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**Algorithmic aspects of core nonemptiness
and core stability**

Dylan LAPLACE MERMOUD, Michel GRABISCH,
Peter SUDHÖLTER

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Algorithmic aspects of core nonemptiness and core stability

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Abstract

Cooperative game theory is the part of game theory that studies the formation of coalitions and the apportionment of the gains among the players. Its main goal is to propose solution concepts satisfying several properties like anonymity, symmetry, efficiency, etc. Two among the most popular solution concept are the *core*, popularized by Gillies, and the *stable sets* defined by von Neumann and Morgenstern in the well-known *Theory of Games and Economic Behavior*. Bondareva and Shapley found a necessary and sufficient condition for a game to have a nonempty core. The first part of this paper is to translate it into a practical computer code, using Peleg's method. In the second part, we implement into a practical computer code the Grabisch-Sudhölter Theorem that formulates necessary and sufficient conditions for a game to have a stable core.

Keywords— Core, stable sets, balanced collections, core stability, cooperative game.

1 Introduction

In 1944, von Neumann and Morgenstern [9] developed the concept of *stable sets* as a solution for cooperative games. Fifteen years later, Gillies [4] popularized the concept

of the *core*, which is a convex polytope. In the next decade, Bondareva [2] and Shapley [13] formulated independently necessary and sufficient conditions for the nonemptiness of the core, using the notion of *minimal balanced collections*.

We start our investigations of the core by implementing Peleg's inductive method [11] developed in 1965 that generates all minimal balanced collections as a computer program, and then, an algorithm that checks if a game admits a nonempty core or not.

Since the popularization of the core concept, many results about the interactions between the core and the stable sets were established. We know since at least 1953 and Shapley's contribution to an informal conference [12] that the core is included in all the stable sets. Seven years later, Aumann and Peleg [1] found an example where a game has a unique stable set, that strictly includes the core (in particular, the core of a game is not the intersection of all its stable sets). In the late 1960's, Lucas [7, 8], presented games that have no stable set and games that have a continuum of stable sets. On the other hand, Shapley [14] showed in 1971 that the core of a convex game is stable. Owen [10] in 1995 demonstrated that a positive fraction of cooperative games have a stable core. More recently, Shellshear and Sudhölter [15] found a sufficient condition in the general case for a game to have a stable core, that is also necessary for some remarkable subclasses of games: matching games, simple flow games, minimum coloring games.

Recently, Grabisch and Sudhölter [5] formulated a theorem describing a necessary and sufficient condition for a game to have a stable core in a general framework, using several mathematical objects and concepts such as *nested balancedness*, *balanced subsets* which generalized the balanced collections, *exact* and *strictly vital-exact* coalitions, etc.

In order to reformulate the aforementioned theorem as an algorithm, a collection of coalitions has to be found that, among other conditions, determines the core of the game. Subsequently, we describe a procedure for checking whether a subset of a given set is balanced. Finally, we implement the algorithm as a working computer program that allows us to check if an arbitrary game has a stable core or not.

In Sect. 2, we introduce the basic definitions of cooperative game theory needed to construct our algorithms. Sect. 3 presents the practical balancedness checking algorithm. Finally, Sect. 4 is devoted to the translation of the Grabisch-Sudhölter theorem into an algorithm. The specific work on the minimal balanced subsets concern the subsection 4.3.

2 Preliminaries

Consider $N = \{1, \dots, n\}$ a set of n players and let 2^N be its power set, the set consisting of all subsets of N . A nonempty subset of N is called a *coalition*. Consider also $v : 2^N \rightarrow \mathbb{R}$ such that $v(\emptyset) = 0$. To each subset $S \subseteq N$, the *characteristic function* v

associates a real *worth* $v(S)$. The pair (N, v) is a (*cooperative*) *game*.

We call an n -dimensional vector $x = (x_i)_{i \in N} \in \mathbb{R}^N$ an *allocation*, which is interpreted as a proposal of how to distribute the payoffs to the players. The coefficient x_i represents what the player $i \in N$ obtains. We denote by $x(S) = \sum_{i \in S} x_i$ what the coalition $S \subseteq N$ receives. An allocation is called *feasible* (for N) if $x(N) \leq v(N)$. A feasible allocation $x \in \mathbb{R}^N$ is *efficient* if $x(N) = v(N)$. These efficient allocations are called *preimputations*, and we denote their set by $X(v)$. We say that a preimputation $x \in X(v)$ is *individually rational* if $x_i \geq v(\{i\})$ for every player $i \in N$. An individually rational preimputation is called an *imputation*, and their set is denoted by $I(v)$.

We say that a preimputation $x \in X(v)$ *dominates via a coalition* $S \subseteq N$ another preimputation $y \in X(v)$ if $x(S) \leq v(S)$, and $x_i > y_i$ for every $i \in S$, and we write $x \text{ dom}_S y$. If there exists a coalition $S \subseteq N$ such that $x \text{ dom}_S y$, we say that x *dominates* y and $x \text{ dom } y$.

Using this notion, von Neumann and Morgenstern [9] defined a type of solution for cooperative games. A set $U \subseteq I(v)$ is a *stable set* in $I(v)$ if it satisfies

1. *internal stability*: if $x \in U$ is dominated by $y \in I(v)$, then $y \notin U$;
2. *external stability*: for all $y \in I(v) \setminus U$, there exists $x \in U$ such that $x \text{ dom } y$.

However, a stable set may or may not exist (Lucas [7]), and is difficult to identify. Moreover, Deng and Papadimitriou [3] proved that knowing if there exists or not a stable set is undecidable. These difficulties have led to the development of other solution concepts. According to Gillies [4], the *core* is defined as the set of *coalitionally rational* preimputations.

$$C(N, v) = \{x \in X(v) \mid x(S) \geq v(S), \forall S \subseteq N\}.$$

The computation of the core is relatively easy, but can be expensive, due to the large number of inequalities defining it.

Both, stable sets and the core have merits as solution concepts. Indeed, the notions of domination and stability are highly intuitive, "coalitional rationality" is a plausible property, and its easy computability supports the core. As mentioned, the core is contained in each stable set. Hence, if the core is (externally) stable, it must be the unique stable set. Therefore, it is an interesting and important problem to characterize the set of TU games for which both mentioned solutions concepts coincide, i.e., to provide necessary and sufficient conditions for external stability of the core. This is what Grabisch and Sudhölter [5] have achieved.

3 Nonemptiness of the core

3.1 Balanced collections & Bondareva-Shapley theorem

Bondareva [2] in 1963 and Shapley [13] in 1965 independently found a necessary and sufficient condition for nonemptiness of the core, by employing the duality theorem of linear programming, and then popularized the notion of minimal balanced collections.

In the sequel, we will use the notation $\mathbb{1}^T$, where $T \subseteq N$, to denote the n -dimensional vector given by $\mathbb{1}_i^T = 1$ if $i \in T$, and $\mathbb{1}_i^T = 0$ if $i \in N \setminus T$. Consider a collection of coalitions $\mathcal{B} \subseteq 2^{N \setminus \{\emptyset\}}$.

Definition 1. We say that \mathcal{B} is a *balanced collection* if there exists a system of positive weights $(w_S)_{S \in \mathcal{B}}$, called *balancing weights*, such that $\sum_{S \in \mathcal{B}} w_S \mathbb{1}^S = \mathbb{1}^N$.

We say that a balanced collection is *minimal* if it does not contain a proper subcollection that is balanced. We denote by \mathbb{B}_N the set of minimal balanced collections on N . Proofs of the following proposition and theorem are due to Bondareva (1963, [2]) and Shapley (1965, [13]).

