Relaxation oscillations in the early development of econometrics: coming (almost) full circle (1929-1951)

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Abstract:

In 1930, Ludwig Hamburger published in Dutch an article on the possibility to account for "economic instability" by using relaxation oscillations, a model of nonlinear vibrations which had recently been developed by the physicist Balthazar van der Pol. Hamburger's article, translated in French in 1931, attracted a strong interest in the nascent Econometric Society and in particular from Ragnar Frisch and François Divisia, who decided to make it one of the theme of the first European meeting of the Econometric Society, scheduled in Lausanne in 1931. Although Hamburger did not provide an economic theory going beyond the analogy with van der Pol's equation, he did propose to integrate shocks in his models in a manner that was completely different from what came after him. His approach was however shunned by other economists because its potentialities were not understood for at least three reasons. It took two decades before Richard Goodwin, spurred by the same physicist that was present at the early meetings of the Econometric Society, finally managed to build a convincing economic model relying on nonlinear oscillations.

Keywords: Relaxation equation, Hamburger, van der Pol, le Corbeiller, economic instability, pendulum, self-sustained oscillations

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Introduction

We would like to draw attention in this paper on a road that was not taken, an approach that was discussed widely among the first macrodynamists, those econometricians that tried to build a dynamic system for the whole economy, before it was abandoned for various reasons. Understanding these reasons, as well as the path that began to form, enlightens us by contrast on the road that was taken, especially on the manner in which Ragnar Frisch stabilized the problem of self-sustained fluctuations and the place of shocks in economic models.

Shocks had a very specific purpose in the approach defended by Ludwig Hamburger, a Dutch engineer who dabbled in the construction of economic models during the interwar. Hamburger's use of relaxation oscillations, a type of non-linear equations that showed self-sustained cycles for a wide range of parameters, showed a lot of promise to explain some phenomena, in particular the fact that the period between each cycle was not rigid, a critique often formulated against other models perceived as too mechanical. Relaxation oscillations also explained why the amplitude of cyclical oscillations could remain unchanged, and offered a (mathematical) explanation for the recurrence of crises.

Relaxation oscillations had been particularly developed by another Dutch, Balthasar van der Pol, who worked at the Philips laboratory in Eindhoven, and Hamburger took up most of his mathematical ideas from the equations presented by van der Pol and his associates. However, Hamburger did not go much beyond the analogy from which he started, although he had a rather impressive knowledge of economic issues. Nevertheless, it was not a mechanical analogy that interested Hamburger but rather the potentiality to express the biological metaphors developed at least since Alfred Marshall, and recently exposed by well-known business cycle researchers such as Ernst Wagemann (1930). Because van der Pol was concerned with models in magnetism and electricity, and applied them to biological problems such as the heartbeat, Hamburger saw this approach as a way out of a purely mechanical analogy such as the pendulum, and as able to represent the rhythms of economic life.

In spite of the fact that this approach was recognized by Frisch himself as a step toward the explanation of cycles, and while renowned scientists such as Philippe Le Corbeiller were invited to present at the first European meeting of the Econometric Society the possibilities offered by relaxation oscillations, the approach did not really took off at the time, either because those involved were unable to relate the mathematical formalism to a truly economic mechanism, or because something else became available at the same time. This does not justify however that the early developments related to relaxation oscillations have not been studied and that the contribution of Hamburger to the early development of macrodynamic analysis remains ignored.[2] Indeed, it is not mentioned in Richard Goodwin's preface to a collection of his articles (Goodwin, 1982) where he reminisced on the development of discussions related to the possibility of building a model demonstrating the existence of regular cycles. In his view, it all started in 1933 with the debate between Michal Kalecki and Frisch on the nature of the solutions of linear models. The same can be said for Paul Samuelson who insisted on the lasting impact of Frisch's confrontation with Kalecki, and the fact that Frisch's vision, he claimed, had won to the point of becoming a "dogma" (1974: 10). Francisco Louçã (2007, chapter 6) has pointed out that Frisch and Hamburger had corresponded in the early 1930s and that Frisch seriously considered the possibility to follow Hamburger's path but without providing any account of Hamburger's analysis. The same is true for Venkatachalam Ragupathy and Kumaraswamy Velupillai (2012) who, in their search for the origins of the nonlinear endogenous mathematical theory of the business cycle, attached little importance to the content of Hamburger's paper. Jean-Marc Ginoux (2017) is

[2] At that time, Frisch had already referred to macrodynamic and microdynamic but is not certain that Hamburger knew of this distinction. Hamburger however knew Frisch's 1928 paper on the statistical treatment of the irregularity of cycles to which he refers.

the only exception; in his investigation into the development of nonlinear oscillation theory, he gives a brief summary of Hamburger's argument but in a narrative concerned by the history of mathematics and physics in which economics has a minor importance.[3]

To understand the development and untimely disappearance of the use of relaxation oscillations in economics, we begin with the transfer between physical sciences and economics that was operated between Hamburger and van der Pol, underlining that what interested the former was the non-mechanical aspects of the theory (section 1). The main objectives of Hamburger are then presented, in particular the attraction of using relaxation oscillations to illustrate the effect of shocks on the frequency rather than amplitude of an oscillation (section 2). Three explanations are advanced as to why the approach was abandoned by other econometricians in the early 1930s and we conclude on the links between these early papers and Goodwin's more well-known approach, that owed more to those first debates than is recognized (section 3).

1. From physical science to economics: a case of non-mechanical analogy

Hamburger claimed that it was only with new mathematical tools that economists would be able to account quantitatively and qualitatively for multiple economic processes. Ultimately, the aim was to transform economics into a science similar to biology,[4] a science capable of understanding the working of social organisms beset by recurrent "pathologies". Hamburger was not the only economist interested in business cycles who was showing some discontent with a mechanical analogy. Wagemann, the German head of the imperial statistical office and

[3] Jovanovic and Ginoux (2020) claim that Hamburger had an impact on the development of Frisch's rocking horse model but provide no account of Hamburger's article.

[4] Hamburger was aware that his research was still at an embryonic level. "Very little of the latter (the biological orientation) is expressed in [Hamburger 1929a and 1929b], although I must immediately add that both in [Hamburger 1929a and 1929b] it is concluded that the apparently disordered picture of market prices in troubled times behaves as a powerful coherent, elastic system, in which organic unity in the multiplicity is unmistakable." (Hamburger 1929: 730).

of the business cycle research institute of Berlin, in a book published in 1928 and translated in 1930 under the title *Economic Rhythm: A Theory of Business Cycles*, called as well for a biological metaphor.

