Marginalism, Egalitarianism and E ciency in Multi-Choice Games

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WP 2115 – September 2021

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JEL codes:
C71

Marginalism, Egalitarianism and Efficiency in Multi-Choice Games

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Abstract

The search for a compromise between marginalism and egalitarianism has given rise to many discussions. In the context of cooperative games, this compromise can be understood as a trade-off between the Shapley value and the Equal division value. We investigate this compromise in the context of multi-choice games in which players have several activity levels. To do so, we propose new extensions of the Shapley value and of the Weighted Division values to multi-choice games. Contrary to the existing solution concepts for multi-choice games, each one of these values satisfies a core condition introduced by Grabisch and Xie (2007), namely Multi-Efficiency. We compromise between marginalism and egalitarianism by introducing the multi-choice Egalitarian Shapley values, computed as the convex combination of our extensions. To conduct this study, we introduce new axioms for multi-choice games. This allows us to provide an axiomatic foundation for each of these values.

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1. Introduction

A situation in which players can obtain payoffs by cooperation can be described by a cooperative game with transferable utility (TU-games henceforth). A payoff vector for a TU-game assigns a payoff to each player. A single-valued solution on a class of TU-games assigns a unique payoff vector to each game in this class. A set-valued solution on a class of TU-games assigns a set of payoff vectors to each game in this class. The Shapley value (see Shapley (1953)) is probably the most prominent single-valued solution for TU-games. It is computed as the average marginal contribution of the players over all possible orders over the player set. The Equal division value is another well-known single-valued solution which divides the worth of the grand coalition equally among the players. Two of the best-known set-valued solutions are the core and the Weber set. The core of a TU-game is defined as the set of payoff vectors satisfying Efficiency and Coalition rationality. A payoff vector is efficient if the sum of all payoffs is equal to the worth of the grand coalition. A payoff vector is coalitionally rational if no coalition of players can achieve, by itself, a better outcome than the one prescribed by the payoff vectors. Additionally, the Weber set of a

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TU-game is defined as the convex hull of all marginal vectors, where a marginal vector collects the marginal contribution of each player with respect to an order over the player set. It is known that the core of a super-modular TU-game is non-empty and coincides with the Weber set. Moreover, because the Shapley value is the centroid of the Weber set, it belongs to the core of super-modular games.

One of the main issues in economic allocation problems is the trade-off between marginalism and egalitarianism. Marginalism supports allocations based on a player's individual performances, while egalitarianism is in favor of an equal allocation at the expense of the differences between the players' performances. In the context of TU-games, this trade-off can be seen as a compromise between the Shapley value and the Equal division value since the two values are often seen as the embodiment of marginalism and egalitarianism, respectively. This compromise can be made by considering convex combinations of the Shapley value and the Equal division value (see Joosten (1996)). These convex combinations of the Shapley value and the Equal division value have been recently studied by van den Brink et al. (2013), Casajus and Huettner (2013), Abe and Nakada (2019) and Béal et al. (2021). In this paper, we investigate the trade-off between marginalism and egalitarianism in the context of multi-choice (cooperative) games.

Multi-choice games, introduced by Hsiao and Raghavan (1992) and van den Nouweland et al. (1993), are a natural extension of TU-games. In TU-games, each player has two choices. It can either cooperate by joining a coalition or not cooperate. In multi-choice games, each player has several activity levels to cooperate within a coalition. Multi-choice games have been successfully applied to economic theory. For instance, Branzei et al. (2009) study multi-choice games that arise from market situations with two corners. One corner consists of a group of powerful players with yes-or-no choices and clan behavior. The other corner consists of non-powerful players with several choices regarding the extent at which cooperation with the clan can be achieved ; Grabisch and Rusinowska (2010) generalize a yes-no influence model to a multi-choice framework. The authors consider a situation in which some agents are part of a social network. Each agent has an ordered set of possible actions and is influenced by its neighbors in the network when choosing its action; and Techer (2021) addresses the social cost problem, originally introduced by Coase (1960), using multi-choice games. The author studies situations in which one polluter interacts with several potential victims, and aims at negotiating a stable agreement regarding the level of pollution. The polluter has several levels at which it wishes to pollute, whereas the victims can either participate or not in the negotiations. In multi-choice games, a coalition is a vector describing each player's activity level within this coalition. A characteristic function for multi-choice games measures the worth of each coalition. Additionally, a (multi-choice) payoff vector describes how much each player's payoff varies according to its activity level. In other words, it associates a payoff to each activity level of each player. A single-valued solution (value for short) on a class of multi-choice games assigns a unique payoff vector to each game in this class. A set-valued solution on a class of multi-choice games assigns a set of payoff vectors to each game in this class.

In multi-choice games, several solution concepts were defined inspired by the Shapley value, the Equal division value, the core and the Weber set. As pointed out by van den Nouweland et al. (1993), there are more than one reasonable extension of the Shapley value from TU-games to multi-choice games. The first extension is introduced by Hsiao and Raghavan (1992). The authors consider multi-choice games in which players all share the same maximal activity level. They use weights on activity levels and extend the idea of the weighted Shapley values (see Kalai and Samet (1987)). Axiomatic characterizations of this extension can be found in Hsiao and Raghavan (1992) and Hwang and Liao (2009). van den Nouweland et al. (1995) consider the full class of multi-choice games and provide a second extension of the Shapley value. Axiomatic characterizations of this extension can be found in Calvo and Santos (2000). Another extension on the full class of multi-choice games is provided by Derks and Peters (1993). Axiomatic characterizations of this value can be found in Klijn et al. (1999). Other extensions can be found in Peters and Zank (2005) and Grabisch and Lange (2007) (see section 2.2 for details). The Equal division value did not receive the same attention as the Shapley value in the multi-choice games framework. To our knowledge, the only single-valued solution extending the Equal division value from TU-games to multi-choice games is the multi-choice constrained egalitarian solution introduced by Branzei et al. (2014).

Regarding set-valued solutions, the core and the Weber set have been extended to multi-choice games in van den Nouweland et al. (1995), Grabisch and Xie (2007) and Hwang and Liao (2011). It should be pointed out that Hwang and Liao (2011) do not provide an extension of the Weber set, and limit themselves to the study of an extension of the core to multi-choice games. Grabisch and Xie (2007) show that their extension of the core and of the Weber set both coincide on the class of super-modular multi-choice games. It should be observed that this property does not hold for the extensions of the core and the Weber set provided by van den Nouweland et al. (1995), since, for each multi-choice game, their extension of the Weber set is strictly included in their extension of the core. For this reason, we consider the core and the Weber set introduced by Grabisch and Xie (2007) (or simply the core and the Weber set afterwards). Precisely, we focus on a necessary condition for a payoff vector to be in the core. This condition, called **Multi-Efficiency**, extends Efficiency from TU-game to multi-choice games. To introduce this condition, assume that all players agree on forming a coalition in which everyone plays the same activity level, let us say j. If a player is unable to play the level j, then it plays its maximal activity level. Such a coalition is referred to a j-synchronized coalition. A payoff vector is Multi-efficient if, for each level j, the sum of the payoffs of all players for their activity levels up to j is equal to the worth of the j-synchronized coalition. We say that a solution on multi-choice games satisfies Multi-Efficiency if it assigns a multi-efficient payoff vector to each game in this class. For instance, assume that the activity levels represent workdays. Multi-Efficiency ensures that the worth generated after a certain number of workdays is fully redistributed among the workers who worked during those days.

In this paper, we introduce two axioms for multi-choice games related to Multi-Efficiency: Efficiency and Independence of Level Reductions. On the one hand, Efficiency is a classical axiom, which indicates that the worth of the grand coalition is distributed among the activity levels of its members. This axiom is weaker than Multi-Efficiency. On the other hand, Independence of Level Reductions ensures that the payoff distributed to a player's activity level is independent from higher activity levels. In particular, Independence of Level Reductions protects players with lower activity levels from being influenced by players with higher activity levels. Axioms similar to Independence of Level Reductions exist in the economic literature. The serial cost sharing method for discrete cost sharing problems introduced by Moulin and Shenker (1992) satisfies a similar axiom if we interpret activity levels as demands. Recently, Albizuri et al. (2020) study solutions for bargaining problems that satisfy a similar axiom if we interpret activity levels as claims. We show that if a value satisfies Independence of Level Reductions and Efficiency, then it satisfies Multi-Efficiency. Therefore, Multi-Efficiency can be seen as a desirable axiom for multi-choice games. First, it is implied by two desirable axioms for multi-choice games. Second, from a technical point of view, it is a necessary condition to be in the core. However, none of the above mentioned single-valued solutions satisfies Multi-Efficiency. For this reason, we propose several solution concepts for multi-choice games satisfying Multi-Efficiency. This allows us to discuss the trade-off between marginalism and egalitarianism by means of a compromise between multi-efficient solutions. To that end, we first study a multi-efficient extension of the Shapley value, which we call the **multi-choice Shapley value**. This value is computed as follows. Assume that the grand coalition forms step by step starting from the empty coalition, in which no player participate at all. At each step, one player increases its activity by one unit, let us say from $j-1$ to j. However, this player cannot increase its activity level until all other players have reached at least level $j-1$, except for those unable to do so, in which case, they play their maximal activity level. We say that this coalition formation process follows a restricted order. The marginal contribution of a player for an activity level to a coalition is the variation in worth that is created when that player reaches that particular level from the level just below. The multi-choice Shapley value assigns to each activity level of each player its expected marginal contribution assuming that each restricted order occurs with equal probability. This value is the centroid of the Weber set and therefore belongs to the core of super-modular multi-choice games. As an additional remark, we show that the multi-choice Shapley value is closely related to the discrete serial cost sharing method for discrete cost sharing problems as introduced by Moulin and Shenker (1992).

Then, we introduce the class of **multi-choice Weighted Division values** for multi-choice games. Each multi-choice Weighted Division value divides the variation in worth between two consecutive synchronized coalitions (e.g. the j-synchronized and the $(j+1)$-synchronized coalitions) among the players able to play the required activity levels. Such division is done proportionally to some exogenous weights on the activity levels of the players. Each value in this class is multi-efficient. Whenever there are only two activity levels (0 and 1), these values coincide with the Weighted Division values for TU-games (see Béal et al. (2016)). One of these values catches our interest. We call this value the **multi-choice Equal division value**: it divides the variation in worth between two consecutive synchronized coalitions equally among the players able to play the required activity levels.

To our knowledge, no previous work has addressed the trade-off between marginalism and egalitarianism in the context of multi-choice games. We address this trade-off by compromising between the multi-choice Shapley value and the multi-choice Equal division value. To that end, we introduce the **multi-choice Egalitarian Shapley values** for multi-choice games. This family of values is composed of convex combinations of the multi-choice Shapley value and the multi-choice Equal division value. Obviously, the multi-choice Egalitarian Shapley values are multi-efficient. Since we consider multi-choice games, we can define a specific convex combination at each activity level. This allows for different types of compromise, depending on the activity level.

We provide several axiomatic characterizations of these new multi-efficient solution concepts. To that end, we invoke classical axioms as well as new axioms for multi-choice games. Among the new axioms, we introduce two axioms that deal with what happens to the payoffs when the maximal activity level of one player is reduced. We show that combining these two axioms implies Independence of Level Reductions. Among the new axioms, we also introduce Equal Sign for Equal Pairs which is a extension of Equal Sign for Equal Agents originally introduced by Casajus (2018) for TU-games. Additionally, we propose Equal Treatment for Equal Pairs which strengthens Equal Sign for Equal Pairs. Furthermore, we introduce Weak Monotonicity. This axiom relaxes the axiom of Strong monotonicity for multi-choice games originally introduced by Klijn et al. (1999)), but also boils down to the axiom of Weak Monotonicity as introduced by van den Brink et al. (2013) for TU-games. Combining classical and new axioms for multi-choice games, we provide two characterizations of the multi-choice Shapley value, one that relies on a classical Additivity axiom for multi-choice games (Theorem 1) and another one that does not (Theorem 2). Furthermore, we show that the multi-choice Shapley value admits an expression in terms of Harsanyi dividends (Corollary 2). Next, we provide an axiomatic characterization of the multi-choice Weighted Division values (Theorem 3) and show that strengthening the Equal Sign for Equal Pairs axiom into the Equal Treatment for Equal Pairs axiom results in a characterization of the multi-choice Equal division value (Corollary 3). Finally, we provide an axiomatic characterization of the Egalitarian Shapley values (Theorem 4).

The rest of the paper is organized as follows. After dealing with preliminaries on multi-choice games in Section 2, we introduce multi-efficient solution concepts in Section 3. Section 3.1 introduces the core as defined by Grabisch and Xie (2007) along with the Multi-Efficiency principle. Subsection 3.2 introduces the multi-choice Shapley value, Section 3.3 introduces the multi-choice Weighted Division values and Section 3.4 introduces the multi-choice Egalitarian Shapley values. We provide the axiomatic characterizations in Section 4. We make some additional remarks regarding the multi-choice Shapley value and its relationship with the serial cost sharing method in Section 5. Finally, Section 6 concludes the paper and Section 7 is an appendix containing the proofs of the results.

2. Preliminaries

We denote by $|A|$ the number of elements in finite set A. For each non-empty $B \subseteq A$, we denote $e_B \in \mathbb{R}^{|A|}$ the vector such that $(e_B)_i = 1$ if $i \in B$ and $(e_B)_i = 0$ otherwise. We denote by $sign(x)$ the sign of a scalar $x \in \mathbb{R}_*$.

