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UMR 8174

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2021.30

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Maison des Sciences Économiques, 106-112 boulevard de L'Hôpital, 75647 Paris Cedex 13
https://centredeconomiesorbonne.cnrs.fr/
ISSN : 1955-611X

Mackey compactness in $B(S)$

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March 19, 2021

Abstract

Let S be a set equipped with the discrete topology and $B(S)$ be the normed space of bounded real mappings on S, endowed with the sup-norm. In this paper, we first prove that $B(S)$ is the nom dual of the space $\mathrm{rca}(S)$ of all regular and bounded Borel measure on S. Then we show that the closed unit ball of $B(S)$ is compact in the Mackey topology $\tau(B(S), \mathrm{rca}(S))$. We also provide a short presentation of an economic application for an intertemporal allocation of resources.

Keywords: Mackey compactness, bounded mappings, regular Borel measure, norm dual
AMS Classification: 46A17, 46A20

1 Introduction

The purpose of this paper is to put into perspective and to provide a synthesis of several results essentially known for the duality pair (ℓ^1, ℓ^∞) (see, e.g. [1]) in the setting of any set S equipped with the discrete topology. Let $B(S)$

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be the normed space of bounded real mappings on S, endowed with the sup-norm. We remark that $B(S)$ is at the same time a space of bounded continuous functions so we can apply the results concerning the spaces of continuous real functions on locally metric spaces and $B(S)$ is also the space of bounded measurable real functions for the Borel sigma-field which is merely the set of all subsets of S. Actually, note that this coincidence of continuity and measurability with respect to the Borel sigma-field happens only with the discrete topology.

So, we can apply the results concerning the spaces L^∞. In particular, L^∞ is the dual of a Banach space. So, in this particular setting, the set $B(S)$ gathers the properties of a space of continuous functions, the properties of a space of bounded measurable functions and the properties of the dual of a Banach space.

We first show that $B(S)$ is isomorphic to the dual of the space of regular and bounded Borel measures on S, $\mathrm{rca}(S)$. Actually, $\mathrm{rca}(S)$ is itself isomorphic to the space of summable functions on S.

Then, we remark that the Mackey topology $\tau(B(S), \mathrm{rca}(S))$ is the strict topology (see [4, 5, 6, 7]) for which a tractable definition of a neighbourhood basis at the origin is available. Thanks to this very particular framework, we show that the closed unit ball of $B(S)$ is Mackey compact using the weak* compactness and the fact that the Mackey topology is locally solid.

This kind of spaces are particularly relevant in economic theory for discrete infinite horizon or for discrete infinite space of events but also in cooperative game theory with an infinite number of players. For example, we can quote the paper of Herves et al. [9] and our previous contribution [2] where the weak-compactness of the unit ball of ℓ^∞ is used. We can also relate our contribution to the paper of Maccheroni and Ruckle [10], which is dedicated to the characterisation of a dual space for a particular space specifically relevant in cooperative game theory.

We provide two examples showing that we cannot expect to get such result neither when S is non finite and equipped with a coarser topology nor for a space like L^∞ or a space of continuous function, even if S is compact.

Finally, we sketch a result for the optimal allocation of ressources over an infinite horizon extensively presented in a companion paper. The proof of the existence of a solution heavily rests upon the Mackey compactness of a subset of ℓ^∞.

2 Preliminaries

This section contains a few definitions and useful properties.

Let S be a nonempty set endowed with the discrete topology. Then S is a locally compact Hausdorf space and $\mathcal{B}(S)$, the Borel sigma-field is merely the set of all subsets of S. Let $B(S)$ be the space of bounded real mapping on S. Note that $B(S)$ is also the space of continuous bounded real mapping on S. We endow the space $B(S)$ with the sup-norm.

It is a simple exercise to see that $\text{rca}(S)$ the linear space of all regular and bounded Borel measures on $\mathcal{B}(S)$ consists of all measures μ such that $\mu = \sum_{n\in\mathbb{N}} \lambda_n \delta_{s_n}$, where (s_n) is a finite or countable subset of S and (λ_n) is an absolutely convergent real series and δ_{s_n} is the point mass (Dirac measure) at s_n.