His 1928 book, published in German, met enough success to warrant its translation in English only two years later, and a prefatory note from Wesley C. Mitchell (Wagemann, 1930). In the foreword to the English edition, he presented his contribution as a small step "toward the repayment of the debt which Europe owes to America in the field of research into economic dynamics." (Wagemann, 1930: v). However, the type of dynamics that was applied remained very empirical; although Wagemann was searching for a theory, he steered resolutely away from abstract constructions which were heavily criticized. His review of existing theories led him to propose that "while the American methods are those of engineering, and the Russian those of astronomy, the German institute represents the medical, or, better, the organic-biological point of view." (*ibid*.: 10). The "organic-biological principle" which he described (with reference to Menger) was meant to capture both the interconnection of the separate parts of an (economic) organism as well as "a peculiarity which may be defined as consisting in the power to regulate its own movement" (*ibid*.: 11), an approach which he emphasized as "anything but mechanical" (*ibid*.).

Another radical opinion on the business cycle was that it was only a "myth". This opinion was shared amongst american economists and statisticians, such as Carl Snyder (1930) and Irving Fisher (1925).[5] While the former based his claim on the fact that compared to the growth of the economy, the amplitude of fluctuations remained in certain limits, the second doubted that "inherent" cyclical regularity in business could be detected. According to Fisher,

[5] "On the subject of economic fluctuations, there exists already a vast literature. Large differences exist between the opinions on the trade cycles. For one great american economist, the Business cycle is a myth." (Hamburger 1931: 3). It is clear that Hamburger referred to Fisher, who entitled a section of his 1925 paper "'The' business cycle a myth?".

even if there existed a simple self-generating cycle similar to that of a pendulum swinging under the influence of the force of gravity, its tendency to materialize would be necessarily "defeated in practice" (Fisher 1925: 192). To show this, he proposed to move away from the pendulum metaphor, toward the "physical analogue" of "the sway of the trees or of their branches." For instance, if a tree is pulled, one would observe a swaying movement similar to that of the cycle: but Fisher did not think that such a movement was actually observed in the woods: "in actual experience [...] twigs or tree tops seldom oscillate so regularly, even temporarily; they register instead, chiefly the variations in wind velocity." (Fisher, 1925: 192). A steady wind as well as any "outside forces"[6] may thus well bend the trees for weeks and annihilate completely its tendency to swing back and forth while changes in its speed and direction will simply modify the angle of the tree with the ground.

The problem is even worse if one assumes the existence of frictions. Fisher pointed out that their existence prevented a pendulum, a tree or a rocking chair to oscillate endlessly, but instead he underlined that they would all stabilize (Fisher, 1925: 192). Writing in the 1920s, Fisher did not see the solution that would be chosen by Frisch some years later, of regularly disturbing by a new shock the oscillating system; instead, he argued that the movement had to be entertained by outside forces acting cyclically and thought that they had to be synchronized with the oscillating system itself, in reference to a clockwork mechanism, however, he added that one "cannot imagine anything analogous to the 'escapement' in a clock which so nicely times the outside force as to keep up the natural swing of the pendulum" (Fisher, 1925: 193).

Hamburger wrote explicitly his paper with reference to Wagemann's "biological principle" (Hamburger, 1931: 13) and to the aim of going away from the mechanical analogy, referring

[6] Fisher did not use the term of "shock" but spoke of "outside forces" or "extraneous forces such as government inflation policies, banking policies, legislation, foreign trade, gold mining, and many other scarcely recurrent, and certainly not periodic factors." (Fisher 1925: 194)

to Fisher's claim that "the" Business cycle is a myth. However, his paper was also an answer to Fisher's line of argument, by integrating in a different way an oscillating system and external forces. Indeed, Van der Pol's way to introduce relaxation oscillations by first considering the working of an oscillator (pendulum) offered Hamburger an ideal way to place his work within Fisher's discussion.

According to Hamburger, he first proposed to extend relaxation oscillations to economics during a conference given on May 7th, 1928 by van der Pol, at the Meeting of the Batavian Society of Logic Empirical Philosophy (Hamburger, 1931: 4).[7] This particular conference was concerned with the heartbeat, a subject that interested van der Pol around this time as he also gave a conference on the same subject in Paris, on May 24th the same year, and published two papers with his assistant van der Mark, in English in the *Philosophical Magazine* and in French in the journal *L'onde électrique*.[8] In both papers, the authors mentioned among other examples of use of relaxation oscillations the fact that they could be used to address the problem of repetitive crises (van der Pol and van der Mark [1928a: 371, 1928b: 765-766], see also Hamburger [1931: 4]).

It was this passing mention to economics from van der Pol that prompted Hamburger to publish his own ideas on the question, in a paper published in 1930 in the Dutch journal *De Economist*. The paper was itself referred to by van der Pol who endorsed it in an article published in 1930 in *L'Onde électrique*. At the same time, Hamburger was asked by the statistician Lucien March,[9] to translate his article in French and it was published in 1931

[7] Van der Pol's breakthrough article on the question of relaxation oscillations was published in 1926 in the *Philosophical Magazine*. Ginoux (2017) has shown that relaxation oscillations were initially introduced by Henri Poincaré for the musical arc (in 1908), by Paul Janet (in 1919) for the series dynamo machine and by André Blondel (in 1919) for the triode while the merit of van der Pol was to have provided a generic equation which captured the relevant dynamic properties shared by earlier models.
[8] The latter journal was published by the *Société des amis de la TSF*, who was at the initiative of inviting the Dutch engineer in Paris for his May conference.
[9] March had been the head of the French statistical research institute until 1920. See Jovanovic and Le Gall (2001) on March's contributions to statistics.

under the title "Analogie des Fluctuations économiques et des Oscillations de relaxation" (Hamburger, 1931).[10]

Hamburger carried on two major objectives on the basis of van der Pol's theory of relaxation oscillations. The first was to provide a new understanding of the causes responsible for the recurrence of crises. The second was to account for the alleged constancy of the amplitude of oscillations while explaining why their frequency was subject to sudden changes. On these two points, Hamburger argued that van der Pol's theory of oscillation offered new insights. First, it could account for the fact that a system may never stabilize and be subject to inner instability.[11] Second, and most importantly, it could account for the effects of regular or irregular "external circumstances." It is precisely with the hope of building new systems accounting for the movement of the economy by integrating in a new way the role played by "external forces," or what Frisch later called impulses, that van der Pol endorsed Hamburger's analysis.[12]

Following van der Pol (1926) almost word for word, Hamburger (1931) began by first considering an equation describing the movement of an oscillatory system showing a simple harmonic motion, a purely sinusoidal oscillation. In the case where there were no frictions, he pointed out that this type of movements could already be found in the literature, for example in Henry L. Moore's work, although the latter had not derived the sinusoidal movement from a particular mechanism.[13]

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[10] Subsequent and earlier quotes of Hamburger are drawn from our own translation of this article.

[11] "The crux of the debate [is] here, as with many other periodic-looking phenomena, to determine whether the mechanism of periodicity is autonomous, endogenous or the result of exogenous causes." (Hamburger 1931: 3).

