Voter coordination in elections : a case for approval voting

François Durand, Antonin Macé, Matias Nunez

VOTER COORDINATION IN ELECTIONS: A CASE FOR APPROVAL VOTING*

FRANÇOIS DURAND[a], ANTONIN MACÉ[b], AND MATÍAS NÚÑEZ[c]

ABSTRACT. We study how voting rules shape voter coordination in large three-candidate elections. We consider three rules, that differ on the number of candidates that voters can support: Plurality (one), Anti-Plurality (two) and Approval Voting (one or two). We show that the Condorcet winner is always elected at some equilibrium under Approval Voting, and that this rule provides better welfare guarantees than Plurality. We then numerically study a dynamic process of political tâtonnement which delivers rich insights. The Condorcet winner is virtually always elected under Approval Voting, but not under the other rules. The dominance of Approval Voting is robust to several alternative welfare criteria and the introduction of expressive voters.

KEYWORDS. Approval voting, Poisson games, Strategic voting, Condorcet consistency, Fictitious play, Expressive voting.

JEL CLASSIFICATION. D72; C72; C63.

* We would like to thank Laurent Bouton, Steve Brams, Élie de Panafieu, Arnaud Dellis, Mehmet Ekmekci, Helios Herrera, Tasos Kalandrakis, Yukio Koriyama, Jérôme Lang, Jean-François Laslier, Gilat Levy, Fabien Mathieu, Klaus Nehring, Yulliy Sannikov, Rafael Treibich and Dimitrios Xefteris for their useful comments as well as participants at the following conferences: *Economics and Computation* (EC'19), *Cyprus Meeting on Behavior and Algorithms* 2019, *Advances in Economic Design* 2019, *Conference on Mechanism and Institution Design* 2020, *GAMES* 2020, *Workshop Political Economy: Theory meets Empirics* 2021, at the seminar at Université Paris 1, at the Polaris seminar at Inria Grenoble, and at the online COMSOC seminar. Financial support from the ANR-14-CE24-0007-01 (CoCoRICoCoDEC) project, the ANR-17-CE26-0003 and the ANR-17-EURE-001 and from the ANR-11-IDEX-0003/Labex Ecodec/ANR11-LABX-0047 is gratefully acknowledged.

[a] NOKIA BELL LABS FRANCE & LINCS, FRANCE (www.lincs.fr). Email address: francois.durand@nokia-bell-labs.com.

[b] CNRS, PARIS SCHOOL OF ECONOMICS & ÉCOLE NORMALE SUPÉRIEURE, FRANCE. Email address: antonin.mace@psemail.eu.

[c] CREST, ÉCOLE POLYTECHNIQUE & CNRS, FRANCE. Email address: matias.nunez@polytechnique.edu.

Date: November 28, 2023.

1. Introduction

First-past-the-post voting remains one of the most common election methods across the world. Under this rule, also known as Plurality rule (henceforth PL), voters vote for just one candidate, and the candidate with the highest support is elected. In the U.S., where the rule is used in most local elections, recent years have seen a surge in popular initiatives aimed to change the local election method away from PL. For instance, in St. Louis, Missouri, the Proposition D initiative, which seeks to replace plurality voting with an approval voting system, obtained the support of 68.15% of the voters in November 2020. Two years earlier, the citizens of Fargo, North Dakota, adopted a similar initiative, and the approval voting system was used there in June 2020.[1] In its simplest form, Approval Voting (henceforth AV) is an extension of PL, whereby voters can vote for (*approve of*) as many candidates as they want, and the candidate with the highest support is elected.[2] Voters thus benefit from increased flexibility in casting their ballots under AV. Beyond this individual benefit, what are the potential collective gains from changing the election method from PL to AV? Would this improve the quality of preference aggregation?

As a first observation, voting rules shape the identity of the election winner beyond the preferences of the electorate. Voting rules not only differ in how they measure the support for each candidate, they also create different incentives for the voters. For instance, PL features a *wasted vote effect*, whereby voters who prefer a candidate with virtually no chance of winning have the incentive to drop their support and opt instead for a more serious candidate. This effect disappears under AV, as voters may simultaneously approve of a non-viable candidate and a more serious one. Taking voter incentives into account, which voting rules select the best candidates?

We address this question by focusing on the simplest setting of a three-candidate

[1]Besides Approval Voting, on which we focus in this paper, several localities have adopted a *ranked-choice voting* system, where ballots are counted by the *instant runoff* procedure (IRV). Examples include San Francisco (in 2002), Berkeley (in 2005), Oakland (in 2006) and New York City (in 2019). While we do not study IRV here, the rule seems ill-suited to selecting the Condorcet winner: experimental results indicate that voters tend to report their preferences truthfully due to the complexity of the system, and that the Condorcet winner may well be overlooked as a result [Van der Straeten et al., 2010].

[2]This rule, first proposed by Weber [1977] and Brams and Fishburn [1978], has attracted the interest of numerous scholars; see Brams and Fishburn [2005], Laslier and Sanver [2010], Laslier [2012] and the review of the literature (Section 7).

election.[3] We consider three main voting rules: AV, PL and Anti-Plurality (henceforth APL), the polar opposite of PL, in which each voter votes for two candidates, and the candidate with the highest support is elected. Our analysis is based on the classical game-theoretical model of Poisson games [Myerson, 2000], where the number of voters is drawn from a Poisson distribution. Voters hold private values and we allow for all possible preference orderings and intensities.

In this setting, the results for PL and APL are clear-cut, and thus serve as benchmarks to evaluate the performance of AV. Under PL, a consequence of the wasted vote effect is *Duverger's law* which asserts that any pair of candidates can concentrate all the votes in equilibrium [Duverger, 1951]. Thus, an equilibrium always exists but its outcome is indeterminate, since the set of equilibrium winners is large. However, we obtain a polar result for APL: generically, no equilibrium exists. This means that, some voters would prefer to change their vote for any belief that the electorate may hold. Hence, this election method appears unstable. Given this panorama, our question becomes as follows: can AV escape from the instability of APL, while selecting a normatively desirable candidate, and thus avoiding the indeterminacy of PL? We offer a qualified but broadly positive answer to that question by combining equilibrium analysis with numerical simulations.

We start by describing voter best replies under AV. Voters either vote for their favorite candidate or for their two preferred candidates. The choice between these two strategies is governed by a *utility threshold*: voters approve of their second candidate only if she yields a utility higher than this threshold. In a technical step, we show that this threshold converges when the expected size of the electorate becomes large. This allows us to define *asymptotic best replies* at the limit and its associated equilibrium notion. We also provide explicit formulas for the limit threshold. These formulas form the core of the Python package "Poisson Approval", designed by one of us, which we extensively use in our numerical analysis.[4]

We use these tools to derive our analytical results. First, we show that under AV, whenever there is a Condorcet winner in the preference profile, she wins in some equilibrium. Yet, this first result does not preclude the existence of other

[3]The case of three candidates is of particular relevance: (i) it constitutes the simplest case of non-trivial multi-candidate elections, and it is thus a central focus in the literature and (ii) a significant share of single-winner elections involve three candidates, notably because barriers to entry prevent other candidates from running.

[4]The package, designed by François Durand, is available at: https://pypi.org/project/poisson-approval. All the commands used for the simulations are here: https://francois-durand.github.io/poisson_approval/notebooks_article/index.html.

equilibria where she might lose. As a first pass, we provide a preliminary set of results indicating that such "bad equilibria" should be rare, relying on a distinction between *cardinal equilibria*, where preference intensities matter for some voters, and *ordinal equilibria*, where they do not. We prove that, if voters' utility distributions satisfy a symmetry assumption, then a Condorcet winner wins at any cardinal equilibrium. In turn, we show numerically that ordinal equilibria failing to select this candidate only exist for an extremely small region of preference profiles. We further investigate another welfare criterion, namely (relative) utilitarian welfare. We prove that AV delivers strictly better welfare guarantees than PL when a Condorcet winner exists.

In the final section, we introduce a model of "political tâtonnement" (a stylized electoral campaign)[5] which can be interpreted as a dynamic micro-foundation for the equilibrium analysis and which we essentially use as an equilibrium selection device. In this adaptive procedure, a poll is released at each stage, and a fraction of voters update their behavior by best replying to a belief incorporating both current and past polls. We simulate the trajectories of the adaptive procedure, and we interpret their long-run behavior as the electoral outcome. We then run Monte-Carlo simulations to obtain relevant statistics on election outcomes under each voting rule. The conclusions we draw from these simulations are broadly consistent with the equilibrium analysis but also offer several additional insights.

First, the procedure converges under AV, provided that a Condorcet winner exists. By contrast, convergence always occurs under PL, whereas it never does under APL. Second, we show that a Condorcet winner virtually always wins under AV, while she often fails to be elected under either PL or APL. We further show that the superiority of AV over both PL and APL extends to other normative criteria. AV surpasses both rules in terms of utilitarian welfare and in terms of Rawlsian (maxmin) and anti-Rawlsian (maxmax) criteria. This is remarkable as these two criteria would respectively be maximized under APL and under PL if voters were not strategic. Finally, we note that the winner under AV is almost always the sole candidate receiving the support of a majority of the electorate. This election method thus tends to confer political legitimacy to the winner.

We conclude by considering the robustness of our results to the assumption of strategic voting. We run simulations in which a fixed share of the electorate is expressive (non-strategic), in line with recent empirical studies (see Section 7).

[5]Myerson and Weber [1993] write "one might expect that voters will ultimately behave in accordance with a voting equilibrium after observing the series of public reports that accompany an extended campaign. (The campaign is, in part, a political tâtonnement, or equilibrium-seeking, process.)"

We find that AV still outperforms both PL and APL.

Overall, our main contribution lies in deriving robust welfare comparisons of voting rules (with respect to preferences, welfare concepts, sincere voters) while tackling the main technical challenge arising (for the analyst) when voters are allowed to vote for several candidates, namely accounting for the correlation in candidates' scores. Furthermore, our work illustrate how theory and simulations can be integrated to address important social issues on which theory alone does not provide complete answers.

The paper is structured as follows. Section 2 presents the main setting. Section 3 establishes the key properties of voters' best replies and describes equilibria under AV. Section 4 studies the equilibrium properties of alternative voting rules. Section 5 presents the numerical results obtained with the adaptive procedure. Section 6 provides a detailed discussion of our equilibrium concept. Notably, we stress that our main theoretical results are not sensitive to the particular choice of equilibrium notion. Section 7 reviews the literature and Section 8 concludes. Three appendices complete the paper. Section A contains the proofs of all results, but the convergence of utility thresholds. This last result is shown in Section B, which constitutes the theoretical grounding for the Python package "Poisson Approval". Section C exposes several robustness checks and additional numerical results.

2. Model

2.1. *Candidates and voters*

We consider an election in which voters elect one candidate into office from a set of candidates $\mathcal{K} = \{a, b, c\}$. Generic candidates will be denoted by i, j or k. Following the model of Myerson [2002], the number of voters is drawn from a Poisson distribution with mean $n \in \mathbb{N}$. Thus, the common knowledge probability that exactly θ voters take part in the election equals $e^{-n}\frac{n^\theta}{\theta!}$.

A voter's type t consists of his von Neumann and Morgenstern preference over lotteries on $\mathcal{K}$. It will be convenient to write $t = (o, u)$ where o corresponds to the voter's ordinal preference over candidates in $\mathcal{K}$, and u corresponds to a cardinal representation. Formally, if a voter has preference $i \succ j \succ k$, his *ordinal type* is denoted by $o = ij$, and we write $o_1 = i$ and $o_2 = j$ to respectively denote the most and the second most preferred candidate of an o-voter. We assume that voters have strict preferences over candidates,[6] so that the set of ordinal types

[6]We make this assumption for the clarity of exposition, but it is not substantive. Note that under AV, it is straightforward for a voter with preference $i \succ j \sim k$ (resp. $i \sim j \succ k$) to vote for i (resp. for both i and j).

is $\mathcal{O} = \{ab, ac, ba, bc, ca, cb\}$. Without loss of generality, a voter attaches utility 1 (resp. 0) to his preferred (resp. worst) candidate, and we denote by $u \in (0,1)$ his utility for his second-best candidate. The set of types can thus be written as $\mathcal{T} = \mathcal{O} \times (0,1)$.

Once the electorate is drawn, voters' types are drawn independently from a distribution ρ on $\mathcal{T}$, that we call the *preference profile*. We denote by $r \in \Delta(\mathcal{O})$ its marginal on $\mathcal{O}$, that we call the *ordinal profile*, so that r_o denotes the share of o-voters in the profile. For any candidate i, we denote by $r_i = r_{ij} + r_{ik}$ her expected share of supporters and by $r_{-i} = r_{kj} + r_{jk}$ her expected share of opponents. We denote by $\Delta^*(\mathcal{O}) = \{r \in \Delta(\mathcal{O}) \mid \forall o \in \mathcal{O}, r_o > 0\}$ the set of ordinal profiles that contain all ordinal types. We denote by ρ_o the conditional distribution of ρ on $(0,1)$ for each ordinal type $o \in \mathcal{O}$, so that ρ_o is the distribution of utilities for candidate o_2 and ordinal type o. We assume that each distribution ρ_o is absolutely continuous with respect to the Lebesgue measure, and we denote by F_o its cumulative distribution function.

2.2. *Strategies, profiles and pivots*

We focus on simple scoring rules, in which candidates are evaluated on the basis of their aggregate support in the electorate. A rule is then defined by the set of messages (or ballots) available to each voter, denoted by $\mathcal{M} \subseteq 2^{\mathcal{K}}$. The Plurality rule is defined by $\mathcal{M}^{PL} = \{a, b, c\}$, the Anti-plurality rule is such that $\mathcal{M}^{APL} = \{ab, ac, bc\}$ and Approval Voting is defined by $\mathcal{M}^{AV} = \{a, b, c, ab, ac, bc\}$. For any ballot $m \in \mathcal{M}$ and candidate k, we note $k \in m$ if k is approved (or supported) in m.

A (pure) *strategy* is a measurable function $\sigma : \mathcal{T} \to \mathcal{M}$. For any ballot m and strategy σ, we denote by τ_m the expected share of voters casting the ballot m:

$$\forall m \in \mathcal{M}, \qquad \tau_m = \rho(\{t \in \mathcal{T} \mid \sigma(t) = m\}) = \sum_{o \in \mathcal{O}} r_o \int_0^1 \mathbb{1}_{\{\sigma(o,u)=m\}} dF_o(u).$$

The (ballot) *profile* $\tau = (\tau_m)_{m \in \mathcal{M}} \in \Delta(\mathcal{M})$ plays a central role in our analysis as it captures all the relevant strategic information contained in the strategy σ. For any candidate k, we denote by γ_k her (normalized) expected score: $\gamma_k = \sum_{m|k \in m} \tau_m$.[7] We say that a candidate k is *elected* under a profile τ if $\gamma_k > \gamma_j$ for any $j \neq k$.

The set $\mathcal{Z} \subseteq \mathbb{N}^{\mathcal{M}}$ describes the possible outcomes of the election, a typical element is a vector $z = (z_m)_{m \in \mathcal{M}}$ where each component z_m denotes the number of

[7]The number of ballots m is a random variable $Z_m \sim \mathcal{P}(n\tau_m)$, while the number of approvals for k is a random variable $S_k \sim \mathcal{P}(n\gamma_k)$. Note that: variables $(Z_m)_{m \in \mathcal{M}}$ are independent by the *independent actions* property [Myerson, 1998]; variables $(S_k)_{k \in \mathcal{K}}$ may be correlated (if $\tau_{ij} > 0$ for some $i, j \in \mathcal{K}$); $\sum_{m \in \mathcal{M}} \tau_m = 1$ and $\sum_{k \in \mathcal{K}} \gamma_k \geq 1$.

voters casting ballot m. The probability of an outcome $z \in \mathcal{Z}$ is:

$$\mathbb{P}[z \mid n\tau] = \prod_{m \in \mathcal{M}} \left(\frac{e^{-n\tau_m}(n\tau_m)^{z_m}}{z_m!} \right).$$

For each outcome z and each candidate k, we denote by $s_k(z)$ the number of approvals for k at z (the score of k) and by $W(z)$ the set of candidates with maximal score at z (the set of winners):

$$s_k(z) = \sum_{m \mid k \in m} z_m \quad \text{and} \quad W(z) = \arg\max_{k \in \mathcal{K}} s_k(z).$$

The expected utility for a voter of type t when he casts a ballot m is given by:

$$U_t[m \mid n\tau] = \sum_{z \in \mathcal{Z}} U_t[m \mid z]\, \mathbb{P}[z \mid n\tau],$$

where $U_t[m \mid z]$ denotes the average utility of a t-voter for candidates in $W(z+m)$, with $z+m$ being the outcome where one ballot m has been added to z.[8]

Finally, we introduce *pivot events* that are key to determine voters' best replies. For each subset of candidates $K \subseteq \mathcal{K}$, we write:

$$\text{piv}_K = \left\{ z \in \mathcal{Z} \mid s_i(z) = s_j(z) > s_k(z), \forall i, j \in K, \forall k \notin K \right\},$$

the event where exactly candidates in K are tied for victory. In such a *pivot event* the ballot cast by an additional voter can affect the outcome in favor of any candidate in K.[9]

3. Best replies and equilibria under AV

This section presents the basic tools for the analysis of strategic voting under AV. Although we solely concentrate on AV, which constitutes our primary focus in this paper, all results (in particular Proposition 1 and Theorem 1 below) can be similarly adapted to PL and APL.