2.1. Multi-choice games

Let $N = \{1, \dots, n\}$ be a fixed **set of players** and $K \in \mathbb{N}$. Each player $i \in N$ has a finite set of pairwise distinct activity levels $M_i := \{0, \dots, m_i\}$ such that $m_i \leq K$. For each player $i \in N$, the set M_i is linearly ordered from the lowest activity level 0 (i does not participate cooperate) to the maximal activity level m_i. Denote by $Q(j) \subseteq N$ the set of **players able** to play activity level j. Formally, the set $Q(j)$ is defined as

$$Q(j) = \Big\{ i \in N : m_i \geq j \Big\}.$$

Without loss of generality, we assume that $Q(1) = N$. Let $\mathcal{M}$ be the cartesian product $\prod_{i \in N} M_i$. Each element $s = (s_1, \dots, s_n) \in \mathcal{M}$ specifies a participation profile for players and is referred to as a (multi-choice) coalition. So, a coalition indicates each player's activity level. Then, $m = (m_1, \dots, m_n) \in \mathcal{M}$ is the players' maximal participation profile that plays the role of the grand coalition, whereas $\Theta = (0, \dots, 0)$ plays the role of the empty coalition. For $s \in \mathcal{M}$, we denote by (s_{-i}, k) the coalition where all players except i play at levels defined in s while i plays at $k \in M_i$. The set $\mathcal{M}$ endowed with the usual binary relation $\leq$ on $\mathbb{R}^n$ induces a (complete) lattice with greatest element m and least element Θ. For any two coalitions $a, b \in \mathcal{M}$, $a \vee b$ and $a \wedge b$ denote their least upper bound and their greatest lower bound over $\mathcal{M}$, respectively. A pair $(i, j) \in M^+$ represents a player and one of its activity levels. We use the notation $M_i^+ = M_i \setminus \{0\}$ for each $i \in N$ and $M^+ = \bigcup_{i \in N} (\{i\} \times M_i^+)$. For $s \in \mathcal{M}$, we introduce the set of **top pairs** $T(s)$ containing players playing the highest activity levels in s. Formally, the set of top pairs in s is defined as

$$\forall s \in \mathcal{M}, \quad T(s) = \Big\{ (i, s_i) \in M^+ : s_i \geq s_k, \ \forall k \in N \Big\}. \tag{1}$$

A (cooperative) **multi-choice game** on N is a couple (m, v) where $v : \mathcal{M} \longrightarrow \mathbb{R}$ is a characteristic function, such that $v(\Theta) = 0$, that specifies the worth $v(s)$ when players participate at profile s. Denote by $\mathcal{G}$ the set of multi-choice games (m, v) on N such that $m_i \leq K$ for each $i \in N$. Notice that TU-games can be viewed as a subclass of multi-choice games satisfying $m = (1, \ldots, 1)$. A multi-choice game is **super-modular** if $v(s \vee t) + v(s \wedge t) \geq v(s) + v(t)$ for each $s, t \in \mathcal{M}$. A multi-choice game (m, v) is the **null game** if $v(s) = 0$ for each $s \in \mathcal{M}$.

Let $(m, v) \in \mathcal{G}$ and a coalition $s \in \mathcal{M}$ such that $s_i = j - 1$. The surplus $v(s + e_i) - v(s)$ refers to the **marginal contribution** of the player i for its activity level j (or simply the marginal contribution of the pair (i, j)) to coalition s. A pair $(i, j) \in M^+$ is a **dummy pair** in $(m, v) \in \mathcal{G}$ if the marginal contribution of each pair (i, j'), such that $j \leq j' \leq m_i$, to each coalition is null. Formally, $(i, j) \in M^+$ is a dummy pair if

$$\forall s \in \mathcal{M}, \forall j \leq l \leq m_i, \quad v(s_{-i}, l) = v(s_{-i}, j - 1). \tag{2}$$

Obviously, in the null game, each pair is a dummy pair. Two distinct pairs containing the same activity level are **equal** if they have the same marginal contributions to coalitions. Formally, $(i, j), (i', j) \in M^+$ are equal pairs if

$$\forall s \in \mathcal{M} : s_i = s_{i'} = j - 1, \quad v(s + e_i) = v(s + e_{i'}). \tag{3}$$

Observe that two dummy pairs in a game are equal. For $(m, v) \in \mathcal{G}$, we define the sub-game $(t, v^t) \in \mathcal{G}$, induced by $t \in \mathcal{M}$, as

$$\forall s \in \mathcal{M}, \quad v^t(s) = \begin{cases} v(s) & \text{if } s \leq t, \\ 0 & \text{else.} \end{cases} \tag{4}$$

When no confusion arises, we simply denote the sub-game (t, v^t) of (m, v) by (t, v) to avoid heavy notation. The sub-game (t, v) corresponds to a cooperative situation in which each player $i \in N$ can play at most the level t_i, where $t_i \leq m_i$. In other words, this describes a situation where the maximal activity level of some players have been reduced. Let $t \in \mathcal{M}$, $t \neq (0, \ldots, 0)$. An analogue of an unanimity TU-game in the multi-choice setting is the concept of **minimal effort game** $(m, u_t) \in \mathcal{G}$ defined as

$$\forall s \in \mathcal{M}, \quad u_t(s) = \begin{cases} 1 & \text{if } s_i \geq t_i \text{ for each } i \in N, \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

For each multi-choice game $(m, v) \in \mathcal{G}$, the characteristic function v admits a unique linear decomposition in terms of minimal effort games (see Hsiao and Raghavan (1992)) as follows

$$v = \sum_{t \leq m} \Delta_v(t) u_t, \quad \text{where} \quad \Delta_v(t) = v(t) - \sum_{s \leq t, s \neq t} \Delta_v(s). \tag{6}$$

For each $t \in \mathcal{M}$, $\Delta_v(t)$ is called the Harsanyi dividend of t.

A **payoff vector** for the game (m, v) is an element $x \in \mathbb{R}^{M^+}$, where $x_{ij} \in \mathbb{R}$ is the payoff received by the pair $(i, j) \in M^+$. A set-valued solution on $\mathcal{G}$ is a map F that assigns a collection of payoff vectors $F(m, v)$ to each $(m, v) \in \mathcal{G}$. A **value** f is a single-valued solution on $\mathcal{G}$ that assigns a unique payoff vector $f(m, v)$ to each $(m, v) \in \mathcal{G}$.

3. Multi-efficient solution concepts

In this section, we discuss a necessary condition for a payoff vector to be in the core of multi-choice games as defined by (Grabisch and Xie, 2007), which we call **Multi-Efficiency**. We propose new multi-efficient solution concepts for multi-choice games. We first provide a new extension of the Shapley value (Shapley (1953)) from TU-games to multi-choice games. Next, we provide new extensions of the Equal division value, the Weighted Division values and the Egalitarian Shapley values from TU-games to multi-choice games.

3.1. Multi-Efficiency

The core of a multi-choice game $(m, v) \in \mathcal{G}$ (see Grabisch and Xie (2007)) is denoted by $\mathcal{C}(m, v)$ and is defined as

$$x \in \mathcal{C}(m,v) \iff \begin{cases} \forall s \in \mathcal{M}, \quad \displaystyle\sum_{i \in N} \sum_{j=1}^{s_i} x_{ij} \geq v(s) \text{ and} & (7) \\ \forall h \leq \displaystyle\max_{k \in N} m_k, \quad \displaystyle\sum_{i \in N} \sum_{j=1}^{h \wedge m_i} x_{ij} = v((h \wedge m_i)_{i \in N}). & (8) \end{cases}$$

The first core condition (7) states that no coalition can achieve, by itself, a better outcome than the one prescribed by the payoff vectors in the core. Observe that, on the class of multi-choice games such that $m = (1, \ldots, 1)$, condition (7) coincides with the coalition rationality core condition for TU-games. Assume that all players agree on forming a coalition in which everyone plays the same activity level, let us say h. Players unable to cooperate at such level play their maximal activity level. We call such coalition a h-synchronized coalition. The second condition (8) states that a h-synchronized coalition achieves the same outcome than the one prescribed by the payoff vectors in the core. Observe that, on the class of multi-choice games such that $m = (1, \ldots, 1)$, condition (8) coincides with the efficiency core condition for TU-games. Let us reformulate (8) as an axiom for solutions on $\mathcal{G}$. Let f be a solution on $\mathcal{G}$.

Multi-Efficiency (ME) For each $(m, v) \in \mathcal{G}$, we have

$$\forall h \leq \max_{k \in N} m_k, \quad \sum_{i \in N} \sum_{j=1}^{h \wedge m_i} f_{ij}(m, v) = v((h \wedge m_i)_{i \in N}). \tag{9}$$

Remark 1. For each $(m, v) \in \mathcal{G}$, (9) can be re-written as

$$\forall h \leq \max_{k \in N} m_k, \quad \sum_{i \in Q(h)} f_{ih}(m, v) = v((h \wedge m_k)_{k \in N}) - v(((h-1) \wedge m_k)_{k \in N}). \tag{10}$$

The sum of the payoffs of all pairs (i, h) containing activity level h is equal to the surplus generated between the h-synchronized coalition and the $(h-1)$-synchronized coalition.

3.2. The multi-choice Shapley value

In this section, we define a multi-efficient value that extends the Shapley value from TU-games to multi-choice games. This value belongs to the core of super-modular multi-choice games.

Let $(m, v) \in \mathcal{G}$, we consider **restricted orders** over the set of pairs M^+, which were originally introduced by Grabisch and Xie (2007). These orders are such that no pair $(i, j) \in M^+$ is ordered before a pair $(i', j') \in M^+$ containing a strictly lower activity level $j' < j$. Formally, a restricted order over the set of pairs is a bijection $\sigma : M^+ \to \{1, \ldots, \sum_{i \in N} m_i\}$ defined as

$$\forall (i, j), (i', j') \in M^+, \quad \big[j < j'\big] \implies \big[\sigma(i, j) < \sigma(i', j')\big].$$

Denote by $\overline{O}$ the set of all restricted orders over the set of pairs. Obviously, we have $\overline{O} \subseteq O$. The number of restricted orders over the set of pairs is given by

$$\prod_{j \leq \max\limits_{k \in N} m_k} |Q(j)|!$$

Let $\sigma \in \overline{O}$ be a restricted order and $h \in \{1, \ldots, \sum_{i \in N} m_i\}$. We denote by $s^{\sigma,h}$ the coalition formed after step h. Formally, it is defined as

$$\forall i \in N, \quad s_i^{\sigma,h} = \max \big\{ j \in M_i : \sigma(i, j) \leq h \big\} \cup \{0\}. \tag{11}$$

We use the convention $s^{\sigma,0} = \Theta$. For each $\sigma \in \overline{O}$, the marginal vector $\eta^\sigma(m, v)$ is defined as

$$\forall (i, j) \in M^+, \quad \eta^\sigma_{ij}(m, v) = v(s^{\sigma,\sigma(i,j)}) - v(s^{\sigma,\sigma(i,j)-1}).$$

Each $\eta^\sigma_{ij}(m, v)$ represents the marginal contribution of (i, j) to the coalition $s^{\sigma,\sigma(i,j)-1}$ formed after $\sigma(i, j) - 1$ steps according to the restricted order σ. We have the material to define our extension of the Shapley value from TU-games to multi-choice games. This value assigns to each pair $(i, j) \in M^+$ its expected marginal contribution assuming that each restricted order over the set of pairs occurs with equal probability.

Definition 1. For each $(m, v) \in \mathcal{G}$, the **multi-choice Shapley value** φ is defined as

$$\forall (i, j) \in M^+, \quad \varphi_{ij}(m, v) = \frac{1}{\prod_{j \leq \max\limits_{k \in N} m_k} |Q(j)|!} \sum_{\sigma \in \overline{O}} \eta^\sigma_{ij}(m, v). \tag{12}$$

Whenever $m = (1, \ldots, 1)$, this value coincides with the Shapley value on TU-games.

Remark 2. Following Grabisch and Xie (2007), for each $(m, v) \in \mathcal{G}$, the Weber set W is the convex hull of all marginal vectors defined as

$$\mathcal{W}(m, v) = co(\{\eta^{\overline{\sigma}}(m, v) \,|\, \overline{\sigma} \in \overline{O}\}).$$

The multi-choice Shapley value is the centroid of the Weber set. By Grabisch and Xie (2007), the Weber set coincides with the core on the class of super-modular multi-choice games. Therefore, for each super-modular multi-choice game $(m, v) \in \mathcal{G}$, it holds that $\varphi(m, v) \in \mathcal{C}(m, v)$.

Next results states that the multi-choice Shapley value admits an alternative expression which requires less orders over the set of pairs to be computed. For each $j \leq \max_{k\in N} m_k$, denote by $M^{+,j} = \{(i,j) \in M^+ : i \in Q(j)\}$ the subset of pairs containing the activity level j. We define orders over the set of pairs $M^{+,j}$. An order over $M^{+,j}$ is a bijection $\sigma_j : M^{+,j} \to \{1,\dots,|Q(j)|\}$. Denote by $\overline{O}_j$ the set of all orders over $M^{+,j}$. These orders can also be interpreted as orders over the set of players in $Q(j)$. For each $\sigma_j \in \overline{O}_j$ and $h \in \{0,\dots,|Q(j)|\}$, define $s^{\sigma_j,h}$ as

$$\forall i \in N, \quad s_i^{\sigma_j,h} = \begin{cases} j & \text{if } i \in Q(j) \text{ and } \sigma_j(i,j) \leq h, \\ j-1 & \text{if } i \in Q(j) \text{ and } \sigma_j(i,j) > h, \\ m_i & \text{if } i \notin Q(j). \end{cases} \tag{13}$$

Observe that $s^{\sigma_j,|Q(j)|} = (j \wedge m_k)_{k\in N}$ and $s^{\sigma_j,0} = ((j-1) \wedge m_k)_{k\in N}$. The coalition $s^{\sigma_j,h} \in \mathcal{M}$ represents a situation in which each player able to play at j and ordered prior to step h, with respect to σ_j, participates at its activity level j, whereas each player able to play j but not ordered prior to step h, with respect to σ_j, participates at its activity level $j-1$. Players unable to play j participate at their maximal activity level.

Proposition 1. For each $(m,v) \in \mathcal{G}$, the multi-choice Shapley value φ admits an alternative expression given by

$$\forall (i,j) \in M^+, \quad \varphi_{ij}(m,v) = \frac{1}{|Q(j)|!} \sum_{\sigma_j \in \overline{O}_j} \Big[v(s^{\sigma_j,\sigma_j(i,j)}) - v(s^{\sigma_j,\sigma_j(i,j)-1}) \Big]. \tag{14}$$

Proof. See Appendix 7.1. □

In the sequel, we will retain expression (14) of the multi-choice Shapley value.

3.3. The multi-choice Equal division value and Weighted Division values

In this section, we propose a new multi-efficient value that extends the Equal division value from TU-games to multi-choice games. This value is referred to as the multi-choice Equal division value. The multi-choice Equal division value divides the surplus generated between two consecutive synchronized coalitions (10) equally among the pairs containing the activity level on which the players in the larger of the two coalitions are synchronized.

Definition 2. For each $(m,v) \in \mathcal{G}$, the **multi-choice Equal division value** ξ is defined as

$$\forall (i,j) \in M^+, \quad \xi_{ij}(m,v) = \frac{1}{|Q(j)|} \Big[v((j \wedge m_k)_{k\in N}) - v(((j-1) \wedge m_k)_{k\in N})) \Big]. \tag{15}$$

Whenever $m = (1,\dots,1)$, the multi-choice Equal division value boils down to the Equal division value on TU-games. In addition, we introduce the class of multi-choice Weighted Division values. Each value in this class divides the surplus generated between two consecutive synchronized coalitions (10) among the pairs containing the activity level on which the players in the larger of the two coalitions are synchronized. Such division is done according to a weight system.