[12] Hamburger did not use the term of shock. He underlined the possibility that "outside forces" which hit a system may be irregular without referring to stochastic "outside forces", the ideas of Eugen Slutsky or the rocking horse of Knut Wicksell and Johann Åkerman.

[13] Hamburger refers here to Moore's book *Economic Cycles, their law and causes*, published in 1914 and his two papers published in 1919 and 1921. This sinusoidal equation can be found for instance in Moore (1921: 232).

But Hamburger remarked, with reference to Fisher's ideas on the question, that the damping influences could not be neglected (Hamburger [1931: 5 and 32]). How then were the fluctuations to be sustained? He argued that a relaxation equation could overcome precisely this difficulty. Adopting it meant however to shift the perspective, and rather than trying to explain how "outside forces" may overcome the tendency of a system to stabilize, the issue was to determine why an unstable system may not eventually break down. To obtain this type of stability meant to determine how a system acted when its resistance changed with the amplitude of fluctuations, that is, the impact of a negative resistance for small oscillations and a positive one for large fluctuations. Such a nonlinear model was at the heart of relaxation oscillations, and of the van der Pol equation that Hamburger reproduced in his paper:

$$\frac{d^2y}{dt^2} - \alpha(1 - y^2)\frac{dy}{dt} + \omega^2 y = 0$$

where y represents the distance to the position of equilibrium, α a coefficient of resistance and ω is proportional to the frequency.

When α is a positive quantity the system has a resistance which for a small amplitude is negative. Therefore, the position $y = 0$ is unstable, and a deviation from equilibrium will lead to increasing fluctuations. But the presence of $1 - y^2$ in the resistance meant that there would be a force limiting the expansion of those increasing fluctuations once they exceeded a certain limit. Hamburger concluded that every time this happened, "The stability of the system is then achieved" (Hamburger 1931: 6).

Moreover, van der Pol had shown that an increase in α would lead to a transformation of the form of oscillations. The cases drawn from van der Pol's 1926 article were presented by Hamburger, with three different values for the resistance. The first case had a very small resistance and it took almost 150 periods until the maximum amplitude was reached and the cycle stabilised. In the second case, where $\alpha = 1$, the maximum amplitude is reached in a few oscillations, as in figure 1, and still looked a lot like a sinusoidal oscillation.

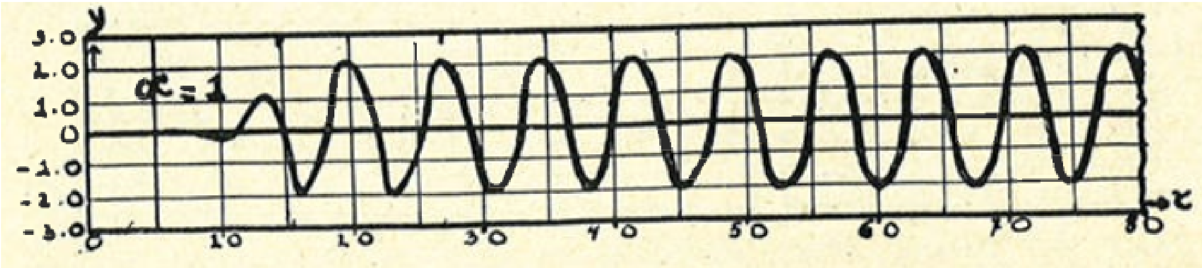

Figure 1. Hamburger (1931: 8)

The third case, for $\alpha = 10$, showed a somewhat similar sequence of events, but a marked departure from the sinusoidal form. Hamburger underlined that the curve first rose asymptotically and after only one period practically reached its final amplitude. The latter then alters very slowly from zero to one and then very suddenly drops to minus two, before slowly increasing to minus one and again suddenly jumping to two, at which point it slowly decreases to one and suddenly drops and the cycle repeats again and again in a self-sustained, periodic, but non-sinusoidal oscillation, where a period of "charge" abc is hence periodically followed by a period of "discharge" cda.

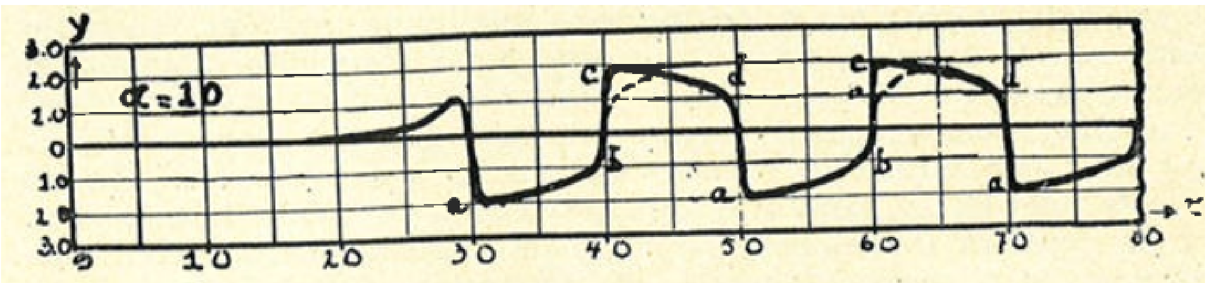

Figure 2. Hamburger (1931: 8)

When α is relatively large, one can hence see that the shape of the movement deviates considerably from the sinusoidal form. The movement is characterized by "sudden jumps" which periodically happen when the nonlinear resistance constrains the deviation of the system from its equilibrium position.

It was this behavior that led Hamburger to argue that relaxation oscillations were well-suited to reflect various types of cumulative processes whether they were rooted in innovative processes (Schumpeter, 1926), in credit (Wicksell, 1898 and Fisher, 1907) or in psychology (Hawtrey 1919).[14] Even better, such oscillations may help to combine - in a way that Hamburger did not specify formally - all these elements in a unifying framework accounting for "the lack of coordination" of the actors implied in the economic process which, in his view, was the main "endogenous cause" of the "instability of business activity".[15]

Hamburger's contribution attracted a strong interest, mainly in France and the Netherlands, on the potentialities offered by relaxation oscillation to interpret the business cycle.[16] One of its main attractions was that shocks in the oscillating system could translate into an irregular periodicity, while leaving the amplitude untouched.

2. Accounting for changing periodicities and constant amplitude

A key advantage of relaxation oscillations lied in their capacity to account for the effects of regular and irregular "external circumstances" in a specific way. Following van der Pol, Hamburger emphasised that "it is remarkable and very important for what follows that this can be done without modifying the amplitude of the oscillating phenomenon. [...] However, we will remark that the frequency of relaxation oscillations, that can be observed in reality, are influenced by secondary circumstances and often irregular." (Hamburger 1931: 9). In fact it was precisely the impact of external forces on the model that interested Hamburger,

[14] Hamburger had an extensive knowledge of the literature on business cycle as evidenced by the numerous references he provides. The references are those given by Hamburger.

