$ and $r = 0.001$. Table (T1) below gives the Bonacich and the intercentrality measures together with the sum of the coordinates of the vector of weighted Bonacich centralities of parameter θ in g_{-i}, with weights $\mathbf{V}' = \left(A_1 y^*\left(g_{-i}, t\right) + B_1, ..., A_n y^*\left(g_{-i}, t\right) + B_n\right)^T$ (i.e., evaluated along the equilibrium trajectories of y resulting from permanently removing criminal i from g) for

[21] $\theta = 0.2$ is consistent with Ballester et al. (2006, Table I) and Ballester et al. (2010, Table I).

the three types of players.

Player Type	b_i	c_i	$b^*_{\mathbf{V}'}(g_{-i}, \theta)$
1	8.3333	41.6670	$40.4108 - 56.0280y^*(g_{-1}, t)$
2	9.1667	40.3337	$41.4853 - 57.4950y^*(g_{-2}, t)$
3	7.7778	32.6670	$47.6837 - 65.9152y^*(g_{-3}, t)$

(T1)

where

$$y^*(g_{-1}, t) = 0.0817 + (y_0 - 0.0817)\exp[-0.0986t],$$

$$y^*(g_{-2}, t) = 0.0811 + (y_0 - 0.0811)\exp[-0.0986t],$$

and

$$y^*(g_{-3}, t) = 0.0818 + (y_0 - 0.0818)\exp[-0.0986t].$$

The trajectories of aggregate crime associated with g_{-1}, g_{-2}, and g_{-3} are given by

$$x^*(g_{-1}, t) = 0.0817 + (1.0986y_0 - 0.0898)\exp[-0.0986t],$$

$$x^*(g_{-2}, t) = 0.0811 + (1.0986y_0 - 0.0891)\exp[-0.0986t],$$

and

$$x^*(g_{-3}, t) = 0.0818 + (1.0986y_0 - 0.0898)\exp[-0.0986t],$$

respectively. Using the definitions of $\underline{y}_i$ and $\overline{y}_i$, we can compute the admissible interval of y where all criminals in g, g_{-1}, g_{-2}, and g_{-3} are active and play nondegenerate Markovian strategies. This interval is given by $(0.01096, 0.70771)$. For any $y_0 \in (0.01096, 0.70771)$, aggregate crime converges to 0.0817 in g_{-1}, to 0.0811 in g_{-2}, and to 0.0818 in g_{-3} as $t \to \infty$. Clearly, in the long run, the key player is Type 2. This is in contrast with Ballester et al. (2006, 2010), in which the key player is Type 1, the one with the highest intercentrality. It can be checked that for some y_0 the trajectories of aggregate crime intersect. Take, for instance, $y_0 = 0.0815$. Aggregate crime in g_{-1} and g_{-3} is increasing, whereas aggregate crime in g_{-2} is decreasing over time. For $t \in [0, 0.95)$, we have $x^*(g_{-3}, t) < x^*(g_{-1}, t) < x^*(g_{-2}, t)$. This implies that, initially, the key player is Type 3. At $t = 0.95$, we have $x^*(g_{-3}, t) = x^*(g_{-1}, t) = x^*(g_{-2}, t)$. For $t \in (0.95, \infty)$, instead, we have $x^*(g_{-2}, t) < x^*(g_{-1}, t) < x^*(g_{-3}, t)$. Hence, after the initial phase where the key player is Type 3, the key player becomes Type 2. Interestingly, Type 1, who is the key player in the static game, is never the key player in the dynamic game (see Figure 6 below, where $x^*(g_{-1}, t)$, $x^*(g_{-2}, t)$, and

$x^*(g_{-3}, t)$ are indicated with xwithout1, xwithout2, and xwithout 3, respectively).