3.1. *Best replies*

For any profile τ, the best reply for a t-voter in an election of expected size n is defined by:

$$BR^n(t \mid \tau) = \arg\max_{m \in \mathcal{M}} U_t[m \mid n\tau].$$

We first observe that for each voter with ordinal type o, any ballot $m \in \mathcal{M} \setminus \{o_1, o_1o_2\}$ is strictly dominated by either o_1 or o_1o_2. Therefore, all voters will approve of their favorite candidate, while some voters will also approve of their

[8]This means that we assume that ties are broken evenly. Formally, for $t = (o, u)$, we write: $U_t[m \mid z] = \frac{1}{\#W(z+m)} \left(\mathbb{1}_{\{o_1 \in W(z+m)\}} + u \times \mathbb{1}_{\{o_2 \in W(z+m)\}} \right)$.

[9]The (cousin) event where a ballot cast by an additional voter can *create* a tie (at the top) between candidates in K is also relevant (see the proof of Proposition 1* in Section A).

second-best one. More precisely, the following result asserts that the choice between the two undominated ballots, o_1 and o_1o_2, monotonically depends on a voter's cardinal utility for his second-best candidate.

Proposition 1. *For each ordinal type $o \in \mathcal{O}$, each $n \in \mathbb{N}$ and each profile $\tau \in \Delta(\mathcal{M})$, there is a threshold $u_o^n(\tau) \in [0,1]$ such that:*

$$\forall u \in (0,1), \qquad BR^n(t = (o,u) \mid \tau) = \begin{cases} o_1 & \text{if} \quad u < u_o^n(\tau) \\ o_1o_2 & \text{if} \quad u > u_o^n(\tau). \end{cases}$$

In words, when deciding whether to approve his second-best candidate, a voter compares his utility for this candidate to the (endogenous) *utility threshold* $u_o^n(\tau)$. If u is low enough ($u < u_o^n(\tau)$), he only votes for his best candidate whereas when u is high enough ($u > u_o^n(\tau)$), he approves of his two preferred candidates. While the explicit formula of $u_o^n(\tau)$ is included in the appendix (Proposition 1*), the underlying intuition can be easily described.

Consider a voter with ordinal type $o = ij$, i.e. preferring $i \succ j \succ k$, in an election of expected size n. The threshold $u_{ij}^n(\tau)$ essentially depends on the relative probabilities of two events: piv_{ij} and piv_{jk}. Informally, if piv_{ij} is much more likely than piv_{jk}, the voter perceives the race between i and j as most important, and thus prefers to cast a ballot i, we have $u_{ij}^n(\tau) \approx 1$. Similarly, if piv_{jk} is much more likely than piv_{ij}, the race between j and k appears as most serious and we have $u_{ij}^n(\tau) \approx 0$. However, if piv_{jk} and piv_{ij} have similar magnitudes, the voter's best reply critically depends on the utility he attaches to j.

3.2. *Equilibria*

We focus on strategic behavior in a large population. Our analysis relies on the following result.

Theorem 1. *For any ordinal type $o \in \mathcal{O}$, for any profile $\tau \in \Delta(\mathcal{M})$, the sequence $(u_o^n(\tau))_{n\geq 0}$ converges when n tends to infinity. We denote by $u_o^\infty(\tau) \in [0,1]$ its limit.*

Theorem 1 is important as it allows to focus on an asymptotic version of the game. In the proof (Section B), we derive precise asymptotic developments of relevant pivot probabilities, from which we obtain both convergence and also explicit formulas for the *asymptotic utility thresholds* $u_o^\infty(\tau)$.[10]

[10] One key ingredient to obtain these developments is the notion of *offset ratios*, which measure for any ballot m the expected share of ballots m cast conditional on a given (pivotal) event to the unconditional expected share. As these ratios are not well-defined in some situations, we introduce in the proof the notion of *pseudo-offset ratios* (Section B.3) and we show that that a suitable version of Myerson [2002]'s Offset Theorem still holds with these ratios (Lemma 7).

Building on Theorem 1, we define asymptotic best replies by:

$$\forall o \in \mathcal{O}, \forall u \in (0,1), \qquad BR^{\infty}(t=(o,u) \mid \tau) = \begin{cases} o_1 & \text{if} \quad u < u_o^{\infty}(\tau), \\ o_1 o_2 & \text{if} \quad u > u_o^{\infty}(\tau). \end{cases}$$

This induces an asymptotic best reply function[11] from the set of profiles onto itself, $\widetilde{BR}^{\infty} : \Delta(\mathcal{M}) \to \Delta(\mathcal{M})$, defined by:

$$\forall \tau \in \Delta(\mathcal{M}), \qquad \widetilde{BR}^{\infty}(\tau) = \left(\rho(\{t \in \mathcal{T} \mid BR^{\infty}(t \mid \tau) = m\})\right)_{m \in \mathcal{M}}.$$

We say that a profile τ is an *equilibrium* if $\tau = \widetilde{BR}^{\infty}(\tau)$. In other words, τ is an equilibrium if, given that voters believe that the aggregate behavior corresponds to τ, the profile arising from their best-replies converges to τ. We discuss this equilibrium notion in detail in Section 6.

3.3. *Pivot magnitudes and discriminatory equilibria*

The magnitude of an event $E \subseteq \mathcal{Z}$ is the rate at which its probability tends to 0 when the population becomes large [Myerson, 2000]:

$$\mu[E] = \lim_{n \to \infty} \frac{\log(\mathbb{P}[E \mid n\tau])}{n}.$$

The main property of magnitudes is that whenever $\mu[E] > \mu[F]$, the event F is infinitely less likely than E when n is large. In the sequel, we extensively apply this concept to three pivot events (piv_{ab}, piv_{ac} and piv_{bc}), and we denote the corresponding *pivot magnitudes* by μ_{ab}, μ_{ac} and μ_{bc}.[12]

Following Myerson [2002], we say that a profile τ is *discriminatory* if there are two candidates $i, j \in \mathcal{K}$ such that $\mu_{ij} > \mu_{ik}, \mu_{jk}$. Essentially, this means that voters perceive the race between candidates i and j as the most serious race in the election. The following result underscores the importance of discriminatory profiles.

Proposition 2. *The set of discriminatory profiles is of (Lebesgue) measure* 1.

Proposition 2 highlights that almost all profiles are discriminatory, so that these profiles constitute a natural benchmark to focus on.[13]

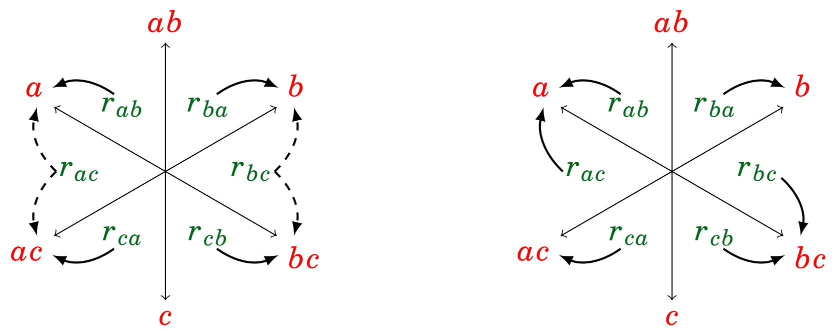
[11]While $\widetilde{BR}^{\infty}$ is defined a priori as a *correspondence*, note that it is in fact a *function* since we assumed that the utility distributions $(\rho_o)_{o \in \mathcal{O}}$ have no atom.

[12]The precise definition is $\mu_{ij} = \mu[\text{piv}_{ij} \cup \text{piv}_{abc}]$ (see Section A.2).

[13]Another way to select discriminatory equilibria is to require that the best-reply function $\widetilde{BR}^{\infty}$ be continuous: non-discriminatory profiles always fail this condition, while generic discriminatory profiles satisfy it. The condition can be interpreted as a (mild) stability refinement, which is essentially the same as that of *asymptotic strict perfection* used in Bouton and Gratton [2015].

3.4. *A typology of discriminatory equilibria*

In this section, we describe qualitative features of discriminatory equilibria that we use later in Section 4.1. Voters' behavior in such equilibria is particularly intuitive, and can be usefully described by classifying them into two types: *cardinal equilibria*, for which $\mu_{ij} > \mu_{ik} = \mu_{jk}$, and *ordinal equilibria*, for which $\mu_{ij} > \mu_{ik} > \mu_{jk}$. We illustrate voters' strategies in these two types of equilibria on Figure 1, with $i = a$, $j = b$ and $k = c$. In each panel, voters with a given ordinal type (in green) may pick any of two undominated ballots (in red), equilibrium choices are indicated by a curly arrow (dashed whenever the choice depends on cardinal utilities).



(A) Cardinal equilibrium
$\mu_{ab} > \mu_{ac} = \mu_{bc}$

(B) Ordinal equilibrium
$\mu_{ab} > \mu_{ac} > \mu_{bc}$

FIGURE 1. Two kinds of discriminatory equilibria.

In both panels of Figure 1, the equilibrium is discriminatory with $\mu_{ab} > \mu_{ac}, \mu_{bc}$. Hence, voters perceive that the most serious race occurs between candidates a and b. This observation determines the behavior of voters not ranking candidate c second: if they prefer $a \succ b \succ c$ (above, left), they only vote for a, while if they prefer $c \succ a \succ b$ (below, left), they vote for both a and c (and similarly if they prefer b to a, on the right).

What differs between the two types of equilibria is the behavior of voters ranking c second. In Figure 1 (A), as $\mu_{ac} = \mu_{bc}$ a voter preferring $a \succ c \succ b$ (middle, left) will always vote for a, and he will also vote for c if she yields a high enough utility. We thus refer to such an equilibrium as *cardinal*, in the sense that preference intensities do matter for some voters, a feature in line with empirical election data [Spenkuch, 2018].

By contrast, in Figure 1 (B), as $\mu_{ac} > \mu_{bc}$ a voter preferring $a \succ c \succ b$ will always vote for a and never for c, since voting for c is much more likely to matter in a race against a rather than against b. Here, preference intensities play no role, and we refer to such an equilibrium as *ordinal*.[14]

A few observations are in order. First, although cardinal equilibria require the equality between two pivot magnitudes, this does not mean that they are not generic. In fact, a positive measure of profiles τ exhibit such an equality (see Lemma 1 in Section A),[15] and we further show that a cardinal equilibrium exists under AV when the preference profile admits a Condorcet winner (see Theorem 2 below). Second, a distinctive feature of ordinal equilibria is that each ordinal type of voters votes in block, so that all ordinal equilibria can be easily computed numerically for a given (ordinal) preference profile. We exploit this feature in our subsequent numerical analysis.

4. Equilibrium Analysis of Voting Rules

In this section, we focus on equilibrium analysis to study the quality of electoral outcomes under alternative voting rules. We first focus on AV, showing that coordination on the election of the Condorcet winner is always feasible. Relying on the dichotomy between cardinal and ordinal equilibria, we argue that selecting a non-Condorcet winner, while possible in some extreme instances, should be a rare phenomenon under AV. Moreover, AV delivers high welfare guarantees. Finally, we point to deficiencies of alternative voting rules: the multiplicity of equilibrium winners and the associated low welfare guarantees under PL and the generic

[14]Laslier [2009]'s model also features ordinal equilibria, where preference intensities are irrelevant. Yet, in the Poisson model, the main race always occurs between the candidates with the highest and lowest scores at an ordinal equilibrium (see Lemma 2 in Section A), in violation of the *ordering condition* [Myerson and Weber, 1993] at the root of Laslier [2009]'s model. Hence, ordinal equilibria do not coincide across the two models.

[15]To illustrate why the magnitude ranking $\mu_{ab} > \mu_{ac} = \mu_{bc}$ is generic, consider the following example. When $\tau_a = \tau_b = 0.4 > 0.2 = \tau_c$, consider the event $\{S_a = S_c \geq S_b\}$ whose magnitude is μ_{ac}. The distribution of S_b/n is concentrated around 0.4, while that of S_a/n conditionally on $\{S_a = S_c\}$ is concentrated around $\sqrt{0.4 \times 0.2} \approx 0.28$. Here, a large deviation principle applies: since conditionally on $\{S_a = S_c\}$, it is likely that $S_b > S_a$, most of the probability of the event $\{S_a = S_c \geq S_b\}$ actually concentrates on $\{S_a = S_c = S_b\}$, which implies that the two events have equal magnitudes. A similar reasoning applies to the event $\{S_b = S_c \geq S_a\}$, so that we obtain $\mu_{ac} = \mu_{bc} = \mu[\{S_a = S_b = S_c\}]$. This equality is generic as the previous reasoning applies for any τ such that $\tau_a, \tau_b > \tau_c = 1 - \tau_a - \tau_b$ and $\min(\tau_a, \tau_b) > \sqrt{\tau_c \max(\tau_a, \tau_b)}$. Lemma 1 further implies that the genericity of the magnitude equality holds more broadly, when τ is 5-dimensional and candidates' scores are correlated.

non-existence of equilibrium under APL.

4.1. *Approval Voting: Condorcet consistency and welfare*

We start by an important theoretical result: a general possibility under AV. We say that a candidate $i \in \mathcal{K}$ is an (asymptotic) *Condorcet winner*[16] if the distribution r ensures that an expected strict majority of voters prefers i to any other candidate j: $\forall k \neq i,\ r_i + r_{ki} > 1/2$.

Theorem 2. *Let ρ be a preference profile with $r \in \Delta^*(\mathcal{O})$. If there is a Condorcet winner in ρ, then this candidate is elected at some equilibrium under AV.*

Theorem 2 establishes two important properties for profiles admitting a Condorcet winner. First, an equilibrium exists under AV. Second, this equilibrium may result in the election of the Condorcet winner. In the language of implementation theory, we may say that the voting rule AV partially implements the Condorcet winner.

The proof is semi-constructive: we show the existence of a cardinal equilibrium where the Condorcet winner is elected. If i denotes the Condorcet winner and j her best contender (losing against i by a smaller margin than k does), the main race occurs between i and j. Voters with ordinal types ik and jk vote on the basis of their preference intensities, the precise shares of such voters voting for their preferred candidate are obtained by a fixed-point argument.[17]

While we have no guarantee that the equilibrium is unique, it is important to know whether other candidates than the Condorcet winner may be elected at equilibrium. To address this question, we divide our inquiry in two parts, by focusing first on cardinal equilibria, for which we provide a formal result, and then on ordinal equilibria, for which we rely on numerical computations. To state the first result, we say that a preference profile ρ satisfies *Symmetry (Assumption S)* if there exists a distribution F on $(0,1)$, symmetric around 1/2, such that for any $o \in \mathcal{O}$, $F_o = F$. Under this assumption, the distribution of voters' utilities for their second candidate does not depend on their ordinal type o and is not biased (for

[16]A Condorcet winner exists with probability 93.7% when r is drawn from the uniform distribution on $\Delta(\mathcal{O})$, as in Section 5. In the literature, Durand [2022] reports that a Condorcet winner exists in 99% of elections taken from a large dataset of local U.S. elections. Van Deemen [2014] surveys the literature: the lack of a Condorcet winner is infrequent but cannot be empirically discarded.