[15] "The final instability that is accepted in any theory of economic crises results in part from the inadequate coordination of the forces moving towards production. It results not only from the aforementioned snowball actions, but also from faltering psychology of the business world which alternates ... between phases of optimism and phases of pessimism." (Hamburger 1931: 11)

[16] Louçã (2007, chapitre 4 and 6) has given an account of the relation between Frisch and Hamburger, emphasizing that Frisch already knew about Hamburger's work as soon as 1930. Frisch was not alone to pay attention to Hamburger's works. Van der Pol, March and later Divisia showed also an interest in what he was doing, a point missed by Ragupathy and Velupillai (2012).

although he did not show formally their impact, but explained their influences in his text and with the help of a mechanical analogy.

Typically, external periodical force applied to a simple harmonic oscillator can dramatically change its amplitude, particularly when its frequency is close to the natural frequency of the vibrating mechanism. In the case of a relaxation system, things work differently. To illustrate that point, van der Pol (1930) referred to a mechanical metaphor also used by Hamburger (1930).[17] Consider a small device consisting of a two-compartment tank, able to tilt around a horizontal axis and held on each side by a stopper. The tank is first pressed against the left stopper, the right compartment gradually fills with water. When the water reaches a certain level the tank tilts, the right compartment empties and the left compartment which is now placed under the tap fills in turn. When full, the tank tilts in the opposite direction and the cycle begins again. As van der Pol and Hamburger summarized, the period of the phenomena will depend on the time required for a tank to be filled which will itself depend on the volume of the tank and the flow of water pouring from the faucet.

[17] The argument was already discussed in 1928. "A third property no less important is the ease with which relaxation oscillations come into synchronization with a periodic external phenomenon acting on them" (van der Pol and van der Mark, 1928a: 372). It is a central mechanism in van der Pol's heartbeat system with which he examined how an oscillator system may impact a relaxation system.

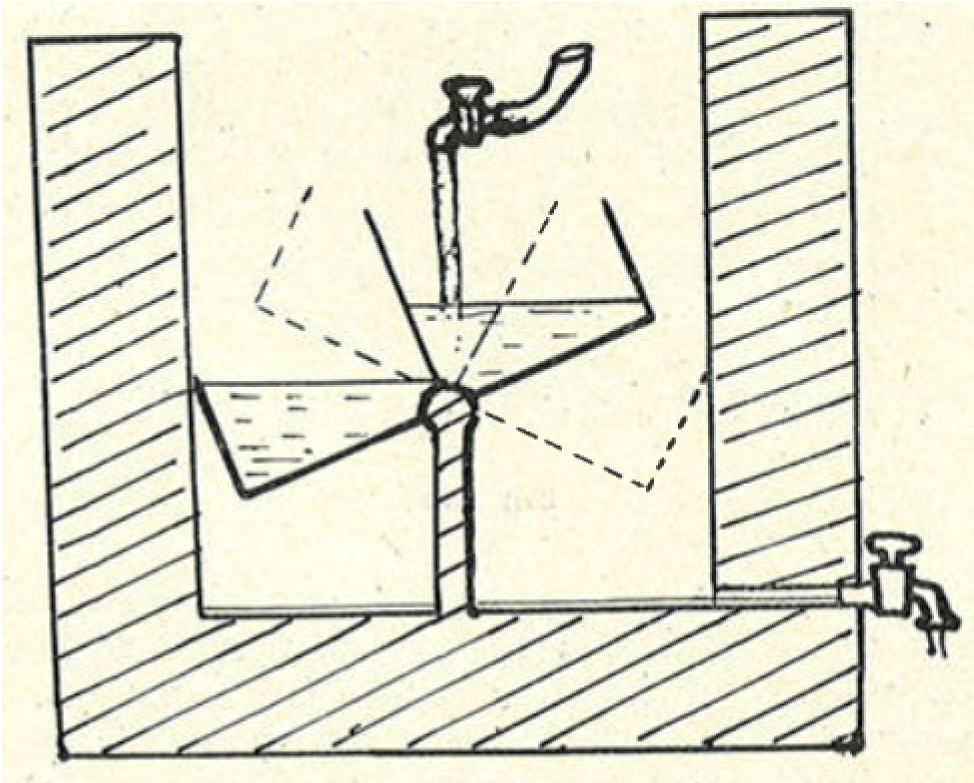

Fig 3. Hamburger (1931: 16)

When the system is subjected to a change in “exterior circumstances,” Van der Pol showed that a periodic opening of the water into the tank would be coordinated with the periodic oscillation of the device: “Tipping takes place only when the amount of water poured into the tank makes it unstable. We therefore see that it is not possible to act on the amplitude of the relaxation oscillation by changing the period of the water supply, since this amplitude is always determined by the same water level inside the tank.” (van der Pol, 1930: 306). But van der Pol remarked that the oscillation system may be exactly synchronized with the period of water supply: “In the example chosen, with the sixth subharmonic, we see [...] that in complete opposition to what occurs in sinusoidal systems, the period of a relaxation system can be easily influenced by external circumstances, while these same circumstances have practically no effect on their amplitude” (van der Pol 1930: 306).

In Hamburger's opinion, it is precisely that property of relaxation oscillation that allowed to provide a new explanation to facts recently established by various statisticians highlighting the constancy of amplitude but the changing period of business fluctuations. Drawing from Carl Snyder’s work, Hamburger provided a figure representing the deviation in percentage of the volume of trade from 1875 to 1924; from these deviations, he concluded that the amplitude of the movements did not exceed 10%, which in his view showed that the fluctuations could be modeled with relaxation oscillations. In addition, Hamburger emphasized that “temporary accidental influences” can have a great impact on the frequency of the oscillation as evidenced by multiple events which occured along the last 50 years. For that very reason, he warned about the risks to use methods consisting in eliminating these “special circumstances”.[18]