The space $\text{rca}(S)$ will be endowed with the bounded variation norm $\|\cdot\|$ defined by: for $\mu \in \text{rca}(S)$:

$$\|\mu\| = \sup\{\sum_{i=1}^{n} |\mu(A_i)|; (A_i) \text{ partition of } S\}$$

Notice (see for instance [8] p. 160 - 161), that the bounded variation norm $\| \ \|$ and the sup-norm $\| \ \|_\infty$ ($\|\mu\|_\infty = \sup\{|\mu(A)|; A \in \mathcal{B}(S)\}$) are equivalent.

Following [4], we can define the strict topology β on the space $B(S)$. It is defined by the neighborhood basis at the origin indexed by the function g in $B_0(S)$ where $B_0(S)$ is the subspace of $B(S)$ consisting of all functions which vanish at infinity. For each $g \in B_0(S)$, the neighborhoud $V_g$is defined by :

$$V_g = \{f \in B(S) : \|fg\|_\infty \leq 1\}$$

Note that β is a locally convex-solid topology since 0 admits a basis of solid neighborhoods, i.e. of neighborhoods V such that $f_1 \in V$, $f_2 \in B(S)$ and $|f_2| \leq |f_1|$ implies $f_2 \in V$.

Recall that [5] (see also [6]) proved that $(B(S), \beta)$ is a Mackey space, i.e., β is the finest locally convex topology on $B(S)$, with the same dual as the dual of $B(S)$ endowed with the norm topology, namely $\text{rca}(S)$.

3 $B(S)$ as a dual space

We first state the isomorphism between the dual space of $(\text{rca}(S), \| \ \|)$ and $B(S)$.

Theorem 1 *$(B(S), \| \ \|_\infty)$ is isomorphic to the nom dual of $(rca(S), \| \ \|)$.*

Proof. We consider the bilinear form ψ on $B(S) \times \text{rca}(S)$ defined for all (f, μ) by:

$$\psi(f, \mu) = \sum_{n \in \mathbb{N}} \lambda_n f(s_n)$$

where $\mu = \sum_{n \in \mathbb{N}} \lambda_n \delta_{s_n}$. ψ is well defined since (λ_n) is an absolutely convergent series and f is bounded on S. Furthermore,

$$|\psi(f, \mu)| \leq \|f\|_\infty \left(\sum_{n \in \mathbb{N}} |\lambda_n| \right) = \|f\|_\infty \|\mu\|$$

So $\psi(f, \cdot)$ is a continuous linear form on $\text{rca}(S)$ and $\|\psi(f, \cdot)\|_{\text{rca}(S)'} \leq \|f\|_\infty$. Furthermore $\psi(f, \cdot) = 0_{\text{rca}(S)'}$ if and only if $f = 0$.

Let us now consider φ in the dual space $\text{rca}(S)'$. We define the function $F(\varphi)$ on S by $F(\varphi)(s) = \varphi(\delta_s)$. Then $F(\varphi)$ is bounded since $\|\delta_s\| = 1$, so $|F(\varphi)(s)| \leq \|\varphi\|$, so $\|F(\varphi)\|_\infty \leq \|\varphi\|$. Then for all $\mu \in \text{rca}(S)$, $\mu = \sum_{n \in \mathbb{N}} \lambda_n \delta_{s_n}$, $\varphi(\mu) = \sum_{n \in \mathbb{N}} \lambda_n \mu(\delta_{s_n}) = \sum_{n \in \mathbb{N}} \lambda_n \mu F(\varphi)(s_n) = \psi(F(\varphi), \mu)$.

So, the linear mapping Ψ from $B(S)$ to $\text{rca}(S)'$ defined by $\Psi(f) = \psi(f, \cdot)$ is a linear isomorphism which preserves the norm. Indeed, from the previous inequalities, since $F(\Psi(f)) = f$,

$$\|\Psi(f)\|_{\text{rca}(S)'} \leq \|f\|_\infty \leq \|\Psi(f)\|_{\text{rca}(S)'}$$

□

According to this result, we can define the weak* topology on $B(S)$ $w^*(B(S), \text{rca}(S))$, for which we know that the closed unit ball is compact as a consequence of the Alaoglu-Bourbaki's Theorem. But we can also consider the Mackey topology $\tau(B(S), \text{rca}(S))$, which is the finest locally convex topology for which $\text{rca}(S)$ is the dual space of $B(S)$. Thanks to the fact that $B(S)$ is also the set of bounded continuous functions on S, we know from [5] that the topology $\tau(B(S), \text{rca}(S))$ is the strict topology β introduced by [4].

4 Mackey compactness of the unit ball of $B(S)$

The purpose of this section is to prove that the closed unit ball of $B(S)$ is compact with respect to the finer Mackey topology β.