Definition 3. Let $\beta = \{\beta^{ij}\}_{i\in N, 1\leq j\leq K}$ be a weight system, such that $\beta^{ij} > 0$ for each $i \in N$ and $1 \leq j \leq K$. For each $(m, v) \in \mathcal{G}$, a **Weighted Division value** ξ^β is defined as

$$\forall (i,j) \in M^+, \quad \xi^\beta_{ij}(m,v) = \frac{\beta^{ij}}{\sum_{k\in Q(j)} \beta^{kj}} \Big[v((j \wedge m_k)_{k\in N}) - v(((j-1) \wedge m_k)_{k\in N}) \Big]. \tag{16}$$

Whenever $m = (1, \dots, 1)$, these values boil down to the Weighted Division values on TU-games introduced by Béal et al. (2016). Obviously, the multi-choice Equal division value is a multi-choice Weighted Division value in which each weight is equal to 1.

3.4. The multi-choice Egalitarian-Shapley values

In this section, we propose a trade-off between marginalism and egalitarianism by considering convex combinations of the multi-choice Shapley value and the multi-choice Equal division value.

Definition 4. Let $\alpha = \{\alpha^j\}_{1\leq j\leq K}$ be a parameter system such that $\alpha^j \in [0, 1]$ for each $1 \leq j \leq K$. For each $(m, v) \in \mathcal{G}$, a **multi-choice Egalitarian Shapley value** χ^α is defined as

$$\forall (i,j) \in M^+, \quad \chi^\alpha_{ij}(m,v) = \alpha^j \varphi_{ij}(m,v) + (1-\alpha^j)\xi_{ij}(m,v). \tag{17}$$

Whenever $m = (1, \dots, 1)$, these values boil down to the Egalitarian Shapley values on TU-games. We illustrate the possibilities offered by multiple convex combinations with an example.

Example 1. Consider $(m, v) \in \mathcal{G}$ and $i \in N$ such that $m_i = 3$. Consider an Egalitarian Shapley value defined by $\alpha^1 = 0.2$, $\alpha^2 = 0.5$ and $\alpha^3 = 0.8$. The payoff of i for an activity level j will be closer to the multi-choice Equal division value if $j = 1$ and closer to the multi-choice Shapley value if $j = 3$. Formally, $\chi^\alpha_{i1}(m, v)$ is a payoff closer to $\xi_{i1}(m, v)$, whereas $\chi^\alpha_{i3}(m, v)$ is a payoff closer to $\varphi_{i3}(m, v)$. Thus, egalitarianism is progressively overtaken by marginalism as the activity level increases. This is due to the fact that $\alpha^1 < \alpha^2 < \alpha^3$. Depending on the parameter system, a multi-choice Egalitarian Shapley value operates different compromises between egalitarianism and marginalism for different activity levels. These differences can be progressive as it is the case in this example.

4. Axiomatic characterizations

In this section, we discuss new and classical axioms for multi-choice games. We also provide axiomatic characterizations of each solution introduced in Section 3.

4.1. Characterizations of the multi-choice Shapley value

We provide two axiomatic characterizations of the multi-choice Shapley value. The first characterization relies on a Linearity axiom, whereas the second does not. We also provide an expression of the multi-choice Shapley value in terms of Harsanyi dividends. Let f be a value on multi-choice games. First, we introduce two classical axioms for multi-choice games.

Efficiency (E). For each $(m, v) \in \mathcal{G}$, we have

$$\sum_{i\in N} \sum_{j\in M_i^+} f_{ij}(m,v) = v(m).$$

Linearity (L). For each $(m, v), (m, w) \in \mathcal{G}$ and $\lambda \in \mathbb{R}$, we have

$$f(m, v + \lambda w,) = f(m, v) + \lambda f(m, w).$$

The next three axioms deal with what happens to the payoffs when the maximal activity level of one or several players is reduced. The first axiom was originally introduced by Hwang and Liao (2009) and Béal et al. (2012). This axiom requires that if the maximal activity level of a player reduces to a certain level, then the payoff of this player for this activity level remains unchanged under the condition that the other players' activities are unchanged. In other words, the payoff of a player for a given activity level is independent from its own higher activity levels.

Independence of Individual Level Reduction (IIR). For each $(m, v) \in \mathcal{G}$, we have

$$\forall (i, j) \in M^+, \quad f_{ij}(m, v) = f_{ij}((m_{-i}, j), v).$$

The next axiom requires that if the maximal activity level of a player reduces to a certain level, then each player is equally impacted.

Equal Loss Under Individual Level Reduction (EL). For each $(m, v) \in \mathcal{G}$, we have

$$\forall (i, j), (i', j) \in M^+, \quad f_{ij}(m, v) - f_{ij}((m_{-i}, j), v) = f_{i'j}(m, v) - f_{i'j}((m_{-i}, j), v).$$

The next axiom requires that if the maximal activity level of each player reduces to a certain level, then the payoff of each player for this activity level remains unchanged. This axiom extends the idea of (IIR) in the sense that the payoff of a player's activity level is not only independent from its own higher activity levels, but also from all the other players' higher activity levels.

Independence of Level Reductions (IR). For each $(m, v) \in \mathcal{G}$, we have

$$\forall (i, j) \in M^+, \quad f_{ij}(m, v) = f_{ij}((j \wedge m_k)_{k \in N}, v).$$

It turns out that (IIR) combined with (EL) implies (IR) and that (IR) combined with (E) implies (ME).

Proposition 2. *If a value f on $\mathcal{G}$ satisfies (IIR) and (EL), then it satisfies (IR).*

Proof. See Appendix 7.2. □

Remark 3. The converse of Proposition 2 is not true. Indeed, consider the value g defined, for each $(m, v) \in \mathcal{G}$, as

$$\forall (i, j) \in M^+, \quad g_{ij}(m, v) = \begin{cases} 1 & \text{if } i = 1, j = 1 \text{ and } m_i \geq m_k, \forall k \in N, \\ 0 & \text{otherwise.} \end{cases}$$

The value g obviously satisfies (IR) but violates (IIR) and (EL). To see this, consider $N = \{1, 2, 3\}$ and $(m, v) \in \mathcal{G}$ such that $m = (3, 2, 3)$. Observe that $g_{11}((2, 2, 3), v) = 0 \neq g_{11}(m, v) = 1$, which show that g violates (IR). Furthermore, $g_{11}(m, v) - g_{11}((2, 2, 3), v) = 1 \neq g_{ij}(m, v) - g_{ij}((2, 2, 3), v) = 0$, for each $(i, j) \neq (1, 1)$, which shows that g violates (EL).

Proposition 3. *If a value f on $\mathcal{G}$ satisfies (E) and (IR), then it satisfies (ME).*

Proof. See Appendix 7.3. $\square$

Corollary 1. *If a value f on $\mathcal{G}$ satisfies (E), (IIR) and (EL), then it satisfies (ME).*

Proof. This results follows directly from Proposition 2 and Proposition 3. $\square$

Remark 4. The converse of Proposition 3 is not true. Indeed, consider the value d defined for each $(m, v) \in \mathcal{G}$ as

$$\forall (i,j) \in M^+, \quad d_{ij}(m,v) = \begin{cases} \dfrac{v((j \wedge m_k)_{k \in N}) - v(((j-1) \wedge m_k)_{k \in N})}{|\{h \in N : m_h \geq m_k, \forall k \in N\}|} & \text{if } m_i \geq m_k, \forall k \in N, \\ 0 & \text{otherwise.} \end{cases}$$

The value d satisfies (ME), but does not verify (IR). To see this, consider $N = \{1,2,3\}$ and $(m,v) \in \mathcal{G}$ such that $m = (3,2,3)$. Observe that $d_{1,1}((2,2,2),v) = \frac{1}{3}v(1,1,1) \neq d_{1,1}(m,v)$ which shows that d violates (IR).

The next two axioms compare the payoffs of equal pairs (see (3) for the definition of equal pairs). First, we introduce the Equal Treatment for Equal Pairs axiom, which states that two equal pairs should receive the same payoff. We also suggest a relaxation of Equal Treatment for Equal Pairs into Equal Sign for Equal Pairs axiom. This axiom states that two equal pairs should receive a payoff of the same sign.

Equal Treatment for Equal Pairs (ET). For each $(m, v) \in \mathcal{G}$ and two distinct equal pairs $(i, j)(i', j) \in M^+$, we have

$$f_{ij}(m, v) = f_{i'j}(m, v).$$

Whenever $m = (1, \ldots, 1)$, (ET) boils down to the classical axiom of equal treatment for equal for TU-games.

Equal Sign for Equal Pairs (ES). For each $(m, v) \in \mathcal{G}$ and two distinct equal pairs $(i, j)(i', j) \in M^+$, we have

$$sign(f_{ij}(m, v)) = sign(f_{i'j}(m, v)).$$

Whenever $m = (1, \ldots, 1)$, (ES) boils down to the equal sign for equal axiom originally introduced by Casajus (2018). The next axiom considers dummy pairs (see (2)). This axiom, originally introduced by Klijn et al. (1999), requires that any dummy pair receives a null payoff.

The Dummy Property (D). For each $(m, v) \in \mathcal{G}$ and each dummy pair $(i, j) \in M^+$, we have

$$f_{ij}(m, v) = 0.$$

Whenever $m = (1 \ldots, 1)$,(D) boils down to the classical null player property for TU-games. We have the material to provide a first axiomatic characterization of the multi-choice Shapley value.

Theorem 1. *A value f on $\mathcal{G}$ satisfies Efficiency (E), Independence of Individual Level Reduction (IIR), Equal Loss Under Individual Level Reduction (EL), Linearity (L), Equal Sign for Equal Pairs (ES) and The Dummy Property (D) if and only if $f = \varphi$.*

Proof. See Appendix 7.4. ☐

By Theorem 1, we provide another alternative expression of the multi-choice Shapley value in terms of Harsanyi dividends.

Corollary 2. *The multi-choice Shapley value admits an alternative expression in terms of Harsanyi dividends. For each game $(m,v) \in \mathcal{G}$, the value is defined as*

$$\forall (i,j) \in M^+, \quad \varphi_{ij}(m,v) = \sum_{\substack{s \in \mathcal{M} \\ (i,j) \in T(s)}} \frac{\Delta_v(s)}{|T(s)|}. \tag{18}$$

Proof. See Appendix 7.5. ☐

We provide a second axiomatic characterization of the multi-choice Shapley value without resorting to (L). In line with Young (1985) and Casajus (2018), we use a Strong monotonicity axiom. This axiom states that, if the marginal contributions to coalitions of a pair increase from a game (m,w) to another game (m,v), then the payoff of this pair also increases.

Strong Monotonicity (SM). For each $(m,v),(m,w) \in \mathcal{G}$, each $(i,j) \in M^+$ and each $s \in \mathcal{M}$ such that $s_i = j-1$, we have

$$v(s+e_i) - v(s) \geq w(s+e_i) - w(s), \text{ then, we have } f_{ij}(m,v) \geq f_{ij}(m,w).$$

Whenever $m = (1,\dots,1)$, (SM) boils down to the strong monotonicity axiom for TU-games introduced by Young (1985). We have the material to provide a second axiomatic characterization of the multi-choice Shapley value.

Theorem 2. *A value f on $\mathcal{G}$ satisfies Efficiency (E), Independence of Individual Level Reduction (IIR), Equal Loss Under Individual Level Reduction (EL), Strong monotonicity (SM) and Equal Sign for Equal Pairs (ES) if and only if $f = \varphi$.*

Proof. See Appendix 7.6. ☐

4.2. Characterization of the multi-choice Weighted Division values

In this section, we characterize the multi-choice Weighted Division values. To that end, we introduce the Non-Negative Lower Bound axiom. This axiom requires that if the surplus generated between two consecutive synchronized coalitions, let us say $((j-1) \wedge m_k)_{k \in N}$ and $(j \wedge m_k)_{k \in N}$, is non-negative, then the payoff of each pair containing the activity level j has to be non-negative.

Non-Negative Lower Bound (NLB). For each $(m,v) \in \mathcal{G}$ and $(i,j) \in M^+$, such that

$$v((j \wedge m_k)_{k \in N}) - v(((j-1) \wedge m_k)_{k \in N}) \geq 0, \quad \text{we have} \quad f_{ij}(m,v) \geq 0.$$

We have the material to provide a axiomatic characterization of the multi-choice Weighted Division values.

Theorem 3. *A value f on $\mathcal{G}$ satisfies Efficiency (E), Independence of Individual Level Reduction (IIR), Equal Loss Under Individual Level Reduction (EL), Linearity (L), Non-Negative Lower Bound (NLB) and Equal Sign for Equal Pairs (ES) if and only if $f = \xi^\beta$, for some β is a weight system.*

Proof. See Appendix 7.7. □

By Theorem 3, we obtain a characterization of the multi-choice Equal division value. The proof of this Corollary is given at the end of Appendix 7.7.

Corollary 3. *A value f on $\mathcal{G}$ satisfies Efficiency (E), Independence of Individual Level Reduction (IIR), Equal Loss Under Individual Level Reduction (EL), Linearity (L), Non-Negative Lower Bound (NLB) and Equal treatment of equal pairs (ET) if and only if $f = \xi$.*

4.3. Characterization of the multi-choice Egalitarian Shapley values

In this subsection we provide an axiomatic characterization of the multi-choice Egalitarian Shapley values. To that end, we introduce two new axioms.

A multi-efficient value shares the surplus generated between two consecutive synchronized coalitions among the pairs containing the required activity level (see (10)). This surplus can eventually be negative. Requiring that the payoff of a pair varies according to its marginal contributions to coalitions regardless of the surplus to be shared is then a strong requirement in (SM). On the contrary, it seems reasonable that the payoff of a pair, let us say $(i,j) \in M^{+,j}$, does not decrease from one game, let us say $(m,v) \in \mathcal{G}$, to another, let us say $(m,w) \in \mathcal{G}$, if the surplus generated between the j-synchronized coalition and the $(j-1)$-synchronized coalition does not decrease from (m,v) to (m,w). The next axiom is a weaker version of (SM) which requires that the surplus generated between two synchronized coalitions should not decrease from one game to another.