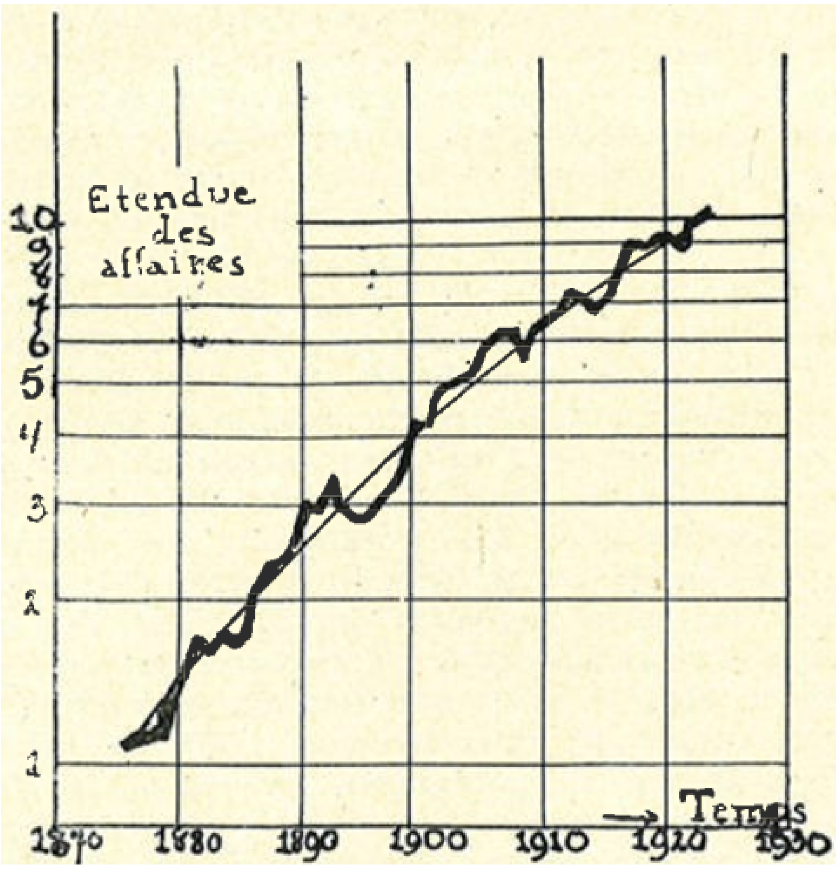

Figure 4.Hamburger (1931: 25)

[18] “we do not sufficiently realize the importance and the effect of these so-called special circumstances.” (Hamburger 1931: 26)

Such conclusions could easily be generalized to multiple if not all time series of sale prices such as those of Frederick Mills[19] (one of the three original associate editors of *Econometrica*). He also found examples of this in time series related to the stock markets, for instance in the work of the French statistician Jean Desserier who argued that the crisis of 1929 "was no more sensational, except in its speed, than many previous American downturns." (Hamburger 1931: 23).

Throughout his article, the problem Hamburger tried to illustrate was thus one where fluctuations showed the same amplitude, but could have different periodicities. He underlined this when referring to Wagemann's work: "Our approach corresponds to the summary definition from Wagemann, who considers the movements of 'conjoncture' as 'rhythmically free' fluctuations determined by the growing economic tensions calling for a resolution. However, the present author contrasts the almost invariable amplitude to the irregularity of the period" (Hamburger 1931: 29-30). Hamburger underlined, as van der Pol and van der Mark, that one of the attraction of relaxation oscillations was that they made it easier to change the frequency by changing the resistance (the first derivative), compared to sinusoidal mechanical oscillations where a change in period implies a change of mass or elasticity (Hamburger [1931: 9]; van der Pol and van der Mark [1928a: 371]).

Hamburger's use of relaxation oscillations in economics was explicitly endorsed by van der Pol (1930),[20] where he reviewed the work that had been done on relaxation oscillations and spent two pages on Hamburger's approach: "The work develops very interesting

[19] "The previous table - as well as fig. 11 corresponding to it - clearly confirm that the stability of the amplitude is greater than that of the period. The relative fixity of the amplitude is maintained in times and under conditions which vary enormously: it is a rather surprising fact." (Hamburger, 1931: 27). Hamburger also refers to a study of V. S. von Szeliski who, on the basis of the "Clearing Index of Business" provided by the Federal Reserve Bank of New York, had shown that in the period 1875-1929, its amplitude did not exceed 1.96% while the amplitude of the prices of public funds did not exceed 16%.

[20] This reference seems to have escaped Hamburger (1934), who complained in his note in *Econometrica* that his ideas were not well received by van der Pol; see also the beginning of his French article where he claims that van der Pol did not refer to him.

considerations on economic fluctuations. As with the sleep of plants, much work has been published on the question of whether periodicity in economic life is due to an internal cause or whether it is caused by external influences." (van der Pol, 1930: 311-312). Again, the importance of the biological metaphor was apparent, and van der Pol agreed that there was a lot of potential in the characteristics of relaxation oscillations that a shock on such a system could left the amplitude unchanged while changing its period: "We have already noticed several times that it is a characteristic of relaxation oscillations that their frequency can easily be influenced, but not their amplitude. Dr. Hamburger also found in discussing published statistical work on business cycles that the amplitudes of fluctuations (corrected for their long-run variations) are much more constant than the periods themselves." (van der Pol 1930: 311).

Hamburger saw clearly the link between his own account of changing frequencies of cyclical movements and Frisch's statistical analysis of time series developed in the late 1920s.[21] Even before his paper was translated in French, Frisch had taken note of it. In a letter sent on May 6th, 1930, he showed his interest in incorporating this approach into his own analysis: "In my own attack to the economic cycle problem I have also been very definitely under the impression that the conception of rigorous harmonic components must be given up [as in the Changing Harmonics paper]. You will see that my approach to the change in the components is purely empirical. In this respect your approach, built on Dr. Van der Pol's idea is more powerful, it seems to me, since it introduces some rationality into the explanation of the change. I want very much to take this idea up through a closer study and try to combine it with my own idea of a 'moving contact approximation'. It seems to me that such a

[21] "The norwegian professor Ragnar Frisch had the same impression regarding this; but so far he has only suggested approximations on a purely empirical basis." (Hamburger 1931: 29). See Andvig (1981) and Morgan (1990: 84) on Frisch's account who, in the mid-1920s, developed a specific approach of changing harmonics capable of accounting for the non sinusoidal shape of the economic movement.

combination ought to furnish a very powerful method." (Frisch to Hamburger, 6 May 1930 quoted by Louçã, 2007: 160).

Frisch's (1933) ultimate solution to the problem of building a model where the period could show a somewhat erratic behavior was to periodically revive damped oscillations with external impulses. Unlike Frisch, what Hamburger had in mind in 1930, was to integrate shocks to show a changing periodicity, although he did not present the result of such a combination (unlike Frisch in his paper). To clarify the idea, we present below a simple simulation of what happens to the same relaxation equation used by van der Pol and Hamburger, when we subject it to random shocks distributed normally over thirty periods. Figure 5 shows the "normal" behavior of the relaxation oscillation for a parameter of $\alpha = 10$, and the values of the different shocks, which are added in figure 6 to the "speed" or the resistance (the first derivative), as in Frisch's model and as it is explained in Hamburger's and van der Pol's papers.