Theorem 2 *The closed unit ball $\bar{B}(0,1)$ of the normed space $(B(S), \| \|_\infty)$ is compact for the Mackey topology $\beta = \tau(B(S), rca(S))$.*

Proof. Let $(f_\alpha)_{\alpha \in \mathcal{A}}$ be a net in $\bar{B}(0,1)$. We just have to prove that there exists a subnet of (f_α) converging for the Mackey topology τ to some element f of $\bar{B}(0,1)$.

From the classical Alaoglu-Bourbaki's Theorem, $\bar{B}(0,1)$ is weak* compact, therefore, there exists a subnet of $(f_\alpha)_{\alpha \in \mathcal{A}}$, again denoted (f_α), which converges to an element $f \in \bar{B}(0,1)$ for $w^*(B(S), \mathrm{rca}(S))$.

We intend to prove that actually, $f_\alpha \overset{\tau}{\to} f$ or equivalently that $|f_\alpha - f| \overset{\tau}{\to} 0$, which will complete the proof.

Note that f_α and f being elements of $\bar{B}(0,1)$ implies that $|f_\alpha(s) - f(s)| \leq 2$ for all $s \in S$. Taking the Dirac measure $\mu = \delta_s$, one gets $f_\alpha(s) \to f(s)$ for all $s \in S$, from $f_\alpha \overset{w^*}{\to} f$.

Let us first prove that there exists a net $a_\alpha \in B(S)$ such that $a_\alpha \geq |f_\alpha - f|$, $a_\alpha \downarrow 0$, i.e. $a_\alpha(s) \downarrow 0$ for all $s \in S$, and $a_\alpha \overset{w^*}{\to} 0$.

So denoting $g_\alpha = |f_\alpha - f|$, let us show that for a fixed s, there exists a net $a_\alpha(s)$ in $\mathbb{R}$ such that $0 \leq g_\alpha(s) \leq a_\alpha(s)$ for all α and such that $a_\alpha(s) \downarrow 0$.

Let $a_\alpha(s) = \sup\{|f_{\alpha'}(s) - f(s)| \mid \alpha' \succeq \alpha\}$. Clearly $(a_\alpha(s))$ is decreasing and $(a_\alpha(s))$ converges to 0 since so does $(|f_\alpha(s) - f(s)|)$.

It remains here to prove that $a_\alpha \overset{w^*}{\to} 0$. Let $\mu \in \mathrm{rca}(S)$, then $\mu = \sum_{n \in \mathbb{N}} \lambda_n \delta_{s_n}$, (λ_n) is absolutely convergent and $\{s_n \mid n \in /N\} \subset S$. Hence $\mu(a_\alpha) = \sum_{n \in \mathbb{N}} \lambda_n a_\alpha(s_n)$. Let $\varepsilon > 0$. There exists $N \in \mathbb{N}$ such that $\sum_{n=N+1}^{\infty} |\lambda_n| < \varepsilon$, so $\sum_{n=N+1}^{\infty} |\lambda_n||a_\alpha(s_n)| \leq 2\varepsilon$. Hence

$$|\mu(a_\alpha)| \leq \left(\sum_{n=1}^{N} |\lambda_n||a_\alpha(s_n)| \right) + 2\varepsilon$$

Since $a_\alpha(s_n) \to 0$ for all $n = 1, \ldots, N$, there exists $\alpha_0 \in \mathcal{A}$ such that $\alpha \succeq \alpha_0$ implies $|\mu(a_\alpha)| \leq 3\varepsilon$. Consequently, one gets $a_\alpha \overset{w^*}{\to} 0$.

Let us now consider V a solid τ-neighborhood of 0 in $B(S)$. We aim at showing that there exists $\alpha_0 \in \mathcal{A}$ such that $\alpha \succeq \alpha_0$ implies $|f_\alpha - f| \in V$.

$(a_\alpha) \overset{w^*}{\to} 0$ implies that 0 belongs to the weak* closed convex hull of $\{a_\alpha \mid \alpha \in \mathcal{A}\}$. But the dual of $B(S)$ for the two locally convex topologies $\sigma(B(S), \mathrm{rca}(S))$ and $\tau(B(S), \mathrm{rca}(S))$ are identical, namely $\mathrm{rca}(S)$, so they have the same closed convex sets. Thus 0 also belongs to the τ-closed convex hull of $\{a_\alpha \mid \alpha \in \mathcal{A}\}$. Henceforth borrowing a similar argument in the proof of Theorem 8.60 of [1], we remark that there exist n indices $\alpha_1, \ldots, \alpha_n$ in $\mathcal{A}$ and positive constants $\lambda_1, \ldots, \lambda_n$ such that $\sum_{i=1}^n \lambda_i = 1$ and $\sum_{i=1}^n \lambda_i a_{\alpha_i} \in V$.