Proposition 1. *A balanced collection is minimal if and only if it has a unique system of balancing weights.*

Theorem (Bondareva-Shapley, sharp form). *A game (N, v) admits a nonempty core if and only if, for any minimal balanced collection \mathcal{B} with balancing weights $(w_S)_{S \in \mathcal{B}}$, we have $v(N) \geq \sum_{S \in \mathcal{B}} w_S v(S)$. Moreover, none of the inequalities is redundant, except the one for $\mathcal{B} = \{N\}$.*

Due to the Bondareva-Shapley theorem, we say that a game with a nonempty core is *balanced*.

The main result of Grabisch and Sudhölter [5] is also based on minimal balanced collections.

3.2 Peleg's method

In his paper, Peleg [11] developed in 1965 an inductive method to construct all minimal balanced collections of a ground set $\bar{N} = N \cup \{p\}$ from those defined on the ground set N , p being a new player added. In this subsection, we will present this result in an algorithmic way, and for this we divide the main result of [11], and slightly reformulate the proofs.

Let us consider $\mathcal{C} = \{S_1, \dots, S_k\}$, a collection of k coalitions of N .

First case. We assume that \mathcal{C} is balanced over N , with balancing weights $(w_S)_{S \in \mathcal{C}}$. Let us consider now a subset $I \subseteq \{1, \dots, k\}$ such that $\sum_{i \in I} w_{S_i} = 1$. We denote by $\bar{\mathcal{C}}$ the new collection in which the coalitions $\{S_i\}_{i \in I}$ contain the new player p as additional member and the other coalitions $\{S_j\}_{j \in \{1, \dots, k\} \setminus I}$ are kept unchanged.

Lemma 1. $\bar{\mathcal{C}}$ is a minimal balanced collection over \bar{N} .

Proof. The equalities $\sum_{S \in \mathcal{C}, i \ni S} w_S = 1$ for any $i \in N$ are already satisfied due to the fact that \mathcal{C} is a minimal balanced collection over N . This equality is also satisfied for the new player because of the construction of the collection. Finally, a system of balancing weights for $\bar{\mathcal{C}}$ is also a system of balancing coefficients for \mathcal{C} , hence unique. \square

Second case. We assume that \mathcal{C} is balanced, with balancing weights $(w_S^{\mathcal{C}})_{S \in \mathcal{C}}$. Let us consider now a subset $I \subseteq \{1, \dots, k\}$ such that $\sum_{i \in I} w_{S_i}^{\mathcal{C}} < 1$. We denote by $\bar{\mathcal{C}}$ the new collection in which the coalition $\{S_i\}_{i \in I}$ contain the new player p as additional member, the other coalitions $\{S_j\}_{j \in \{1, \dots, k\} \setminus I}$ are kept unchanged, and in which we add the coalition $\{p\}$ with $w_{\{p\}}^{\bar{\mathcal{C}}} = 1 - \sum_{i \in I} w_{S_i}^{\mathcal{C}}$.

Lemma 2. $\bar{\mathcal{C}}$ is a minimal balanced collection over \bar{N} .

Proof. The equalities $\sum_{S \in \mathcal{C}, i \ni S} w_S^{\bar{\mathcal{C}}} = 1$ for any $i \in N$ are already satisfied due to the fact that \mathcal{C} is a minimal balanced collection over N . By construction, we have that

$$\sum_{\substack{S \in \mathcal{C} \\ p \ni S}} w_S^{\bar{\mathcal{C}}} = \sum_{i \in I} w_{S_i} + w_{\{p\}}^{\bar{\mathcal{C}}} = \sum_{i \in I} w_{S_i} + 1 - \sum_{i \in I} w_{S_i} = 1.$$

Finally, \mathcal{C} being minimal, we cannot remove a coalition different than $\{p\}$ to create another balanced collection. Furthermore, if we remove $\{p\}$, we do not longer have $\sum_{S \in \mathcal{C}, p \ni S} w_S = 1$. So $\bar{\mathcal{C}}$ is also minimal. \square

Third case. We assume that \mathcal{C} is balanced, with balancing weights $(w_S^{\mathcal{C}})_{S \in \mathcal{C}}$. Let us consider now a subset $I \subseteq \{1, \dots, k\}$ such that there exists an index $\delta \in \{1, \dots, k\} \setminus I$ such that $1 > \sum_{i \in I} w_{S_i}^{\mathcal{C}} > 1 - w_{S_\delta}^{\mathcal{C}}$. We denote by $\bar{\mathcal{C}}$ the new collection in which the coalitions $\{S_i\}_{i \in I}$ contain the new player p as additional member and the other coalitions $\{S_j\}_{j \in \{1, \dots, k\} \setminus I}$ are kept unchanged, and in which we add the coalitions

- $S'_\delta = S_\delta \cup \{p\}$ with $w_{S'_\delta}^{\bar{\mathcal{C}}} = 1 - \sum_{i \in I} w_{S_i}^{\mathcal{C}}$ and
- S_δ with $w_{S_\delta}^{\bar{\mathcal{C}}} = w_{S_\delta}^{\mathcal{C}} + \sum_{i \in I} w_{S_i}^{\mathcal{C}} - 1$.

Lemma 3. The collection $\bar{\mathcal{C}}$ is minimally balanced over \bar{N} .

Proof. Consider a player $i \in N$. We have

$$\sum_{\substack{S \in \bar{\mathcal{C}} \\ i \ni S}} w_S^{\bar{\mathcal{C}}} = \sum_{\substack{S \in \mathcal{C} \setminus \{S_\delta\} \\ i \ni S}} w_S^{\mathcal{C}} + 1 - \sum_{i \in I} w_{S_i}^{\mathcal{C}} + w_{S_\delta}^{\mathcal{C}} + \sum_{i \in I} w_{S_i}^{\mathcal{C}} - 1 = w_{S_\delta}^{\mathcal{C}} + \sum_{\substack{S \in \mathcal{C} \setminus \{S_\delta\} \\ i \ni S}} w_S^{\mathcal{C}}$$

that is equal to 1 because \mathcal{C} is balanced over N . For the player p , we have

$$\sum_{\substack{S \in \mathcal{C} \\ p \ni S}} w_S^{\bar{\mathcal{C}}} = \sum_{i \in I} w_{S_i}^{\mathcal{C}} + w_{S'_\delta}^{\bar{\mathcal{C}}} = 1.$$

Finally, \mathcal{C} being minimal, we cannot remove a coalition different than S'_δ . We cannot remove the coalition S'_δ neither, because without it, we don't have $\sum_{S \in \mathcal{C}, p \ni S} w_S = 1$. So $\bar{\mathcal{C}}$ is also minimal. \square

Last case. In this case, we assume that \mathcal{C} is the union of two different minimal balanced collections over N , \mathcal{C}^1 and \mathcal{C}^2 , with balancing weights $(w_S^{\mathcal{C}^1})_{S \in \mathcal{C}^1}$ and $(w_S^{\mathcal{C}^2})_{S \in \mathcal{C}^2}$. We define two systems of balancing weights for \mathcal{C} , by

$$\mu_S^{\mathcal{C}} = \begin{cases} w_S^{\mathcal{C}^1} & \text{if } S \in \mathcal{C}^1; \\ 0 & \text{otherwise.} \end{cases} \quad \nu_S^{\mathcal{C}} = \begin{cases} w_S^{\mathcal{C}^2} & \text{if } S \in \mathcal{C}^2; \\ 0 & \text{otherwise.} \end{cases}$$

Now, consider a subset $I \subseteq \{1, \dots, k\}$ such that $\sum_{i \in I} \nu_{S_i}^{\mathcal{C}} \neq \sum_{i \in I} \mu_{S_i}^{\mathcal{C}}$ and

$$t_I = \frac{1 - \sum_{i \in I} \mu_{S_i}^{\mathcal{C}}}{\sum_{i \in I} \nu_{S_i}^{\mathcal{C}} - \sum_{i \in I} \mu_{S_i}^{\mathcal{C}}} \in]0, 1[;$$