[17]Note that the result is different from that in Laslier [2009], where an ordinal equilibrium electing the Condorcet winner is constructed. This equilibrium does not exist here, as it satisfies the ordering condition (by construction), while all ordinal equilibria violate this condition in Poisson games under AV (see Footnote 14).

each level $\overline{u} \leq 1/2$, there is the same proportion of voters with utility below $\overline{u}$ or above $1-\overline{u}$). For instance, Assumption S is satisfied if each voter's utility for her second candidate is drawn uniformly on $(0,1)$. While arguably restrictive, deriving the properties of AV under Assumption S for arbitrary ordinal preference profiles serves as a good benchmark.[18]

Proposition 3. *Let ρ be a preference profile satisfying Assumption S, with $r \in \Delta^*(\mathscr{O})$. If there is a Condorcet winner in ρ, then this candidate is elected at any cardinal equilibrium under AV.*

Proposition 3 offers a (conditional) reciprocal statement to Theorem 2 but remains silent on the class of ordinal equilibria. In fact, earlier work [Núñez, 2010] shows that ordinal and cardinal equilibria may coexist under AV, and that a non-Condorcet winner may be an ordinal equilibrium outcome. We thus turn to ordinal equilibria, a class than we can exhaustively compute for a given preference profile.

We run Monte-Carlo simulations, generating $10,000$ ordinal profiles drawn independently from the uniform distribution on $\Delta(\mathscr{O})$. For each draw, we compute all ordinal equilibria. We first obtain that when a Condorcet winner exists, a non-negligible proportion of ordinal profiles, estimated at 29.4%, admit an ordinal equilibrium. This implies that the (cardinal) equilibrium built in Theorem 2 is not generically unique. Second, we find that the overall prevalence of ordinal equilibria where the Condorcet winner is not elected is extremely rare, as they appear for only 0.1% of ordinal profiles admitting a Condorcet winner.[19]

Finally, we consider an alternative welfare criterion, utilitarian welfare, defined by:

$$W_i = r_i + r_{ki}\int_0^1 u dF_{ki}(u) + r_{ji}\int_0^1 u dF_{ji}(u).$$

As (von Neumann and Morgenstern) utilities have been normalized to lie between 0 and 1 for each voter, this welfare criterion corresponds to *relative utilitarianism*, which has a solid theoretical foundation [Dhillon and Mertens, 1999, Borgers and Choo, 2017]. We show that AV provides high welfare guarantees under Assumption S, when a Condorcet winner exists (so that equilibrium existence is assured).

[18]Note that Assumption S is a sufficient condition in Proposition 3, but it is by no means necessary. For instance, the result from Proposition 3 holds if either (i) one assumes that the Condorcet winner is ranked first by at least half of the voters; or (ii) utility distributions exhibit *weak cardinal support for the Condorcet winner* (named i): $\forall \overline{u} \in (0,1)$, $F_{ji}(\overline{u}) + F_{ki}(1-\overline{u}) \leq 1$, i.e. the Condorcet winner has sufficient support from voters that rank her second.

[19]In that case, the elected candidate is the non-Condorcet loser.

Proposition 4. *Let ρ be a preference profile satisfying Assumption S, admitting a Condorcet winner, and with $r \in \Delta^*(\mathcal{O})$. For any candidate i elected at a discriminatory equilibrium under AV, we have $W_i > 1/2$.*

Note that under Assumption S, 1/2 corresponds to the average welfare of the candidates. Proposition 4 thus asserts that when a Condorcet winner exists, AV guarantees the election of an above-average candidate in terms of welfare. The proof relies on the dichotomy between ordinal and cardinal equilibria, it shows in particular that even when a non-Condorcet winner is elected, this candidate must nevertheless perform well in terms of welfare.

The conclusion we draw from this section is positive for AV: when a Condorcet winner exists, the rule partially implements it, the election of a non-Condorcet winner seems unlikely and any elected candidate yields above-average welfare under a symmetry assumption. In the sequel, we address the same questions for PL and APL, as these alternative rules offer a benchmark to which we can compare the performance of AV.

4.2. *Plurality Voting: Duverger's law*

In this section, we recall Duverger's law for PL. For any pair of candidates $i, j \in \mathcal{K}$, the belief that all voters vote for either i or j is self-fulfilling. Hence there is an equilibrium where one of these two candidates is elected: the one preferred by a majority in the race between i and j.

Proposition 5. *(Duverger, 1951; Myerson, 2002) Under PL, for a generic ordinal profile r, for any candidates i and j, there is an equilibrium τ with $\gamma_i, \gamma_j > \gamma_k = 0$. For any such profile r, at least two candidates can be elected at equilibrium.*

We observe that PL always admits an equilibrium where the Condorcet winner is elected, exactly as for AV. However, contrary to AV, there always is a multiplicity of equilibrium winners. Moreover, some equilibrium winners can be significantly below-average in terms of welfare.

Proposition 6. *For any $\varepsilon > 0$, there are generic preference profiles ρ, satisfying Assumption S and admitting a Condorcet winner, such that there exists an equilibrium winner i under PL with $W_i < 1/4 + \varepsilon$.*

Hence, under the conditions for which AV provides the high welfare guarantee of 1/2, elected candidates under PL can yield welfare arbitrarily close to 1/4, that is, only half the average welfare.

4.3. *Anti-Plurality Voting: Coordination failure*

In this section, we show that, generically, no equilibrium exists under APL.

Proposition 7. *Under APL, for any utility distributions* $(\rho_o)_{o\in\mathcal{O}}$*, the set of ordinal profiles r for which an equilibrium exists is of measure* 0.

The result implies that coordination failures are pervasive under APL. Proposition 7 falls in line with the non-existence of discriminatory equilibria proven by Myerson [2002]. The intuition is simple: if two candidates i and j were expected to form the most serious race of the election, all voters would vote against their most disliked candidate between i and j, thus pushing the score of the third candidate, k, above both i and j's scores, a contradiction.[20] What we add to this intuition is the observation that other, non-discriminatory equilibria cannot generically exist (see Section 6 for a further discussion on equilibrium non-existence).

5. A MODEL OF POLITICAL TÂTONNEMENT

To gain more insights on strategic behavior in large elections, we consider a dynamic process from which equilibria may plausibly arise: an adaptive procedure of "political tâtonnement". The procedure simulates a repeated sequence of polls,[21] in which a fraction of voters updates its behavior after the publication of each poll, and we focus on long-run outcomes. The virtues of this alternative modeling device are twofold. First, we avoid the difficulties associated with equilibrium multiplicity or non-existence, highlighted in the previous section. Indeed, if multiple equilibria exist, any converging trajectory selects one particular equilibrium. If no equilibrium exists, the long-run outcome of a trajectory provides relevant information on voters' behavior. Second, the trajectories of the procedure can be numerically computed. We thus perform Monte-Carlo simulations of the procedure to describe various features of voter behavior in large elections and to assess the robustness of the results to expressive voting.

5.1. *Adaptive procedure*

The procedure is based on the concept of *fictitious play* [Brown, 1951]. Formally, we consider a sequence of profiles $(\tau^p)_{p\geq 0}$, with initial profile τ^0 drawn from the uniform distribution on $\Delta(\mathcal{M})$, and whose dynamic is described by the following

[20] This logic is at play in the lab experiment reported in Dellis et al. [2011]: voters often use non-sincere ballots under APL, and close three-way ties emerge as a result.

[21] We refer to a ballot profile τ as a poll for illustrative purposes here. Note that τ corresponds to an idealized poll, as there is no randomness arising from sample selection.

equations:

$$\begin{cases} \hat{\tau}^1 &= \tau^0 \\ \tau^1 &= \widetilde{BR}^{\infty}(\tau^0) \end{cases} \quad \text{and} \quad \forall p \geq 2, \quad \begin{cases} \hat{\tau}^p &= (1-\alpha^p)\hat{\tau}^{p-1} + \alpha^p \tau^{p-1} \\ \tau^p &= (1-\beta^p)\tau^{p-1} + \beta^p \widetilde{BR}^{\infty}(\hat{\tau}^p), \end{cases}$$

where $\alpha^p = \beta^p = \frac{1}{\log(p+1)}$ for each $p \geq 2$.

This set of equations can be read as follows. The profile $\hat{\tau}^p$ corresponds to the *perceived profile* at period p. At $p = 1$, the perceived profile coincides with the initial profile τ^0. At later periods, the perceived profile $\hat{\tau}^p$ is a weighted average of the perceived profile at $p-1$ and of the *actual profile* being played at $p-1$, α^p being the weight attached to that latter term. The actual profile τ^p initially corresponds to the best reply to τ^0. At later periods, a share β^p of the electorate updates its behavior and plays a best reply to the perceived profile $\hat{\tau}^p$. We choose $\alpha^p = \beta^p = \frac{1}{\log(p+1)}$ to optimize (numerically) the convergence rate of the procedure. In terms of beliefs, the procedure is thus intermediate between the best-reply dynamic ($\alpha^p = 1$) and the classical fictitious play ($\alpha^p = \frac{1}{p+1}$). Intuitively, all previous periods are taken into account, but later ones get a disproportionate weight.[22]

We choose to focus on a uniform draw of the initial poll τ^0, as this assumption provides a transparent account of all the possible outcomes that may happen under a certain voting rule at a given preference profile. In practice, there are many reasons why polls at the beginning of an election campaign may be only remotely related to preferences: preferences might evolve during the campaign as information and debates unfold, polls may be strategically used to favor certain candidates, primaries might first occur in localities whose preferences are at odds with those of the electorate at large, etc. We discuss in Section C.4 alternative specifications for the distribution of τ^0.

We note that the steady states of the procedure coincide with the equilibria analyzed in Section 4:

Remark 1. $(\hat{\tau}, \tau)$ *is a steady state of the adaptive procedure if and only if* $\hat{\tau} = \tau$ *is an equilibrium.*

In the sequel, we report the results of Monte-Carlo simulations of the adaptive procedure. Each observation consists of two independent draws, one for the ordinal preference profile r and the other for the initial poll τ^0. The profile r is drawn from the uniform distribution on the simplex $\Delta(\mathcal{O})$, unless otherwise specified. Moreover, we assume that for each ordinal type o, the distribution of utilities for

[22]More precisely, for any fraction $f \in (0,1)$, the total weight assigned to the fraction f of latest periods converges to one when the total number of periods diverges to infinity. Note that this is a property of the *weights*, which does not preclude behavior in earlier periods of the campaign to affect its long-run outcome.

the second candidate, ρ_o, is uniform on $(0,1)$.[23] All simulations reported below rely on a number of 10,000 draws.[24]

5.2. *Comparisons of voting rules*

5.2.1. *Convergence.* We consider that the sequence $(\tau^p)_{p\geq 0}$ *converges* (numerically) if there is a period p such that the two following conditions are satisfied: $\|\tau^p - \widetilde{BR}^{\infty}(\hat{\tau}^p)\|_{\infty} \leq 10^{-9}$ and $\|\hat{\tau}^p - \widetilde{BR}^{\infty}(\hat{\tau}^p)\|_{\infty} \leq 10^{-9}$. We report in Table 1 the convergence rate of the procedure in the Monte-Carlo simulations, for each voting rule and with a maximum of $P = 1{,}000$ iterations. We decompose this rate for profiles with and without a Condorcet winner.

	PL	APL	AV
general	100.0%	0.0%	95.2%
$\exists$ CW			99.97%
$\nexists$ CW			19.3%

TABLE 1. Percentage of observations for which the procedure converges.

We observe that the procedure always converges under PL, but that it never does under APL, in line with the theoretical results of Section 4: coordination always occurs under PL but never arises under APL. Convergence is very frequent under AV in general, but appears to be rare in the absence of a Condorcet winner.

When the sequence $(\tau^p)_{p\geq 1}$ converges at a period p^*, we consider that τ^{p^*} is the outcome of the adaptive procedure. When this is not the case, we use in the sequel the following definition of the outcome of the trajectory, which corresponds to its average long-run behavior, computed for $P = 1{,}000$. For any outcome function $w : \Delta(\mathscr{M}) \to X$ (candidates' scores, identity of the winner, etc.), we define inductively $w^P \in \Delta(X)$, the (average, long-run) outcome of the sequence $(\tau^p)_{1\leq p\leq P}$, by:

$$w^1 = w(\tau^1) \quad \text{and} \quad \forall p \geq 2, \quad w^p = (1-\lambda^p)w^{p-1} + \lambda^p w(\tau^p),$$

where $\lambda^p = \frac{1}{\log(p+1)}$, so that all periods are accounted for but later ones get a disproportionate weight.

Note that, since the definition of the outcome of the procedure may be probabilistic, the percentages we report in the sequel corresponds to averages taken

[23]Note that utilities are not drawn. Instead, for each ordinal type o, we compute the share of voters casting a ballot o_1 (resp. o_1o_2) by explicitly computing the asymptotic utility threshold $u_o^{\infty}(\tau)$.

[24]With 10,000 independent draws, the boundaries of the 95% confidence interval are always within one percentage point of the point estimate, and are typically more narrow when the point estimate approaches 0% or 100%. To lighten notations, we do not report the (asymmetric) confidence intervals in the tables, but we provide the boundaries of the 95% confidence interval attached to all possible point estimates in Section C.8.

over three non-deterministic elements: the draw of the ordinal preference profile r, the draw of the initial profile τ^0 and the outcome w^P (only if the trajectory does not converge).

5.2.2. *Condorcet consistency.* In this section, we focus on preference profiles with a Condorcet winner. We report in Table 2 below the frequency with which the Condorcet winner is elected for each voting rule.

	PL	APL	AV
∃ CW	66.1%	49.5%	99.96%

TABLE 2. Percentage of observations for which the Condorcet winner is elected.

The main result of Table 2 is that AV almost always implements the Condorcet winner, whereas PL and APL often elect a non-Condorcet winner.

For PL, the equilibrium multiplicity of Duverger's law (Proposition 5) translates in a significant chance to miss the Condorcet winner, which occurs if voters coordinate on the "wrong equilibrium" where the Condorcet winner is not a serious contender.[25] For APL, the generic nonexistence of an equilibrium (Proposition 7) hinders voters to coordinate to elect the Condorcet winner.

5.2.3. *Welfare comparisons.* In this section, we extend the comparison of voting rules to other normative criteria than the selection of the Condorcet winner. We start with utilitarian welfare, introduced in Section 4.1. For each election, we compute the welfare loss, defined as the difference between the welfare of the highest-welfare candidate and that of the elected candidate. We report on Figure 2 the cumulative distribution of utilitarian welfare losses under the three voting rules.[26]

We observe a clear ordering of the three voting rules, in the sense of first-order stochastic dominance, the comparison of AV and PL matching the analytical results on welfare guarantees (Proposition 4 and Proposition 6). Welfare losses are significantly lower under AV than under the other rules, and are smaller under PL compared to APL. In the appendix (Section C.6), we further show that the dominance of AV also holds when comparing absolute welfare levels rather than

[25]Under PL, updating voters only vote for the two candidates appearing in the pivot with highest magnitude, so that this pivot always remains the same from the first period onward. Hence, the Condorcet winner fails to be elected when she does not appear in the main pivot in the initial poll τ^0 (this arises with a probability of 1/3).

[26]For this figure as for subsequent ones, the grey area around each line corresponds to a width of 2 percentage points, which contains the 95% confidence interval associated to each reported estimate (see Section C.8).