Now fix some α_0 such that $\alpha_0 \succeq \alpha_i$ for each $i = 1, \ldots, n$. If $\alpha \succeq \alpha_0$, then $0 \leq a_\alpha = \sum_{i=1}^n \lambda_i a_\alpha \leq \sum_{i=1}^n \lambda_i a_{\alpha_i} \in V$. Since V is solid, a_α belongs to V for each $\alpha \succeq \alpha_0$ and $|f_\alpha - f| \leq a_\alpha$ also implies that $|f_\alpha - f| \in V$ for each $\alpha \succeq \alpha_0$, hence $f_\alpha \overset{\tau}{\to} f$. □

Corollary 1 *The closed unit ball $\bar{B}(0,1)$ of ℓ^∞ is compact in the Mackey topology $\tau(\ell^\infty, \ell^1)$.*

Proof. Taking $S = \mathbb{N}$, clearly $\ell^\infty = B(S)$ and $\ell^1 = \mathrm{rca}(S)$. □

Remark. To emphasise the very particular properties of the set $B(S)$ presented above, we show by means of two examples that these properties are no longer true if we endow the set S with a coarser topology than the discrete topology.

Let $S = [0,1]$ with the usual topology on $\mathbb{R}$ and λ be the Lebesgue measure on S. For the first example, we consider $L^\infty(S)$, the space of the essentially bounded measurable functions on S with sup-norm $\| \; \|_\infty$. Then, it is identifiable with the dual of $L^1(S)$ and, so, according to the Alaoglu-Bourbaki's Theorem, the unit ball is compact for the weak* topology $\sigma(L^\infty(S), L^1(S))$. But, the following example shows that the unit ball is not compact for the Mackey topology $\tau(L^\infty(S), L^1(S))$.

Let (f_n) be the sequence of functions in $L^\infty(S)$ defined by:

$f_n(t) = -1$ if $t \in [\frac{2k}{2^n}, \frac{2k+1}{2^n}[$, $k = 0, \ldots, 2^n - 2$;

$f_n(t) = 1$ if $t \in [\frac{2k+1}{2^n}, \frac{2k+2}{2^n}[$, $k = 0, \ldots, 2^n - 2$;

Then (f_n) converges for the w*-topology to 0. Indeed, for all continuous function g on S, one shows that $\lim_{n\to\infty} \int_0^1 f_n(t) g(t) d\lambda = 0$ and the set of continuous functions is dense in $L^1(S)$.

We now prove the sequence (f_n) has no convergent subsequence for the Mackey topology $\tau(L^\infty(S), L^1(S))$. Indeed, since the sequence (f_n) converges for the w*-topology to 0, the only possible cluster point of (f_n) for $\tau(L^\infty(S), L^1(S))$ is 0. Indeed, if a subnet converges for the Mackey topology, it also converges for the weak* topology since the Mackey topology is finer than the weak* topology. So the cluster point is equal to the limit for the weak* topology, that is 0.

Now, we remark that the closed unit ball for the uniform norm $\bar{B}_\infty(0,1)$ is a relatively weakly compact convex circled subset of $L^1(S)$. Indeed, it is bounded since it is included in the unit ball for the L^1 norm and uniformly integrable: for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $f \in \bar{B}_\infty(0,1)$, for all E measurable subset of S, $\mu(E) \leq \varepsilon$ implies $\int_E |f| d\lambda \leq \varepsilon$. So, according the the definition of the Mackey topology (See for example Aliprantis-Border 5.18), the semi-norm φ on $L^\infty(S)$ defined by:

$$\varphi(g) = \sup \left\{ \int_S fg d\mu \mid f \in \bar{B}_\infty(0,1) \right\}$$

is continuous for the topology $\tau(L^\infty(S), L^1(S))$. and we remark that $\varphi(g) = \|g\|_{L^1} = 1$ for all $g \in L^\infty(S)$.