Define the procedure SPECIALEXTEND by

Algorithm 1 Second extension algorithm

Require: The collection \mathcal{C} , the set I , the balancing weights $\nu^{\mathcal{C}}$ and $\mu^{\mathcal{C}}$, and t_I

Ensure: The extension $\bar{\mathcal{C}}$ and its system of balancing weights $(w_S^{\bar{\mathcal{C}}})_{S \in \bar{\mathcal{C}}}$

```

1: procedure SPECIALEXTEND( $\mathcal{C}, I, \nu^{\mathcal{C}}, \mu^{\mathcal{C}}, t_I$ )
2:   for  $i \in \{1, \dots, k\}$  do
3:     if  $i \in I$  then
4:        $\bar{\mathcal{C}} \leftarrow S_i \cup \{p\}$ 
5:     else
6:        $\bar{\mathcal{C}} \leftarrow S_i$ 
7:     if  $S_i \notin \mathcal{C}^2$  then
8:        $w_{S_i}^{\bar{\mathcal{C}}} \leftarrow (1 - t_I) \mu_{S_i}^{\mathcal{C}}$ 
9:     else if  $S_i \notin \mathcal{C}^1$  then
10:       $w_{S_i}^{\bar{\mathcal{C}}} \leftarrow t_I \nu_{S_i}^{\mathcal{C}}$ 
11:    else
12:       $w_{S_i}^{\bar{\mathcal{C}}} \leftarrow (1 - t_I) \mu_{S_i}^{\mathcal{C}} + t_I \nu_{S_i}^{\mathcal{C}}$ 
13:  return  $\bar{\mathcal{C}}$  with its balancing weights  $(w_S^{\bar{\mathcal{C}}})_{S \in \bar{\mathcal{C}}}$ 

```

We call the $(n \times k)$ -matrix the columns of which are the indicators of $\{S_j\}_{j=1, \dots, k}$ *incidence matrix* of \mathcal{C} and denote this matrix by $A^{\mathcal{C}}$, i.e., $A_{ij}^{\mathcal{C}} = 1$ if $i \in S_j$ and $A_{ij}^{\mathcal{C}} = 0$ if $i \in N \setminus S_j$ for $j = 1, \dots, k$.

Lemma 4. *The extension $\bar{\mathcal{C}}$ constructed by SPECIALEXTEND is a minimal balanced collection over \bar{N} if $\text{rank}(A^{\mathcal{C}}) = k - 1$ and if there exists a unique system of balancing weights w^* of \mathcal{C} such that $\sum_{i \in I} w_i^* = 1$.*

Proof. Let us consider a player $i \in N$. We have

$$\begin{aligned} \sum_{\substack{S \in \bar{\mathcal{C}} \\ i \ni S}} w_S^{\bar{\mathcal{C}}} &= \sum_{\substack{S \notin \mathcal{C}^2 \\ i \ni S}} (1 - t_I) \nu_{S_i}^{\mathcal{C}} + \sum_{\substack{S \notin \mathcal{C}^1 \\ i \ni S}} t_I \mu_{S_i}^{\mathcal{C}} + \sum_{\substack{S \in \mathcal{C}^1 \cap \mathcal{C}^2 \\ i \ni S}} (1 - t_I) \nu_{S_i}^{\mathcal{C}} + t_I \mu_{S_i}^{\mathcal{C}} \\ &= (1 - t_I) \sum_{\substack{S \in \mathcal{C}^1 \\ i \ni S}} \nu_{S_i}^{\mathcal{C}} + t_I \sum_{\substack{S \in \mathcal{C}^2 \\ i \ni S}} \mu_{S_i}^{\mathcal{C}} \end{aligned}$$

Because \mathcal{C}^1 and \mathcal{C}^2 are balanced, we have that $\sum_{S \in \bar{\mathcal{C}}, i \ni S} w_S^{\bar{\mathcal{C}}} = 1$. For the new player p , we have that $\sum_{S \in \bar{\mathcal{C}}, p \ni S} w_S^{\bar{\mathcal{C}}} = 1$ by construction. Because w^* is unique by hypothesis, $\bar{\mathcal{C}}$ is minimal over \bar{N} . \square

3.3 Final algorithm

We are now able to construct, from a set of minimal balanced collections over a ground set N , a set of minimal balanced collections over another ground set $\bar{N} = N \cup \{p\}$ in which we add a new player.

Algorithm 2 Construction algorithm

Require: A set of minimal balanced collection \mathbb{B}_N over a ground set N

Ensure: A set of minimal balanced collection $\mathbb{B}_{\bar{N}}$ over a ground set $\bar{N} = N \cup \{p\}$

```

1: procedure ADDINGNEWPLAYER( $\mathbb{B}_N, N$ )
2:   for  $(\mathcal{C}^1, \mathcal{C}^2) \in \mathbb{B}_N \times \mathbb{B}_N$  do
3:     Define  $\nu^{\mathcal{C}}, \mu^{\mathcal{C}}$ 
4:     for  $I \subseteq \{1, \dots, k\}$  do
5:       if  $\sum_{i \in I} \nu_{S_i}^{\mathcal{C}} \neq \sum_{i \in I} \mu_{S_i}^{\mathcal{C}}$  and  $t_I \in ]0, 1[$  then
6:         if  $\text{rank}(A^{\mathcal{C}}) = k - 1$  and  $\exists! w^*$  such that  $\sum_{i \in I} w_i^* = 1$  then
7:            $\mathbb{B}_{\bar{N}} \leftarrow \text{SPECIALEXTEND}(\mathcal{C}^1 \cup \mathcal{C}^2, I, \nu^{\mathcal{C}}, \mu^{\mathcal{C}}, t_I)$ 
8:   for  $\mathcal{C} \in \mathbb{B}_N$  do
9:     for  $I \subseteq \{1, \dots, k\}$  do
10:      if  $\sum_{i \in I} w_{S_i} = 1$  then
11:         $\bar{\mathcal{C}} \leftarrow$  empty collection
12:        for  $i \in \{1, \dots, k\}$  do
13:          if  $i \in I$  then
14:             $\bar{\mathcal{C}} \leftarrow S_i \cup \{p\}$  with weight  $w_{S_i}^{\mathcal{C}}$ 
15:          else
16:             $\bar{\mathcal{C}} \leftarrow S_i$  with same weight
17:         $\mathbb{B}_{\bar{N}} \leftarrow \bar{\mathcal{C}}$ 
18:      else if  $\sum_{i \in I} w_{S_i} < 1$  then

```

```

19:       $\bar{\mathcal{C}} \leftarrow$  empty collection
20:      for  $i \in \{1, \dots, k\}$  do
21:          if  $i \in I$  then
22:               $\bar{\mathcal{C}} \leftarrow S_i \cup \{p\}$  with weight  $w_{S_i}^{\mathcal{C}}$ 
23:          else
24:               $\bar{\mathcal{C}} \leftarrow S_i$  with same weight
25:       $\bar{\mathcal{C}} \leftarrow \{p\}$  with weight  $1 - \sum_{i \in I} w_i$ 
26:       $\mathbb{B}_{\bar{\mathcal{C}}} \leftarrow \bar{\mathcal{C}}$ 
27:      for  $\delta \in \{1, \dots, k\} \setminus I$  do
28:          if  $\sum_{i \in I} w_{S_i} > 1 - w_{S_\delta}$  then
29:               $\bar{\mathcal{C}} \leftarrow$  empty collection
30:              for  $i \in \{1, \dots, k\}$  and  $i \neq \delta$  do
31:                  if  $i \in I$  then
32:                       $\bar{\mathcal{C}} \leftarrow S_i \cup \{p\}$  with weight  $w_{S_i}^{\mathcal{C}}$ 
33:                  else
34:                       $\bar{\mathcal{C}} \leftarrow S_i$  with same weight
35:               $\bar{\mathcal{C}} \leftarrow S_\delta \cup \{p\}$  with weight  $1 - \sum_{i \in I} w_i^{\mathcal{C}}$ 
36:               $\bar{\mathcal{C}} \leftarrow S_\delta$  with weight  $w_{S_\delta} + \sum_{i \in I} w_{S_i} - 1$ 
37:               $\mathbb{B}_{\bar{N}} \leftarrow \bar{\mathcal{C}}$ 
38:      return  $\mathbb{B}_{\bar{N}}$ 

```

Theorem. *The procedure ADDINGNEWPLAYER, which takes as an input the set of all minimal balanced collections over a ground set N , generates all the minimal balanced collections over the ground set \bar{N} .*

This result is an immediate application of Peleg's theorem about the minimal balanced collections, and the four previous lemmas. With this procedure used recursively, we are able to generate all the minimal balanced collections for a fixed ground set N starting from $\{1, 2\}$. After that, we will be able to check the nonemptiness of the core.