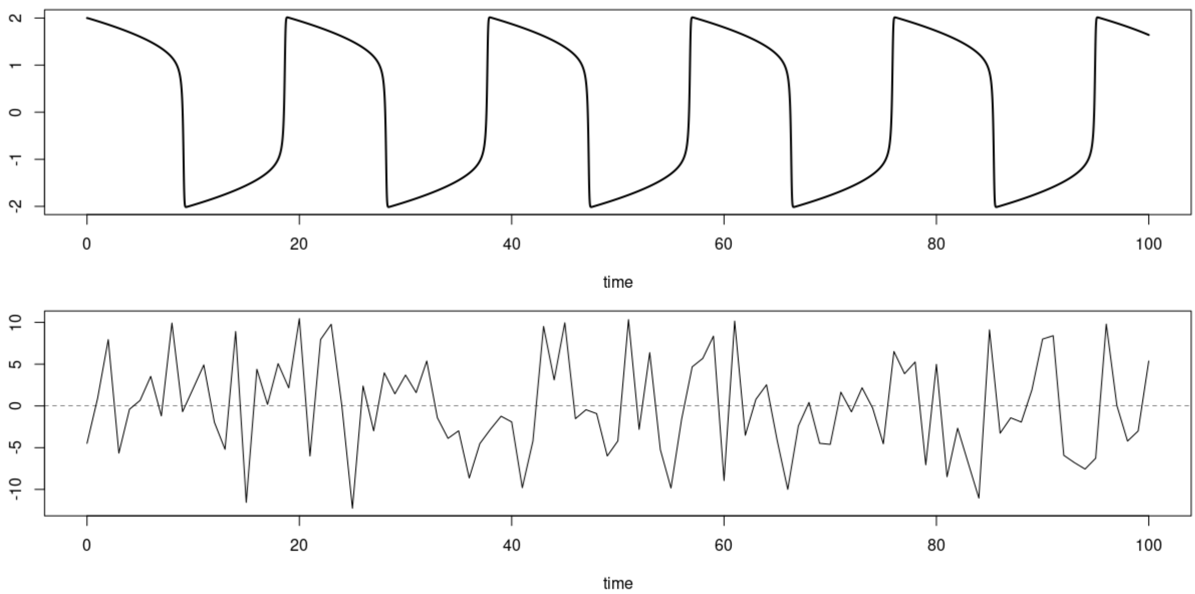

Figure 5

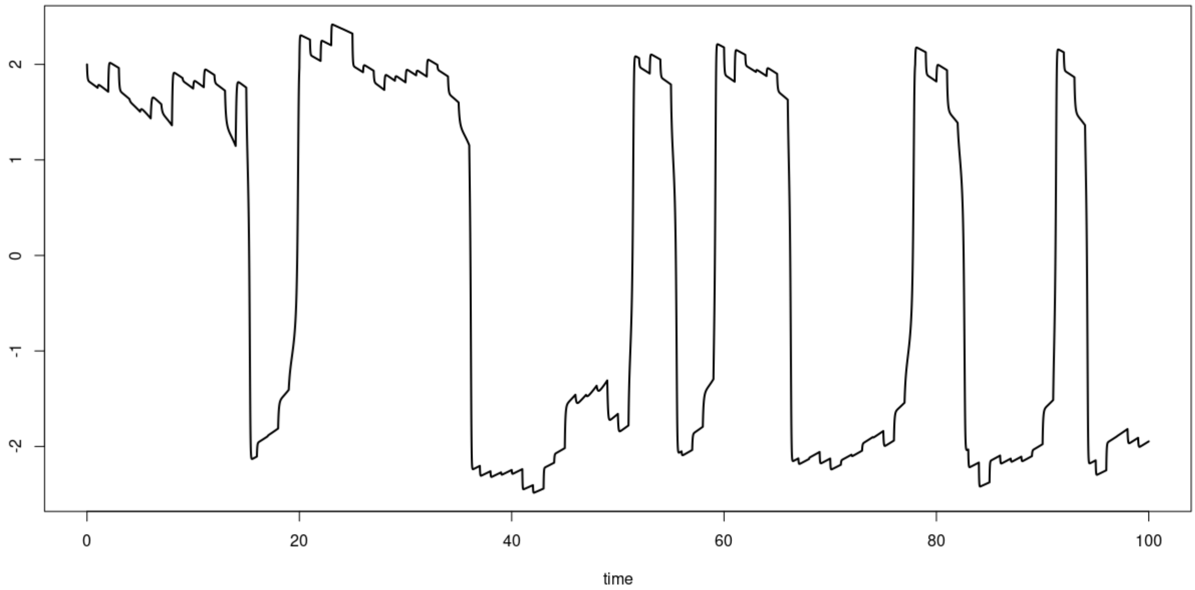

Figure 6

We see clearly on figure 6 that the shocks do not impact significantly the amplitude of the cyclical fluctuations, but that they can accelerate or hinder the "charge" and "discharge" of the model, thus changing the duration of a period from peak to peak in a random fashion.

Frisch's enthusiasm was shared by François Divisia, then first vice-President of the Econometric Society who was presided by Irving Fisher. With the purpose to explore Hamburger's line of research, both agreed to spend part of the first Econometric Society meeting on relaxation oscillations and to invite Philippe Le Corbeiller, who had collaborated with van der Pol when the latter was in Paris in 1928 and in 1930. Relaxation oscillations were also discussed among the members of X-crise, a group of alumni and student engineers from the École Polytechnique that sought answers to the deepening crisis. In his account for their monthly periodical of the Paris meeting of the Econometric Society, Divisia placed Hamburger as one of the main contributors to macrodynamic analysis, next to Frisch, Kalecki and Jan Tinbergen, and mentioned the work of a young polytechnician on the question (Divisia, 1934: 12).

3. Coming (almost) full circle

When preparing together the programme of the first econometric meeting scheduled to take place in Lausanne, Frisch and Divisia both agreed to ask Le Corbeiller to give a conference on relaxation oscillations. At that time, Le Corbeiller - who was a close friend of Divisia[22] - was already recognized as an expert of relaxation oscillations, and he agreed to meet the members of the nascent Econometric society in Lausanne in 1931.

In his 1931 communication, Le Corbeiller introduced the economists to the engineer's approach of the theory of oscillations, and considered the conditions for which different devices - a clock showing harmonic movements or a pile driver - may show self-sustained oscillations. Like van der Pol and Hamburger, he started by considering the working of a simple pendulum. When frictions are negligible, Le Corbeiller noted that the movements will be described by a regular sinusoidal curve, and thus concluded that for a device like a pendulum to keep swinging, no energy must be dissipated. This required, in the presence of frictions, that a source of energy should be added which may exactly compensate for the energy dissipated. In the case of a clock, this was made possible by the addition of weights which, as long as they had not gone down, maintained unchanged the swinging of the pendulum. Thus, contrary to Fisher, Le Corbeiller thought that the escapement mechanism could be used as a useful model for economic fluctuations.