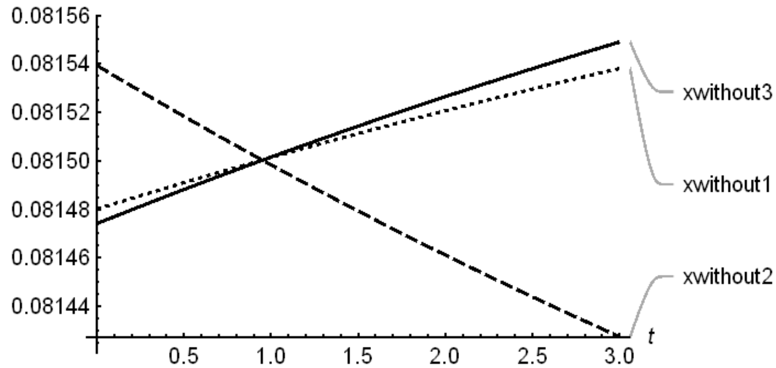

Figure 6: Key player over time

It can be checked that the necessary condition for $i^* \neq i_S^*$ given in Theorem 2 is satisfied. Indeed, for $t \in [0, \infty)$, we have

$$\frac{b^*_{\mathbf{V}'}(g_{-1}, \theta)}{b(g_{-1}, \theta)} < \frac{b^*_{\mathbf{V}'}(g_{-3}, \theta)}{b(g_{-3}, \theta)},$$

where $b^*_{\mathbf{V}'}(g_{-i}, \theta)$ is given in Table (T1) and $b(g_{-i}, \theta) = b(g, \theta) - c_i(g, \theta)$, with $i = 1, 3$.

6 Concluding Remarks

In this paper, we have taken a novel approach, namely, a differential game approach, to the study of criminal networks, with the aim to reconsider some results derived in the static literature, and to answer a new set of questions related to the network structure and its impact on the evolution of crime.

The existing literature on criminal networks abstracts from dynamic intertemporal considerations. Both the benefits and the costs of crime for criminals are assumed to be static, thus precluding the analysis of important topics such as the impact of network structure on the evolution of crime and the relationship between productivity shocks, crime and growth. Besides theoretical interest, these topics have real-world relevance and their understanding is of paramount importance for designing effective policies.

An established result in the static literature is that the vector of Nash equilibrium crime efforts is proportional to the vector of Bonacich centralities. We have challenged this result by showing that such an established proportionality between the Nash equilibrium and the Bonacich centrality does not hold in general in a dynamic setting.

One of the key lessons that can be drawn from the static literature on criminal networks is the existence of a social multiplier effect: networks with a higher number of criminals or links or both are associated with higher levels of aggregate crime. This lesson is valid as long as time does not play any role. Indeed, our dynamic analysis, which, to our knowledge, is novel in the network theory literature, has shown that more criminals or more connected criminals or both may lead to the counterintuitive opposite result, i.e., a decrease in aggregate crime. This holds true not only in the short run, but also at the steady state. Intuitively, the intertemporal cost of committing crime, which our dynamic framework is able to capture, may increase as a result of an increase in either network size or network density or both to such an extent that aggregate crime is reduced. Conditions exist under which forward looking criminals anticipate that an increase in network size or density or both will lead to an increase in crime by all the other criminals, and, therefore, to a decrease in total wealth in the economy. Consequently, given symmetry, each criminal will find it optimal to decrease their own crime efforts (since equilibrium crime efforts are increasing in total wealth), leading to an equilibrium in which aggregate crime is lower.

Another lesson that can be drawn from the static literature on criminal networks is that the impact of an increase in the marginal expected punishment on aggregate crime can be either positive or negative (or nil) depending on the interplay between two effects, namely, a direct and an indirect effect. Our dynamic analysis has shown that, together with these (static) effects, there exists also a dynamic effect, which, in some cases, outweighs the static effects, thus profoundly changing policy recommendations aimed at reducing aggregate crime.

In this paper, we have also highlighted the presence of a voracity effect, occurring when the implicit growth rate of total wealth in the economy is increased and, as a consequence of that, economic growth is reduced. This finding points to the counterintuitive conclusion that, in the presence of crime, positive productivity shocks may have a detrimental effect on economic growth.