Algorithm 3 Balancedness checking algorithm

Require: A game (N, v)

Ensure: An answer to the question: is the game balanced?

```

1: procedure BALANCEDNESSCHECK( $N, v$ )
2:    $n \leftarrow \text{card}(N)$ 
3:    $\mathbb{B}_N \leftarrow \{\mathcal{C}^1 = \{\{1\}, \{2\}\}, \mathcal{C}^2 = \{\{1, 2\}\}\}$ 
4:   for  $\iota \in \{3, \dots, n\}$  do
5:        $\mathbb{B}_N \leftarrow \text{ADDINGNEWPLAYER}(\mathbb{B}_N, N)$ 
6:   for  $\mathcal{B} \in \mathbb{B}_N$  with weight  $w^{\mathcal{B}}$  do
7:       if  $\sum_{S \in \mathcal{B}} w_S^{\mathcal{B}} v(S) > v(N)$  then
8:           return False
9:   return True

```

3.4 Example

Consider $N = \{1, 2, 3, 4\}$ and $\bar{N} = \{1, 2, 3, 4, 5\}$. Let $S_1 = \{1, 2\}, S_2 = \{1, 3\}, S_3 = \{1, 4\}, S_4 = \{2, 3, 4\}$. Then $\mathcal{C} = \{S_1, \dots, S_4\}$ is a minimal balanced collection with the following system of balancing weights $w = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{2}{3}\right)$.

First case. We remark that the set $I = \{1, 4\}$ satisfies the equation $\sum_{i \in I} w_{S_i} = 1$. Therefore, we can construct a minimal balanced collections over \bar{N} .

$$\bar{\mathcal{C}} = \{\{1, 2, 5\}, \{1, 3\}, \{1, 4\}, \{2, 3, 4, 5\}\}, \text{ with } \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{2}{3}\right).$$

Second case. Consider now the set $I = \{4\}$. We have $\sum_{i \in I} w_{S_i} = \frac{2}{3} < 1$. Therefore

$$\bar{\mathcal{C}} = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3, 4, 5\}, \{5\}\}, \text{ with } \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{1}{3}\right)$$

is a minimal balanced collection over \bar{N} .

Third case. Consider $I = \{1, 2\}$ and $\delta = 4$. We have that $\sum_{i \in I} w_{S_i} = w_{S_1} + w_{S_2} = \frac{2}{3}$ and $1 - w_{S_\delta} = \frac{1}{3}$. Therefore, $1 > \sum_{i \in I} w_{S_i} > 1 - w_{S_\delta}$ and we can construct a collection:

$$\bar{\mathcal{C}} = \{\{1, 2, 5\}, \{1, 3, 5\}, \{1, 4\}, \{2, 3, 4\}, \{2, 3, 4, 5\}\} \text{ with } \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right).$$

Last case. For the last case, we will consider another environment. Let us consider a ground set $N = \{1, 2\}$, and the only two minimal balanced collections over N : $\mathcal{C}^1 = \{\{1\}, \{2\}\}$ and $\mathcal{C}^2 = \{\{1, 2\}\}$. Consider now the union $\mathcal{C} = \{\{1\}, \{2\}, \{1, 2\}\}$. We have two systems of balancing weights

$$\nu^{\mathcal{C}} = (1, 1, 0) \text{ and } \mu^{\mathcal{C}} = (0, 0, 1).$$

By studying the adjacency matrix, we remark that

$$\text{rank}(A^{\mathcal{C}}) = \text{rank} \left[\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \right] = 2 = k - 1.$$

Now, consider the set $I = \{1, 2\}$. We have that $\sum_{i \in I} \nu_{S_i}^{\mathcal{C}} = 2$, $\sum_{i \in I} \mu_{S_i}^{\mathcal{C}} = 0$, and

$$t_I = \frac{1 - \sum_{i \in I} \mu_{S_i}^{\mathcal{C}}}{\sum_{i \in I} \nu_{S_i}^{\mathcal{C}} - \sum_{i \in I} \mu_{S_i}^{\mathcal{C}}} = \frac{1}{2},$$

that is indeed strictly between 0 and 1. Finally, we can construct the collection:

$$\bar{\mathcal{C}} = \{\{1, 3\}, \{2, 3\}, \{1, 2\}\} \text{ with } \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right).$$

4 Stability of the core

There already exist several sufficient conditions for core stability in the general case (extendability [6]), or necessary and sufficient conditions for restricted classes of games (matching games, simple flow games and minimum coloring games [15]). Thanks to Grabisch and Sudhölter, we have a necessary and sufficient condition in the general case [5]. Their main result is based on a new notion of balancedness, that we can understand as a double balancedness condition, or a nested balancedness condition.

First, we need a specific collection $\mathcal{F}(v)$, or simply \mathcal{F} , in which we take nonempty subcollections \mathcal{S} . For each subcollection \mathcal{S} , we select a set of families of *minimal balanced subsets* that have specific properties. Once we have all these objects, we check a balancedness property for all these families.

4.1 The collection \mathcal{F}

Consider a coalition S of N .

Definition 2. We say that S is *exact* (at v) if there exists a core element $y \in C(N, v)$ such that $y(S) = v(S)$. We denote the set of exact coalitions by $\mathcal{E}(v)$.

Definition 3. We say that S is *strictly vital-exact* if there exists a core element $x \in C(N, v)$ such that $x(S) = v(S)$ and $x(T) > v(T)$ for all $T \in 2^S \setminus \{\emptyset, S\}$. We denote by $\mathcal{VE}(v)$ the set of strictly vital-exact coalitions.

We remark that $\mathcal{VE}(v) \subseteq \mathcal{E}(v)$. We impose the following properties for \mathcal{F} .

$$\mathcal{F} \subseteq \mathcal{E}(v), \tag{1}$$

$$C(N, v) = \{x \in X(v) \mid x(S) \geq v(S), \forall S \in \mathcal{F}\}, \tag{2}$$

$$\mathcal{F} \supseteq \mathcal{VE}(v). \tag{3}$$

We can take any \mathcal{F} that satisfies these three conditions to apply the main result of Grabisch and Sudhölter. For instance, we could take $\mathcal{F} = \mathcal{E}(v)$. Indeed, (1) and (3) are satisfied and (2) can be shown in a straightforward manner. However, this is the largest possible collection, and because we make calculations for every subcollection of it, we want it to be of smallest cardinality.