Weak Monotonicity (WM). For each $(m,v),(m,w) \in \mathcal{G}$ and $(i,j) \in M^+$, such that

$$v((j \wedge m_k)_{k\in N}) - v(((j-1) \wedge m_k)_{k\in N}) \geq w((j \wedge m_k)_{k\in N}) - w(((j-1) \wedge m_k)_{k\in N}),$$

and for each $s \in \mathcal{M}$, such that $s_i = j-1$, we have

$$v(s+e_i) - v(s) \geq w(s+e_i) - w(s), \text{ then, we have } f_{ij}(m,v) \geq f_{ij}(m,w).$$

Whenever $m = (1,\ldots,1)$, (WM) boils down to the Weak Monotonicity axiom for TU-games introduced by van den Brink et al. (2013). Obviously, (SM) implies (WM).

Consider $(m,v) \in \mathcal{G}$ and two distinct pairs $(i,j),(i',j) \in M^{+,j}$. We say that the pair (i,j) is more **desirable** than the pair (i',j) in (m,v) if its has better marginal contributions to coalitions. Formally, (i,j) more **desirable** than (i',j) if for each $s \in \mathcal{M}$, such that $s_i = s_{i'} = j-1$, we have

$$v(s+e_i) \geq v(s+e_{i'}).$$

The next axiom requires that a pair receives a greater payoff than other less desirable pairs.

Level Desirability (LD). For each $(m,v) \in \mathcal{G}$ and two distinct pairs $(i,j),(i',j) \in M^+$, such that (i,j) is desirable over (i',j) in (m,v), we have

$$f_{ij}(m,v) \geq f_{i'j}(m,v).$$

Whenever $m = (1,\ldots,1)$, (LD) boils down to the classical desirability axiom for TU-games. We have the material to provide a characterization of the multi-choice Egalitarian Shapley values.

Theorem 4. *A solution f on $\mathcal{G}$ satisfies Efficiency (E), Independence of Individual Level Reduction (IIR), Equal loss under level reduction (EL), Linearity (L), Weak Monotonicity (WM) and Level Desirability (LD) if and only if $f = \chi^\alpha$, for some parameter system α.*

Proof. See Appendix 7.8. $\square$

Remark 5. As mentioned in Example 1, in some economic situations, it may be interesting to consider families of specific parameter systems, for instance parameter systems that operate a progressive compromise between marginalims and egalitarianism. In this case, it is possible to refine Theorem 4 in order to characterize multi-choice Egalitarian Shapley values endowed with such parameter systems.

5. Additional remarks

In this last section, we make three remarks regarding the solutions introduced in this paper. First, we discuss the relationship between the multi-choice Shapley value and the discrete serial cost sharing method introduced by Moulin and Shenker (1992) for discrete cost sharing problems. Then, we discuss a possible refinement of the Weighted Division values. Finally, we discuss a potential application of the multi-choice Egalitarian Shapley values.

The class of discrete cost sharing problems is introduced by Moulin and Shenker (1992) and studied by Moulin (1995), Albizuri et al. (2003), Sprumont (2005) and Bahel and Trudeau (2013) to cite a few. Fix $N = \{1, \dots, n\}$ a set of n different goods produced in indivisible units. A discrete cost sharing problem is a couple (q, C), where $q = (q_1, \dots, q_n)$. Each $q_i \in \mathbb{N}$ represent the demand in good i, and C is a non decreasing real-valued function on $\prod_{i \in N}\{0, 1, \dots, q_i\}$ such that $C(\Theta) = 0$. The total cost to be shared is given by $C(q)$. As shown by Calvo and Santos (2000) and Albizuri et al. (2003), one can view discrete cost sharing problems as a sub-class of multi-choice games. Indeed, q can be interpreted as a vector of maximal activity levels and C can be interpreted as characteristic function. Since C is a non decreasing real-valued function, it follows that discrete cost sharing problems can be viewed as the subclass of multi-choice games with a non decreasing real-valued characteristic function. We denote by $\mathrm{C} \subseteq \mathcal{G}$ the class of discrete cost sharing problems. In the cost sharing literature, a method on C is a map S that associates to each problem $(q, C) \in \mathrm{C}$ a vector $S(q, C) \in \mathbb{R}^n$ satisfying the budget balanced condition i.e. $\sum_{i \in N} S_i(q, C) = C(q)$. In this sense, a method on the class of discrete cost sharing problems differs from a value, which distributes a payoff to each pair in M^+. A popular cost sharing method for cost sharing problems is the **discrete serial cost sharing method** (denoted SCS afterward) introduced by Moulin and Shenker (1992).

In order to present the discrete serial cost sharing method, we define a specific TU-game. Consider $(q, C) \in \mathrm{C}$ and $j \leq \max_{k \in N} m_k$. Define the TU-game $(Q(j), w_j^{(q,C)})$ as

$$\forall E \subseteq Q(j), \quad w_j^{(q,C)}(E) = C\Big(((j-1) \wedge q_k)_{k \in N} + e_E\Big) - C\Big(((j-1) \wedge q_k)_{k \in N}\Big).$$

The worth $w_j^{(q,C)}(E)$ can be interpreted as the additional costs generated when each player (good) in E increases its activity level (demand) from $j-1$ to j while all the other players play either the activity level $j-1$ or their maximal feasible activity level if they are unable to do so. We denote by Sh the Shapley value (see Shapley (1953)) for TU-games. Albizuri et al. (2003) show that the discrete serial cost sharing admits the following expression

$$\forall i \in N, \quad SCS_i(q, C) = \sum_{j=1}^{q_i} Sh_i\Big(Q(j), w_j^{(q,C)}\Big). \tag{19}$$

Consider $(q, C) \in \mathrm{C}$. We say that a value f is **consistent** with a method S on C if we have

$$\forall i \in N, \quad S_i(q, C) - S_i(q - e_i, C) = f_{iq_i}(q, C).$$

In other words, f is consistent with S if it describes the variation in cost share a good undergoes when its demand increases by one unit. Whereas S describes the total cost share allocated to each good.

Proposition 4. *The multi-choice Shapley value is consistent with the discrete serial cost sharing method proposed by Moulin and Shenker (1992).*

Proof. See Appendix 7.9. □

Regarding the Weighted Division values, the weight system determining the value is independent from the variables of the problem. One can also consider weight systems that depends on (m, v). Fix $\beta^{ij}(m, v) = v(((j-1) \wedge m_k)_{k \in N \setminus i}, j)$ for each $i \in Q(j)$. This particular weight represents i's contribution for its activity level j to the synchronized coalition $(((j-1) \wedge m_k)_{k \in N})$. The resulting multi-choice Weighted Division value generalizes the proportional division value introduced by Moriarity (1975); Banker (1981) and studied by Zou et al. (2019) from TU-games to multi-choice games.

To conclude this section, we discuss a potential application of the multi-choice Egalitarian Shapley values. Consider a wage assignment problem in a firm as discussed in Abe and Nakada (2019). In a firm, each worker may receive a base salary in addition to a reward for its contribution to the firm. This wage assignment may be more secure than an assignment without a base salary given the possibility that employees cannot contribute due to raising children, for instance. Such assignment can obviously be viewed as a compromise between marginalism and egalitarianism. Abe and Nakada (2019) point out that the wage may be affected by exogenous variables independent of one's contributions, such as seniority or educational background. A way to address this problem is to model it with a multi-choice game and endow each employee with a maximal activity level representing its seniority or education background. In this case, if we assume that the assignment of an employee is equal to the total payoff she receives by a multi-choice Egalitarian Shapley value, then the base salary a worker receive corresponds to the egalitarian part of the value. Observe that, the base salary of an employee will increase with respect to her seniority or education. In addition, this increase depends on the parameter system used for the computation of the value. For instance, one could select a parameter system that operates a progressive compromise between marginalism and egalitarianism.

6. Conclusion

In this paper we proposed several multi-efficient values for multi-choice games. We introduced the multi-choice Shapley value and the multi-choice Weighted Division values. We also introduced the multi-choice Equal division value as a specific Weighted Division value. This paper is in line with the literature which deals with the trade-off between marginalism and egalitarianism using cooperative game theory since we introduce multi-choice Egalitarian Shapley values for multi-choice games. These values are computed as the convex combination of the multi-choice Shapley value and the multi-choice Equal division value. We provided at least one axiomatic characterization for each one of these values and families of values.

Some questions remain of interest for future research. It would be interesting to characterize multi-efficient solutions for multi-choice games with a structure. Regarding multi-choice games with a structure, several studies have already been conducted. Albizuri (2009) study multi-choice games with a coalition structure, Béal et al. (2012) study multi-choice games with communication constraints, and Lowing (2021) studies multi-choice games with a permission structure. The solution concepts proposed in these studies are not multi-efficient. It can be interesting to look for multi-efficient values for such games, since non-multi-efficient values cannot belong to the core.

Acknowledgments We thank Stéphane Gonzalez, Justin Leroux and Philippe Solal for useful comments. We thank Encarnación Algaba and the EURO21 organizing committee for letting us present this paper at the EURO21 conference. The authors want to thank GRDF (Gaz Réseau Distribution France) for their financial support through the ANRT (Association Nationale Recherche Technologie) doctoral program CIFRE (Conventions Industrielles de Formation par la REcherche).

7. Appendix

This section contains all the proofs of our results. In order to carry out our proofs, we introduce some definitions and remarks.

Remark 6. For $t \in \mathcal{M}$, $t \neq (0, \dots, 0)$, we formulate two distinct remarks regarding a minimal effort game (m, u_t). Each pair $(i, j) \in M^+$, such that $j > t_i$, is a dummy pair in (m, u_t). Let $(i, j), (i', j) \in M^+$ be two distinct pairs such that $j \leq t_i$ and $j \leq t_{i'}$. Both pairs are equal in (m, u_t).

Consider $t \in \mathcal{M}$, $t \neq (0, \dots, 0)$. The **Dirac game** $(m, \delta_t) \in \mathcal{G}$, induced by t, is defined as

$$\forall s \in \mathcal{M}, \quad \delta_t(s) = \begin{cases} 1 & \text{if } s_i = t_i \text{ for each } i \in N, \\ 0 & \text{otherwise.} \end{cases} \tag{20}$$

Remark 7. For $t \in \mathcal{M}$, $t \neq \Theta$, we formulate two distinct remarks regarding a Dirac game (m, δ_t). Each pair $(i, j) \in M^+$, such that $j > t_i + 1$, is a dummy pair in (m, δ_t). If there exists two distinct players $i, i' \in N$ such that $t_i = t_{i'}$, then (i, t_i) and $(i', t_{i'})$ are equal in (m, δ_t).

For each multi-choice game $(m, v) \in \mathcal{G}$, the characteristic function v admits a unique linear decomposition in terms of Dirac games as follows

$$v = \sum_{t \leq m} v(t)\delta_t. \tag{21}$$

7.1. Proof of Proposition 1

We show that the multi-choice Shapley value admits an alternative expression given by (14). Observe that there are $|\overline{O}_l| = |Q(l)|!$ ways to order the pairs in $M^{+,l}$, for each $l \leq \max_{k\in N} m_k$. Additionally, there are $\prod_{l<j} |Q(l)|!$ ways to order the pairs in $M^{+,1}$, then the pairs in $M^{+,2}$, and so on, until the pairs in $M^{+,l-1}$. Similarly, there are $\prod_{l>j} |Q(l)|!$ ways to order the pairs in $M^{+,j+1}$, then the pairs in $M^{+,j+2}$, and so on. Observe that, for each $\sigma \in \overline{O}$, there exists exactly one order $\sigma_j \in \overline{O}_j$ such that $s^{\sigma,\sigma(i,j)} = s^{\sigma_j,\sigma_j(i,j)}$. Additionally, for each $\sigma_j \in \overline{O}_j$, there are $\prod_{l<j} |Q(l)|! \times \prod_{l>j} |Q(l)|!$ orders $\sigma \in \overline{O}$ such that $s^{\sigma,\sigma(i,j)} = s^{\sigma_j,\sigma_j(i,j)}$. It follows that, for each $(m, v) \in \mathcal{G}$ and $(i, j) \in M^+$, we have

$$\begin{aligned}\varphi_{ij}(m,v) &= \frac{1}{\prod_{l\leq \max\limits_{k\in N} m_k} |Q(l)|!} \sum_{\sigma\in\overline{O}} \Big[v(s^{\sigma,\sigma(i,j)}) - v(s^{\sigma,\sigma(i,j)-1})\Big] \\ &= \frac{(\prod_{l<j} |Q(l)|!)(\prod_{l>j} |Q(l)|!)}{\prod_{\leq \max\limits_{k\in N} m_k} |Q(l)|!} \sum_{\sigma_j\in\overline{O}_j} \Big[v(s^{\sigma_j,\sigma_j(i,j)}) - v(s^{\sigma_j,\sigma_j(i,j)-1})\Big].\end{aligned}$$

The first line comes from the definition of the multi-choice Shapley value, the second line follows from (13) and the fact that there are $\prod_{l<j} |Q(l)|! \times \prod_{l>j} |Q(l)|!$ orders $\sigma \in \overline{O}$ such that $s^{\sigma,\sigma(i,j)} = s^{\sigma_j,\sigma_j(i,j)}$, for each $\sigma_j \in \overline{O}_j$. Simplifying the expression, we obtain the desired result

$$\forall (i,j) \in M^+, \quad \varphi_{ij}(m,v) = \frac{1}{|Q(j)|!} \sum_{\sigma_j\in\overline{O}_j} \Big[v(s^{\sigma_j,\sigma_j(i,j)}) - v(s^{\sigma_j,\sigma_j(i,j)-1})\Big].$$

□

7.2. Proof of Proposition 2

Let $(m, v) \in \mathcal{G}$ and f a value satisfying (IIR) and (EL). For each $(i, j) \in M^+$, (IIR) implies

$$f_{ij}(m,v) = f_{ij}((m_{-i}, j), v). \tag{22}$$

Combining (22) with (EL) we obtain, for each $(i', j) \in M^+$ such that $i' \neq i$,

$$\begin{aligned} & f_{ij}(m,v) - f_{ij}((m_{-i},j),v) = f_{i'j}(m,v) - f_{i'j}((m_{-i},j),v) \\ \iff \quad & f_{ij}((m_{-i},j),v) - f_{ij}((m_{-i},j),v) = f_{i'j}(m,v) - f_{i'j}((m_{-i},j),v) \\ \iff \quad & f_{i'j}(m,v) = f_{i'j}((m_{-i},j),v). \end{aligned}$$

Therefore, by successive applications of (EL) and (IR), we obtain the desired result. □

7.3. Proof of Proposition 3

Let $(m, v) \in \mathcal{G}$, $h \leq \max_{k\in N} m_k$ and f a value satisfying (E) and (IR). Consider the sub-game $((h \wedge m_k)_{k\in N}, v)$. By (E), it holds that

$$\sum_{i\in N} \sum_{j=1}^{h\wedge m_i} f_{ij}((h \wedge m_k)_{k\in N}, v) = v((h \wedge m_k)_{k\in N}). \tag{23}$$

By (IR), we have

$$\sum_{i\in N} \sum_{j=1}^{h\wedge m_i} f_{ij}((h \wedge m_k)_{k\in N}, v) = \sum_{i\in N} \sum_{j=1}^{h\wedge m_i} f_{ij}(m, v). \tag{24}$$

Combining (23) with (24), we obtain the desired result. □

7.4. Proof of Theorem 1

The proof is divided in two steps.