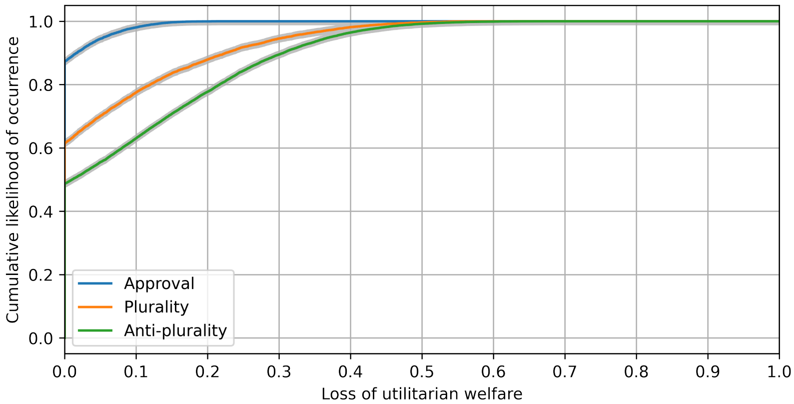
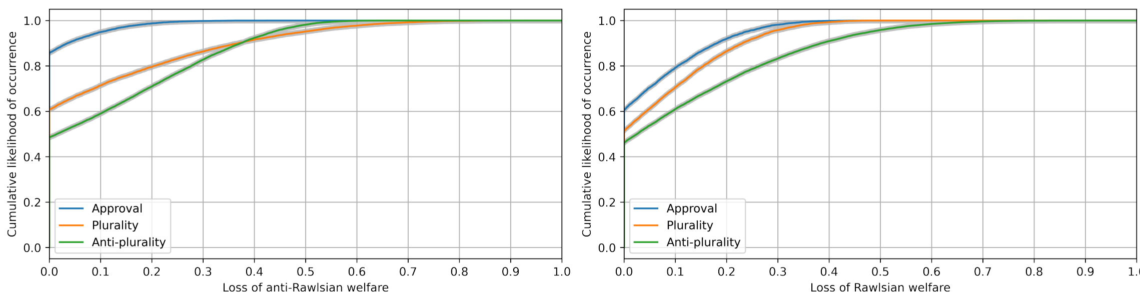


FIGURE 2. Cumulative distributions of utilitarian welfare losses.

welfare losses.

We then consider two other criteria: W^{PL} measures the share of voters having their first choice elected, while W^{APL} measures the share of voters having either their first choice or their second choice elected.[27] These two criteria are of particular interest with respect to the rules we consider: W^{PL} would be maximized under PL if all voters voted expressively (for their first choice), and W^{APL} would be maximized under APL if all voters voted expressively (against their last choice). What happens when voters are strategic? We report on Figure 3 the cumulative distributions of welfare losses under the three voting rules, computed for W^{PL} in the left-panel and for W^{APL} in the right panel.



(A) Anti-Rawlsian welfare W^{PL} (B) Rawlsian welfare W^{APL}

FIGURE 3. Cumulative distributions of welfare losses.

Remarkably, AV still dominates both PL and APL in the sense of first-order stochastic dominance, for both welfare criteria. Overall, Figure 2 and Figure 3 indicate that the dominance of AV over the other voting rules is robust, and not sensitive to the retained normative criterion.

[27]Formally: $W_i^{PL} = r_i$ and $W_i^{APL} = 1 - r_{-i}$. Note that W^{APL} may be interpreted as Rawlsian welfare (maxmin), while W^{PL} may be interpreted as anti-Rawlsian welfare (maxmax).

5.3. *Features of voter behavior and election outcomes under AV*

We report statistics on voter behavior and election outcomes under AV in the simulations.

As emphasized in Section 3, the main parameter governing the choice of a voter is his asymptotic utility threshold. Averaging over all simulations and types, we find that 47.2% of voters have a utility threshold of 1, and thus always vote for their favorite candidate only, while 18.5% of voters have a threshold of 0, and thus always vote for their two favorite candidates. The remaining share of 34.3% of voters vote for their two favorite candidates only if they have a high enough utility for the second candidate. This confirms that utility-dependent voting is a robust phenomenon in Poisson games, and that the approximate Condorcet consistency of AV is not driven by the same behavior as in Laslier [2009], where preference intensities do not play any role. As a result of this threshold distribution, we obtain that, over all simulations, 31.9% of voters approve of two candidates, while 68.1% of voters only vote for one candidate. Hence, the positive welfare results under AV are obtained despite a majority of voters do not use the possibility to vote for multiple candidates.

In relation to Condorcet consistency and welfare comparisons, an arguably important criterion for the acceptability of election results concerns majoritarian legitimacy. While it is important that the election winner be approved by a majority of the electorate, this might not be sufficient to ensure legitimacy of the winner, as more than one candidate may receive majoritarian support under AV. The legitimacy of the winner would be stronger if this candidate was the only one to be approved by a majority. It turns out that this is what happens in almost all simulations under AV.[28] We report on Figure 4 the empirical cumulative distribution of the score of each candidate as a function of her rank in the election.

[28]In that case, the elected candidate may be deemed legitimate, as her election is both *internally* and *externally* consistent with a principle of majoritarian support (see Patty and Penn [2011]).

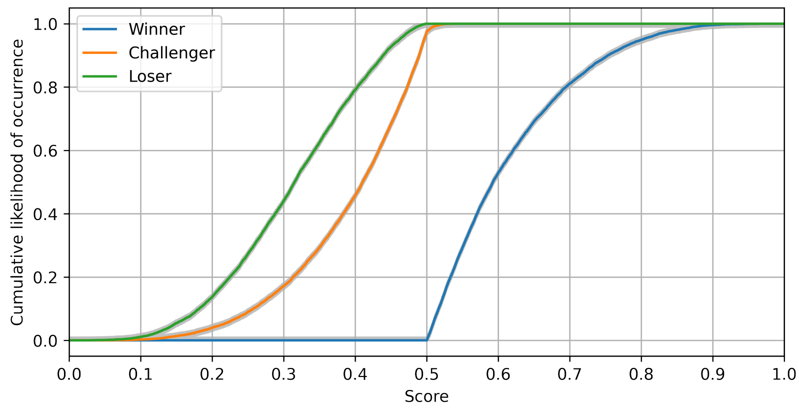


FIGURE 4. Cumulative distribution of candidates' scores under AV.

We observe that the winner always gets above 50% of approvals, while both the challenger (ranked second) and the loser (ranked third) are almost always approved by a minority of the electorate.[29] Hence, not only AV performs well with respect to the Condorcet and welfare criteria, but it also provides a form of majoritarian legitimacy to the election winner, at least in a strategic electorate.

5.4. *Robustness to expressive voting*

We assumed in the analysis above that all voters were strategic. This is an arguably strong assumption that has been disputed in the empirical literature (see Section 7). While strategic voting models do capture features of behavior observed in experimental and empirical data, there is ample evidence that some voters vote expressively, casting a ballot for the candidate they prefer, independently of her chance of winning. We consider in this section simulations where only a fraction of the electorate is strategic, while remaining voters are expressive.

While expressiveness (or sincerity) is easy to define under PL and APL, it may be polysemic under AV [Merill and Nagel, 1987]. We start by adopting the view, both empirically plausible and well-defined theoretically, that an expressive voter votes as if he were the only voter in the electorate.[30] Hence, we first assume that a (model 1-) expressive voter simply approves of his favorite candidate under AV. We report on Figure 5 the frequency of the election of the Condorcet winner when she exists, as a function of the fraction of (model 1-) expressive voters, under each voting rule.

[29]While running the simulations, we found rare exceptions, where the challenger was approved by a majority of the electorate. One such example is described in Section C.7.

[30]Formally, for any rule $R \in \{\text{AV, PL, APL}\}$, for any t, $\sigma_t^{R,E1} \in \arg\max_{m \in \mathcal{M}^R} U_t[m \mid n = 0]$. For $t = (o,u)$, we obtain $\sigma_t^{AV,E1} = \sigma_t^{PL,E1} = o_1$ and $\sigma_t^{APL,E1} = o_1 o_2$.

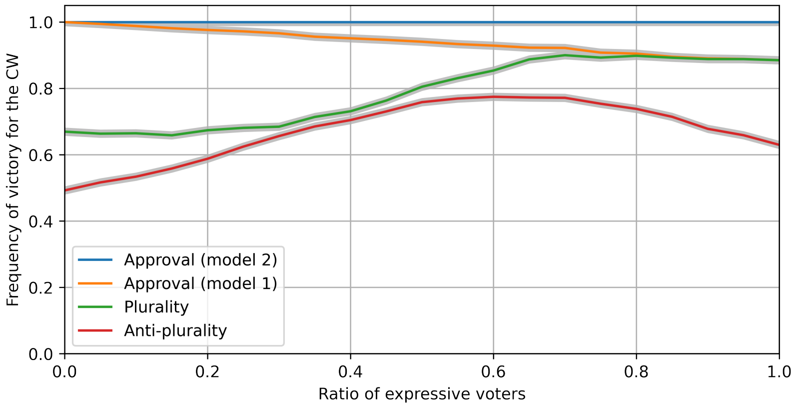


FIGURE 5. Winning frequency of the Condorcet winner as a function of the fraction of expressive voters.

We observe that the superiority of AV over PL and APL is robust to the introduction of expressive voters. Even if half of the electorate behaves (model 1-) expressively, AV still clearly dominates the two alternative rules.

We emphasize that this comparison is conservative as we implicitly assumed that the share of expressive voters is independent of the voting rule. However, moving from expressive to strategic behavior is plausibly more costly under PL than under AV. Indeed, when expressive and strategic behaviors differ under PL, a strategic voter should give up voting for his favorite candidate, which might be difficult to accept. On the other hand, when expressive and strategic behaviors differ under AV, a strategic voter should just approve of his second candidate in addition to his favorite one, which appears a milder departure from expressive behavior.

We thus consider a second model of expressive voting. We now assume that a (model 2-) expressive voter is restricted to cast a *weakly sincere* ballot, i.e. to support a candidate deemed better than another supported candidate. Yet, we allow such a voter to (optimally) choose his ballot if several ballots are weakly sincere.[31] This second model of expressive voting is not different from the first one under PL or APL. However, the constraint of weak sincerity is moot under AV, as the (unconstrained) optimal ballot is always weakly sincere, so that the expressive model 2 coincides with the strategic one. As a result, we obtain a stronger argument for AV: we observe on Figure 5 that AV clearly dominates both PL and APL for *any fraction* of (model 2-) expressive voters.

[31]Formally, for any $t = (o, u)$, we let $\mathscr{M}_t^{ws} = \{o_1, o_1 o_2, abc\}$ be the set of *weakly sincere* ballots for type t. For any rule $R \in \{\text{AV, PL, APL}\}$, for any profile τ, for any t, a (model-2) expressive voter casts a ballot $\sigma_t^{R,E2} \in \arg\max_{m \in \mathscr{M}^R \cap \mathscr{M}_t^{ws}} U_t[m \mid n \to +\infty, \tau]$. For $t = (o, u)$, we obtain $\sigma_t^{AV,E2} = \widetilde{BR}^{\infty}(t \mid \tau)$, $\sigma_t^{PL,E2} = o_1$ and $\sigma_t^{APL,E2} = o_1 o_2$.

Finally, we close this section by focusing on utilitarian welfare. We report on Figure 6 the average utilitarian welfare loss as a function of the fraction of (model 1 or model 2-) expressive voters, under each voting rule.[32]

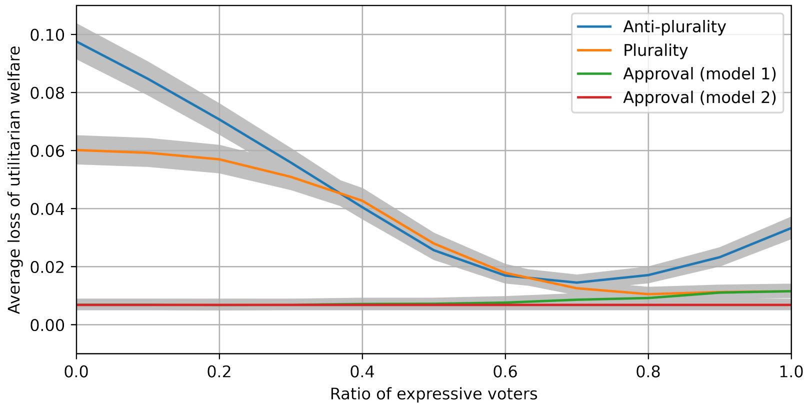


FIGURE 6. Average utilitarian welfare loss as a function of the fraction of expressive voters.

Again, we conclude that the welfare-superiority of AV over PL and APL is robust to the introduction of expressive voters, particularly so when these voters represent no more than two thirds of the electorate.

6. DISCUSSION ON THE EQUILIBRIUM CONCEPT

In this section, we discuss the equilibrium concept employed in this study and how it relates to the literature. We defined an *equilibrium* τ as a fixed point of the asymptotic best reply function $\widetilde{BR}^{\infty}$. This definition may be contrasted with the equilibrium concept introduced by Myerson [2002]. In that work, a *large equilibrium* τ is defined as a limit of profiles $(\tau_n)_{n\geq 1}$, where each τ_n is a fixed-point of the best-reply function $\widetilde{BR}^{n}$, associated to an electorate of expected size n. As a first remark, we note that all of our theoretical results on AV and PL remain valid for this alternative equilibrium concept under a mild technical condition.[33] In the sequel, we argue that our equilibrium notion is well-grounded theoretically and practical to use for the analyst, we discuss the interpretation of potential non-existence, and we explore the relation between the two equilibrium notions.

---

[32]For this figure, as most values are close to 0, we report the precise 95% confidence intervals estimated in Section C.8.

[33]Precisely, the preference profiles must be such that no voter has a utility for his second candidate arbitrarily close to either 0 or 1. As a consequence, the article could be alternatively written with the notion of *large equilibrium*. Yet, we stick to our equilibrium notion for the reasons we outline in this section.

First, what purpose does an equilibrium concept serve? In general, an equilibrium concept can help describe plausible long-run behavior in a situation of interest. This view is particularly relevant for *voting equilibria*, as emphasized in the early works on this topic [Palfrey, 1989, Myerson and Weber, 1993]: these equilibria capture the steady states of an electoral campaign, whereby voters adjust their behavior after observing pre-election polls. We closely follow this approach here: we introduce a tâtonnement model that simulates a stylized electoral campaign within a large electorate, and our equilibrium concept coincides with the steady states of the dynamic (Remark 1). By contrast, it seems unclear whether large equilibria coincide with steady states of an explicit dynamic.

Second, this dynamic micro-foundation for our equilibrium concept also confers advantages for studying the voting game. When multiple equilibria exist, simulating the dynamic is useful to discover likely long-run outcomes of the game, that is, to select equilibria, as we illustrate in Section 5 for PL and AV. Note that equilibria may not exist with our concept (see Proposition 7 for APL), while large equilibria always exist for the voting games we study [Myerson, 2002]. When this arises, for instance under APL, the non-existence reveals that voters may fail to coordinate on a given election outcome in practice, although coordination is permitted in large equilibria by having the analyst fine-tuning a converging sequence of fixed points. Indeed, when we simulate the dynamic under APL, we find that it never converges (Table 1). Yet, we may use the simulations to predict likely winners in these unstable elections, for instance, the likelihood of electing the Condorcet winner under APL is estimated at 49.5% (Table 2). We illustrate this point in further detail in Section C.5.

Third, the equilibrium notion we employ is convenient to use for the analyst: to check whether a ballot profile τ is an equilibrium, one only needs to verify that it coincides with $\widetilde{BR}^{\infty}(\tau)$. This is less cumbersome than checking whether τ is a large equilibrium, which implies (at least in principle) checking the existence of a sequence of fixed points $(\tau_n)_{n\geq 1}$ whose limit is τ.

Finally, the precise relationship between the two equilibrium concepts remains a natural open question whose general answer is beyond the scope of the current paper. Yet, we can make a few observations. First, some large equilibria do not satisfy our equilibrium requirement, for instance any large equilibrium under APL (for generic preference profiles). Second, we have not been able to find equilibria (in our sense) that were not large equilibria, but we can rule out this possibility for a large class of them. Under a mild technical condition (see footnote 33), any ordinal equilibrium is a large equilibrium. As for cardinal equilibria, we ran simulations of the dynamic for $10,000$ randomly drawn profiles under AV.

The dynamic converged to a cardinal equilibrium for 83.1% of the simulations, and in all those cases it was also a large equilibrium.[34]

7. Review of the literature

Our work closely relates to the literature on multi-candidate elections relying on the model of Poisson games, established in the pioneering papers of Myerson [1998, 2000, 2002]. In this model, the endogenous relative probabilities of *pivot events* drive voters' behavior. Myerson [2002] provides two illuminating examples of stylized preference profiles for which AV performs better than other scoring rules.[35] Yet, Núñez [2010] and Bouton and Castanheira [2012] exhibit preference profiles admitting an equilibrium where the Condorcet winner is not elected under AV, suggesting that the equilibrium analysis of this rule is subtle. Our results show that, despite these examples, Myerson's initial insights are particularly robust. The selection of the Condorcet winner under AV is always possible at equilibrium but it also most often occurs in practice for a wide variety of preference configurations, and the dominance of AV over PL and APL extends to several prominent welfare criteria.[36,37]

Some other studies on Poisson games are also connected to our work. The model has been useful to describe the properties of *runoff elections* [Bouton, 2013, Bouton and Gratton, 2015], which are quite similar to PL, as we discuss in the conclusion. While we do not include these rules in our study, we borrow from Bouton and Gratton [2015] the general setting in which all possible preference ordering

---

[34]For any cardinal equilibrium, non-degenerate asymptotic utility thresholds are defined by an equation of the form $G(r,u) = u$, where the function G is computed using the formulas in Lemma 3 (ii). When $\frac{\partial G}{\partial u}(r,u) \neq 1$, this ensures that the ballot profile is indeed a large equilibrium (intuitively, each function close to $G(r,\cdot)$ has a fixed point close to u, which implies the existence of the desired sequence).