Proposition 2 (Grabisch-Sudhölter). *Let (N, v) be a balanced game. The core is stable only if*

$$C(N, v) = \{x \in X(v) \mid x(S) \geq v(S), \forall S \in \mathcal{VE}(v)\}.$$

Proof. Consider $x \in X(v) \setminus C(N, v)$. Because the core is stable, we can take a core element $y \in C(N, v)$ such that $y \text{ dom } x$. Choose a minimal (w.r.t inclusion) coalition S such that $y \text{ dom}_S x$. Then, $y(T) > v(T)$ for all $T \in 2^S \setminus \{\emptyset, S\}$. Therefore, S is strictly vital-exact and because $y \text{ dom}_S x$, we have that $x(S) < v(S)$. \square

Therefore, for the algorithm, we will only consider games for which their strictly

vital-exact coalitions determine the core. Indeed, if it is not the case the game does not have a stable core, and it is the smallest collection \mathcal{F} possible (smaller collections violate (3) by definition).

4.2 Association, admissibility and outvoting

Consider a collection $\mathcal{F} \subseteq 2^N \setminus \{N, \emptyset\}$, a coalition $S \in \mathcal{F}$ and a minimal balanced collection $\mathcal{B} \subseteq 2^N \setminus \{N, \emptyset\}$.

Definition 4. We say that \mathcal{B} is *associated to* S if there exists a player $i \in S$ such that $\{i\} \in \mathcal{B}$ and

$$\mathcal{B} \subseteq \{\{j\} \mid j \in S\} \cup \{N \setminus S\} \cup (\mathcal{F} \setminus 2^S).$$

We denote by \mathbb{B}_S the set of minimal balanced collections over N . Here is a procedure that check if a minimal balanced collection \mathcal{B} is associated to a coalition S .

```

1: procedure CHECKASSOCIATION( $\mathcal{B}, S$ )
2:    $bool \leftarrow$  False
3:   for  $i \in S$  do
4:     if  $\{i\} \in \mathcal{B}$  then
5:        $bool \leftarrow$  True
6:   for  $B \in \mathcal{B}$  do
7:     if  $B \neq N \setminus S$  and  $B \notin \mathcal{F} \setminus 2^S$  and  $B \not\subseteq \{\{i\} \mid i \in S\}$  then
8:       return False
9:   return  $bool$ 

```

Example. Consider $N = \{1, 2, 3, 4\}$, $\mathcal{F} = 2^N \setminus \{N, \emptyset\}$ and $S = \{1, 2\}$. Therefore, the minimal balanced collection $\mathcal{B} = \{\{1\}, \{2\}, \{3, 4\}\}$ is included in \mathbb{B}_S . Indeed, coalitions $\{1\}$ and $\{2\}$ are singletons of S , and $\{3, 4\}$ is the complement of S . Moreover, consider now $T = \{1, 2, 3\}$. \mathcal{B} is also included in \mathbb{B}_T . Coalitions $\{1\}$ and $\{2\}$ are still singletons of T , and $\{3, 4\} \notin 2^T$, so $\{3, 4\} \in \mathcal{F} \setminus 2^T$.

We now define a sub-relation of the domination relation that is transitive.

Definition 5. Consider two preimputations x and $y \in X(v)$ and a coalition $P \in \mathcal{F}$. We say that x *outvotes* y via P , written $x \succ_P y$, if $x \text{ dom}_P y$ (i.e. $x_i > y_i$ for every $i \in P$ and $x(P) \leq v(P)$), and $x(S) \geq v(S)$ for all $S \in \mathcal{F} \setminus 2^P$. Also, we say that x outvotes y ($x \succ y$) if there exists $P \in \mathcal{F}$ such that $x \succ_P y$.

We denote by $M(v) = \{x \in X(v) \mid y \not\succeq x, \forall y \in X(v)\}$ the set of preimputations that are maximal w.r.t. outvoting.

Consider a coalition S and a minimal balanced collection \mathcal{B}_S associated to it. We denote by \mathcal{B}_S^* the collection $\mathcal{B}_S^* = \mathcal{B}_S \setminus \{\{i\} \mid i \in S\}$ and define a new characteristic function v^S by $v^S(K) = v(K)$ if $K \neq N \setminus S$ and $v^S(N \setminus S) = v(N) - v(S)$. Thanks to the notion of association and outvoting, the following important result follows.

Theorem (Grabisch-Sudhölter, 2020). *Let (N, v) be a balanced game, $y \in X(v)$, and $S \in \mathcal{F}$. Then y is outvoted by some preimputation via S if and only if*

$$\forall \mathcal{B} \in \mathbb{B}_S, \quad \sum_{\substack{i \in S \\ \{i\} \in \mathcal{B}}} w_{\{i\}}^{\mathcal{B}} y_i + \sum_{T \in \mathcal{B}^*} w_T^{\mathcal{B}} v^S(T) < v(N). \quad (4)$$

For each collection \mathcal{S} , we define its associated *region* $X_{\mathcal{S}}(v)$ by

$$X_{\mathcal{S}}(v) = \{x \in X(v) \mid x(S) < v(S) \text{ if and only if } S \in \mathcal{S}\}.$$

We say that \mathcal{S} is *feasible* if $X_{\mathcal{S}}(v) \neq \emptyset$. From the foregoing theorem the following lemma can be deduced.

Lemma 5 (Grabisch-Sudhölter, 2020). *The balanced game (N, v) has a stable core if and only if for each feasible collection \mathcal{S} for v and for each $y \in X_{\mathcal{S}}(v)$, there exists $S \in \mathcal{S}$ such that, for all $\mathcal{B} \in \mathbb{B}_S$, (4) holds.*

We now define admissibility for a nonempty collection $\mathcal{S} \subseteq \mathcal{F}$.

Definition 6. Consider a coalition $S \in \mathcal{S}$ and a minimal balanced collection $\mathcal{B}_S \in \mathbb{B}_S$ associated to S . We say that \mathcal{B}_S is *admissible* for \mathcal{S} if it satisfies one of the following properties.

- $\mathcal{B}_S^* \cap \mathcal{S} \neq \emptyset$;
- $\mathcal{B}_S^* \cap \{N \setminus S \mid S \in \mathcal{S}\} = \emptyset$.

We denote by $\mathbb{B}_{\mathcal{S}}^{\mathcal{S}}$ the set of minimal balanced collections associated to S and admissible for \mathcal{S} .

For each nonempty collection $\mathcal{S} \subseteq \mathcal{F}$, we denote

$$\mathbb{C}(\mathcal{S}) = \{(\mathcal{B}_S)_{S \in \mathcal{S}} \mid \mathcal{B}_S \in \mathbb{B}_S^{\mathcal{S}}, \forall S \in \mathcal{S}\} = \prod_{S \in \mathcal{S}} \mathbb{B}_S^{\mathcal{S}}.$$

The following procedure checks if a minimal balanced collection \mathcal{B} is associated to S and admissible for \mathcal{S} .

```

1: procedure CHECKADMISSIBILITY( $\mathcal{B}_S, \mathcal{S}, S$ )
2:   if not CHECKASSOCIATION( $\mathcal{B}_S, S$ ) then
3:     return False
4:    $\mathcal{B}^* \leftarrow \mathcal{B} \setminus \{\{i\} \mid i \in S\}$ 
5:   if  $\mathcal{B}_S^* \cap \mathcal{S} = \emptyset$  and  $\mathcal{B}_S^* \cap \{N \setminus R \mid R \in \mathcal{S}\} \neq \emptyset$  then
6:     return False
7:   return True

```

Thanks to the previous procedure, an algorithm that constructs $\mathbb{C}(\mathcal{S})$ can be built.

```

1: procedure GENERATEADMISSIBLE( $\mathcal{S}, \mathbb{B}_N$ )
2:    $(\mathcal{E}_S)_{S \in \mathcal{S}} \leftarrow$  collection of empty lists
3:   for  $S \in \mathcal{S}$  do
4:     for  $\mathcal{B} \in \mathbb{B}_N$  do
5:       if CHECKADMISSIBILITY( $\mathcal{B}, \mathcal{S}, S$ ) then
6:          $\mathcal{E}_S \leftarrow \mathcal{B}$ 
7:   return  $\times_{S \in \mathcal{S}} \mathcal{E}_S$ 

```

Example. Consider $N = \{1, 2, 3, 4\}$, $\mathcal{F} \subseteq 2^N$, and $S = \{1, 4\}$. Therefore, the collection $\mathcal{B}_S = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{4\}\}$ is associated to S , and $\mathcal{B}_S^* = \mathcal{B}_S \setminus \{4\}$. Consider $\mathcal{S} = \{\{2, 3\}, \{1, 4\}\}$. The first condition on the definition is satisfied: $\mathcal{B}_S^* \cap \mathcal{S} = \{\{2, 3\}\} \neq \emptyset$, therefore \mathcal{B}_S is admissible for \mathcal{S} .