But, now we remark that $\varphi(f_n) = \|f_n\|_{L^1} = 1$ for all n, so, no subnet of $(\varphi(f_n))$ converges to 0, hence (f_n) has no convergent subsequence for the Mackey topology, which is enough to show that the closed unit ball of $L^\infty(S)$ is not $\tau(L^\infty(S), L^1(S))$ compact.

For our second example, we consider the space $\mathcal{C}([0,1])$ of continuous functions on $S = [0,1]$ for the uniform norm. Then, we know that the space $\mathrm{rca}(S)$ of all regular and bounded Borel measures on $\mathcal{B}(S)$ is identifiable with the dual space of $\mathcal{C}([0,1])$. But, the closed unit ball of $\mathcal{C}([0,1])$ is not weakly compact and, thus, not compact for the Mackey topology $\tau(\mathcal{C}([0,1]), \mathrm{rca}(S))$, which is the strict topology, finer than the weak topology.

Indeed, let us consider a sequence (f_ν) in $\mathcal{C}([0,1])$ such that for all $\nu \in \mathbb{N}$, $f_\nu(\frac{1}{n+1})$ is equal to $(-1)^{n+1}$ if $n \leq \nu$ and 0 if $n > \nu$. Let us assume that this sequence converges to $\bar{f} \in \mathcal{C}(([0,1])$ for the weak topology. Let us consider the measure $m \in \mathrm{rca}(S)$ defined by:

$$m = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2^{n+1}} \delta_{\frac{1}{n+1}}$$

where $\delta_{\frac{1}{n+1}}$ is the Dirac measure. Then, $\int_0^1 f_\nu dm = 1 - \frac{1}{2^{\nu+1}}$. So, at the limit, we get $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2^{n+1}} \bar{f}(\frac{1}{n+1}) = \int_0^1 \bar{f} dm = 1$. This implies that $\bar{f}(\frac{1}{n+1}) = (-1)^{n+1}$ for all $n \in \mathbb{N}$, which is contradiction with the fact that $\bar{f}$ is continuous and $\lim_{n\to\infty} \bar{f}(\frac{1}{n+1}) = f(0)$.

5 Economic application

In order to illustrate the potential economic usefulness of our results, we describe briefly below the content of a companion paper [3]. In it, we consider a rare ressource, let us say oil, that for ecological and precautionary motives, one forecasts to be exploited (extracted, consumed) in quantity $x(n)$ for each countable future years $n = 1, 2, \ldots, \infty$ between upper and lower bounds denoted respectively $\beta_n \geq \alpha_n \geq 0$. Normalising at 1 the total ressource available at time 0, $x(0) = 1$. Having $x(n)$ be the consumption of the ressource x during year n, it turns out that in order that the flow $((x(n))_{n=1}^{\infty}$ satisfying the given bounds, trivially, β and α must satisfy $\sum_{n=1}^{\infty} \beta_n \geq 1$ and $\sum_{n=1}^{\infty} \alpha_n \leq 1$. We show that by merely requiring the time discounting to be strictly decreasing, there exists a unique optimal intertemporal allocation $\hat{x}(n)$ with an explicit formula, which turns out to be independent of the exact values of the time discounting.

Our main result. Let $\mu \in \ell^\infty$ with $\mu(n)$ strictly decreasing. From the Mackey compactness in ℓ^∞ of $K = \{x \in \ell^\infty \mid \alpha(n) \leq x(n) \leq \beta(n), \forall n \geq 1, \sum_{n=1}^{\infty} x(n) = 1\}$ and from the continuity of $x \to \sum_{n=1}^{\infty} \mu_n x(n)$ on the compact K, we obtain that $\max\{\sum_{n=1}^{\infty} \mu_n x(n) \mid x \in K\}$ is attained for some $\hat{x}$, where $\hat{x}$ does not depend on the exact values of the μ_n but only of the values of the $\beta(n)$ and $\alpha(n)$ and the fact that $\mu(n)$ is strictly decreasing.

Namely, we got that $\hat{x}(n)$ is given by the following formula:

$$\hat{x}(n) = \max\{1 - \sum_{k\leq n-1} \beta_k, \sum_{k\geq n} \alpha_k\} - \max\{1 - \sum_{k\leq n} \beta_k, \sum_{k\geq n+1} \alpha_k\}$$

where, indeed, $\sum_{k\leq n-1} \beta_k$ is equal to 0 by convention when $n = 1$.

Aknowledgments: This paper has been supported by the Agence nationale de la recherche through the Investissements d?Avenir program (ANR-17-EURE-01).

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