Step 1: we show that φ satisfies all the axioms of the statement of Theorem 1.

For each $(m,v) \in \mathcal{G}$, we have

$$\begin{aligned}\sum_{i\in N}\sum_{j\in M_i^+}\varphi_{ij}(m,v) &= \sum_{j\leq \max\limits_{k\in N} m_k}\sum_{i\in Q(j)}\varphi_{ij}(m,v)\\ &\overset{(14)}{=} \sum_{j\leq \max\limits_{k\in N} m_k}\frac{1}{|Q(j)|!}\sum_{\sigma_j\in\overline{O}_j}\sum_{i\in Q(j)}\Big[v(s^{\sigma_j,\sigma_j(i,j)})-v(s^{\sigma_j,\sigma_j(i,j)-1})\Big].\end{aligned}$$

Observe that, for each $\sigma_j \in \overline{O}_j$, we have

$$\sum_{i\in Q(j)}\Big[v(s^{\sigma_j,\sigma_j(i,j)})-v(s^{\sigma_j,\sigma_j(i,j)-1})\Big] = v(s^{\sigma_j,|Q(j)|})-v(s^{\sigma_j,0}).$$

By (13), for each $\sigma_j \in \overline{O}_j$, we have

$$s^{\sigma_j,|Q(j)|} = (j\wedge m_k)_{k\in N}, \quad \text{and} \quad s^{\sigma_j,0} = ((j-1)\wedge m_k)_{k\in N}.$$

It follows that

$$\sum_{i\in N}\sum_{j\in M_i^+}\varphi_{ij}(m,v) = \sum_{j\leq \max\limits_{k\in N} m_k}\frac{1}{|Q(j)|!}\sum_{\sigma_j\in\overline{O}_j}\Big[v((j\wedge m_k)_{k\in N})-v(((j-1)\wedge m_k)_{k\in N})\Big]. \tag{25}$$

Since the quantity $v((j\wedge m_k)_{k\in N})-v(((j-1)\wedge m_k)_{k\in N})$ is independent from any order $\sigma_j \in \overline{O}_j$, it follows that it is summed as many times in (25) as there are orders in $\overline{O}_j$. Therefore, we have

$$\begin{aligned}\sum_{i\in N}\sum_{j\in M_i^+}\varphi_{ij}(m,v) &= \sum_{j\leq \max\limits_{k\in N} m_k}\frac{1}{|Q(j)|!}Q(j)!\Big[v((j\wedge m_k)_{k\in N})-v(((j-1)\wedge m_k)_{k\in N}\Big]\\ &= \sum_{j\leq \max\limits_{k\in N} m_k}\Big[v((j\wedge m_k)_{k\in N})-v(((j-1)\wedge m_k)_{k\in N})\Big]\\ &= v(m),\end{aligned}$$

which shows that the value satisfies (E). By definition of the multi-choice Shapley value (see (14)), the payoff of a pair is independent from any activities different from the activity level contained in this pair. Therefore, we have that φ satisfies (IIR) and (EL). (L) follows directly from (14). By (14), by definition of equal pairs (see (3)) and by definition of dummy pairs (see (2)), we have that φ satisfies (ES) and (D). This concludes Step 1.

Step 2: To complete the proof, it remains to show that there is at most one value satisfying all the axioms of the statement of Theorem 1. Take any f satisfying all the axioms of the statement of Theorem 1. Consider any $(m,v) \in \mathcal{G}$. We know that each multi-choice game admits a unique linear decomposition in terms of minimal effort games $\{u_s\}_{s\in\mathcal{M}}$. Consider $s \in \mathcal{M}$ such that $s \neq \Theta$. The set of top pairs $T(s)$ (see (1)) can be re-written as

$$T(s) = \{(i,s^T) \in M^{+,s^T} : s_i = s^T\},$$

where $s^T = \max_{i\in N} s_i$. Let us show that $f(m, u_s)$ is uniquely determined. We divide this Step 2 into several sub-steps.

Step 2.1. Let us show that, for each $(i,j) \in M^+$ such that $j \neq s^T$, $f_{ij}(m, u_s)$ is uniquely determined.

Step 2.1.1. If $j < s^T$, then $(j \wedge m_k)_{k\in N} \ngeq s$. It follows that $((j \wedge m_k)_{k\in N}, u_s)$ is the null game since $u_s(t) = 0$ for each $t \leq (j \wedge m_k)_{k\in N}$. Recall that each pair is a dummy pair (see (2)) in the null game. Since f satisfies (IIR) and (EL), by Proposition 2 it also satisfies (IR). Combining these observations with (D), for each $(i,j) \in M^+$ such that $j < s^T$, we obtain

$$f_{ij}(m, u_s) \overset{(IR)}{=} f_{ij}((j \wedge m_k)_{k\in N}, u_s) \overset{(D)}{=} 0.$$

Step 2.1.2. If $j > s^T$ then, by Remark 6, (i,j) is a dummy pair in (m, u_s). By (D), for each $(i,j) \in M^+$ such that $j > s^T$, we have

$$f_{ij}(m, u_s) \overset{(D)}{=} 0.$$

We have shown that $f_{ij}(m, u_s) = 0$, and so is uniquely determined for each $(i,j) \in M^+$ such that $j \neq s^T$.

Step 2.2. Now, we show that, for each pair $(i,j) \in M^+$ such that $j = s^T$ i.e. each pair $(i, s^T) \in M^{+,s^T}$, $f_{is^T}(m, u_s)$ is uniquely determined.

To that end, consider the game $(m, w) \in \mathcal{G}$ defined as

$$\forall t \leq m, \quad w(t) = u_s(t) - \sum_{(i,s^T)\in T(s)} \varphi_{is^T}(m, u_s) u_{(0_{-i}, s^T)}(t). \tag{26}$$

Step 2.2.1. We show that

$$\sum_{(i,s^T)\in M^{+,s^T}} f_{is^T}(m, w) = 0.$$

We consider pairs in M^{+,s^T}. By definition of M^{+,s^T}, observe that

$$\sum_{i\in Q(s^T)} f_{is^T}(m, w) = \sum_{(i,s^T)\in M^{+,s^T}} f_{is^T}(m, w).$$

We have that any pair $(i, s^T) \in M^{+,s^T}$ is either in $T(s)$ or not. Since f satisfies (E) and (IR), by Proposition 3, f also satisfies (ME). Therefore we have

$$\begin{aligned}
\sum_{(i,s^T)\in M^{+,s^T}} f_{is^T}(m, w) &\overset{(ME)}{=} w((s^T \wedge m_k)_{k\in N}) - w(((s^T - 1) \wedge m_k)_{k\in N}) \\
&\overset{(26)}{=} u_s((s^T \wedge m_k)_{k\in N}) - \sum_{(i,s^T)\in T(s)} \varphi_{is^T}(m, u_s) u_{(0_{-i}, s^T)}((s^T \wedge m_k)_{k\in N}) \\
&\quad - u_s(((s^T - 1) \wedge m_k)_{k\in N}) \\
&\quad + \sum_{(i,s^T)\in T(s)} \varphi_{is^T}(m, u_s) u_{(0_{-i}, s^T)}(((s^T - 1) \wedge m_k)_{k\in N}).
\end{aligned} \tag{27}$$

Observe that $((s^T \wedge m_k)_{k\in N}) \geq s \geq ((0_{-i}, s^T)$, $(((s^T-1) \wedge m_k)_{k\in N}) \not\geq s$ and $(((s^T-1) \wedge m_k)_{k\in N}) \not\geq (0_{-i}, s^T)$, where $(i, s^T) \in T(s)$. By definition of a minimal effort game (5), we have

$$u_s((s^T \wedge m_k)_{k\in N}) = 1, \qquad \text{and,} \quad \forall (i, s^T) \in M^{+,s^T}, \quad u_{(0_{-i},s^T)}((s^T \wedge m_k)_{k\in N}) = 1,$$
$$u_s(((s^T-1) \wedge m_k)_{k\in N}) = 0, \qquad \text{and,} \quad \forall (i, s^T) \in M^{+,s^T}, \quad u_{(0_{-i},s^T)}(((s^T-1) \wedge m_k)_{k\in N}) = 0.$$

It follows that (27) becomes

$$\sum_{(i,s^T)\in M^{+,s^T}} f_{is^T}(m, w) = 1 - \sum_{(i,s^T)\in T(s)} \varphi_{is^T}(m, u_s) - 0 + 0. \tag{28}$$

Observe that, since $(i, s^T) \notin T(s)$ if and only if $s^T > s_i$, then each $(i, s^T) \notin T(s)$ is a dummy pair in (m, u_s). Since φ satisfies (D), we have $\varphi_{is^T}(m, u_s) = 0$, for each $(i, s^T) \notin T(s)$. Since φ satisfies (E), (IIR) and (EL), by Corollary 1 the value satisfies (ME). Therefore, we have

$$\begin{aligned}
\sum_{(i,s^T)\in T(s)} \varphi_{is^T}(m, u_s) &\overset{(D)}{=} \sum_{(i,s^T)\in T(s)} \varphi_{is^T}(m, u_s) + \sum_{(i,s^T)\notin T(s)} \varphi_{is^T}(m, u_s) \\
&= \sum_{(i,s^T)\in M^{+,s^T}} \varphi_{is^T}(m, u_s) \\
&\overset{(ME)}{=} u_s((s^T \wedge m_k)_{k\in N}) \\
&= 1.
\end{aligned}$$

Therefore, (28) becomes

$$\sum_{(i,s^T)\in M^{+,s^T}} f_{is^T}(m, w) = 1 - 1 = 0, \tag{29}$$

which concludes Step 2.2.1.

Step 2.2.2. We show that, for each $(i, s^T) \in M^{+,s^T}$, we have

$$f_{is^T}(m, w) = 0.$$

We know that each pair $(i, s^T) \notin T(s)$ is a dummy pair in (m, u_s). Moreover, each pair $(i, s^T) \notin T(s)$ is a dummy pair in each $(m, u_{0_{-i'},s^T})$, $(i', s^T) \in T(s)$. Indeed, in $(m, u_{0_{-i'},s^T})$, (i', s^T) is the only productive pair and all other pairs are dummy pairs. It follows that each pair $(i, s^T) \notin T(s)$ is a dummy pair in (m, w). By (D), for each $(i, s^T) \notin T(s)$, we have

$$f_{is^T}(m, w) = 0. \tag{30}$$

It follows that

$$\begin{aligned}
\sum_{(i,s^T)\in M^{+,s^T}} f_{is^T}(m, w) &= \sum_{(i,s^T)\in T(s)} f_{is^T}(m, w) + \sum_{(i,s^T)\notin T(s)} f_{is^T}(m, w) \\
&\overset{(D)}{=} \sum_{(i,s^T)\in T(s)} f_{is^T}(m, w) + 0 \\
&\overset{(29)}{=} 0.
\end{aligned} \tag{31}$$

To complete the proof of Step 2.2.2 and to apply (ET), it remains to show that if there exist two distinct pairs $(i, s^T), (i', s^T) \in T(s)$, then these pairs are equal. By Remark 6 two distinct pairs $(i, s^T), (i', s^T) \in T(s)$ are equal in (m, u_s). Since φ satisfies (ET), it follows that $\varphi_{is^T}(m, u_s) = \varphi_{i's^T}(m, u_s)$. By definition of a minimal effort game, for each $t \in \mathcal{M}$ such that $t_i = t_{i'} = s^T - 1$, we have

$$u_{(0_{-i},s^T)}(t) = u_{(0_{-i'},s^T)}(t) = 0, \tag{32}$$

$$\text{and} \quad u_{(0_{-i},s^T)}(t + e_i) = u_{(0_{-i'},s^T)}(t + e_{i'}) = 1. \tag{33}$$

Therefore, for each $t \in \mathcal{M}$ such that $t_i = t_{i'} = s^T - 1$, we have

$$\begin{aligned}
\sum_{(k,s^T)\in T(s)} \varphi_{ks^T}(m, u_s) u_{(0_{-k},s^T)}(t + e_i) &= \sum_{(k,s^T)\in T(s)} \varphi_{ks^T}(m, u_s) u_{(0_{-k},s^T)}(t) + \varphi_{is^T}(m, u_s) \\
&= \sum_{(k,s^T)\in T(s)} \varphi_{ks^T}(m, u_s) u_{(0_{-k},s^T)}(t) + \varphi_{i's^T}(m, u_s) \\
&= \sum_{(k,s^T)\in T(s)} \varphi_{ks^T}(m, u_s) u_{(0_{-k},s^T)}(t + e_{i'}).
\end{aligned}$$

Where the first equality and third equality follow from (32) and (33), and the second equality follows from $\varphi_{is^T}(m, u_s) = \varphi_{i's^T}(m, u_s)$, since φ satisfies (ET). It follows that

$$w(t + e_i) = w(t + e_{i'}),$$

for each $t \in \mathcal{M}$ such that $t_i = t_{i'} = s^T - 1$, showing that $(i, s^T), (i', s^T) \in T(s)$ are equal pairs in (m, w). By (ES) we have $sign(f_{is^T}(m, w)) = sign(f_{i's^T}(m, w))$. It follows from (31) that, for each $(i, s^T) \in T(s)$, we have

$$f_{is^T}(m, w) = 0. \tag{34}$$

Combining (30) with (34), the proof of Step 2.2.2 is complete.