The concept of admissibility allows us to reduce the algorithmic complexity of the core stability checking, thanks to the following result. Before, recall that $M(v) = \{x \in X(v) \mid y \not\prec x, \forall y \in X(v)\}$.

Corollary 1. *Let (N, v) be a balanced game and consider a nonempty collection $\mathcal{S} \subseteq \mathcal{F}$. Then $M(v) \cap X_{\mathcal{S}}(v) \neq \emptyset$ if and only if there exists a system of balanced collection $(\mathcal{B}_S)_{S \in \mathcal{S}} \in \mathbb{C}(\mathcal{S})$ and $x \in X(v)$ such that*

$$\forall S \in \mathcal{S}, \quad \sum_{\substack{i \in S \\ \{i\} \in \mathcal{B}_S}} w_{\{i\}}^{\mathcal{B}_S} x_i + \sum_{T \in \mathcal{B}_S^*} w_T^{\mathcal{B}_S} v^S(T) \geq v(N).$$

4.3 Minimal balanced subsets

In this section we expand the notion of balancedness, defined for collections of coalitions till now, to finite subsets of the positive orthant of the Euclidean space.

Definition 7. Let $Z \subseteq \mathbb{R}_+^N \setminus \{0\}$ be a finite set. We say that $Z' \subseteq Z$ is *balanced* (in Z) if there exists a system $(w_z)_{z \in Z'}$ of positive weights (called *balancing weights*) such that $\sum_{z \in Z'} w_z z = \mathbb{1}^N$.

We say that a balanced subset is *minimal* if it does not contain a proper subset that is balanced.

Lemma 6. *A balanced subset is minimal if and only if it has a unique system of balancing weights.*

Proof. Consider a balanced subset $Z' \subseteq Z$. **Necessity:** Suppose that Z' is not minimal. Then there exists $Z^* \subsetneq Z'$ that is balanced with a system of balancing weights $(w_z^*)_{z \in Z^*}$. Then Z' has infinitely many systems of balancing weights $(w_z^\alpha)_{z \in Z'}$, defined by

$$w_z^\alpha = \begin{cases} \alpha w_z + (1 - \alpha) w_z^*, & \text{if } z \in Z^*, \\ \alpha w_z, & \text{if } z \in Z' \setminus Z^*. \end{cases}$$

with $0 < \alpha < 1$.

Sufficiency: Suppose that Z' has two different systems of balancing weights $(w_z)_{z \in Z'}$ and $(w'_z)_{z \in Z'}$. Then there exists $z \in Z'$ such that $w'_z > w_z$, and we put

$$\tau = \min \left\{ \frac{w_z}{w'_z - w_z} : w'_z > w_z \right\}.$$

We define the system of balancing weights $(\tilde{w}_z)_{z \in Z'}$ by

$$\tilde{w}_z = (1 + \tau)w_z - \tau w'_z \quad \text{for } z \in Z'.$$

Then $Z^* = \{z \in Z' \mid \tilde{w}_z > 0\}$ is a proper subset of Z' that is balanced with system of balancing weights $(\tilde{w}_z)_{z \in Z'}$. \square

Consider the convex polytope

$$F(Z) = \left\{ w \in \mathbb{R}_+^Z : \sum_{z \in Z} w_z z = \mathbb{1}^N, w_z \geq 0, \forall z \in Z \right\}.$$

Lemma 7. *Let $w \in F(Z)$ and consider $Z' = \{z \in Z \mid w_z > 0\}$. Then w is an extremal point of $F(Z)$ if and only if Z' is a minimal balanced subset.*

Proof. Necessity: If Z' is not minimal, then there exists $Z^* \subsetneq Z'$ that is balanced, with a weight vector $(w_z^*)_{z \in Z^*}$. We set

$$\gamma_z = (1 - t)w_z + tw_z^*, \quad \gamma'_z = (1 + t)w_z - tw_z^*$$

for all $z \in Z'$, letting $w_z^* = 0$ if $z \in Z^*$, with $t > 0$ small enough to ensure $\gamma_z, \gamma'_z > 0$ for all $z \in Z'$. Then $(\gamma_z)_{z \in Z'}$, $(\gamma'_z)_{z \in Z'}$ are systems of balancing weights for Z' that are different, because $\gamma_z < \gamma'_z$ for all $z \in Z' \setminus Z^*$. Moreover, $w_z = \frac{1}{2}(\gamma_z + \gamma'_z)$ for all $z \in Z'$, hence w is not an extremal point.

Sufficiency: Suppose that Z' is a minimal balanced subset. If w is not an extremal point, there exist distinct $\gamma, \gamma' \in F(Z)$ such that

$$w_z = \frac{\gamma_z + \gamma'_z}{2}, \quad z \in Z'.$$

Since γ, γ' are nonnegative, $w_z = 0$ implies that $\gamma_z = \gamma'_z = 0$. Therefore γ, γ' define distinct systems of balancing weights for sets W, W' , subsets of Z' , which by lemma 6 contradicts the minimality of Z' . \square

Corollary 2. *A minimal balanced subset contains at most n elements.*

Proof. Consider a minimal balanced subset $Z' \subseteq Z$, with balancing weights w . By lemma 7, w is an extremal point of $F(Z)$. Therefore, w is the intersection of at least $\text{card}(Z)$ facets of $F(Z)$, hence the (unique) solution of a system of at least $\text{card}(Z)$ equalities, among the system $\{\sum_{z \in Z} w_z z = \mathbb{1}^N, w_z \geq 0, \forall z \in Z\}$. Since the number of equalities in this system is $n + \text{card}(Z) - \text{card}(Z')$, the above condition yields $\text{card}(Z') \leq n$. \square

In Sect. 3 we constructed the minimal balanced collections on a ground set N by induction on the cardinality of N , finding a proper way to add the new player to the existing coalitions. The idea here for the new algorithm is completely different because we do not only handle indicator functions, but also ordinary real vectors. To this end, we solve a linear system to check if we find a unique system of balancing weight, thanks to Lemma 6. In order to improve efficiency, we can add some conditions on the subset that we are considering: for instance, the subset Z' must cover N .

In order to implement our procedure for checking minimal balancedness, we define the *weighted incidence matrix* $A^{Z'}$ of a subset Z' . Consider $Z' = \{z_1, \dots, z_k\}$. According to this notation, we define the weighted adjacency $(n \times k)$ -matrix $A^{Z'}$ the columns of which are the vectors z_j of Z' . For the collection Z' , we define a *system of balancing coefficients* a vector $(\gamma_z)_{z \in Z'}$ such that $\sum_{z \in Z'} \gamma_z z = \mathbb{1}^N$. The difference here between the coefficients and the weights is that the coefficients can be negative.