In a system of relaxation, Le Corbeiller argued that the problem was completely different. In that case, the system will keep oscillating precisely because the source of energy acting on it is periodically and suddenly dissipated (the "discharge"). This happened every time a device

[22] Divisa and Le Corbeiller met in 1910 at the *École Polytechnique*, France's most prestigious engineering school where Divisia entered in 1909 and Le Corbeiller in 1910. In the late 1920s, the latter collaborated with van der Pol at the occasion of his multiple stays in Paris. At that time, it is highly probable that Le Corbeiller had heard of Hamburger's works. Having read and helped van der Pol to translate in French his two 1930 papers, it is hard to imagine that he had missed the 2-pages development of van der Pol on Hamburger

was characterized by a period of charge and a period of discharge of energy as it is the case for the Wimshurst machine[23] or for a pile driver. In the latter case, a heavy mass is pulled up along a pile by a steam engine acting on a piston. Once it has reached a given height, a tap is opened which lets the steam escape and the mass fall on the head of the pile: “If therefore we consider on the one hand a clock, on the other hand a pile driver, we see that in both cases there is a source of energy which maintains a periodic movement by providing the energy dissipated during each period” (Le Corbeiller, 1933: 330).

In the second part of his presentation, Le Corbeiller focused on forced oscillations and considered the interaction between two systems, a small system A and a large system B assumed to oscillate sinusoidally. For instance, if an engine (B) is placed on a table (A), and the engine is doing n turns per minute, then, one may expect the table to vibrate with the frequency n. Now, if, due to its mass and the elasticity of its legs, the frequency of the table is equal to n, in accordance to the “well known resonance phenomena,” (*ibid.*) the table will oscillate with a great amplitude and may even break down after some time. If system B is not only the engine but the whole electric plant producing energy, the oscillations will really be forced as the table does not react back on the plant. If, however, both systems are of the same size, “it is highly probable that the final movement of the whole system does not admit a unique period. We are therefore in a much more complicated case than that of forced oscillations.” (*ibid.*: 331). But Le Corbeiller thought that the economy was precisely in this case, where different systems interacted and reacted together, which would be a problem “of extreme difficulty” (*ibid.*).

From his short presentation, it is apparent that, at this time, Le Corbeiller did not conclude on the superiority of relaxation oscillations over forced oscillations, and not even on purely

[23] The Wimshurst machine is an electrostatic generator machine for generating high voltages developed between 1880 and 1883 by the British inventor James Wimshurst

sinusoidal fluctuations. In doing so, he must have given more confidence to the econometricians and macrodynamists who built models based on other kinds of oscillations. Neither did he examine in this note the interaction of external oscillations or external shocks with a relaxation system as Hamburger wanted to, or the effect of irregular changes on a harmonic system.

When preparing the program of the second Econometric meeting that was expected to take place in Paris, Divisia suggested to Frisch to invite Hamburger in order to continue the discussion started in Lausanne. Frisch and Divisia also agreed to turn again to Le Corbeiller. But this time, Frisch insisted on the need to go into more detail:

> "I think it would be exceedingly interesting if Corbeiller could tell us something more about the oscillations he spoke about at the Lausanne meeting. This time I think he ought to go into the matter with more detail, not being afraid of making the paper a mathematical and technical one. If he could indicate those aspects of the problem that would be of importance so far as the statistical treatment of our economic problems (sic) is concerned – so much the better." (Frisch to Divisia, June 11th, 1932, RFA)

However, Le Corbeiller did not participate in the second meeting of the Econometric society for unknown reasons, and the Hamburger-van der Pol-Le Corbeiller line of research was slowly abandoned. Several reasons can explain it.

The first was that from 1931, Tinbergen had already made important progress in the building of dynamic economic models capable of accounting for "the theory of economic oscillations." It was clear to him that existing economic models, although they were in an embryonic state and could only apply to specific markets, may lead to the "discovery of the causes and of the mechanism of the general cyclical movement of trade." (Tinbergen 1933: 36). With that ambition, Tinbergen worked toward extending Hanau's "pork cycle" (a cobweb model) in multiple ways[24] and succeeded in generating various cycles. Most importantly, Tinbergen managed to build a model likely to apply to durable goods, and to determine the conditions of existence of new types of cycles. His innovation was to combine the influence of rate of changes and lags on economic activity within a model. Kalecki saw the potential of this approach and transposed Tinbergen's argument to the whole economy, thus building the first macrodynamic model.[25] Meanwhile, Frisch (1933) saw in Tinbergen's approach a possibility to interpret the economic data produced by the supporters of the "economic barometers". Because the solutions of models mixing difference and differential equations can be decomposed into a trend and cyclical components, he saw in particular a possibility to make sense of the cycle periodicity studied in the 1920s.[26]

By not going beyond drawing an analogy between relaxation oscillations and the movements of the economy, Hamburger's plea for the use of relaxation oscillation paled in comparison to Tinbergen's early theories where mathematical formalism was subordinated to an economic theory. For that very reason, Frisch even doubted that Hamburger's participation in the second econometric meeting could be of any help: "With regard to Hamburger – I don't think that he will be able to give us anything particularly interesting. You remember of course that he wrote a paper some time ago on van der Pol's Theory of Oscillations. That paper in itself seemed promising but nothing more seems to have come from him so, on the whole, I am a little bit disappointed with him." (Frisch to Divisia, June 11th, 1932, RFA). Divisia was less

[24] The first consisted in adding a lag on the demand side. The second amounted to building a model in which demand ("purchasing power") could result from the activity of the market, a key step toward building a "closed economic system" accounting for the working of the whole economy. The third, finally, was to introduce a role of speculation on the demand side by adding the rate of change of price as an argument of the demand. See Boumans (2013) for an examination of the different mathematical "mouldings" chosen by Tinbergen.
[25] See Tinbergen (1935: 268) on Kalecki's model and his place in the history of early macrodynamic modeling.
[26] See Carret (2020) on the development of Frisch's model, its roots in the problem of statistical decomposition and the role of mixed difference-differential equations in his modeling strategy.

severe than Frisch as evidenced by his 1933 report of the Leiden conference in which Hamburger is mentioned on an equal footing with Kalecki and Frisch.[27]

Hamburger thus fell short of the expectations of mathematical economists, by failing to work out an economic system from which a relaxation equation could be derived.[28] Contrary to Le Corbeiller, Hamburger was at the Paris meeting of the Econometric Society, where he presented a communications on “Some general considerations on an economic system of equilibria.” In the report of the meeting, Lutfalla formulated an opinion that was gaining traction, that analogy alone could not be sufficient to build a theory:[29]

> “Another point which seems to us to be established is the futility of the research carried out by following too closely the models provided by the natural sciences. All of M. Hamburger's knowledge of chemistry ... can only confirm our remark: analogy implies a whole philosophy of classification. It can be fruitful for two neighboring sciences, but between economics and physics, there is the whole world of biology.” (Lutfalla, 1933: 192)

While we do not have the text of Hamburger’s conference, it is apparent that he did not manage to convince the econometricians that he was able to go beyond the simple analogy, even if his approach was much closer to biology than mechanics.