Step 2.2.3. We show that for each $(i, s^T) \in M^{+,s^T}$, we have

$$f_{is^T}(m, u_s) = \varphi_{is^T}(m, u_s).$$

By (26), (34) and (L), for each $(i, s^T) \in M^{+,s^T}$, we have

$$\begin{aligned}
f_{is^T}(m, w) &\overset{(26),(L)}{=} f_{is^T}(m, u_s) - f_{is^T}\Big(m, \sum_{(k,s^T)\in T(s)} \varphi_{ks^T}(m, u_s) u_{(0_{-k},s^T)}\Big) \\
\iff f_{is^T}(m, u_s) &\overset{(34)}{=} f_{is^T}\Big(m, \sum_{(k,s^T)\in T(s)} \varphi_{ks^T}(m, u_s) u_{(0_{-k},s^T)}\Big) \\
&\overset{(L)}{=} \sum_{(k,s^T)\in T(s)} \varphi_{ks^T}(m, u_s) f_{is^T}\Big(m, u_{(0_{-k},s^T)}\Big).
\end{aligned}$$

Additionally, by (D) and (ME), we have that $f_{is^T}(m, u_{(0_{-i},s^T)}) = 1$ since (i, s^T) is the only productive pair in $(m, u_{(0_{-i},s^T)})$. Therefore, for each $(i, s^T) \in M^{+,s^T}$ we have

$$\varphi_{ks^T}(m, u_s) f_{is^T}\Big(m, u_{(0_{-k},s^T)}\Big) = \begin{cases} \varphi_{ks^T}(m, u_s) & \text{if } k = i, \\ 0 & \text{otherwise.} \end{cases}$$

It follows that, for each $(i, s^T) \in M^{+,s^T}$, we have

$$f_{is^T}(m, u_s) = \varphi_{is^T}(m, u_s),$$

therefore $f_{is^T}(m, u_s)$ is uniquely determined. This concludes Step 2.2.3.

From Step 2.1 and Step 2.2, we conclude that $f(m, u_s)$ is uniquely determined. By (L), we have that $f(m, v)$ is uniquely determined, which concludes the proof of Theorem 1. $\square$

7.5. Proof of Corollary 2

By the the proof of Theorem 1, φ satisfies (E), (IIR), (EL), (L), (D) and (ET). Consider $(m, u_s) \in \mathcal{G}$, $s \in \mathcal{M}$ such that $s \neq \Theta$. Similarly to (30), for each $(i, j) \notin T(s)$, (D) and (ME) imply

$$\varphi_{ij}(m, u_s) = 0. \tag{35}$$

All pairs in $T(s)$ pairs are equal in (m, u_s), thus, by (ET), we have

$$\varphi_{is^T}(m, u_s) = \ldots = \varphi_{i's^T}(m, u_s). \tag{36}$$

By (E) and (L), we obtain the desired result. $\square$

7.6. Proof of Theorem 2

From Theorem 1, we know that φ satisfies (E), (IIR), (EL) and (ES). By definition (see (14)), the multi-choice Shapley value satisfies (SM)

Next, we show that φ is the unique value satisfying all the axioms of the statement of Theorem 2. Take any f satisfying all the axioms of the statement of Theorem 2 and consider any $(m, v) \in \mathcal{G}$. Recall that $(m, v) \in \mathcal{G}$ can be rewritten as $(m, \sum_{t \in M} \Delta_v(t) u_t)$. We define the set of coalitions for which the Harsanyi dividend is non-null as

$$\mathcal{T}(v) = \{t \in \mathcal{M} \mid \Delta_v(t) \neq 0\}.$$

By induction on the cardinality of $\mathcal{T}(v)$, we show that

$$f(m, v) = \varphi(m, v).$$

Initialization: If $|\mathcal{T}(v)| = 0$, then each dividend is null. The only game $(m, v) \in \mathcal{G}$ such that $|\mathcal{T}(v)| = 0$ is the null game. Recall that $M^{+,j} = \{(i, j) \in M^+ : i \in Q(j)\}$. Since f satisfies (E), (IIR) and (EL), by Corollary 1, it satisfies (ME). It follows that, for each $j \leq \max_{k \in N} m_k$, we have

$$\begin{aligned}\sum_{(i,j) \in M^{+,j}} f_{ij}(m, v) &\overset{(ME)}{=} v((j \wedge m_k)_{k \in N}) - v(((j-1) \wedge m_k)_{k \in N}) \\ &= 0. \end{aligned} \tag{37}$$

Recall that any two distinct pairs $(i, j)(i', j) \in M^{+,j}$ are equal in the null game (m, v). Therefore, by (ES), we have

$$sign(f_{ij}(m, v)) = sign(f_{i'j}(m, v)). \tag{38}$$

Combining (37) and (38), for each $j \leq \max_{k \in N} m_k$ and each $(i, j) \in M^{+,j}$, we obtain

$$f_{ij}(m, v) = 0.$$

Recall also that each pair is a dummy pair in the null game. Since φ satisfies (D), for each $j \leq \max_{k \in N} m_k$ and each $(i,j) \in M^{+,j}$, we have

$$\varphi_{ij}(m,v) \overset{(D)}{=} 0 = f_{ij}(m,v).$$

This concludes the initialization.

Hypothesis: Fix $r \in \mathbb{N}$ such that $r < |\mathcal{M}| - 1$. We assume that, for each $(m,v) \in \mathcal{G}$ such that $|\mathcal{T}(v)| \leq r$, we have

$$f(m,v) = \varphi(m,v).$$

Induction: Consider any $(m,v) \in \mathcal{G}$ such that $|\mathcal{T}(v)| = r+1$. Let us show that

$$f(m,v) = \varphi(m,v).$$

We define the minimum of the set $\mathcal{T}(v)$ as

$$p = \bigwedge_{t \in \mathcal{T}(v)} t.$$

Two cases can be distinguished. First, assume that $p \neq \Theta$. Consider any pair $(i,j) \in M^+$ such that $j > p_i$. By definition of p, there exists a $t \in \mathcal{T}(v)$ such that $j > t_i$. For such t, consider the game $(m, v - \Delta_v(t)u_t)$. By definition of a minimal effort game (5) and Remark 1, we have that (i,j) is a dummy pair in $(m, \Delta_v(t)u_t)$. Therefore (i,j) has the same marginal contributions in (m,v) and in $(m, v - \Delta_v(t)u_t)$. Moreover, observe that $|\mathcal{T}(v)| > |\mathcal{T}(v - \Delta_v(t)u_t)|$. Therefore, we have $r \geq |\mathcal{T}(v - \Delta_v(t)u_t)|$. By the induction hypothesis and (SM), for each $(i,j) \in M^+$ such that $j > p_i$, we have

$$f_{ij}(m,v) \overset{SM}{=} f_{ij}(m, v - \Delta_v(t)u_t) \overset{Hyp}{=} \varphi_{ij}(m, v - \Delta_v(t)u_t) \overset{SM}{=} \varphi_{ij}(m,v). \tag{39}$$

Next, we assume that $p = \Theta$. For each $(i,j) \in M^+$, there exists a $t \in \mathcal{T}(v)$ such that $j > t_i$. In this case, (39) holds for each $(i,j) \in M^+$ and the proof is complete.

It remains to show that, if $p \neq \Theta$, then for each $(i,j) \in M^+$ such that $j \leq p_i$, we have

$$f_{ij}(m,v) = \varphi_{ij}(m,v).$$

We proceed in two steps.

Step 1. We define the game $(m,w) \in \mathcal{G}$ as

$$w = v - \sum_{\substack{(i,j) \in M^+ \\ j \leq p_i}} \varphi_{ij}(m,v) u_{(0_{-i},j)}, \tag{40}$$

and we show that, for each $(i,l) \in M^+$ such that $l \leq p_i$, we have

$$f_{il}(m,w) = 0. \tag{41}$$

Step 1.1. To that end, we show that

$$\sum_{\substack{(i,l) \in M^{+,l} \\ l \leq p_i}} f_{il}(m,w) = 0.$$

By Corollary 1, f satisfies (ME). By (ME) and (40), for each $l \leq \max_{k\in N} m_k$, we have

$$
\begin{aligned}
&\sum_{(i,l)\in M^{+,l}} f_{il}(m,w) \overset{ME}{=} w((l\wedge m_k)_{k\in N}) - w((l-1\wedge m_k)_{k\in N}) \\
\iff &\sum_{\substack{(i,l)\in M^{+,l}\\ l\leq p_i}} f_{il}(m,w) = w((l\wedge m_k)_{k\in N}) - w((l-1\wedge m_k)_{k\in N}) - \sum_{\substack{(i,l)\in M^{+,l}\\ l>p_i}} f_{il}(m,w) \\
&\qquad\overset{(40)}{=} v((l\wedge m_k)_{k\in N}) - v((l-1\wedge m_k)_{k\in N}) - \sum_{\substack{(i,j)\in M^{+}\\ j\leq p_i}} \varphi_{ij}(m,v)u_{(0_{-i},j)}((l\wedge m_k)_{k\in N}) \\
&\qquad\quad + \sum_{\substack{(i,j)\in M^{+}\\ j\leq p_i}} \varphi_{ij}(m,v)u_{(0_{-i},j)}((l-1\wedge m_k)_{k\in N}) - \sum_{\substack{(i,l)\in M^{+,l}\\ l>p_i}} f_{il}(m,w).
\end{aligned} \tag{42}
$$

Before proceeding further into the computation of (42), observe that

$$
\begin{aligned}
&- \sum_{\substack{(i,j)\in M^{+}\\ j\leq p_i}} \varphi_{ij}(m,v)u_{(0_{-i},j)}((l\wedge m_k)_{k\in N}) + \sum_{\substack{(i,j)\in M^{+}\\ j\leq p_i}} \varphi_{ij}(m,v)u_{(0_{-i},j)}((l-1\wedge m_k)_{k\in N}) \\
=& - \sum_{\substack{(i,j)\in M^{+}\\ j\leq p_i\\ j<l}} \varphi_{ij}(m,v)u_{(0_{-i},j)}((l\wedge m_k)_{k\in N}) - \sum_{\substack{(i,j)\in M^{+}\\ j\leq p_i\\ j=l}} \varphi_{ij}(m,v)u_{(0_{-i},j)}((l\wedge m_k)_{k\in N}) \\
&- \sum_{\substack{(i,j)\in M^{+}\\ j\leq p_i\\ j>l}} \varphi_{ij}(m,v)u_{(0_{-i},j)}((l\wedge m_k)_{k\in N}) \\
&+ \sum_{\substack{(i,j)\in M^{+}\\ j\leq p_i\\ j<l}} \varphi_{ij}(m,v)u_{(0_{-i},j)}((l-1\wedge m_k)_{k\in N}) + \sum_{\substack{(i,j)\in M^{+}\\ j\leq p_i\\ j\geq l}} \varphi_{ij}(m,v)u_{(0_{-i},j)}((l-1\wedge m_k)_{k\in N})
\end{aligned}
$$

By definition, $p_i \leq m_i$, for each $i \in N$. For each $i \in N$ and $j \leq p_i \leq m_i$, we have

$$
u_{(0_{-i},j)}((l\wedge m_k)_{k\in N}) \begin{cases} 1 & \text{if } j \leq (l\wedge m_i), \\ 0 & \text{otherwise.} \end{cases}
$$

It follows that

$$
\begin{aligned}
&- \sum_{\substack{(i,j)\in M^{+}\\ j\leq p_i}} \varphi_{ij}(m,v)u_{(0_{-i},j)}((l\wedge m_k)_{k\in N}) + \sum_{\substack{(i,j)\in M^{+}\\ j\leq p_i}} \varphi_{ij}(m,v)u_{(0_{-i},j)}((l-1\wedge m_k)_{k\in N}) \\
=& - \sum_{\substack{(i,j)\in M^{+}\\ j\leq p_i\\ j<l}} \varphi_{ij}(m,v) - \sum_{\substack{(i,j)\in M^{+}\\ j\leq p_i\\ j=l}} \varphi_{ij}(m,v) + \sum_{\substack{(i,j)\in M^{+}\\ j\leq p_i\\ j<l}} \varphi_{ij}(m,v) \\
=& - \sum_{\substack{(i,j)\in M^{+}\\ j\leq p_i\\ j=l}} \varphi_{ij}(m,v) = - \sum_{\substack{(i,l)\in M^{+,l}\\ l\leq p_i}} \varphi_{il}(m,v).
\end{aligned} \tag{43}
$$

By (43), (42) becomes

$$\sum_{\substack{(i,l)\in M^{+,l}\\ l\leq p_i}} f_{il}(m,w) = v((l\wedge m_k)_{k\in N}) - v((l-1\wedge m_k)_{k\in N} - \sum_{\substack{(i,l)\in M^{+,l}\\ l\leq p_i}} \varphi_{il}(m,v) - \sum_{\substack{(i,l)\in M^{+,l}\\ l> p_i}} f_{il}(m,w). \tag{44}$$

By (39), for each $(i,l) \in M^{+,l}$ such that $l > p_i$, we have

$$f_{il}(m,w) = \varphi_{il}(m,w). \tag{45}$$

Combining (44) and (45), we obtain

$$\sum_{\substack{(i,l)\in M^{+,l}\\ l\leq p_i}} f_{il}(m,w) = v((l\wedge m_k)_{k\in N}) - v((l-1\wedge m_k)_{k\in N} - \sum_{\substack{(i,l)\in M^{+,l}\\ l\leq p_i}} \varphi_{il}(m,v) - \sum_{\substack{(i,l)\in M^{+,l}\\ l> p_i}} \varphi_{il}(m,w). \tag{46}$$

Moreover, each $(i,l) \in M^{+,l}$, such that $l > p_i$, is a dummy pair in $(m, u_{0_{-i},j})$, for each $(i,j) \in M^+$ such that $j \leq p_i$. By definition of (m,w) (see (40)), it follows that each pair (i,l), such that $l > p_i$, has the same marginal contributions in (m,w) and in (m,v). Since φ satisfies (SM), for each $(i,l) \in M^{+,l}$ such that $l > p_i$, we have

$$\varphi_{il}(m,w) = \varphi_{il}(m,v). \tag{47}$$

Combining (46), (47) and the fact that φ satisfies (ME), we obtain

$$\begin{aligned}\sum_{\substack{(i,l)\in M^{+,l}\\ l\leq p_i}} f_{il}(m,w) =& v((l\wedge m_k)_{k\in N}) - v((l-1\wedge m_k)_{k\in N} - \sum_{(i,l)\in M^{+,l}} \varphi_{il}(m,v)\\ \overset{(ME)}{=}& 0.\end{aligned} \tag{48}$$

This concludes Step 1.1.

Step 1.2. We show that all the pairs $(i,l) \in M^{+,l}$, such that $l \leq p_i$, are equal in (m,w).