[35]In the first example (*Above the Fray*), one candidate is ranked first by all voters. This candidate is always elected under AV, but may be overlooked under PL. In the second example (*Bad Apple*), one candidate is disliked by all voters. The candidate is never elected under AV but she remains a serious candidate under APL, elected with positive probability.

[36]Myerson [2002] establishes that all equilibria under AV select the Condorcet winner for the domain of *bipolar elections with corruption* (see also Myerson 2006). For three candidates, this domain is one-dimensional, while the results we obtain hold for either the Condorcet domain or the universal domain (both are 5-dimensional).

[37]Beyond voting rules, more complex mechanisms with transfers, such as Quadratic Voting with many alternatives, may deliver even higher welfare guarantees, such as the implementation of the utilitarian optimum [Eguia et al., 2023]. Note though that these guarantees are obtained under restrictive assumptions [Eguia and Xefteris, 2021].

and intensities are considered feasible.[38] Our results are thus not driven by a specific domain restriction. Poisson games have also proved tractable to study information aggregation in elections, notably under AV. While Bouton and Castanheira [2012] argue that the presence of common values is essential to generate coordination under AV,[39] our results show that AV facilitates coordination even in a pure preference aggregation setting.

Our work also relates to models of large elections where the probabilities of pivot events are not explicitly computed, but are instead assumed to follow natural restrictions [Myerson and Weber, 1993, Laslier, 2009].[40] When applied to AV, these models may be interpreted as behavioral, in the sense that voters neglect both the correlation in candidates' scores and pivot events involving more than two candidates. Laslier [2009] proves that if a Condorcet winner exists, there is a unique equilibrium under AV, and it is such that the Condorcet winner is elected. The substantive conclusion thus coincides with our results, although the precise behavior of voters differs. In the equilibrium of Laslier [2009], voters' choices do not depend on preference intensities, whereas they do matter in our model. While we believe that the latter feature seems more behaviorally plausible, we

[38]With three options available, it seems empirically plausible to encounter many different preference orderings. For instance, Eggers [2020] reports that in the 2018 mayoral election of San Francisco held under ranked-choice voting, each of the 6 possible orderings among the top three candidates is expressed by more than 5% of the voters.

[39]Bouton and Castanheira [2012] focus on the specific domain of a *divided majority*, where a majority group is split between two candidates. In this setting, see also Ekmekci [2009] who shows that coordination issues under PL can be palliated (and also exploited) by a *political endorser* endowed with private information over the true preferences of the electorate. On the topic of information aggregation, see also Goertz and Maniquet [2011] and Ahn and Oliveros [2016], who argue that AV performs better than other scoring rules.

[40]The main restriction is the *ordering condition*, stating that, if the expected score of candidate j is above that of k, then the pivot event where candidates i and j are tied for victory is infinitely more likely than the one where i and k are tied for victory.

are comforted that the two models reach the same normative conclusion.[41]

In our last section, we simulate the trajectories of an adaptive procedure (a plausible microfoundation for the equilibrium notion) to select the likely outcomes of the election. This approach has a long tradition in game theory [Fudenberg and Levine, 1998] and it is also well suited to the study of strategic voting. As mentioned in the concluding remarks of one of the early papers on the topic [Palfrey, 1989], this perspective presents at least two benefits. First, it highlights the importance of pre-election polls to generate the coordination of voters. This role of polls has indeed been put forth experimentally by Forsythe et al. [1993]. Second, the dynamic approach can help resolve indeterminacies when multiple equilibria exist, by selecting the ones that are more likely to arise.[42] For instance, Fey [1997] leverages this perspective to underline the instability of an equilibrium for which three candidates receive positive vote shares under PL. Here, we use this approach more generally by applying the same dynamic procedure to several voting rules and for a wide variety of preference profiles. In the computer science literature, the convergence of a related iterative voting process under PL has been explored by Meir et al. [2017], who show that it depends on the nature of the tie-breaking rule in small electorates. By contrast, we focus on large electorates for which tie-breaking rules are less important but we find significant differences of convergence rates across voting rules.[43]

[41]Besides large election models, our paper is also connected to the literature that studies AV and other scoring rules in small electorates. Buenrostro et al. [2013] and Courtin and Núñez [2017] show that, in models of strategic reasoning (i.e. iterated dominance), AV tends to select the Condorcet winner whenever the game is dominance solvable. This *Condorcet efficiency* of AV is also supported experimentally. In Van der Straeten et al. [2010], the Condorcet winner (centrist candidate with single-peaked preferences) is most often selected by AV, but not by the other voting rules under study. Forsythe et al. [1996] focus on a *divided majority* and show that the Condorcet loser tends to be elected less frequently under AV than under PL. In Bol et al. [2023], when experienced voters face a choice between AV and PL under a veil of ignorance on their preferences, they choose AV more often than PL.

[42]Note that this approach of *equilibrium selection* is conceptually distinct from that of *equilibrium refinements*, which is sometimes used in voting games [Bouton and Gratton, 2015].

[43]Andonie and Diermeier [2019] introduce a somewhat less related dynamic model, in which voters update their behavior after receiving individual-specific shocks but not as a function of others' behavior (as in our model). The comparison of voting rules then differs from ours, as AV is found to be typically intermediate between PL and APL [Andonie and Diermeier, 2022]. As the authors note, the model may be difficult to interpret under AV since a voter's state describes his propensity to vote for each candidate without taking correlations into account.

Our Monte-Carlo simulations bring to mind the literature assessing statistical properties of voting rules. A relevant example is Gehrlein and Lepelley [2015], which measures the probability of the election of the Condorcet winner (when she exists) under the probabilistic model of *impartial anonymous culture*.[44] In that paper, voter behavior under AV is *mechanic* as a fixed randomly drawn share of the electorate casts a vote for two candidates. Here PL is found to select the Condorcet winner more often than AV, which is itself better than APL for that purpose. What we add to this literature is that we take into account voter strategic behavior for each given draw of the preference profile. As a result, the ordering of the rules is reversed with AV dominating both PL and APL.

Our final set of results on the robustness to expressive voting is motivated by a recent empirical literature assessing the relevance of the strategic voting model. Kawai and Watanabe [2013] estimate a relaxed version of the model of Myerson and Weber [1993] on data from Japanese general elections, held under PL in multiple districts. They evaluate that 64% to 89% of the electorate can be considered as strategic. Spenkuch [2018] uses rich data from German parliamentary elections, and estimates that, under PL, at least one third of the electorate is not expressive, while around two thirds of voters are not strategic. Pons and Tricaud [2018] study French elections held under PL with quasi-experimental methods, they document voting patterns that are consistent with expressive voting. While not entirely consistent on the share of expressive voters, these works clearly point to the importance of taking expressive voting into account. We thus opt to report our robustness results for any possible fraction of expressive voters.

8. Concluding remarks

This work underlines the potential of AV as a promising alternative voting method to PL for large elections. The flexibility voters have to cast their ballots under AV translates to significant gains in the quality of the collective decision. Our results converge to show that voters will profitably coordinate to elect the Condorcet winner (when she exists). This implies that, despite the theoretically plausible but unlikely coordination failures under AV, they rarely arise as the outcome of a political campaign understood as a tâtonnement process. The dominance of AV is robust to alternative welfare criteria and the existence of expressive voters.

Besides AV, another usual idea to ease coordination is to add a runoff to a first-stage election held under PL. In its most common version, which is used in many

[44]See also Apesteguia et al. [2011] for a focus on welfare, similar in spirit to what we do in Section 5.2.3.

democracies, the runoff is only held if no candidate obtains the support of a majority in the first round. Alas, as shown in Bouton [2013] and Bouton and Gratton [2015], this rule creates the same Duvergerian equilibria as under PL, and the coordination problem remains.[45] Another possibility, put forth by Tsakas and Xefteris [2021] in a context of information aggregation, would be to *always* conduct a runoff between the top-two candidates, even if a majority supports a candidate in the first round. Interestingly, the strategic situation of the first round becomes similar to that of APL.[46] Hence, we conjecture that it would lead to the same instability. Finally, while we have shown that AV tends to confer legitimacy to the elected candidate when voters are strategic, there is no guarantee that this will always occur in practice. Combining AV with a runoff would provide such a guarantee.[47] Understanding the effect of strategic behavior under this rule is outside the scope of the current paper but indeed seems a promising direction for future research.

References

D.S Ahn and S. Oliveros. Approval voting and scoring rules with common values. *Journal of Economic Theory*, 166:304–310, 2016.

C. Andonie and D. Diermeier. Impressionable voters. *American Economic Journal: Microeconomics*, 11(1):79–104, 2019.

C. Andonie and D. Diermeier. Electoral institutions with impressionable voters. *Social Choice and Welfare*, 59, 2022.

J. Apesteguia, M.A. Ballester, and R. Ferrer. On the justice of decision rules. *The Review of Economic Studies*, 78(1):1–16, 2011.

D. Bol, A. Blais, M. Coulombe, J.-F. Laslier, and J.-B. Pilet. Choosing an electoral rule: Values and self-interest in the lab. 2023.

T. Borgers and Y.M. Choo. Revealed relative utilitarianism. 2017.

L. Bouton. A theory of strategic voting in runoff elections. *American Economic Review*, 103(4):1248–88, 2013.

[45]Bouton and Gratton [2015] identify another equilibrium where three candidates obtain positive vote shares. Yet, even in this equilibrium, the Condorcet winner may not be selected for the second round.

[46]Under this rule, each voter votes for one candidate in the first round, and two out of three are selected. Symmetrically, each voter vote for two candidates under APL (or equivalently, against the third one) and one is selected (two are eliminated).

[47]Note that the voting rule adopted in St. Louis, Missouri in November 2020 precisely corresponds to AV with an *automatic* runoff, held between the two candidates obtaining the highest support in the first round.

L. Bouton and M. Castanheira. One person, many votes: Divided majority and information aggregation. *Econometrica*, 80(1):43–87, 2012.

L. Bouton and G. Gratton. Majority runoff elections: strategic voting and Duverger's hypothesis. *Theoretical Economics*, 10(2):283–314, 2015.

S.J. Brams and P.C. Fishburn. Approval voting. *The American Political Science Review*, 72:831–847, 1978.

S.J. Brams and P.C. Fishburn. Going from theory to practice: the mixed success of approval voting. *Social Choice and Welfare*, 25:457–474, 2005.

G.W. Brown. Iterative solution of games by fictitious play. 1951.

L. Buenrostro, A. Dhillon, and P. Vida. Scoring rule voting games and dominance solvability. *Social Choice and Welfare*, 40(2):329–352, 2013.

S. Courtin and M. Núñez. Dominance solvable approval voting games. *Journal of Public Economic Theory*, 19(6):1055–1076, 2017.

A. Dellis, S. D'Evelyn, and K. Sherstyuk. Multiple votes, ballot truncation and the two-party system: an experiment. *Social Choice and Welfare*, 37(2):171–200, 2011.

A. Dhillon and J-F. Mertens. Relative utilitarianism. *Econometrica*, 67:471–498, 1999.

F. Durand. Coalitional manipulation of voting rules: Simulations on empirical data. *Preprint*, 2022.

F. Durand and É. de Panafieu. Limit distribution of two Skellam distributions, conditionally on their equality. *arXiv*, 2102.10835, 2021.

M. Duverger. *Les partis politiques*. 1951.

A.C. Eggers. A diagram for analyzing ordinal voting systems. *Social Choice and Welfare*, 56:1–29, 2020.

J. X. Eguia and D. Xefteris. Implementation by vote-buying mechanisms. *American Economic Review*, 111(9):2811–28, 2021.

J.X. Eguia, N. Immorlica, S.P. Lalley, K. Ligett, G. Weyl, and D. Xefteris. Efficiency in collective decision-making via quadratic transfers. *arXiv preprint arXiv:2301.06206*, 2023.

M. Ekmekci. Manipulation through political endorsements. *Journal of Economic Theory*, 144(3):1227–1248, 2009.

M. Fey. Stability and coordination in Duverger's law: A formal model of preelection polls and strategic voting. *American Political Science Review*, 91(1): 135–147, 1997.

R. Forsythe, R.B. Myerson, T.A. Rietz, and R.J. Weber. An experiment on coordination in multi-candidate elections: The importance of polls and election histories. *Social Choice and Welfare*, 10(3):223–247, 1993.

R. Forsythe, T. Rietz, R. Myerson, and R. Weber. An experimental study of voting rules and polls in three-candidate elections. *International Journal of Game Theory*, 25(3):355–383, 1996.

D. Fudenberg and D.K. Levine. *The theory of learning in games*, volume 2. MIT press, 1998.

W.V. Gehrlein and D. Lepelley. The Condorcet efficiency advantage that voter indifference gives to approval voting over some other voting rules. *Group Decision and Negotiation*, 24(2):243–269, 2015.

J. Goertz and F. Maniquet. On the informational efficiency of approval voting. *Journal of Economic Theory*, 146:1464–1480, 2011.

K. Kawai and Y. Watanabe. Inferring strategic voting. *American Economic Review*, 103(2):624–62, 2013.

W. Kulpa. The Poincaré-Miranda theorem. *The American Mathematical Monthly*, 104(6):545–550, 1997.

J.-F. Laslier. And the loser is... plurality voting. In *Electoral systems*, pages 327–351. Springer, 2012.

J.-F. Laslier and M.R. Sanver. *Handbook on approval voting*. Springer Science & Business Media, 2010.

J.F. Laslier. The leader rule: Strategic approval voting in a large electorate. *Journal of Theoretical Politics*, 21:113–136, 2009.

R. Meir, M. Polukarov, J.S. Rosenschein, and N.R. Jennings. Iterative voting and acyclic games. *Artificial Intelligence*, 252:100–122, 2017.

S. Merill and J. Nagel. The effect of approval balloting on strategic voting under alternative decision rules. *American Political Science Review*, 81:509–524, 1987.

R. Myerson. Extended Poisson games and the Condorcet jury theorem. *Games and Economic Behavior*, 25:111–131, 1998.

R. Myerson. Large Poisson games. *Journal of Economic Theory*, 94:7–45, 2000.

R. Myerson. Comparison of scoring rules in Poisson voting games. *Journal of Economic Theory*, 103:219–251, 2002.

R. Myerson. Bipolar multicandidate elections with corruption. *The Scandinavian Journal of Economics*, 108(4):727–742, 2006.

R. Myerson and R.J. Weber. A theory of voting equilibria. *American Political Science Review*, 87:102–114, 1993.

M. Núñez. Condorcet consistency of approval voting: A counter example on large Poisson games. *Journal of Theoretical Politics*, 22:64–84, 2010.

T. R. Palfrey. A mathematical proof of Duverger's law. In P. C. Ordeshook, editor, *Models of Strategic Choice in Politics*, pages 69–92. Ann Arbor: University of

Michigan Press, 1989.

J.W Patty and E.M. Penn. A social choice theory of legitimacy. *Social Choice and Welfare*, 36(3-4):365–382, 2011.

V. Pons and C. Tricaud. Expressive voting and its cost: Evidence from runoffs with two or three candidates. *Econometrica*, 86(5):1621–1649, 2018.

J.L. Spenkuch. Expressive vs. strategic voters: An empirical assessment. *Journal of Public Economics*, 165:73–81, 2018.

N. Tsakas and D. Xefteris. Information aggregation with runoff voting. *Journal of Economic Theory*, 191:105–130, 2021.

A. Van Deemen. On the empirical relevance of condorcet's paradox. *Public Choice*, 158(3):311–330, 2014.

K. Van der Straeten, J.-F. Laslier, N. Sauger, and A. Blais. Strategic, sincere, and heuristic voting under four election rules: an experimental study. *Social Choice and Welfare*, 35(3):435–472, 2010.