Proposition 3. *Consider a finite nonempty set $Z \subseteq \mathbb{R}_+^N$, a subset $Z' \subseteq Z$ with cardinality $\text{card}(Z') = k$, its weighted incidence $(n \times k)$ -matrix $A^{Z'}$ and its augmented matrix $A_{\mathbb{1}}^{Z'} = [A^{Z'} \mid \mathbb{1}^N]$. Z' has a unique system of coefficients if and only if $k \leq n$ and $\text{rank}(A_{\mathbb{1}}^{Z'}) = \text{rank}(A^{Z'}) = k$. If all these coefficients are nonnegative, Z' is minimally balanced.*

Proof. Consider a finite nonempty set $Z \subseteq \mathbb{R}_+^N$, a subset $Z' \subseteq Z$ with cardinality $\text{card}(Z') = k$. Due to corollary 2, we know that $k \leq n$. If the system $A^{Z'} w = \mathbb{1}^N$ admits a unique solution, then Z' is minimal balanced and w is the balancing weights. Because $k \leq n$, the system is overdetermined, and thanks to the Rouché-Capelli theorem, there exists a solution if and only if $\text{rank}(A^{Z'}) = \text{rank}(A_{\mathbb{1}}^{Z'})$, and the dimension of the set of solution is $k - \text{rank}(A^{Z'})$. Therefore, to have a unique solution, we need to have $\text{rank}(A^{Z'}) = k$. \square

For the study of the core stability, minimal balanced collections of specific sets Z must be computed. Consider a nonempty collection $\mathcal{S} \subseteq \mathcal{F}$ and a collection of minimal collections $(\mathcal{B}_S)_{S \in \mathcal{S}} \in \mathcal{C}(\mathcal{S})$. For $S \in \mathcal{S}$, let $z^S \in \mathbb{R}^N$ be given by

$$z_j^S = \begin{cases} w_{\{i\}}^{\mathcal{B}_S}, & \text{if } j = i \text{ for some } i \in S \text{ such that } \{i\} \in \mathcal{B}_S, \\ 0, & \text{for all other } j \in N. \end{cases}$$

We define the set

$$Z(\mathcal{S}, (\mathcal{B}_S)_{S \in \mathcal{S}}) = \left\{ \mathbb{1}^{N \setminus S} \mid S \in \mathcal{S} \right\} \cup \left\{ \mathbb{1}^T \mid T \in \mathcal{F} \setminus \mathcal{S} \right\} \cup \left\{ z^S \mid S \in \mathcal{S} \right\}.$$

Consider a procedure `LINALGSOLVE` that takes as input a (n, k) -matrix A and returns a k -dimensional vector w such that $Aw = \mathbb{1}^N$. We can then build the procedure generating all the minimal balanced subsets of Z , and we denote their set by \mathbb{B}^Z .

Algorithm 4 Minimal balanced subsets generation algorithm

Require: The set Z

Ensure: The set of all minimal balanced subsets of Z

```

1: procedure ISMINIMALBALANCED( $Z'$ )
2:    $k \leftarrow \text{card}(Z')$ 
3:   if  $k \leq n$  then
4:     if  $\text{rank}(A^{Z'}) = \text{rank}(A_{\mathbb{1}}^{Z'}) = k$  then
5:        $w \leftarrow \text{LINALGSOLVE}(A^{Z'})$ 
6:       if  $w \gg 0$  then
7:         return True, w
8:   return False
9: procedure BALANCEDSUBSETS( $Z$ )
10:   $\mathbb{B}^Z \leftarrow \text{empty list}$ 
11:  for  $Z' \subseteq Z$  do
12:    if  $Z'$  covers  $N$  then
13:       $bool, w \leftarrow \text{ISMINIMALBALANCED}(Z')$ 
14:      if  $bool$  then
15:         $\mathbb{B}^Z \leftarrow Z'$  with weights  $w$ 
16:  return  $\mathbb{B}^Z$ 

```

4.4 Final algorithm

Finally, we have defined almost all the objects needed for the Grabisch-Sudhölter theorem, with their algorithms that generate them. The last one follows. For each $z \in Z$, we define $a_z = a_z(\mathcal{S}, (\mathcal{B}_S)_{S \in \mathcal{S}}) = \max(A \cup B \cup C)$, where

$$A = \{v(N) - v(S) \mid S \in \mathcal{S}, \mathbb{1}^{N \setminus S} = z\}, \quad B = \{v(T) \mid T \in \mathcal{F} \setminus \mathcal{S}, \mathbb{1}^T = z\},$$

$$C = \left\{ v(N) - \sum_{T \in \mathcal{B}_S^*} w_T^{\mathcal{B}_S} v^S(T) : S \in \mathcal{S}, z = z^S \right\}.$$

A and B are singletons or empty, but C can contain multiples values, because distinct coalitions can generate the same z . Consider $N = \{1, 2, 3\}$, $S = \{1, 2\}$, $T = \{1, 3\}$ and $\mathcal{B}_S = \mathcal{B}_T = \{\{1\}, \{2, 3\}\}$. Then,

$$z_1^S = 1, \quad z_2^S = 0, \quad z_3^S = 0; \quad z_1^T = 1, \quad z_2^T = 0, \quad z_3^T = 0.$$

For the coefficient a_z , we can summarize as

$$a_z = \begin{cases} \max C & \text{if } C \neq \emptyset = A; \\ \max\{A, C\} & \text{if } C \neq \emptyset \neq A; \\ v(N) - v(S) & \text{if } z = \mathbb{1}^{N \setminus S} \text{ for some } S \in \mathcal{S}, C = \emptyset; \\ v(T) & \text{if } z = \mathbb{1}^T \text{ for some } T \in \mathcal{F} \setminus \mathcal{S}, A = \emptyset = C. \end{cases}$$

and therefore build an algorithm to compute it for every $z \in Z$.

Recall that \mathbb{B}^Z is the set of all minimal balanced subsets of Z and define $\mathbb{B}_0^Z = \mathbb{B}_0^Z(\mathcal{S}, (\mathcal{B}_S)_{S \in \mathcal{S}})$, the subset of \mathbb{B}^Z such that, for all $Z' \in \mathbb{B}_0^Z$, there exists $S \in \mathcal{S}$ such that $z = 1^{N \setminus S} \in Z'$ and $a_z = v(N) - v(S)$. With this notation, we are now able to formulate the Grabisch-Sudhölter theorem from [5].

Theorem (Grabisch-Sudhölter). *A balanced game (N, v) has a stable core if and only if for every nonempty collection $\mathcal{S} \subseteq \mathcal{F}$, for every $(\mathcal{B}_S)_{S \in \mathcal{S}} \in \mathbb{C}(\mathcal{S})$,*

$$\exists Z' \in \mathbb{B}^Z \setminus \mathbb{B}_0^Z \text{ such that } \sum_{z \in Z'} w_z^{Z'} a_z > v(N) \text{ holds, or}$$

$$\exists Z' \in \mathbb{B}_0^Z \text{ such that } \sum_{z \in Z'} w_z^{Z'} a_z \geq v(N) \text{ holds.}$$

Consider that the WEIGHTEDSUM procedure compute the sum $\sum_{z \in Z'} w_z^{Z'} a_z$ taking Z' as input, and the ZCONSTRUCTION procedure constructs the set Z from \mathcal{S} and $(\mathcal{B}_S)_{S \in \mathcal{S}}$. All necessary small procedures like these two will be described in the appendix. Below is the general procedure for testing stability of the core.