By definition of $M^{+,l}$, we have that $l \geq 1$. Consider two pairs $(i,l), (i',l) \in M^{+,l}$ such that $l \leq p_i$ and $l \leq p_{i'}$. Since $p = \bigwedge_{t\in\mathcal{T}(v)} t$, we have that each $t \in \mathcal{T}(v)$ verifies $t_i \geq l$ and $t_{i'} \geq l$. In other words, for each $s \in \mathcal{M}$, such that $s_i < l$ or $s_{i'} < l$, we have $\Delta_v(s) = 0$. Therefore, for each $s \in \mathcal{M}$ such that $s_i = s_{i'} = l-1$, we have

$$v(s+e_i) = v(s+e_{i'}) = 0. \tag{49}$$

Therefore, (i,l) and (i',l) are equal in (m,v). Since φ satisfies (ET) and by (40), for each $s \in \mathcal{M}$ such that $s_i = s_{i'} = l-1$, we have

$$w(s+e_i) \overset{(40)}{=} v(s+e_i) - \sum_{\substack{(h,j)\in M^{+}\\ j\leq p_h}} \varphi_{hj}(m,v) u_{(0_{-h},j)}(s+e_i)$$

$$= v(s+e_i) - \sum_{\substack{(h,j)\in M^+\\ j\le p_h\\ h\neq i,i'}} \varphi_{hj}(m,v)u_{(0_{-h},j)}(s+e_i) - \sum_{\substack{j\le p_{i'}\\ j\le l-1}} \varphi_{i'j}(m,v)u_{(0_{-i'},j)}(s+e_i)$$

$$- \sum_{\substack{j\le p_i\\ j\le l-1}} \varphi_{ij}(m,v)u_{(0_{-i},j)}(s+e_i) - \varphi_{il}(m,v)u_{(0_{-i},l)}(s+e_i)$$

$$= v(s+e_i) - \sum_{\substack{(h,j)\in M^+\\ j\le p_h\\ h\neq i,i'}} \varphi_{hj}(m,v)u_{(0_{-h},j)}(s+e_i) - \sum_{\substack{j\le p_{i'}\\ j\le l-1}} \varphi_{i'j}(m,v)u_{(0_{-i'},j)}(s+e_i)$$

$$- \sum_{\substack{j\le p_i\\ j\le l-1}} \varphi_{ij}(m,v)u_{(0_{-i},j)}(s+e_i) - \varphi_{il}(m,v)$$

$$\overset{(ET),(49)}{=} v(s+e_{i'}) - \sum_{\substack{(h,j)\in M^+\\ j\le p_h\\ h\neq i,i'}} \varphi_{hj}(m,v)u_{(0_{-h},j)}(s+e_{i'}) - \sum_{\substack{j\le p_{i'}\\ j\le l-1}} \varphi_{i'j}(m,v)u_{(0_{-i'},j)}(s+e_{i'})$$

$$- \sum_{\substack{j\le p_i\\ j\le l-1}} \varphi_{ij}(m,v)u_{(0_{-i},j)}(s+e_{i'}) - \varphi_{i'l}(m,v)$$

$$\overset{(40)}{=} w(s+e_{i'}).$$

Therefore, two pairs (i,l) and (i',l), such that $l \le p_i$ and $l \le p_{i'}$, are equal in (m,w). This concludes Step 1.2

By (ES), we have

$$sign(f_{il}(m,w)) = sign(f_{i'l}(m,w)). \tag{50}$$

Combining (48) and (50), for each $(i,l) \in M^+$ such that $l \le p_i$, we obtain

$$f_{il}(m,w) = 0,$$

which concludes Step 1.

Step 2. For each $(i,j) \in M^+$, such that $0 < j \le p_i$, we define the game $(m,w^{ij}) \in \mathcal{G}$ as

$$w^{ij} = v - \varphi_{ij}(m,v)u_{(0_{-i},j)}. \tag{51}$$

In this step, we first show that, for each $(i,j) \in M^+$ such that $j \le p_i$, we have

$$\varphi_{ij}(m,v) = f_{ij}(m,v) - f_{ij}(m,w^{ij}).$$

The game (m,w^{ij}) is defined in such a way that the pair (i,j) has the same marginal contribution in (m,w) as in (m,w^{ij}). Indeed, observe that the pair (i,j) has null marginal contributions to coalition in each game $(m,u_{(0_{-i'},j')})$ such that $i' \neq i$ or $i' = i$ and $j' \neq j$. Therefore, by (SM), for each $(i,j) \in M^+$ such that $j \le p_i$, we have

$$f_{ij}(m,w) = f_{ij}(m,w^{ij}). \tag{52}$$

Additionally, by (ME), (51) and the definition of a minimal effort game (see (5)), we have

$$\begin{aligned}
\sum_{(k,j)\in M^{+,j}} f_{kj}(m, w^{ij}) &\overset{ME}{=} w^{ij}((j \wedge m_k)_{k\in N}) - w^{ij}(((j-1) \wedge m_k)_{k\in N}) \\
&\overset{(51)}{=} v((j \wedge m_k)_{k\in N}) - v(((j-1) \wedge m_k)_{k\in N}) - \varphi_{ij}(m,v)u_{(0_{-i},j)}((j \wedge m_k)_{k\in N}) \\
&\quad + \varphi_{ij}(m,v)u_{(0_{-i},j)}(((j-1) \wedge m_k)_{k\in N}) \\
&\overset{(5)}{=} v((j \wedge m_k)_{k\in N}) - v(((j-1) \wedge m_k)_{k\in N}) - \varphi_{ij}(m,v). \qquad (53)
\end{aligned}$$

Each pair in $M^{+,j} \setminus \{(i,j)\}$ is dummy in $(m, u_{0_{-i},j})$. Therefore, by (51), each pair in $M^{+,j} \setminus \{(i,j)\}$ has the same marginal contribution in (m, w^{ij}) and in (m, v). It follows that, by (SM), each pair in $M^{+,j} \setminus \{(i,j)\}$ receives the same payoff in (m, w^{ij}) and in (m, v). Then, we have

$$\begin{aligned}
\sum_{(k,j)\in M^{+,j}} f_{kj}(m, w^{ij}) &= \sum_{\substack{(k,j)\in M^{+,j} \\ k\neq i}} f_{kj}(m, w^{ij}) + f_{ij}(m, w^{ij}) \\
&\overset{SM}{=} \sum_{\substack{(k,j)\in M^{+,j} \\ k\neq i}} f_{kj}(m, v) + f_{ij}(m, w^{ij}) \\
&\overset{ME}{=} v((j \wedge m_k)_{k\in N}) - v(((j-1) \wedge m_k)_{k\in N} - f_{ij}(m,v) + f_{ij}(m, w^{ij}). \qquad (54)
\end{aligned}$$

Combining (53) and (54), for each $(i,j) \in M^+$ such that $j \leq p_i$, we obtain

$$\varphi_{ij}(m,v) = f_{ij}(m,v) - f_{ij}(m, w^{ij}), \qquad (55)$$

which concludes Step 2.

We have the material to conclude the proof of the Induction step. By (52), we have $f_{ij}(m, w^{ij}) = f_{ij}(m, w)$ and by (41) we have $f_{ij}(m, w) = 0$, for each $(i,j) \in M^+$ such that $j \leq p_i$. By (55), for each $(i,j) \in M^+$ such that $j \leq p_i$, we have

$$f_{ij}(m,v) = \varphi_{ij}(m,v).$$

Therefore, for each $(m,v) \in \mathcal{G}$ and each $(i,j) \in M^+$, we have $f_{ij}(m,v) = \varphi_{ij}(m,v)$. The proof is complete. $\square$

7.7. Proof of Theorem 3

The proof is divided in two steps.

Step 1: We show the existence of a solution. Let $\beta = \{\beta^{ij}\}_{i\in N, 1\leq j\leq K}$ be a weight system, such that $\beta^{ij} > 0$ for each $i \in N$ and $1 \leq j \leq K$. We show that ξ^β satisfies all the axioms of the statement of Theorem 3.

For each $(m,v) \in \mathcal{G}$, we have

$$\begin{aligned}
\sum_{i\in N}\sum_{j\in M_i^+} \xi^\beta_{ij}(m,v) &= \sum_{j\leq \max_{k\in N} m_k} \sum_{i\in Q(j)} \frac{\beta^{ij}}{\sum_{k\in Q(j)} \beta^{kj}} \Big[v((j \wedge m_k)_{k\in N}) - v(((j-1) \wedge m_k)_{k\in N}) \Big] \\
&= \sum_{j\leq \max_{k\in N} m_k} v((j \wedge m_k)_{k\in N}) - v(((j-1) \wedge m_k)_{k\in N}) \\
&= v(m).
\end{aligned}$$

The second equality follows from the fact that $\sum_{i\in Q(j)} \beta^{ij} = 1$. This shows that the value satisfies (E). By definition of the Weighted Division values (see (16)), the payoff of a pair does not depends on activity levels different from the one contained in this pair. Therefore, ξ^β satisfies both (IIR) and (EL). (L) is direct from the definition of ξ^β. By (16) and the fact that the weight system consists of strictly positive weights, we have that ξ^β satisfies (NLB). This concludes Step 1.

Step 2: We show the uniqueness of the solution. Let f be a value satisfying all the axioms of the statement of Theorem 3. We show that there exists a weight system such that f coincides with a multi-choice Weighted Division value. We know that each characteristic function v admits a linear decomposition in terms of Dirac games. By (L), for each $(m, v) \in \mathcal{G}$, we have

$$f(m, v) = \sum_{s\leq m} v(s) f(m, \delta_s).$$

For each $s \in \mathcal{M}$, we show that

$$f(m, \delta_s) = \xi^\beta_{ij}(m, \delta_s),$$

for some weight system β. We consider several cases.

Case 1. Suppose that $s \in \mathcal{M}$ is not a synchronized coalition, that is $s \neq ((l \wedge m_k)_{k\in N})$, for each $l \leq \max_{k\in N} m_k$. In other words, s is not a synchronized coalition. Since f satisfies (E), (IIR) and (EL), by Corollary 1 it satisfies (ME). Therefore, by (ME), for each $j \leq \max_{k\in N} m_k$, we have

$$\sum_{(i,j)\in M^{+,j}} f_{ij}(m, \delta_s) = \delta_s((j \wedge m_k)_{k\in N}) - \delta_s(((j-1) \wedge m_k)_{k\in N}).$$

Since $s \neq ((j \wedge m_k)_{k\in N})$ and $s \neq (((j-1) \wedge m_k)_{k\in N})$, by definition of a Dirac game, we have

$$\sum_{(i,j)\in M^{+,j}} f_{ij}(m, \delta_s) = 0. \tag{56}$$

Since $\delta_s((j \wedge m_k)_{k\in N}) - \delta_s(((j-1) \wedge m_k)_{k\in N}) \geq 0$, by (NLB) and (56), for each $(i, j) \in M^{+,j}$, we have

$$f_{ij}(m, \delta_s) \geq 0. \tag{57}$$

Combining (56) and (57), for each $(i, j) \in M^{+,j}$, we have

$$f_{ij}(m, \delta_s) = 0 = \xi^\beta_{ij}(m, \delta_s),$$

for any weight system β.

Case 2. Suppose that $s \in \mathcal{M}$ is a synchronized coalition, that is $s = (l \wedge m_k)_{k\in N}$, where $l \leq \max_{k\in N} m_k$. In other words, s is a synchronized coalition. Take any activity level j such that $j < l$. By (ME), we have

$$\sum_{(i,j)\in M^{+,j}} f_{ij}(m, \delta_s) = \delta_s((j \wedge m_k)_{k\in N}) - \delta_s(((j-1) \wedge m_k)_{k\in N}).$$

Since $s \neq ((j \wedge m_k)_{k\in N})$ and $s \neq (((j-1) \wedge m_k)_{k\in N})$, by definition of a Dirac game, we have

$$\sum_{(i,j)\in M^{+,j}} f_{ij}(m, \delta_s) = 0.$$

Observe that $\delta_s((j \wedge m_k)_{k\in N}) - \delta_s(((j-1) \wedge m_k)_{k\in N}) \geq 0$. Similarly to Case 1, by (NLB) and (56), for each pair $(i,j) \in M^{+,j}$ such that $j < l$, we have

$$f_{ij}(m, \delta_s) = 0 = \xi^{\beta}_{ij}(m, \delta_s),$$

for any weight system β.

Case 3. Suppose that $s \in \mathcal{M}$ is a synchronized coalition such that $s = (l \wedge m_k)_{k\in N}$, where $l \leq \max_{k\in N} m_k$. Similarly to Case 2, for each $(i,j) \in M^+$ such that $j > l+1$, we have

$$f_{ij}(m, \delta_s) = 0 = \xi^{\beta}_{ij}(m, \delta_s),$$

for any weight system β.

Case 4. Suppose that $s \in \mathcal{M}$ is a synchronized coalition such that $s = (l \wedge m_k)_{k\in N}$, where $l \leq \max_{k\in N} m_k$. Consider the pairs $(i,j) \in M^+$ such that $j = l$, that is the pairs in $M^{+,l}$. By (ME) and the definition of a Dirac game, we have

$$\begin{aligned}\sum_{(i,l)\in M^{+,l}} f_{il}(m, \delta_s) &= \delta_s((l \wedge m_k)_{k\in N}) - \delta_s((l-1 \wedge m_k)_{k\in N}) \\ &= 1. \end{aligned} \tag{58}$$

Two distinct pairs $(i,l), (i',l) \in M^{+,l}$ are equal in (m, δ_s). Therefore, by (ES) we have

$$Sign(f_{il}(m, \delta_s)) = Sign(f_{i'l}(m, \delta_s)). \tag{59}$$

From (58) and (59), it follows that there exists a set of real weights $\{\beta^{kl}\}_{k\in Q(l)}$ verifying $\beta^{kl} > 0$, for each $(k,l) \in M^{+,l}$, and such that, for each $(i,l) \in M^{+,l}$, we have

$$f_{il}(m, \delta_s) = \frac{\beta^{il}}{\sum_{k\in Q(l)} \beta^{kl}} = \xi^{\beta}_{ij}(m, \delta_s).$$

Case 5. Consider $s \in M$, such that $s = (l \wedge m_k)_{k\in N}$, where $l < \max_{k\in N} m_k$. Consider the pairs $(i, l+1) \in M^{+,l+1}$. By (ME) and the definition of a Dirac game, we have

$$\sum_{(i,l+1)\in M^{+,l+1}} f_{i(l+1)}(m, \delta_s) = \delta_s(((l+1) \wedge m_k)_{k\in N}) - \delta_s((l \wedge m_k)_{k\in N}) = 0 - 1 = -1.$$

Similarly to Case 4, there exists a set of real weights $\{\beta^{i(l+1)}\}_{i\in Q(l+1)}$ verifying $\beta^{i(l+1)} > 0$, and such that, for each $(i, l+1) \in M^{+,l+1}$, we have

$$f_{i(l+1)}(m, \delta_s) = -\frac{\beta^{i(l+1)}}{\sum_{k\in Q(l+1)} \beta^{k(l+1)}} = \xi^{\beta}_{i(l+1)}(m, \delta_s).$$

Therefore, for each $s \in \mathcal{M}$, we have $f(m, \delta_s) = \xi^{\beta}(m, \delta_s)$, for some weight system $\beta = \{\beta^{ij}\}_{(i,j)\in M^+}$. By (L), we conclude the proof of Theorem 3. $\square$

Observe that in the uniqueness part of the proof of Theorem 3, the existence of a weight system follows directly from (ES) (see (59)). By strengthening (ES) into (ET), we characterize the specific Weighted Division value ξ^{β} where $\beta^{ij} = \beta^{i'j}$ for each $1 \leq j \leq K$ and any two $i, i' \in N$. This specific Weighted Division value is the multi-choice Equal division value. This proves Corollary 3.