R.J. Weber. Comparison of voting systems. Technical report, Cowles Foundation Discussion Paper No. 498. Yale University, 1977.

APPENDIX A. TECHNICAL APPENDIX: PROOFS

This appendix contains the proofs to all results from the article, except the convergence of utility thresholds (Theorem 1), relegated to Section B. We start by introducing notations that will be useful throughout the proofs. We define a *tie event* as follows: for each $K \subseteq \mathscr{K}$, we let

$$\text{tie}_K = \left\{ z \in \mathcal{Z} \mid s_i(z) = s_j(z), \forall i, j \in K \right\},$$

stand for the event where all candidates in K are tied. For any event $E \subseteq \mathcal{Z}$ and ballot m, we denote by $E - m = \{z \in \mathcal{Z} \mid z + m \in E\}$ the event where adding a ballot m creates an outcome in E. For each pair ij of candidates and each ordinal type $o \in \mathcal{O}$, we denote by

$$\text{piv}^o_{ij} = (\text{piv}_{ij} - o_1) \cup (\text{piv}_{ij} - o_1 o_2),$$

the event where casting a ballot o_1 or $o_1 o_2$ can create a pivot between candidates i and j (see Section 2.2 for the definition of a pivot event). Finally, we denote by

$$\text{tie}^{o,1}_{abc} = \text{tie}_{abc} - o_1 \quad \text{and} \quad \text{tie}^{o,2}_{abc} = \text{tie}_{abc} - o_1 o_2,$$

the events where casting a ballot o_1 or $o_1 o_2$ (respectively) can create a tie between the three candidates.

A.1. *Proof of Proposition 1*

We prove a stronger version of Proposition 1 in which a formula for the utility thresholds is explicitly provided.

Proposition 1*. *For each ordinal type $o \in \mathcal{O}$, each $n \in \mathbb{N}$ and each profile $\tau \in \Delta(\mathscr{M})$, best replies are such that:*

$$\forall u \in (0,1), \qquad BR^n(t = (o,u) \mid \tau) = \begin{cases} o_1 & \text{if} \quad u < u^n_o(\tau) \\ o_1 o_2 & \text{if} \quad u > u^n_o(\tau), \end{cases}$$

where the utility threshold $u^n_o(\tau) \in [0,1]$ is given by:

$$u^n_o(\tau) = \frac{3\mathbb{P}[\text{piv}^o_{o_1 o_2} \mid n\tau] + 2\mathbb{P}[\text{tie}^{o,1}_{abc} \mid n\tau] + \mathbb{P}[\text{tie}^{o,2}_{abc} \mid n\tau]}{3\mathbb{P}[\text{piv}^o_{o_1 o_2} \mid n\tau] + 3\mathbb{P}[\text{piv}^o_{o_2 o_3} \mid n\tau] + 4\mathbb{P}[\text{tie}^{o,1}_{abc} \mid n\tau] + 2\mathbb{P}[\text{tie}^{o,2}_{abc} \mid n\tau]}. \tag{1}$$

Proof. We study in-depth the decision problem of a voter with ordinal type $o = ij$.

We first note that any ballot $m \in \mathscr{M} \setminus \{i, ij\}$ is strictly dominated by either i or ij. Indeed, k and ik are weakly dominated by the ballot i for any outcome $z \in \mathcal{Z}$, while j and jk are weakly dominated by the ballot ij for any outcome $z \in \mathcal{Z}$. Moreover, each of these dominance relations become strict for the outcome $z = 0$, which arises with probability $e^{-n} > 0$. We may thus restrict our attention to ballots i and ij.

Writing $\mathbb{P}[z]$ for $\mathbb{P}[z \mid n\tau]$ for each $z \in \mathcal{Z}$ for ease of notation, the expected utility of a voter with type $t = (o, u)$ from casting a ballot i equals:

$$U_t[i \mid n\tau] = \overbrace{\mathbb{P}[\text{piv}_i - i] \cdot 1 + \mathbb{P}[\text{piv}_j - i] \cdot u + \mathbb{P}[\text{piv}_k - i] \cdot 0}^{\text{Single-winner events}}$$
$$+ \overbrace{\mathbb{P}[\text{piv}_{ij} - i]\left(\frac{1+u}{2}\right) + \mathbb{P}[\text{piv}_{ik} - i]\left(\frac{1}{2}\right) + \mathbb{P}[\text{piv}_{jk} - i]\left(\frac{u}{2}\right)}^{\text{Two-way pivots}}$$
$$+ \overbrace{\mathbb{P}[\text{tie}_{abc} - i]\left(\frac{1+u}{3}\right)}^{\text{Three-way tie}}.$$

The expected utility of a voter with type $t = (o, u)$ from casting a ballot ij equals:

$$U_t[ij \mid n\tau] = \overbrace{\mathbb{P}[\text{piv}_i - ij] \cdot 1 + \mathbb{P}[\text{piv}_j - ij] \cdot u + \mathbb{P}[\text{piv}_k - ij] \cdot 0}^{\text{Single-winner events}}$$
$$+ \overbrace{\mathbb{P}[\text{piv}_{ij} - ij]\left(\frac{1+u}{2}\right) + \mathbb{P}[\text{piv}_{ik} - ij]\left(\frac{1}{2}\right) + \mathbb{P}[\text{piv}_{jk} - ij]\left(\frac{u}{2}\right)}^{\text{Two-way pivots}}$$
$$+ \overbrace{\mathbb{P}[\text{tie}_{abc} - ij]\left(\frac{1+u}{3}\right)}^{\text{Three-way tie}}.$$

Thus, the difference $\Delta_t(n,\tau) := U_t[i \mid n\tau] - U_t[ij \mid n\tau]$ equals:

$$\left(\mathbb{P}[\text{piv}_i - i] + \mathbb{P}[\text{piv}_j - i]u + \mathbb{P}[\text{piv}_{ij} - i]\frac{1+u}{2} + \mathbb{P}[\text{piv}_{ik} - i]\frac{1}{2} + \mathbb{P}[\text{piv}_{jk} - i]\frac{u}{2} + \mathbb{P}[\text{tie}_{abc} - i]\frac{1+u}{3}\right) -$$
$$\left(\mathbb{P}[\text{piv}_i - ij] + \mathbb{P}[\text{piv}_j - ij]u + \mathbb{P}[\text{piv}_{ij} - ij]\frac{1+u}{2} + \mathbb{P}[\text{piv}_{ik} - ij]\frac{1}{2} + \mathbb{P}[\text{piv}_{jk} - ij]\frac{u}{2} + \mathbb{P}[\text{tie}_{abc} - ij]\frac{1+u}{3}\right).$$

The previous expression can be simplified due to the three following observations. Note first that $(\text{piv}_i - i) = (\text{piv}_i - ij) \uplus (\text{piv}_{ij} - ij)$, where the symbol $\uplus$ expresses a union with an empty intersection. Hence,

$$\mathbb{P}[\text{piv}_i - i] - \mathbb{P}[\text{piv}_i - ij] - \mathbb{P}[\text{piv}_{ij} - ij]\frac{1+u}{2} = \mathbb{P}[\text{piv}_{ij} - ij]\frac{1-u}{2}.$$

Second, observe that $(\text{piv}_j - ij) = (\text{piv}_j - i) \uplus (\text{piv}_{ij} - i) \uplus (\text{piv}_{jk} - i) \uplus (\text{tie}_{abc} - i)$. Thus,

$$\mathbb{P}[\text{piv}_j - i]u + \mathbb{P}[\text{piv}_{ij} - i]\frac{1+u}{2} + \mathbb{P}[\text{piv}_{jk} - i]\frac{u}{2} + \mathbb{P}[\text{tie}_{abc} - i]\frac{1+u}{3} - \mathbb{P}[\text{piv}_j - ij]u$$
$$= \mathbb{P}[\text{piv}_{ij} - i]\frac{1-u}{2} + \mathbb{P}[\text{piv}_{jk} - i]\left(-\frac{u}{2}\right) + \mathbb{P}[\text{tie}_{abc} - i]\frac{1-2u}{3}.$$

Third, observe that $(\text{piv}_{ik} - i) = (\text{piv}_{ik} - ij) \uplus (\text{tie}_{abc} - ij)$. Therefore,

$$\mathbb{P}[\text{piv}_{ik} - i]\frac{1}{2} - \mathbb{P}[\text{piv}_{ik} - ij]\frac{1}{2} - \mathbb{P}[\text{tie}_{abc} - ij]\frac{1+u}{3} = \mathbb{P}[\text{tie}_{abc} - ij]\left(\frac{1}{2} - \frac{1+u}{3}\right)$$
$$= \mathbb{P}[\text{tie}_{abc} - ij]\frac{1-2u}{6}.$$

Combining the three observations, the difference $\Delta_t(n,\tau)$ can be rewritten as :

$$\Big(\mathbb{P}[\text{piv}_{ij}-i]+\mathbb{P}[\text{piv}_{ij}-ij]\Big)\frac{1-u}{2}+\Big(\mathbb{P}[\text{piv}_{jk}-i]+\mathbb{P}[\text{piv}_{jk}-ij]\Big)\Big(-\frac{u}{2}\Big)$$
$$+\mathbb{P}[\text{tie}_{abc}-i]\frac{1-2u}{3}+\mathbb{P}[\text{tie}_{abc}-ij]\frac{1-2u}{6}.$$

Finally, observing that $\text{piv}^o_{ij}=(\text{piv}_{ij}-i)\uplus(\text{piv}_{ij}-ij)$, $\text{piv}^o_{jk}=(\text{piv}_{jk}-i)\uplus(\text{piv}_{jk}-ij)$, $\text{tie}^{o,1}_{abc}=(\text{tie}_{abc}-i)$ and $\text{tie}^{o,2}_{abc}=(\text{tie}_{abc}-ij)$, we may write:

$$\Delta_t(n,\tau)=\mathbb{P}[\text{piv}^o_{ij}]\left(\frac{1-u}{2}\right)+\mathbb{P}[\text{piv}^o_{jk}]\Big(-\frac{u}{2}\Big)+\mathbb{P}[\text{tie}^{o,1}_{abc}]\left(\frac{1-2u}{3}\right)+\mathbb{P}[\text{tie}^{o,2}_{abc}]\left(\frac{1-2u}{6}\right).$$

We thus obtain the desired formulas for the best replies and the utility threshold.

□

A.2. *Two lemmata on pivot magnitudes*

In the sequel, we use the notion of *weak pivot events*, defined for $i,j,k\in\mathcal{K}$ by

$$\widetilde{\text{piv}_{ij}}=\left\{z\in\mathcal{Z}\mid s_i(z)=s_j(z)\geq s_k(z)\right\}=\text{piv}_{ij}\uplus\text{tie}_{abc}.$$

The magnitudes of these events and of the three-candidate tie will be used throughout the appendix, we denote them by $\mu_{ij}:=\mu[\widetilde{\text{piv}_{ij}}]$ and $\mu_{abc}:=\mu[\text{tie}_{abc}]$. Moreover, we extensively use the notion of *offset ratios*, which is important to derive these magnitudes. The offset-ratio ϕ_m of a ballot m at an event E corresponds to the most likely number of ballots m when E occurs, divided by its expected value $n\tau_m$. For instance, the offset ration of ballot m in the event $\{z\}$ is simply $\frac{z_m}{n\tau_m}$. The notion of offset ratios is introduced by Myerson [2000, 2002], who applies *large deviations* techniques to show that the probability of a cone event E is concentrated in outcomes for which the proportion of either ballot is (approximatively) fixed. The offset ratios can then be obtained as solutions of a constrained minimization program, whose value corresponds to the magnitude of the event E (*Dual Magnitude Theorem* in Myerson [2002]). By application of these techniques, we obtain the following result.

Lemma 1. *Let τ be a profile such that $\tau_i+\tau_{ij}>0$ for all $i,j\in\mathcal{K}$. For each ordinal type o, for any pair (i,j), pivot magnitudes are such that:*

- $\mu_{ij}=\mu[\text{piv}^o_{ij}]=\begin{cases}\mu[\text{tie}_{ij}] & \text{if } \delta_{ij}(\tau)>0,\\ \mu[\text{tie}_{abc}] & \text{if } \delta_{ij}(\tau)\leq 0,\end{cases}$ *with:*

$$\delta_{ij}(\tau)=\tau_i\sqrt{\frac{\tau_j+\tau_{jk}}{\tau_i+\tau_{ik}}}+\tau_{ij}-\tau_{jk}\sqrt{\frac{\tau_i+\tau_{ik}}{\tau_j+\tau_{jk}}}-\tau_k,$$

- $\mu_{abc}=\mu[\text{tie}^{o,1}_{abc}]=\mu[\text{tie}^{o,2}_{abc}]$.

Lemma 1 establishes two important properties: (i) the magnitudes of pivot events do not depend on types (superscript o); and (ii) the magnitude of a pivot ij equals either the magnitude of tie_{ij} or that of tie_{abc}. The intuition for the second point stems from the interpretation of the threshold $\delta_{ij}(\tau)$ as the difference between the expected score of i (which coincides with that of j) and that of k, conditionally on tie_{ij}. For $E = \text{tie}_{ij}$, it is easy to see that the offset-ratios satisfy $\phi_k = \phi_{ij} = 1$ (ballots ij and k do not influence the relative scores of i and j), while $\phi_i = 1/\phi_{jk} = \sqrt{(\tau_j + \tau_{jk})/(\tau_i + \tau_{ik})}$. Therefore, the most likely score difference between i and k in the event tie_{ij} can be written as $\delta_{ij}(\tau) = \tau_i\phi_i + \tau_{ij}\phi_{ij} - \tau_{jk}\phi_{jk} + \tau_k\phi_k$. If conditionally on tie_{ij}, the expected score of k is below that of i (i.e. $\delta_{ij}(\tau) > 0$), then a typical realization of tie_{ij} is actually also in piv_{ij}. This explains why the two events have the same magnitude. On the other hand, if conditionally on tie_{ij}, the expected score of k is above that of i (i.e. $\delta_{ij}(\tau) \le 0$), then piv_{ij} becomes much less likely than tie_{ij}, and in fact, piv_{ij} has the same magnitude as tie_{abc}.

Proof. Let τ be a profile such that $\tau_i + \tau_{ij} > 0$ for all $i, j \in \mathcal{K}$. As a direct application of the *Magnitude Equivalence Theorem* by Núñez [2010], we have that for any ordinal type o, any profile τ, and any pair ij of candidates:

$$\mu[\text{piv}^o_{ij}] = \mu[\widetilde{\text{piv}_{ij}}] = \mu_{ij} \qquad \text{and} \qquad \mu[\text{tie}^{o,1}_{abc}] = \mu[\text{tie}^{o,2}_{abc}] = \mu[\text{tie}_{abc}] = \mu_{abc}.$$

Applying the *Dual Magnitude Theorem* [Myerson, 2002], we can write:

$$\mu_{abc} = \min_{x,y\in\mathbb{R}} F(x,y), \qquad \mu_{ij} = \min_{x\in\mathbb{R}, y\ge 0} F(x,y), \qquad \mu[\text{tie}_{ij}] = \min_{x\in\mathbb{R}, y=0} F(x,y)$$

where

$$F(x,y) = \tau_i e^{x+y} + \tau_{ik}e^x + \tau_j e^{-x} + \tau_{jk}e^{-(x+y)} + \tau_k e^{-y} + \tau_{ij}e^y - 1.$$

The function F is strictly convex as $\dfrac{\partial^2 F}{\partial x^2} = e^x(\tau_i e^y + \tau_{ik}) + e^{-x}(\tau_j + \tau_{jk}e^{-y}) > 0$ and $\Delta := \dfrac{\partial^2 F}{\partial x^2} \times \dfrac{\partial^2 F}{\partial y^2} - \left(\dfrac{\partial^2 F}{\partial x \partial y}\right)^2$ is such that:

$$\begin{aligned}
\Delta = &\left[e^x(\tau_i e^y + \tau_{ik}) + e^{-x}(\tau_j + \tau_{jk}e^{-y})\right] \times \\
&\qquad \left[e^y(\tau_i e^x + \tau_{ij}) + e^{-y}(\tau_k + \tau_{jk}e^{-x})\right] - \left(\tau_i e^{x+y} + \tau_{jk}e^{-(x+y)}\right)^2 \\
&= e^{x+y}(\tau_i\tau_{ik}e^x + \tau_i\tau_{ij}e^y + \tau_{ij}\tau_{ik}) + e^{y-x}(\tau_i\tau_j e^x + \tau_j\tau_{ij} + \tau_{ij}\tau_{jk}e^{-y}) \\
&\quad + e^{x-y}(\tau_i\tau_k e^y + \tau_k\tau_{ik} + \tau_{ik}\tau_{jk}e^{-x}) + e^{-x-y}(\tau_j\tau_{jk}e^{-x} + \tau_j\tau_k + \tau_k\tau_{jk}e^{-y}) > 0.
\end{aligned}$$

Since $\tau_i + \tau_{ij} > 0$ for all $i, j \in \mathcal{K}$, it can be checked that $|F(x,y)|$ tends to infinity when either $|x|$ or $|y|$ tends to infinity.[48] Therefore, the strictly convex function F admits a unique critical point $(x^{**}, y^{**}) \in \mathbb{R}^2$. The value y^{**} solves for

[48] We obtained this by considering in turn the five possible cases: (i) $\tau_i\tau_j\tau_k > 0$; (ii) $\tau_{ij}\tau_{ik}\tau_{jk} > 0$; (iii) $\tau_i\tau_j\tau_{ik}\tau_{jk} > 0$; (iv) $\tau_i\tau_k\tau_{ij}\tau_{jk} > 0$ and (v) $\tau_j\tau_k\tau_{ij}\tau_{ik} > 0$.