Finally, we have reconsidered the problem of identifying the key player in the network, i.e., the player who, if removed, leads to the largest drop in aggregate crime. A well-known result in the static literature is that the key player is the player with the highest intercentrality measure, defined as the difference between the sum of Bonacich centralities in the original network and the sum of Bonacich centralities in the network without the removed player. We have shown that conditions exist under which the key player in the static and the dynamic setting differ: the key player in the dynamic setting is not necessarily the player with the highest intercentrality measure. We have also shown that the key player in the dynamic setting may change over time. The policy implication of this finding is that it might be optimal for a planner seeking to minimize aggregate crime at each point in time to remove (through imprisonment) some criminals up to a certain point, at which they should be reintegrated into society. From this point onwards, other criminals should be removed, either temporarily or ad infinitum, depending on the specific network structure and the parameter values.

Acknowledgments

The authors thank Francis Bloch, Dinko Dimitrov, Stéphane Gonzalez, Antoine Mandel, Julien Salanié, Philippe Solal, Yves Sprumont, Ching-Jen Sun, Xavier Venel, and Georges Zaccour for fruitful discussions. The usual disclaimer applies.

Funding

Agnieszka Rusinowska acknowledges the financial support by the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No 956107, "Economic Policy in Complex Environments (EPOC)".

Appendix A. Proof of Theorem 1

For the proof of the necessary and sufficient condition for $[\mathbf{I}-\theta\mathbf{G}]^{-1}$ to be well-defined and non-negative, see the proof of Theorem 1 in Ballester et al. (2006).

By standard arguments (see Starr and Ho, 1969), MPE strategies must satisfy the following HJB equations ($i = 1, ...n$):

$$rV_i(y) = \max_{x_i \geq 0} \left\{ u_i(x_i, \boldsymbol{\psi}^*_{-i}, g) + V_i'(y) \left[\mu y - x_i - \sum_{j=1, j\neq i}^{n} \psi_j^*(y) \right] \right\},$$

where $V_i'(y) = \partial V_i(y)/\partial y$ denotes the shadow price of total wealth for criminal i. Assuming that $\theta\rho(g) < 1$, maximization of the RHS of the above HJB implies that[22]

$$[\delta\mathbf{I} + \delta\mathbf{U} - \pi\phi\mathbf{G}]\boldsymbol{\psi}^* = (1-\pi)\,\mathbf{1} - \mathbf{V'},$$

where $\mathbf{U}$ is the n-square matrix of ones, or equivalently,

$$\boldsymbol{\psi}^* = [\delta\mathbf{I} + \delta\mathbf{U} - \pi\phi\mathbf{G}]^{-1}\left[\mathbf{1} - \boldsymbol{\pi} - \mathbf{V'}\right],$$

where $\mathbf{V'} = (V_1', ..., V_n')^T$.

Recall that $\theta = \pi\phi/\delta$. Since $\mathbf{U}\boldsymbol{\psi}^* = \psi^*\mathbf{1}$, where $\psi^* = \sum_{i=1}^n \psi_i^*$, then

$$\delta[\mathbf{I} - \theta\mathbf{G}]\boldsymbol{\psi}^* = [1 - \pi - \delta\psi^*]\,\mathbf{1} - \mathbf{V'},$$

and

$$\delta\boldsymbol{\psi}^* = [1 - \pi - \delta\psi^*]\,[\mathbf{I} - \theta\mathbf{G}]^{-1}\mathbf{1} - [\mathbf{I} - \theta\mathbf{G}]^{-1}\mathbf{V'}.$$

Using the definitions of $\mathbf{b}(g,\theta)$ and $\mathbf{b}_{\mathbf{V'}}(g,\theta)$, we obtain

$$\delta\boldsymbol{\psi}^* = [1 - \pi - \delta\psi^*]\,\mathbf{b}(g,\theta) - \mathbf{b}_{\mathbf{V'}}(g,\theta),$$

and since $\psi^* = \mathbf{1}^T\boldsymbol{\psi}^*$ it follows that, at an interior solution,

$$\boldsymbol{\psi}^* = \frac{(1-\pi)\,\mathbf{b}(g,\theta) - \mathbf{b}_{\mathbf{V'}}(g,\theta)}{\delta\,[1 + b(g,\theta)]}.$$