7.8. Proof of Theorem 4

Before starting the proof, which is divided in two steps, we provide a useful remark.

Remark 8. By definition, (LD) implies (ET). If $(m, v) \in \mathcal{G}$ is the null game, then (ME) and (LD) imply $f_{ij}(m, v) = 0$, for each $(i, j) \in M^+$.

Consider any parameter system α. By definition and the fact that multi-choice Egalitarian Shapley values are convex combinations of the multi-choice Shapley value and the multi-choice Equal division value (see (17)), χ^α satisfies all the axioms of the statement of Theorem 4.

Next, we show that the multi-choice Egalitarian Shapley values are the only values satisfying all the axioms of the statement of Theorem 4. Consider a value f satisfying all the axioms of the statement of Theorem 4. To prove uniqueness, we show that, for each $(m, v) \in \mathcal{G}$, there exists a parameter system α such that we have

$$f(m, v) = \chi^\alpha(m, v).$$

Similarly to the proof of Theorem 1 and Theorem 3, for each $(m, v) \in \mathcal{G}$, we have

$$f(m, v) \overset{(L)}{=} \sum_{t \leq m} \Delta_t(v) f(m, u_t).$$

For each $t \in \mathcal{M}$, we introduce the notation $t^T = \max_{i \in N} t_i$. Consider any $1 \leq x \leq m^T$. Let us show that f can be written, for each (m, u_t) such that $t^T = x$, as

$$\forall (i, j) \in M^+, \quad f_{ij}(m, u_t) = \begin{cases} \alpha^x \varphi_{ix}(m, u_t) + (1 - \alpha^x) \xi_{ix}(m, u_t) & \text{if } j = x, \\ 0 & \text{otherwise,} \end{cases}$$

for some $0 \leq \alpha^x \leq 1$. Consider all pairs $(i, j) \in M^+$ such that $j < x$. Since f satisfies (IIR) and (EL), by Corollary 2, f satisfies (IR). Therefore, we have

$$f_{ij}(m, u_t) = f_{ij}((j \wedge m_k)_{k \in N}, u_t).$$

Since $((j \wedge m_k)_{k \in N}, u_t)$ is the null game, by Remark 8, for each $(i, j) \in M^+$ such that $j < x$, we have

$$f_{ij}(m, u_t) = 0. \tag{60}$$

Consider all pairs $(i, j) \in M^+$ such that $j > x$. These pairs are all dummy pairs in (m, u_t) and thus are equal. From Remark 8 and (ME), for each $(i, j) \in M^+$ such that $j > x$, we have

$$f_{ij}(m, u_t) = 0. \tag{61}$$

Now, consider all pairs $(i, x) \in M^{+,x}$. We show that f can be written, for each (m, u_t) such that $t^T = x$, as

$$\forall (i, x) \in M^{+,x}, \quad f_{ix}(m, u_t) = \alpha^x \varphi_{ix}(m, u_t) + (1 - \alpha^x) \xi_{ix}(m, u_t),$$

for some $0 \leq \alpha^x \leq 1$. We proceed by induction on $Q^t(x)$ the number of players that play x in the coalition t.

Initialization: Assume that $Q^t(x) = 1$. In this case, there is exactly one player, let us say $k \in N$, that plays x in t. Any two distinct pairs $(i,x),(i',x) \in M^{+,x}$ such that $i,i' \neq k$ are dummy pairs in (m,u_t) and thus are equal. It follows that, for each pair $(i,x) \in M^{+,x}$ such that $i \neq k$, we have

$$f_{ix}(m,u_t) = f_{i'x}(m,u_t) = c,$$

for some $c \in \mathbb{R}$. The pair (k,x) being the only non-dummy pair in (m,u_t) and $u_t(m) \geq 0$, (LD) implies that $f_{kx}(m,u_t) \geq cx$, which can also be written $f_{kx}(m,u_t) = c + \alpha^x$ for some $\alpha^x \geq 0$. (ME) implies that $f_{kx}(m,u_t) = 1 - (Q(x)-1)c$, and thus $\alpha^x = 1 - Q(x)c$. We obtain

$$c = \frac{1-\alpha^x}{Q(x)}.$$

By (8), each pair receives a zero payoff in the null game. Observe that each pair has better marginal contribution to coalitions in (m,u_t) than in the null game. By (WM), it follows that $f_{ix}(m,u_t) \geq 0$, for each $(i,x) \in M^{+,x}$. It follows that

$$c = \frac{1-\alpha^x}{Q(x)} \geq 0 \implies \alpha^x \leq 1.$$

Therefore, for each $(i,x) \in M^{+,x}$, we have

$$f_{ix}(m,u_t) = \begin{cases} \dfrac{1-\alpha^x}{Q(x)} & \text{if } j = x \text{ and } i \neq k \\ \dfrac{1-\alpha^x}{Q(x)} + \alpha^x & \text{if } j = x \text{ and } i = k, \end{cases} \tag{62}$$

for some $0 \leq \alpha^x \leq 1$. Observe that, for each $(i,x) \in M^{+,x}$, we have

$$\xi_{ix}(m,u_t) = \frac{1}{Q(x)}, \quad \varphi_{ix}(m,u_t) = \begin{cases} 0 & \text{if } i \neq k, \\ 1 & \text{if } i = k. \end{cases}$$

Comparing $\xi_{ix}(m,u_t)$ and $\varphi_{ix}(m,u_t)$ with (62), we have

$$\forall (i,x) \in M^{+,x}, \quad f_{ix}(m,u_t) = \alpha^x \varphi_{ix}(m,u_t) + (1-\alpha^x)\xi_{ix}(m,u_t).$$

for some $0 \leq \alpha^x \leq 1$. This concludes the initialization step.

Hypothesis: Consider $W \in \mathbb{N}$ such that $1 \leq W < |Q(x)|$. Consider any t such that $Q^t(x) = W$. In this case, there are W players that play x in t. We suppose that we have

$$\forall (i,x) \in M^{+,x}, \quad f_{ix}(m,u_t) = \alpha^x \varphi_{ix}(m,u_t) + (1-\alpha^x)\xi_{ix}(m,u_t).$$

Induction: Consider any t such that $Q^t(x) = W+1$. Let $s = t - e_h$, for some $h \in N$ such that $t_h = x$. Obviously, we have $Q^s(x) = W$. Recall that $(i,x) \notin T(t)$ if and only if $t_i < x$. Observe that if $(i,x) \notin T(t)$ then $(i,x) \notin T(s)$. If $(i,x) \notin T(t)$ then (i,x) is a dummy pair in (m,u_t) and is also a dummy pair in (m,u_s). Therefore, each $(i,x) \notin T(t)$ has the same contributions in both games (m,u_t) and (m,u_s). Then by double application of (WM), the induction hypothesis and by definition of φ and ξ, for each $(i,x) \notin T(t)$, we have

$$f_{ix}(m,u_t) \overset{(WM)}{=} f_{ix}(m,u_s) \overset{Hyp}{=} \alpha^x \varphi_{ix}(m,u_s) + (1-\alpha^x)\xi_{ix}(m,u_s) = \frac{(1-\alpha^x)}{Q(x)}. \tag{63}$$

By (ME), (63) and the definition of a minimal effort game, we have

$$\sum_{(i,x)\in T(t)} f_{ix}(m,u_t) = u_t((x\wedge m_k)_{k\in N}) - u_t(((x-1)\wedge m_k)_{k\in N}) - \sum_{(i,x)\notin T(t)} f_{ix}(m,u_t)$$
$$= 1 - 0 - (|Q(x)| - |T(t)|)\frac{1-\alpha^x}{|Q(x)|}. \tag{64}$$

Additionally, any two distinct pairs $(i,x),(i',x)\in M^{+,x}$, such that $(i,x),(i',x)\in T(t)$, are equal in (m,u_t). By Remark 8, f satisfies (ET), therefore for each $(i,x)\in T(t)$, we have

$$f_{ix}(m,u_t) = c,$$

for some $c\in\mathbb{R}$. It follows that

$$\sum_{(i,x)\in T(t)} f_{ix}(m,u_t) = |T(t)|c. \tag{65}$$

Therefore, combining (64) and (65), for each $(i,x)\in T(t)$, we obtain

$$c = \frac{1-(|Q(x)|-|T(t)|)\dfrac{1-\alpha^x}{|Q(x)|}}{|T(t)|}.$$

It follows that, for each $(i,x)\in T(t)$, we have

$$\begin{aligned} f_{ix}(m,u_t) &= \frac{1-(|Q(x)|-|T(t)|)\dfrac{1-\alpha^x}{|Q(x)|}}{|T(t)|} \\ &= \frac{\alpha^x}{|T(t)|} + \frac{1-\alpha^x}{|Q(x)|} \\ &= \alpha^x\varphi_{ix}(m,u_t) + (1-\alpha^x)\xi_{ix}(m,u_t). \end{aligned} \tag{66}$$

Combining (63) and (66), if $s^T = x$, then for each $(i,x)\in M^{+,x}$, we have

$$f_{ix}(m,u_t) = \alpha^x\varphi_{ix}(m,u_t) + (1-\alpha^x)\xi_{ix}(m,u_t).$$

This concludes the induction.

We have shown that there exists a parameter system α such that f can be written, for each (m,u_t) such that $t^T = x$, as

$$\forall (i,j)\in M^+,\quad f_{ij}(m,u_t) = \begin{cases} \alpha^x\varphi_{ix}(m,u_t) + (1-\alpha^x)\xi_{ix}(m,u_t) & \text{if } j = x, \\ 0 & \text{otherwise.} \end{cases}$$

By definition of multi-choice Egalitarian Shapley values (see (17)), for such a parameter systems α, there is a χ^α such that, for each (m,u_t), we have

$$f(m,u_t) = \chi^\alpha(m,u_t).$$

We conclude by (L) that there exists parameter system α such that, for each $(m,v)\in\mathcal{G}$, we have

$$f(m,v) = \chi^\alpha(m,v).$$

This concludes the proof. □

7.9. Proof of Proposition 4

Observe that, for each $i \in N$ and $j < q_i$, we have

$$\forall E \subseteq Q(j), \quad w_j^{(q,C)}(E) = w_j^{(q-e_i,C)}(E).$$

Therefore, we have

$$\forall i \in N, j < q_i, \quad Sh_i\Big(N, w_j^{(q,C)}\Big) = Sh_i\Big(N, w_j^{(q-e_i,C)}\Big). \tag{67}$$

Additionally, recall that, for each $j \leq \max_{k \in N} m_k$, the set of orders $\overline{O}_j$ over $M^{+,j}$ can be interpreted as the set of orders over the set of players in $Q(j)$. An order over $Q(j)$ is a bijection $\sigma_j^N : Q(j) \rightarrow \{1, \ldots, |Q(j)|\}$. We denote by $\overline{Q}(j)$ the set of of orders over $Q(j)$. Consider an order $\sigma_j^N \in \overline{Q}_j$ and $h \in \{1, \ldots, |Q(j)|\}$. Recall that, for each $B \subseteq N$, the vector $e_B \in \mathbb{R}^{|A|}$ is defined by $(e_B)_i = 1$ if $i \in B$ and $(e_B)_i = 0$ otherwise. We denote by

$$((j-1) \wedge q_k)_{k \in N} + e_{E^{\sigma_j^N,h}}$$

the coalition in which each player in $Q(j)$ ordered prior to step h with respect to σ_j^N, participates at its activity level j, whereas each player in $Q(j)$ ordered after step h with respect to σ_j^N, participates at its activity level $j-1$. Each player not in $Q(j)$ participates at its maximal activity level. Obviously, this coalition coincides with $s^{\sigma_j,h}$, where σ_j is the counterpart of σ_j^N among the orders in $\overline{O}_j$. We use the convention $((j-1) \wedge q_k)_{k \in N} + e_{E^{\sigma_j^N,0}} = ((j-1) \wedge q_k)_{k \in N}$. Consider an order $\sigma_j^N \in \overline{Q}_j$. For each $i \in Q(j)$, we denote by

$$\mu_i^{\sigma_j^N}(q,C) = C\big(((j-1) \wedge q_k)_{k \in N} + e_{E^{\sigma_j^N,\sigma_j^N(i)}}\big) - C\big(((j-1) \wedge q_k)_{k \in N} + e_{E^{\sigma_j^N,\sigma_j^N(i)-1}}\big), \tag{68}$$

the marginal contribution of player i for its activity level j with respect to the order σ_j^N. By (13), (14) and (68), for each $(q,C) \in \mathrm{C}$, the multi-choice Shapley value can be re-written as

$$\forall (i,j) \in M^{+,j}, \quad \varphi_{ij}(q,C) = \frac{1}{|Q(j)|!} \sum_{\sigma_j^N \in \overline{Q}_j} \mu_i^{\sigma_j^N}(q,C).$$

By definition of the Shapley value for TU-games (see Shapley (1953)), for each $j \leq \max_{k \in N} m_k$, we have

$$\forall i \in Q(j), \quad Sh_i\Big(N, w_j^{(q,C)}\Big) = \frac{1}{|Q(j)|!} \sum_{\sigma_j^N \in \overline{Q}_j} \mu_i^{\sigma_j^N}(q,C) = \varphi_{ij}(q,C). \tag{69}$$

It follows that the multi-choice Shapley value is consistent with the discrete serial cost sharing method since, for each $i \in N$, we have

$$\begin{aligned} SCS_i(q,C) - SCS_i(q-e_i,C) &= \sum_{j=1}^{q_i} Sh_i\Big(N, w_j^{(q,C)}\Big) - \sum_{j=1}^{q_i-1} Sh_i\Big(N, w_j^{(q-e_i,C)}\Big) \\ &= Sh_i\Big(N, w_{q_i}^{(q,C)}\Big) \\ &= \varphi_{iq_i}(q,C), \end{aligned}$$

where the second equality follows from (19) and (67), and the third equality follows from (69). $\square$

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