$\min_{y\in\mathbb{R}} F(x^*(y), y)$, where $x^*(y) = \arg\min_{x\in\mathbb{R}} F(x, y)$ is uniquely defined. We consider two cases:

- Either $\frac{\partial F}{\partial y}(x^*(0), 0) > 0$. In that case, $y^{**} < 0$, and we obtain that $\mu_{ij} > \mu_{abc}$ and $\mu[\text{tie}_{ij}] > \mu_{abc}$. Moreover, we have:

$$\mu_{ij} = \min_{x\in\mathbb{R}, y\geq 0} F(x, y) = \min_{y\geq 0} F(x^*(y), y) = F(x^*(0), 0) = \min_{x\in\mathbb{R}, y=0} F(x, y) = \mu[\text{tie}_{ij}].$$

- Or $\frac{\partial F}{\partial y}(x^*(0), 0) \leq 0$. In that case, $y^{**} \geq 0$, and we obtain that $\mu_{ij} = \mu_{abc}$.

To conclude, the value $x^*(0)$ is obtained by setting

$$0 = \frac{\partial F}{\partial x}(x^*(0), 0) = e^{x^*(0)}(\tau_i + \tau_{ik}) - e^{-x^*(0)}(\tau_j + \tau_{jk}),$$

and we get $e^{x^*(0)} = \sqrt{\frac{\tau_j + \tau_{jk}}{\tau_i + \tau_{ik}}}$. Finally, we write

$$\frac{\partial F}{\partial y}\left(x^*(0), 0\right) = \tau_i e^{x^*(0)} + \tau_{ij} - \tau_k - \tau_{jk} e^{-x^*(0)} = \delta_{ij}(\tau).$$

As desired, we have shown that μ_{ij} is equal to $\mu[\text{tie}_{ij}]$ when $\delta_{ij}(\tau) > 0$ or to μ_{abc} otherwise. □

We now state a second lemma, relating candidates' expected scores to pivot magnitudes, which can be viewed as a corollary of Lemma 1.

Lemma 2. *Let τ be a profile such that $\tau_i + \tau_{ij} > 0$ for all $i, j \in \mathcal{K}$. The expected scores $\gamma = (\gamma_k)_{k\in\mathcal{K}}$ are such that:*

(i) If $\gamma_a, \gamma_b > \gamma_c$, then $\mu_{ab} > \mu_{abc}$,

(ii) If $\gamma_a \geq \gamma_b, \gamma_c$, then $\mu_{bc} = \mu_{abc}$.

Proof. In order to prove (i), let us assume w.l.o.g. that $\gamma_b \geq \gamma_a$, so that $\tau_b + \tau_{bc} \geq \tau_a + \tau_{ac}$. Building on Lemma 1, we write:

$$\delta_{ab}(\tau) = \tau_a \sqrt{\frac{\tau_b + \tau_{bc}}{\tau_a + \tau_{ac}}} + \tau_{ab} - \tau_{bc} \sqrt{\frac{\tau_a + \tau_{ac}}{\tau_b + \tau_{bc}}} - \tau_c \geq \tau_a + \tau_{ab} - \tau_{bc} - \tau_c = \gamma_a - \gamma_c > 0.$$

Hence, $\mu_{ab} > \mu_{abc}$, as desired.

For (ii), let us assume w.l.o.g. that $\gamma_c \leq \gamma_b$. We obtain similarly that $\delta_{bc}(\tau) \leq \gamma_b - \gamma_a \leq 0$. Hence, $\mu_{bc} = \mu_{abc}$, as desired. □

A.3. *Proof of Proposition 2*

Proposition 2. *The set of discriminatory profiles is of (Lebesgue) measure* 1.

Proof. The set of profiles τ such that there exist $i, j \in \mathcal{K}$ for which $\tau_i + \tau_{ij} = 0$ is of measure 0. We may thus focus on profiles τ for which $\tau_i + \tau_{ij} > 0$ for all $i, j \in \mathcal{K}$.

Let τ be a non-discriminatory profile. If the magnitude ordering is of the form $\mu_{ij} = \mu_{ik} > \mu_{jk}$, then we know from Lemma 1 that $\mu_{ij} = \mu[\text{tie}_{ij}]$ and $\mu_{ik} = \mu[\text{tie}_{ik}]$. Denoting by Z_m the random variable counting the number of ballots m cast in the election, we have $\text{tie}_{ij} = \{Z_i + Z_{ik} = Z_j + Z_{jk}\}$, where $Z_i + Z_{ik} \sim \mathscr{P}((\tau_i + \tau_{ik})n)$ and $Z_j + Z_{jk} \sim \mathscr{P}((\tau_j + \tau_{jk})n)$ are independent. A symmetric formula applies for tie_{ik}. It is well-known that the magnitude of the event that two independent Poisson variables of respective parameters λn and $\lambda' n$ are equal has magnitude $-(\lambda - \lambda')^2$.[49] Thus, $\mu_{ij} = \mu_{ik}$ implies:

$$-(\tau_i + \tau_{ik} - \tau_j - \tau_{jk})^2 = -(\tau_i + \tau_{ij} - \tau_k - \tau_{jk})^2.$$

In any small neighborhood of τ, the inequalities $\delta_{ij}(\tau) > 0$ and $\delta_{ik}(\tau) > 0$ are preserved, so that the formulas for μ_{ij} and μ_{ik} remain valid. As the previous equality is generically violated in such a neighborhood, we conclude that the set of profiles τ with pivot ordering $\mu_{ij} = \mu_{ik} > \mu_{jk}$ is of (Lebesgue) measure 0.

If the pivot ordering is of the form $\mu_{ij} = \mu_{ik} = \mu_{jk}$, then this magnitude is equal to μ_{abc}, by application of Lemma 1 and Lemma 2 (ii). By Lemma 2, (i), there is not a unique candidate with the lowest expected score, so it must be that $\gamma_i \geq \gamma_j = \gamma_k$. In any neighborhood of τ, there generically exists a unique candidate with the lowest expected score, so that, by application of Lemma 2, (i), one pivot magnitude is strictly higher than μ_{abc}. Hence, the set of profiles τ with pivot ordering $\mu_{ij} = \mu_{ik} = \mu_{jk}$ is of (Lebesgue) measure 0. □

A.4. *One lemma on discriminatory profiles*

We now focus on the class of profiles τ for which $\tau_{ij} = \tau_k = 0$. These profiles are the only relevant candidates for discriminatory equilibria under which $\mu_{ij} > \mu_{ik}, \mu_{jk}$. Intuitively, if the most serious race occurs between candidates i and j, all voters want to bear on this race, so that no one casts a ballot ij or a ballot k. The next lemma provides explicit formulas for the pivot magnitudes and asymptotic utility thresholds at such profiles. The formulas for the asymptotic utility thresholds are provided for offsets $\phi_m \neq 1$, but they remain true at the limit when some offsets are equal to 1.

Lemma 3. *For any profile τ with $\tau_{ab} = \tau_c = 0$ and $\tau_a, \tau_b, \tau_{ac}, \tau_{bc} > 0$,*

(i) Pivot magnitudes are such that:

$$\begin{cases} \mu_{ab} > \mu_{abc} & \Leftrightarrow \quad \tau_a \tau_b > \tau_{ac} \tau_{bc}, \\ \mu_{ac} > \mu_{abc} & \Leftrightarrow \quad \tau_{ac} > \tau_b, \\ \mu_{bc} > \mu_{abc} & \Leftrightarrow \quad \tau_{bc} > \tau_a. \end{cases}$$

[49] See for instance Lemma 4, (iii) in Section B.

In particular, $\mu_{abc} = \min(\mu_{ab}, \mu_{ac}, \mu_{bc})$.

(ii) If $\mu_{ab} > \mu_{ac} = \mu_{bc}$*, then, noting* $\phi_a = \sqrt{\frac{\tau_{bc}}{\tau_a}}$, $\phi_b = \sqrt{\frac{\tau_{ac}}{\tau_b}}$, $\phi_{ac} = \frac{1}{\phi_b}$ *and* $\phi_{bc} = \frac{1}{\phi_a}$*, we have that for any ordinal type* $o \in \{ac, bc\}$*:*

$$u_o^\infty(\tau) = \frac{g_o(\phi_{o_1o_2})}{g_o(\phi_{o_1o_2}) + g_o(1/\phi_{o_1})} \quad \text{with} \quad g_o(y) = \phi_{o_1} + (\phi_{o_1} + \phi_{o_1o_2})\left(1 + \frac{3}{y-1}\right).$$

Proof. The proof of (i) is a direct application of Lemma 1.

Let τ be a profile such that $\tau_{ab} = \tau_c = 0$, $\tau_a, \tau_b, \tau_{ac}, \tau_{bc} > 0$ and $\mu_{ab} > \mu_{ac} = \mu_{bc}$. By application of (i), it is impossible to have simultaneously $\mu_{ab}, \mu_{ac}, \mu_{bc} > \mu_{abc}$. Thus, we have $\mu_{ac} = \mu_{bc} = \mu_{abc}$. By (i), it follows that $\tau_a \geq \tau_{bc}$ and $\tau_b \geq \tau_{ac}$.

To determine the best replies of ordinal types ac and bc, we compute the relative probabilities of pivots ac, bc and tie_{abc}. As shown in the sequel, the offset ratios in the event tie_{abc} constitute sufficient statistics to compute the relative likelihood of any relevant pivot event with respect to tie_{abc}. The offset ratios are obtained as follows. Applying the *Dual Magnitude Theorem* [Myerson, 2002], the magnitude $\mu[\text{tie}_{abc}]$ equals the optimal value of

$$\min_{x,y \in \mathbb{R}} \tau_a e^{x+y} + \tau_{ac} e^x + \tau_b e^{-x} + \tau_{bc} e^{-(x+y)} - 1.$$

Moreover, denoting by (x^*, y^*) the values of (x, y) at the optimum, the offset ratios are given by:

$$\phi_a = e^{x^*+y^*} = \sqrt{\frac{\tau_{bc}}{\tau_a}}, \quad \phi_b = e^{-x^*} = \sqrt{\frac{\tau_{ac}}{\tau_b}}, \quad \phi_{ac} = 1/\phi_b \quad \text{and} \quad \phi_{bc} = 1/\phi_a.$$

Claim: The offset ratios are identical in the events piv_{ac}, piv_{bc} and tie_{abc}.

We prove the claim for piv_{ac} and tie_{abc}, a similar argument can be made for piv_{bc}. We write $\text{piv}_{ac} = \{z \in \mathcal{Z} \mid z_a = z_{bc}\} \cap \{z \mid z_{ac} > z_b\}$ and $\text{tie}_{abc} = \{z \in \mathcal{Z} \mid z_a = z_{bc}\} \cap \{z \in \mathcal{Z} \mid z_{ac} = z_b\}$. We first observe that the offset ratios of ballots a and bc will be the same for the two events: both are equal to the corresponding offset ratio in the event $\{z \in \mathcal{Z} \mid z_a = z_{bc}\}$. Moreover, as $\tau_b \geq \tau_{ac}$, we have that for any n, the most likely outcome in the event $\{z \in \mathcal{Z} \mid z_{ac} > z_b\}$ belongs to the event $\{z \in \mathcal{Z} \mid z_{ac} = z_b + 1\}$, which implies that the two events admit the same offset ratios. In turn, the offset ratios of ballots ac and b are the same for events $\{z \in \mathcal{Z} \mid z_{ac} = z_b + 1\}$ and $\{z \in \mathcal{Z} \mid z_{ac} = z_b\}$, which proves the claim.

From now on, we can thus write ϕ_m for the offset ratio of ballot m. Since $\tau_a, \tau_b, \tau_{ac}, \tau_{bc} > 0$, we have $\phi_m > 0$ for any $m \in \{a, b, ac, bc\}$. The *Offset Theorem*

[Myerson, 2000] implies that:

$$\lim_{n\to\infty} \frac{\mathbb{P}[\text{piv}^{o,1}_{abc} \mid n\tau]}{\mathbb{P}[\text{tie}_{abc} \mid n\tau]} = \phi_{o_1} \quad \text{and} \quad \lim_{n\to\infty} \frac{\mathbb{P}[\text{piv}^{o,2}_{abc} \mid n\tau]}{\mathbb{P}[\text{tie}_{abc} \mid n\tau]} = \phi_{o_1 o_2}.$$

For the pivots with two candidates, we focus on pivot ac. We have:

$$\lim_{n\to\infty} \frac{\mathbb{P}[\text{piv}^{o}_{ac} \mid n\tau]}{\mathbb{P}[\text{piv}_{ac} \mid n\tau]} = \lim_{n\to\infty} \left(\frac{\mathbb{P}[\text{piv}_{ac} - o_1 \mid n\tau]}{\mathbb{P}[\text{piv}_{ac} \mid n\tau]} + \frac{\mathbb{P}[\text{piv}_{ac} - o_1 o_2 \mid n\tau]}{\mathbb{P}[\text{piv}_{ac} \mid n\tau]} \right) = \phi_{o_1} + \phi_{o_1 o_2}.$$

What remains to be done is to compare the likelihood of events piv_{ac} and tie_{abc}. We rely on two observations: first, $\text{piv}_{ac} = (\text{piv}_{ac} \uplus \text{tie}_{abc}) - b$, as $\{z \in \mathcal{Z} \mid z_{ac} > z_b\} = \{z \in \mathcal{Z} \mid z_{ac} \geq z_b\} - b$. Second, ϕ_b is the same in piv_{ac} and in tie_{abc}: hence, ϕ_b is the the offset ratio of ballot b in $\text{piv}_{ac} \uplus \text{tie}_{abc}$. Applying the offset theorem, we can write:

$$\lim_{n\to\infty} \frac{\mathbb{P}[\text{piv}_{ac} \mid n\tau]}{\mathbb{P}[\text{piv}_{ac} \mid n\tau] + \mathbb{P}[\text{tie}_{abc} \mid n\tau]} = \lim_{n\to\infty} \frac{\mathbb{P}[(\text{piv}_{ac} \uplus \text{tie}_{abc}) - b \mid n\tau]}{\mathbb{P}[\text{piv}_{ac} \uplus \text{tie}_{abc} \mid n\tau]} = \phi_b.$$

From this we obtain that $\lim_{n\to\infty} \frac{\mathbb{P}[\text{piv}_{ac} \mid n\tau]}{\mathbb{P}[\text{tie}_{abc} \mid n\tau]} = \frac{\phi_b}{1-\phi_b}$,[50] and we conclude that

$$\lim_{n\to\infty} \frac{\mathbb{P}[\text{piv}^{o}_{ac} \mid n\tau]}{\mathbb{P}[\text{tie}_{abc} \mid n\tau]} = \frac{\phi_b}{1-\phi_b}(\phi_{o_1} + \phi_{o_1 o_2}) = \frac{1}{\phi_{ac} - 1}(\phi_{o_1} + \phi_{o_1 o_2}).$$

By symmetry, we also have:

$$\lim_{n\to\infty} \frac{\mathbb{P}[\text{piv}^{o}_{bc} \mid n\tau]}{\mathbb{P}[\text{tie}_{abc} \mid n\tau]} = \frac{\phi_a}{1-\phi_a}(\phi_{o_1} + \phi_{o_1 o_2}) = \frac{1}{1/\phi_a - 1}(\phi_{o_1} + \phi_{o_1 o_2}).$$

Applying expression (1) from Proposition 1*, we obtain at the limit the desired formula:

$$u_o^\infty(\tau) = \frac{3\frac{1}{\phi_{o_1 o_2} - 1}(\phi_{o_1} + \phi_{o_1 o_2}) + 2\phi_{o_1} + \phi_{o_1 o_2}}{3\frac{1}{\phi_{o_1 o_2} - 1}(\phi_{o_1} + \phi_{o_1 o_2}) + 3\frac{1}{1/\phi_{o_1} - 1}(\phi_{o_1} + \phi_{o_1 o_2}) + 4\phi_{o_1} + 2\phi_{o_1 o_2}}. \quad (2)$$

□

A.5. *Proof of Theorem 2*

Theorem 2. *For any preference profile ρ, with $r \in \Delta^*(\mathcal{O})$ and with a Condorcet winner i, there exists an equilibrium τ under AV such that the candidate i is elected at τ.*

Proof. Let r be an ordinal profile admitting a Condorcet winner denoted by a. Let b be such that $r_{ba} \geq r_{ca}$, that is, b is the *best contender* to a as she loses her duel against a with the lowest margin. We construct an equilibrium τ for which $\mu_{ab} > \mu_{ac} = \mu_{bc}$.