Hence, we have

$$\psi_i^* = \frac{(1-\pi)\,b_i(g,\theta) - b_{\mathbf{V'},i}(g,\theta)}{\delta\,[1 + b(g,\theta)]},$$

which, using $x^*_{S,i}$ given in Theorem 1, can be rewritten as

$$\psi_i^* = x^*_{S,i}\left[1 - \frac{1}{1-\pi}\frac{b_{\mathbf{V'},i}(g,\theta)}{b_i(g,\theta)}\right],$$

where $b_{\mathbf{V'},i}(g,\theta)$ is the $i-th$ coordinate of the vector $\mathbf{b}_{\mathbf{V'}}(g,\theta)$, with $\mathbf{V'} = (V_1', ..., V_n')^T = (A_1 y + B_1, ..., A_n y + B_n)^T$.

[22] This solution represents a maximum since the expression in curly brackets in (7) is concave in x_i.

Appendix B. Proof of Corollary 1

From $\boldsymbol{\psi}^*$ in Theorem 1, for interior solutions, we get

$$\mathbf{1}^T \boldsymbol{\psi}^* = \frac{(1-\pi)\,\mathbf{1}^T \mathbf{b}\,(g,\theta) - \mathbf{1}^T \mathbf{b}_{\mathbf{V}'}\,(g,\theta)}{\delta\,[1+b\,(g,\theta)]},$$

implying that

$$x^* = \frac{(1-\pi)\,b\,(g,\theta) - b_{\mathbf{V}'}\,(g,\theta)}{\delta\,[1+b\,(g,\theta)]} = x_S^* \left[1 - \frac{1}{1-\pi}\frac{b_{\mathbf{V}'}\,(g,\theta)}{b\,(g,\theta)}\right],$$

where

$$x_S^* = \frac{(1-\pi)\,b\,(g,\theta)}{\delta\,[1+b\,(g,\theta)]}.$$

Appendix C. Proof of Corollary 2

The trajectory of y is the solution to the following first-order linear differential equation

$$\dot{y}(t) = \mu y(t) - x_S^* \left[1 - \frac{b_{\mathbf{A}}\,(g,\theta)\,y\,(t) + b_{\mathbf{B}}\,(g,\theta)}{(1-\pi)\,b\,(g,\theta)}\right],$$

with initial condition $y(0) = y_0 \in (\underline{y}_{(n)}, \overline{y}_{i_{\min}})$. Routine calculations lead to $y^*\,(t)$ given in Corollary 2. We have $\lim_{t\to\infty} y^*\,(t) = \widehat{y}$ provided that $\mu + b_{\mathbf{A}}\,(g,\theta)\,/\{\delta[1+b\,(g,\theta)]\} < 0$. The trajectory of x, $x^*\,(t)$, can be computed from x^* given in Corollary 1 by evaluating $b_{\mathbf{V}'}\,(g,\theta) = b_{\mathbf{A}}\,(g,\theta)\,y + b_{\mathbf{B}}\,(g,\theta)$ at $y = y^*\,(t)$.

Appendix D. Proof of Theorem 2

By definition, i_S^* is such that $x_S^*\left(g_{-i_S^*}\right) < x_S^*\,(g_{-i})$ and $b\left(g_{-i_S^*},\theta\right) < b\,(g_{-i},\theta)$, for all $i \neq i_S^*$. Moreover, we have

$$x^*\left(g_{-i_S^*}\right) = x_S^*\left(g_{-i_S^*}\right)\left[1 - \frac{1}{1-\pi}\frac{b_{\mathbf{V}'}^*\left(g_{-i_S^*},\theta\right)}{b\left(g_{-i_S^*},\theta\right)}\right].$$