A second factor explaining why most of Hamburger’s contemporaries did not explore relaxation oscillations is that with the aim to account for the Great Depression, Tinbergen

[27] Although we cannot agree with Ragupathy and Velupillai that Hamburger’s work “did not attract much – or, indeed, any – attention” (2012: 59), they are right to point out that he did not manage “to actually build a formal mathematical model of aggregate fluctuations from economic principles, encapsulated in the dynamics of a nonlinear (or even a linear) system of equations capable of relaxation oscillations” (*ibid.*: 59-60).

[28] Tinbergen was among the few to refer to Hamburger’s article as evidenced by his 1935 *Econometrica* survey or his 1936 Wagemann festschrift article.

[29] On this opinion, see in particular the “Creedy episode” reported by Louçã (2001: 171 ff.), which is revealing of Tinbergen’s discomfort with physical analogies that did not go beyond a simple transposition, an opinion that Tinbergen shared privately with Frisch in his position as associate editor of *Econometrica*.

(1934, 1936) and Frisch (1934) seriously considered - for a short time - the possibility that an economy may collapse instead of oscillate endlessly around an equilibrium. From 1934, for instance, Tinbergen examined macrodynamic systems with two equilibria with damped or self-sustained cycles around the high equilibrium and a collapse around the low equilibrium. Along that line, Tinbergen came close to Fisher's 1933 metaphor of a capsizing boat used to set out his debt deflation theory as a way to interpret the potential of a crisis to trigger the collapse of the economy.[30] At the same time, Frisch (1934, 1935) developed a simple nonlinear dynamic model with the aim to question the self-adjustment capacity of the economy (see Louçã 2007, chapter 10) in which collapse may result from too much price flexibility. For Frisch as for Tinbergen, models with only local instability who showed a limit cycle but no possibility of collapse, such as the one described by Hamburger, may thus have been of little help.

Finally, a third reason may be advanced for the abandonment of relaxation oscillations in the early 1930s. At that time, Frisch was already involved in the building of his linear propagation and impulse approach. Unlike van der Pol and Hamburger - and Le Corbeiller to a lesser extent - who suggested abandoning the metaphor of the pendulum, Frisch chose to show how that metaphor, once combined with Slutsky's analysis, could help to explain the regularity of the cycle. Clearly, his approach proved highly successful among economists who almost unanimously adopted it. For instance, Kalecki, who had built a linear model choosing the (parametrically) unstable solution of self-sustained oscillation, asked Frisch if he could change his still unpublished paper after he read Frisch's critique of his approach, and accepted "the legitimate reproach that my assumption of a constant amplitude is not well founded" (Kalecki to Frisch, May 16th, 1935, RFA). But this linear approach, while proving

[30] In his 1933 *Econometrica* article, Fisher referred to the metaphor of the ship which may keep floating as long as its tipping point has not been reached (Fisher, 1933: 339). See Assous and Carret (2020) on Tinbergen's early macrodynamic systems and its connection to Fisher.

successful for the handling of empirical data, and much more tractable for the econometricians, was eventually castigated as a "dogma" by Samuelson (1974: 10) and others, who forgot that it had been built at a time when other nonlinear approaches were available, because those early endeavours did not manage to make it into the official stories.

However, by a twist of fate, Le Corbeiller, who had not shown at first an overly enthusiastic interest for the use in economics of relaxation oscillations, was the one crucial link between these early works and the revival of nonlinear limit cycle models in the 1950s. Indeed, while Le Corbeiller refrained to advocate for a specific approach in the early 1930s, he was the one who eventually allowed the path pioneered by Hamburger to be finally taken by Goodwin (1951). Goodwin, who had taught physics during the war, returned to economics at the end of the conflict and met Le Corbeiller, then a physicist at Harvard. Following his advice, as he made it clear in the opening pages of his seminal article, he discovered the literature on relaxation oscillations: "My debt to Professor Le Corbeiller is very great, not only for the original stimulation to search for the essential nonlinearities, but also for his patient insistence, in the face of the many difficulties which turned up, that this type of analysis *must* somehow be worked out." (Goodwin 1951: 2).

Goodwin placed himself in the context of the work of Hicks (1950) on the trade cycle, which hinged upon some nonlinearities, but also the work of Tinbergen, who had published during the war a paper working out the effects of high and low limits on an accelerator multiplier model. In this little known paper (translated in Tinbergen [1959]), Tinbergen explored the implications of multiple equilibria in a Keynesian cross diagram and showed how, depending on the slope of the expenditure curve, different types of equilibrium could be defined.[31]

[31] See Assous and Carret (2021) on the different nonlinear models that were built by Tinbergen during the second half of the 1930s and how they shaped his approach to policy.

Two decades after Hamburger's first paper, macroeconomics had now established itself as an autonomous field of research structured around various models. Nevertheless, the issues remained similar. Goodwin began his paper by reminding the reader of the "dilemma" between explosive or damped cycles shown by a linear model, rejecting the middle case of undamped fluctuations as a "classroom" solution without any economic content (Goodwin, 1951: 1). However, he pointed out two potential ways out of this dilemma, using external forces: the "synchronized systems" such as a clock and its escapement mechanism (a clear reference to Le Corbeiller's own distinction), or the random impulses approach developed by Frisch after the debate on relaxation oscillations (*ibid.*: 2). Although Frisch's approach had eclipsed the other way out, Goodwin put them back on the same level, and proposed to follow the odd way out, that of relaxation oscillations.

Even if Goodwin was completely unaware of Hamburger's articles, his contribution met some of his most important objectives: self-sustained oscillations accounting for the trajectory of an economy and accounting for a movement with an amplitude which remains unchanged. As Goodwin pointed out, an essential feature of any model based on relaxation oscillations was to be "frequency converters" so that any steady change of technical progress for instance may change the period of the movement but not its amplitude. In this manner, Goodwin argued that one may "easily introduce an historical event, prolonging or shortening the boom or depression" but without changing the trajectory of the economy (*ibid.*: 8).

Finally, both Hamburger and Goodwin wanted to explain fluctuations around a growing economy. In 1930, Hamburger illustrated his argument with reference to Snyder's figures on the relatively small oscillations around an upward trend, and he prolonged this approach in several papers and communications during the following years (Hamburger, 1933). It is striking that Goodwin succeeded in integrating in his different cases some that allowed the economy to grow while showing self-sustained oscillations. In particular, his approach was successful to encapsulate the asymmetry between booms and depressions.

By giving a new content to a set of ideas that were laboriously presented two decades before, Goodwin managed to revive the approach and this shows that even if the authors may be forgotten, the ideas still linger, waiting for someone to give them a new life.

Archives

Ragnar Frisch Archives (RFA): some available at
https://www.sv.uio.no/econ/english/about/facts-and-figures/nobel-prize-winners/ragnar-frisch/
(consulted in March-April 2021) and personal copies

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