Algorithm 5 Nested balancedness checking algorithm

Require: The game (N, v)

Ensure: The answer to the question: does the game have a stable core ?

```

1: procedure CORESTABILITYCHECKING( $N, v$ )
2:   for nonempty  $\mathcal{S} \subseteq \mathcal{F}$  do
3:     for  $(\mathcal{B}_S)_{S \in \mathcal{S}} \in \mathbb{C}(\mathcal{S})$  do
4:       if not ISNESTEDBALANCED( $v, \mathcal{S}, (\mathcal{B}_S)_{S \in \mathcal{S}}$ ) then
5:         return The core is not stable.
6:   return The core is stable.
7: procedure ISNESTEDBALANCED( $v, \mathcal{S}, (\mathcal{B}_S)_{S \in \mathcal{S}}$ )
8:    $Z \leftarrow$  ZCONSTRUCTION( $\mathcal{S}, (\mathcal{B}_S)_{S \in \mathcal{S}}$ )
9:    $\mathbb{B}^Z \leftarrow$  BALANCEDSUBSETS( $Z$ )
10:  for  $Z' \in \mathbb{B}^Z$  do
11:    if  $Z' \in \mathbb{B}_0^Z$  then
12:      if WEIGHTEDSUM( $Z'$ )  $\geq v(N)$  then
13:        return True
14:    else
15:      if WEIGHTEDSUM( $Z'$ )  $> v(N)$  then
16:        return True
17:  return False

```

A Appendix

Algorithm 6 Complete nested balancedness checking algorithm

Require: The game (N, v)

Ensure: The answer to the question: does the game have a stable core ?

```

1: procedure CHECKASSOCIATION( $\mathcal{B}, S$ )
2:    $bool \leftarrow \mathbf{False}$ 
3:   for  $i \in S$  do
4:     if  $\{i\} \in \mathcal{B}$  then
5:        $bool \leftarrow \mathbf{True}$ 
6:   for  $B \in \mathcal{B}$  do
7:     if  $B \neq N \setminus S$  and  $B \notin \mathcal{F} \setminus 2^S$  and  $B \notin \{\{i\} \mid i \in S\}$  then
8:       return  $\mathbf{False}$ 
9:   return  $bool$ 

10: procedure CHECKADMISSIBILITY( $\mathcal{B}_S, \mathcal{S}, S$ )
11:    $\mathcal{B}^* \leftarrow \mathcal{B} \setminus \{\{i\} \mid i \in S\}$ 
12:   if  $\mathcal{B}_S^* \cap \mathcal{S} = \emptyset$  and  $\mathcal{B}_S^* \cap \{N \setminus R \mid R \in \mathcal{S}\} \neq \emptyset$  then
13:     return  $\mathbf{False}$ 
14:   return CHECKASSOCIATION( $\mathcal{B}_S, S$ )

15: procedure GENERATEADMISSIBLE( $\mathcal{S}, \mathbb{B}_N$ )
16:    $(\mathcal{E}_S)_{S \in \mathcal{S}} \leftarrow$  collection of empty lists
17:   for  $S \in \mathcal{S}$  do
18:     for  $\mathcal{B} \in \mathbb{B}_N$  do
19:       if CHECKADMISSIBILITY( $\mathcal{B}, \mathcal{S}, S$ ) then
20:          $\mathcal{E}_S \leftarrow \mathcal{B}$ 
21:   return  $\times_{S \in \mathcal{S}} \mathcal{E}_S$ 

22: procedure COMPUTATIONZC( $S$ )
23:    $z \leftarrow 0_{\mathbb{R}^N}$ 
24:   for  $i \in N$  do
25:     if  $i \in S$  and  $\{i\} \in \mathcal{B}_S$  then
26:        $z_i \leftarrow w_{\{i\}}^{\mathcal{B}_S}$ 
27:   return  $z$ 

28: procedure COMPUTATIONZ( $\mathcal{S}, (\mathcal{B}_S)_{S \in \mathcal{S}}$ )
29:    $Z_A \leftarrow$  empty list,  $Z_B \leftarrow$  empty list,  $Z_C \leftarrow$  empty list
30:   for  $S \in \mathcal{F}$  do
31:     if  $S \in \mathcal{S}$  then
32:        $Z_A \leftarrow \mathbf{1}^{N \setminus S}$ 
33:        $Z_C \leftarrow$  (COMPUTATIONZC( $S$ ),  $S$ )

```

```

34:     else
35:          $Z_B \leftrightarrow \mathbf{1}^S$ 
36:     return  $(Z_A, Z_B, Z_C)$ 

37: procedure ISMINIMALBALANCED( $Z'$ )
38:      $k \leftarrow \text{card}(\mathcal{B})$ 
39:     if  $k > n$  then
40:         return False
41:      $A^{Z'} \leftarrow$  adjacency matrix of  $Z'$ 
42:      $A_{\mathbb{1}}^{Z'} \leftarrow [A^{Z'} \mid \mathbb{1}^N]$ 
43:     if  $\text{rank}(A^{Z'}) \neq \text{rank}(A_{\mathbb{1}}^{Z'})$  then
44:         return False
45:      $w \leftarrow \text{LINALGSOLVE}(A^{Z'})$ 
46:     if  $w \gg 0$  then
47:         return True,  $w$ 
48:     else
49:         return False

50: procedure BALANCEDSUBSETS( $Z$ )
51:      $\mathbb{B}^Z \leftarrow$  empty list
52:     for  $Z'$  in  $\mathcal{Z}$  do
53:         if  $Z'$  covers  $N$  then
54:              $bool, w \leftarrow \text{ISMINIMALBALANCED}(Z')$ 
55:             if  $bool$  then
56:                  $\mathbb{B}^Z \leftarrow (Z', w)$ 
57:     return  $\mathbb{B}^Z$ 

58: procedure COEFFCOMPUTATION( $z, S, \mathcal{S}, (\mathcal{B}_S)_{S \in \mathcal{S}}$ )
59:      $A \leftarrow$  empty list
60:     if  $S \in \mathcal{S}$  then
61:         if  $z = \mathbf{1}^{N \setminus S}$  then
62:              $A \leftarrow v(N) - v(S)$ 
63:         for  $(z', T) \in Z_C$  do
64:             if  $z = z'$  then
65:                  $A \leftarrow \sum_{U \in \mathcal{B}_S^*} w_U^{\mathcal{B}_S} v^S(U)$ 
66:         return  $\max(A)$ 
67:     else
68:         if  $z = \mathbf{1}^S$  then
69:             return  $v(S)$ 

70: procedure SUMWEIGHTSCOEFF( $Z'$ )
71:      $sum \leftarrow 0$ 

```

```

72:   for  $z \in Z'$  do
73:      $a_z \leftarrow \text{COEFFCOMPUTATION}(z, S, \mathcal{S}, (\mathcal{B}_S)_{S \in \mathcal{S}})$ 
74:      $sum \leftarrow sum + w_z^{Z'} a_z$ 
75:   return  $sum$ 

76: procedure CHECKNESTEDBALANCEDNESS( $v, \mathcal{S}, (\mathcal{B}_S)_{S \in \mathcal{S}}$ )
77:    $\mathbb{B}^Z \leftarrow \text{BALANCEDSUBSETS}(Z)$ 
78:   for  $Z' \in \mathbb{B}^Z \setminus \mathbb{B}_0^Z$  do
79:     if  $\text{SUMWEIGHTSCOEFF}(Z') > v(N)$  then
80:       return True
81:   for  $Z' \in \mathbb{B}_0^Z$  do
82:     if  $\text{SUMWEIGHTSCOEFF}(Z') \geq v(N)$  then
83:       return True
84:   return False

85: procedure CHECKCORESTABILITY( $v$ )
86:   for  $\emptyset \neq \mathcal{S} \subseteq \mathcal{F}$  do
87:     for  $(\mathcal{B}_S)_{S \in \mathcal{S}} \in \mathbb{C}(\mathcal{S})$  do
88:       if not  $\text{CHECKCONDITIONS}(v, \mathcal{S}, (\mathcal{B}_S)_{S \in \mathcal{S}})$  then
89:         return The core is not stable.
90:   return The core is stable.

```

A python computer code will be available soon on github and the personal webpage of the author.

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