(i) A necessary (but not sufficient) condition for $x^*\left(g_{-i_S^*}\right) > x^*\,(g_{-i})$ is

$$1 - \frac{1}{1-\pi}\frac{b_{\mathbf{V}'}^*\left(g_{-i_S^*},\theta\right)}{b\left(g_{-i_S^*},\theta\right)} > 1 - \frac{1}{1-\pi}\frac{b_{\mathbf{V}'}^*\,(g_{-i},\theta)}{b\,(g_{-i},\theta)}$$

which simplifies to

$$\frac{b_{\mathbf{V}'}^*\left(g_{-i_S^*},\theta\right)}{b\left(g_{-i_S^*},\theta\right)} < \frac{b_{\mathbf{V}'}^*\,(g_{-i},\theta)}{b\,(g_{-i},\theta)}.$$

(ii) $x^*\left(g_{-i_S^*}\right) < x^*\left(g_{-i}\right)$ if

$$\frac{b^*_{\mathbf{V}'}\left(g_{-i_S^*},\theta\right)}{b\left(g_{-i_S^*},\theta\right)} > \frac{b^*_{\mathbf{V}'}\left(g_{-i},\theta\right)}{b\left(g_{-i},\theta\right)}.$$

A sufficient (but not necessary) condition for the above inequality to hold is

$$b^*_{\mathbf{V}'}\left(g_{-i_S^*},\theta\right) > b^*_{\mathbf{V}'}\left(g_{-i},\theta\right).$$

References

[1] Allum, F, Clough Marinaro, I., Sciarrone, R., 2019, Italian mafias today: Territory, business and politics. Edward Elgar Publishing, Cheltenham UK.

[2] Ballester, C., Calvó-Armengol, A., Zenou, Y., 2006, Who's who in networks. Wanted: the key player, *Econometrica* 74, 1403-1417.

[3] Ballester, C., Calvó-Armengol, A., Zenou, Y., 2010, Delinquent networks, *Journal of the European Economic Association* 8, 34-61.

[4] Başar, T., Olsder, G.J., 1995, *Dynamic Noncooperative Game Theory*, San Diego, Academic Press.

[5] Banerjee, A., Chandrasekhar, A., Duflo, E., Jackson., M., 2013, Diffusion of microfinance, *Science* 341, 1236498

[6] Banerjee, A., Chandrasekhar, A., Duflo, E., Jackson., M., 2019, Using gossips to spread information: Theory and evidence from two randomized controlled trials, *The Review of Economic Studies* 86, 2453-2490.

[7] Bayer, P., Hjalmarsson, R., Pozen, D., 2009, Building criminal capital behind bars: Peer effects in juvenile corrections, *Quarterly Journal of Economics* 124, 105-147.

[8] Becker, G., 1968, Crime and punishment: An economic approach, *Journal of Political Economy* 76, 169-217.

[9] Benchekroun, H., 2003, Unilateral production restrictions in a dynamic duopoly, *Journal of Economic Theory* 111, 214-239.

[10] Benchekroun, H., 2008, Comparative dynamics in a productive asset oligopoly, *Journal of Economic Theory* 138, 237-261.

[11] Benchekroun, H., Gaudet, G., Lohoues, H., 2014, Some effects of asymmetries in a common pool natural resource oligopoly, *Strategic Behavior and the Environment* 4, 213-235.

[12] Bimpikis, K., Ozdaglar, A., Yildiz., E., 2016, Competitive targeted advertising over networks, *Operations Research* 64, 705-720.

[13] Bloch, F., 2016, Targeting and pricing in social networks, In: Bramoullé, Y. Galeotti, A. Rogers, B.W. (eds) *The Oxford Handbook of the Economics of Networks*, Oxford University Press, pages 776-791.

[14] Bloch, F., Querou, N., 2013, Pricing in social networks, *Games and Economic Behavior* 80, 263-281.

[15] Bonacich, P., 1987, Power and centrality: A family of measures, *American Journal of Sociology* 92, 1170-1182.

[16] Calderoni, F., 2012, The structure of drug trafficking mafias: the 'Ndrangheta and cocaine, *Crime Law and Social Change* 58, 321-349.

[17] Calvó-Armengol, A., Zenou, Y., 2004, Social networks and crime decisions. The role of social structure in facilitating delinquent behavior, *International Economic Review* 45, 939-958.

[18] Candogan, O, Bimpikis, K., Ozdaglar., A., 2012, Optimal pricing in networks with externalities, *Operations Research* 60, 883-905.

[19] Colombo, L., Labrecciosa, P., 2015, On the Markovian efficiency of Bertrand and Cournot equilibria, *Journal of Economic Theory* 155, 332-358.

[20] Cortés, D., Friebel, G., Maldonado, D. 2019, Crime and education in a model of information transmission, *Annals of Public Cooperative Economics* 91, 71-93.

[21] Crettez, B., Hayek, N., Zaccour, G., 2020, Non-deceptive counterfeiting and consumer welfare: A differential game approach, In: Ramsey D.M., Renault J. (eds) *Advances in Dynamic Games. Annals of the International Society of Dynamic Games*, vol 17. Birkhäuser, Cham.

[22] Damm, A., Dustmann, C., 2014, Does growing up in a high crime neighborhood affect youth criminal behavior?, *American Economic Review* 104, 1806-1832.

[23] Dawid, H., Feichtinger, G., 1996, Optimal allocation of drug control efforts: A differential game analysis, *Journal of Optimization Theory and Applications* 91, 279-297.

[24] Demange., G., 2017, Optimal targeting strategies in a network under complementarities, *Games and Economic Behavior* 105, 84-103.

[25] Dockner, E.J., Long, N.V., 1993, International pollution control: Cooperative versus non-cooperative strategies, *Journal of Environmental Economics and Management* 24, 13-29.

[26] Dockner, E.J., Sorger, G., 1996, Existence and properties of equilibria for a dynamic game on productive assets, *Journal of Economic Theory* 171, 201-227.

[27] Dockner, E.J, Jørgensen, S., Long, N.V., Sorger, G., 2000, *Differential Games in Economics and Management Science*, Cambridge, Cambridge University Press.

[28] Dubovik, A., Parakhonyak, A., 2014, Drugs, guns, and targeted competition, *Games and Economic Behavior* 87, 497-507.

[29] Faria, J.R., Mixon, F.G.J., Upadhyaya, A., Upadhyaya, K.P., 2019, Gang rivalry and crime: A differential game approach, *Review of Law and Economics* 15, 1-30.

[30] Feichtinger, G., 1983, A differential game solution to a model of competition between a thief and the police, *Management Science* 29, 686-699.

[31] Fershtman, C., Kamien, M., 1987, Dynamic duopolistic competition with sticky prices, *Econometrica* 55, 1151-1164.

[32] Galeotti, A., Golub, B., Goyal, S., 2020, Targeting interventions in networks, *Econometrica* 88, 2445-2471.

[33] Galeotti, A. Goyal., S., 2009, Influencing the influencers: A theory of strategic diffusion, *The RAND Journal of Economics* 40, 509-532.

[34] Glaeser, E., Sacerdote, B., Scheinkman, J.A., 1996, Crime and social interactions, *Quarterly Journal of Economics* 111, 507-548.

[35] Grabisch, M., Mandel, A., Rusinowska, A., Tanimura, E., 2018, Strategic influence in social networks, *Mathematics of Operations Research* 43, 29-50.

[36] Haurie, A., Krawczyk, J.B., Zaccour, G., 2012, *Games and Dynamic Games*, World Scientific.

[37] Haynie, D.L., 2001, Delinquent peers revisited: Does network structure matter?, *American Journal of Sociology* 106, 1013-1057.

[38] Jordan, J.S., 2006, Pillage and property, *Journal of Economic Theory* 131, 26-44.

[39] Jun, B., Vives, X., 2004, Strategic incentives in dynamic duopoly, *Journal of Economic Theory* 116, 249-281.

[40] Katz, L., 1953, A new status index derived from sociometric analysis, *Psychometrika* 18, 39-43.

[41] Kemp, M.C., Long, N.V., 2009, Foreign aid in the presence of corruption: Differential games among donors, *Review of International Economics* 17, 230-243.

[42] Kling, J.R., Ludwig, J., Katz, L.F., 2005, Neighborhood effects on crime for female and male youth: Evidence from a randomized housing voucher experiment, *Quarterly Journal of Economics* 120, 87-130.

[43] Lee, L.-F., Liu, X., Patacchini, E., Zenou, Y., 2021, Who is the key player? A network analysis of juvenile delinquency, *Journal of Business and Economic Statistics* 39, 849-857.

[44] König, M.D., Tessone, C., Zenou, Y., 2014, Nestedness in networks: A theoretical model and some applications, *Theoretical Economics* 9, 695-752.

[45] Lindquist, M.J., Zenou, Y., 2019, Crime and networks: 10 policy lessons, *Oxford Review of Economic Policy* 35, 746-771.

[46] Liu, X., Patacchini, E., Zenou, Y., Lee, L.-F., 2012, Criminal networks: Who is the key player? FEEM Working Paper No. 39.2012, Available at SSRN: https://ssrn.com/abstract=2089267 or http://dx.doi.org/10.2139/ssrn.2089267

[47] Long, N.V., 2010, *A Survey of Dynamic Games in Economics*, World Scientific Publishing, Singapore.

[48] Long, N.V., Sorger, G., 2006, Insecure property rights and growth: The role of appropriation costs, wealth effects, and heterogeneity, *Economic Theory* 28, 513-529.

[49] Ludwig, J., Duncan, G.J., Hirschfield, P., 2001, Urban poverty and juvenile crime: Evidence from a randomized housing-mobility experiment, *Quarterly Journal of Economics* 116, 655-679.

[50] Maskin, E., Tirole, J., 2001, Markov perfect equilibrium I. Observable actions, *Journal of Economic Theory* 100, 191-219.

[51] Ngendakuriyo, F., Zaccour, G., 2013, Fighting corruption: To precommit or not?, *Economics Letters* 120, 149-154.

[52] Ngendakuriyo, F., Zaccour, G., 2017, Should civil society organizations cooperate or compete in fighting a corrupt government?, *Mathematical Social Sciences* 85, 30-36.

[53] Nova, A.J., Feichtinger, G., Leitmann, G., 2010, A differential game related to terrorism: Nash and Stackelberg strategies, *Journal of Optimization Theory and Applications* 144, 533-555.

[54] Patacchini, E., Zenou, Y., 2012, Juvenile delinquency and conformism, *Journal of Law, Economics, and Organization* 28, 1-31.

[55] Piccione, M., Rubinstein, A., 2004, The curse of wealth and power, *Journal of Economic Theory* 117, 119-123.

[56] Piccione, M., Rubinstein, A., 2007, Equilibrium in the jungle, *The Economic Journal* 117, 883-896.

[57] Sah, R., 1991, Social osmosis and patterns of crime, *Journal of Political Economy* 99, 1272-1295.

[58] Sarnecki, J., 2001, *Delinquent Networks: Youth Co-offending in Stockholm*, Cambridge, Cambridge University Press.

[59] Starr, A.W., Ho, Y.C., 1969, Further properties of nonzero-sum differential games, *Journal of Optimization Theory and Applications* 3, 207-219.

[60] Sutherland, E.H., 1947, *Principles of Criminology*, 4th Edition, Chicago, J.B. Lippincott.

[61] Tornell, A., Lane, P.R., 1999, The voracity effects, *American Economic Review* 89, 22-46.

[62] Tsutsui, S., Mino, K., 1990, Nonlinear strategies in dynamic duopolistic competition with sticky prices, *Journal of Economic Theory* 52, 136-161.

[63] Van der Ploeg, F., 2011, Natural resources: Curse or blessing?, *Journal of Economic Literature* 49, 366-420.

[64] Warr, M., 2002, *Companions in Crime: The Social Aspects of Criminal Conduct*, Cambridge, Cambridge University Press.

[65] Wrzaczek, S., Kaplan, E.H., Caulkins, J.P., Seidl, A., Feichtinger, G., 2017, Differential terror queue games, *Dynamic Games and Applications* 7, 578-593.

[66] Zenou, Y., 2016, Key players, In: Bramoullé, Y. Galeotti, A. Rogers, B.W. (eds) *The Oxford Handbook of the Economics of Networks*, Oxford University Press, pages 244-274.