[50]Note that, if $\phi_b = 1$, the formula holds true at the limit.

We denote by $\alpha \in (0,1)$ the share of voters with ordinal type ac casting a ballot a, and by $\beta \in (0,1)$ the share of voters with ordinal type bc casting a ballot b. This is, we consider a profile $\tau = \tau(\alpha,\beta)$ of the form $\tau_{ab} = \tau_c = 0$ and:

$$\tau_a = r_{ab} + \alpha r_{ac}, \quad \tau_b = r_{ba} + \beta r_{bc}, \quad \tau_{ac} = r_{ca} + (1-\alpha) r_{ac}, \quad \tau_{bc} = r_{cb} + (1-\beta) r_{bc}.$$

We are interested in the set $\mathscr{D}_r \subseteq [0,1]^2$ of couples (α,β) that are consistent with the pivot ordering $\mu_{ab} > \mu_{ac} = \mu_{bc}$. As $r \in \Delta^*(\mathscr{O})$, we have $\tau_a, \tau_b, \tau_{ac}, \tau_{bc} > 0$. We may thus apply Lemma 3 and write:

$$\mathscr{D}_r = \left\{(\alpha,\beta) \in [0,1]^2 \mid \tau_a \geq \tau_{bc}, \tau_b \geq \tau_{ac}, \tau_a\tau_b > \tau_{ac}\tau_{bc}\right\}.$$

We have

$$\tau_a \geq \tau_{bc} \quad \Leftrightarrow \quad \alpha r_{ac} + \beta r_{bc} \geq r_{-a} - r_{ab}$$

and

$$\tau_b \geq \tau_{ac} \quad \Leftrightarrow \quad \alpha r_{ac} + \beta r_{bc} \geq r_{-b} - r_{ba}.$$

Since a is the Condorcet winner, there is a strict majority of voters preferring a to b, so that $r_{-b} - r_{ba} > r_{-a} - r_{ab}$. Hence, $\tau_b \geq \tau_{ac} \Rightarrow \tau_a > \tau_{bc}$, which leads to

$$\mathscr{D}_r = \left\{(\alpha,\beta) \in [0,1]^2 \mid \tau_b \geq \tau_{ac}\right\} = \left\{(\alpha,\beta) \in [0,1]^2 \mid \alpha r_{ac} + \beta r_{bc} \geq r_{-b} - r_{ba}\right\}.$$

By assumption, we have $r_{ba} \geq r_{ca}$, and it follows that $r_{ac} \geq r_{-b} - r_{ba}$, hence we obtain that $(1,0) \in \mathscr{D}_r$.

We define an asymptotic best-reply function from the set of couples (α,β) to itself by:

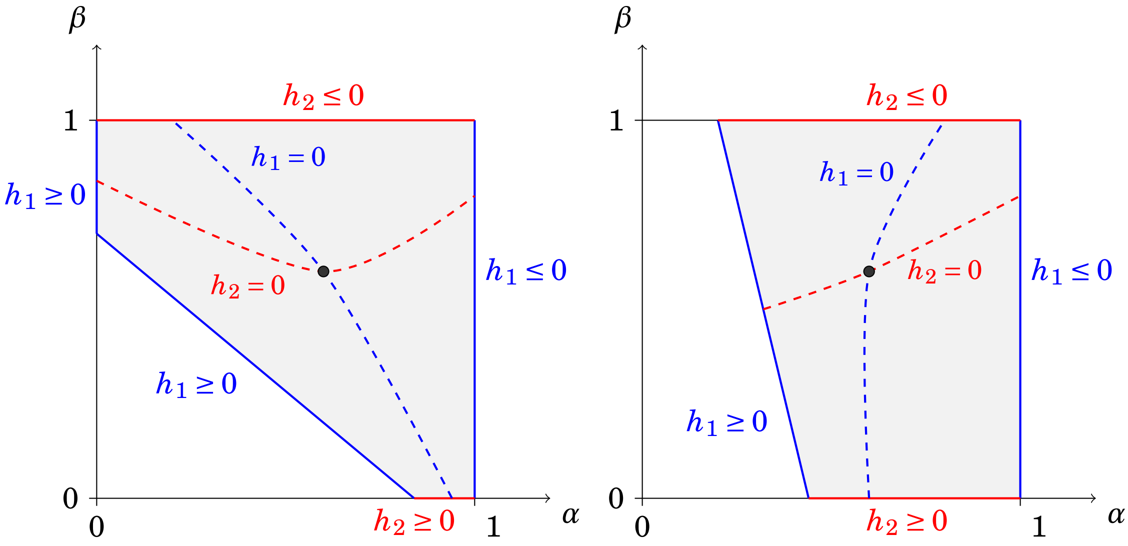
$$\mathrm{br}: \left| \begin{array}{lcl} \mathscr{D}_r & \to & [0,1]^2 \\ (\alpha,\beta) & \mapsto & \mathrm{br}(\alpha,\beta) = \left(F_{ac}\left(u_{ac}^{\infty}(\tau(\alpha,\beta))\right), F_{bc}\left(u_{bc}^{\infty}(\tau(\alpha,\beta))\right)\right). \end{array} \right.$$

We will show that br admits a fixed point (α^*,β^*) on the set $\mathscr{D}_r$, which will imply the existence of an equilibrium $\tau(\alpha^*,\beta^*)$ with pivot ordering $\mu_{ab} > \mu_{ac} = \mu_{bc}$. For that, we define $h(\alpha,\beta) = \mathrm{br}(\alpha,\beta) - (\alpha,\beta)$. Let us show that there exists $(\alpha^*,\beta^*) \in \mathscr{D}_r$ for which $h(\alpha^*,\beta^*) = 0$.

First, we observe that the function h is continuous, as each function F_o is continuous (by assumption) and each function $u_o^{\infty}(\cdot)$ is continuous (from expression (2) in Lemma 3).

Second, we compute the value of h along the (possibly empty) segment $\mathscr{S}_r = \{(\alpha,\beta) \in [0,1]^2 \mid \alpha r_{ac} + \beta r_{bc} \geq r_{-b} - r_{ba}\alpha\}$, on the boundary of $\mathscr{D}_r$. On this segment, we have that $\tau_b = \tau_{ac}$ and $\tau_a > \tau_{bc}$. We thus have $\phi_a < 1$ (hence $\phi_{bc} > 1$) while $\phi_b = 1$ (hence $\phi_{ac} = 1$). By application of (2) in Lemma 3, we obtain $u_{ac}^{\infty}(\tau) = 1$ and $u_{bc}^{\infty}(\tau) = 0$. Thus, we have that $\forall (\alpha,\beta) \in \mathscr{S}_r, \mathrm{br}(\alpha,\beta) = (1,0)$. It follows that for any $(\alpha,\beta) \in \mathscr{S}_r$, $h_1(\alpha,\beta) \geq 0$ and $h_2(\alpha,\beta) \leq 0$.

We draw on Figure 7 the two possible forms that the domain $\mathscr{D}_r$ can take, using the fact that $(1,0) \in \mathscr{D}_r$. We observe that, by definition of the function h, the sign of at least one function h_i is known at any point on the boundaries of $\mathscr{D}_r$.



(A) Case $(0,1) \in \mathscr{D}_r$ (B) Case $(0,1) \notin \mathscr{D}_r$

FIGURE 7. Application of the Poincaré-Miranda theorem.

These conditions on the sign of h_1 and h_2 allow us to apply the Poincaré-Miranda theorem [Kulpa, 1997]. There exists a $(\alpha^*, \beta^*) \in \mathscr{D}_r$ such that $h(\alpha^*, \beta^*) = 0$, it is represented by a black point on Figure 7. Hence, we have constructed an equilibrium with $\mu_{ab} > \mu_{ac} = \mu_{bc}$.

Since a is the Condorcet winner and $\mu_{ab} > \mu_{ac}, \mu_{bc}$, we have that $\gamma_a > \gamma_b$. We observe that it cannot be that $\gamma_c > \gamma_a > \gamma_b$, otherwise we would have $\mu_{ab} = \mu_{abc}$ by application of Lemma 2, (ii). Moreover, it cannot be that $\gamma_c = \gamma_a > \gamma_b$, otherwise we would have $\mu_{ac} > \mu_{ab} = \mu_{abc}$ by application of Lemma 2, (i) and (ii). Thus it must be that $\gamma_a > \gamma_b, \gamma_c$. This means that a is elected under the constructed equilibrium, as desired. □

A.6. *Proof of Proposition 3*

Proposition 3. *Let ρ be a preference profile satisfying Assumption S, with $r \in \Delta^*(\mathscr{O})$. If there is a Condorcet winner in ρ, then this candidate is elected at any cardinal equilibrium under AV.*

Proof. Let ρ be a preference profile with $r \in \Delta^*(\mathscr{O})$, with a Condorcet winner, say candidate a, and assume that it satisfies Assumption S. Let τ be a cardinal equilibrium, with magnitude ordering $\mu_{ij} > \mu_{ik} = \mu_{jk}$. If $i = a$ or $j = a$, we can apply the same reasoning as in the last paragraph of the proof of Theorem 2 to obtain that $\gamma_a > \gamma_b, \gamma_c$, in which case a is elected under τ.

We thus consider the remaining case where $\mu_{bc} > \mu_{ab} = \mu_{ac}$. As each voter votes for his favorite candidate between b and c, we must have $\gamma_b + \gamma_c = 1$, and we may assume without loss of generality that $\gamma_c \le 1/2$. Let $\alpha \in [0,1]$ be the fraction of voters with ordinal type $o = ca$ voting for c and let $\beta \in [0,1]$ be the fraction of voters with ordinal type $o = ba$ voting for b. As τ is an equilibrium, we have $\alpha = F_{ca}(u^{\infty}_{ca}(\tau))$ and $\beta = F_{ba}(u^{\infty}_{ba}(\tau))$.

To relate α and β, we apply Lemma 3 (note that we must have $\tau_{bc} = \tau_a = 0$ and $\tau_b, \tau_c, \tau_{ab}, \tau_{ac} > 0$, so that the lemma can be applied). Since τ is a cardinal equilibrium, we have for any $o \in \{ca, ba\}$, $u^{\infty}_o(\tau) = \frac{g_o(1/\phi_{o_3})}{g_o(1/\phi_{o_3}) + g_o(1/\phi_{o_1})}$, where $g_o(y) = \phi_{o_1} + (\phi_{o_1} + 1/\phi_{o_3})(1 + \frac{3}{y-1})$. Denoting by $-o$ the ordinal type having the reversed order from o, we may write:

$$g_{-o}(y) = \phi_{o_3} + (\phi_{o_3} + 1/\phi_{o_1})(1 + \frac{3}{y-1}) = \frac{\phi_{o_3}}{\phi_{o_1}} g_o(y).$$

We obtain, for any $o \in \{ca, ba\}$,

$$\begin{aligned} u^{\infty}_o(\tau) = \frac{g_o(1/\phi_{o_3})}{g_o(1/\phi_{o_3}) + g_o(1/\phi_{o_1})} &= \frac{g_{-o}(1/\phi_{o_3})}{g_{-o}(1/\phi_{o_3}) + g_{-o}(1/\phi_{o_1})} \\ &= 1 - \frac{g_{-o}(1/\phi_{o_1})}{g_{-o}(1/\phi_{o_1}) + g_{-o}(1/\phi_{o_3})} = 1 - u^{\infty}_{-o}(\tau). \end{aligned}$$

By application of Assumption S, we obtain that:

$$\alpha + \beta = F_{ca}(u^{\infty}_{ca}(\tau)) + F_{ba}(u^{\infty}_{ba}(\tau)) = F(u^{\infty}_{ca}(\tau)) + F(1 - u^{\infty}_{ca}(\tau)) = 1.$$

Now, we may write the score of candidate a as:

$$\begin{aligned} \gamma_a = r_a + (1-\alpha) r_{ca} + (1-\beta) r_{ba} &= r_a + \beta r_{ca} + (1-\beta) r_{ba} \\ &= \beta(r_a + r_{ca}) + (1-\beta)(r_a + r_{ba}) > 1/2, \end{aligned}$$

where we use the fact that a is the Condorcet winner, and thus $r_a + r_{ca} > 1/2$ and $r_a + r_{ba} > 1/2$. Hence, we have that $\gamma_a > 1/2 \ge \gamma_c$.

If $\gamma_b > 1/2$, we have $\gamma_a, \gamma_b > \gamma_c$, and by application of Lemma 2 (i), we get $\mu_{ab} > \mu_{abc}$, a contradiction.

If $\gamma_b = 1/2$, we have $\gamma_a > \gamma_b = \gamma_c$, and by application of Lemma 2 (ii), we get $\mu_{bc} = \mu_{abc}$, a contradiction.

We conclude that there can be no cardinal equilibrium with $\mu_{bc} > \mu_{ab} = \mu_{ac}$. Hence, any cardinal equilibrium must elect the Condorcet winner a. □

A.7. *Proof of Proposition 4*

Proposition 4. *Let ρ be a preference profile satisfying Assumption S, admitting a Condorcet winner, and with $r \in \Delta^*(\mathscr{O})$. For any candidate i elected at a discriminatory equilibrium under AV, we have $W_i > 1/2$.*

Proof. In the sequel, we note for any candidate k, $\Delta_k = r_k - r_{-k}$. Note that

$$\Delta_k = \frac{1}{2}\left(r_k + r_{ik} - r_{-k} - r_{jk}\right) + \frac{1}{2}\left(r_k + r_{jk} - r_{-k} - r_{ik}\right),$$

so that Δ_k measures of the average net margin of candidate k against the other candidates.

Under Assumption S, the welfare attached to a candidate k may be written as:

$$W_k = r_k + (r_{ik} + r_{jk}) \int_0^1 u dF(u) = r_k + \frac{r_{ik} + r_{jk}}{2} = \frac{1}{2} + \frac{r_k - r_{-k}}{2} = \frac{1}{2} + \frac{\Delta_k}{2},$$

where we use Assumption S and then the fact that $r_k + r_{ik} + r_{jk} + r_{-k} = 1$.

If a candidate i is elected at a cardinal equilibrium, we know from Proposition 3 that it must be the Condorcet winner. As Δ_i represents the average net margin of candidate i, we must have $\Delta_i > 0$. Therefore, $W_i > 1/2$.

If a candidate i is elected at an ordinal equilibrium with pivot ordering $\mu_{ab} > \mu_{ac} > \mu_{bc}$, then by Lemma 2 (ii), neither b nor c can have the highest expected score, so that $\gamma_a > \gamma_b, \gamma_c$, and thus $i = a$. We know from the equilibrium description in the panel (B) of Figure 1 that $\gamma_a = r_a$ and $\gamma_c = r_{-a}$. We thus have $W_i = W_a = \frac{1}{2} + \frac{\gamma_a - \gamma_c}{2} > 1/2$. □

A.8. *Proof of Proposition 5*

Proposition 5. *Under PL, for a generic ordinal profile r, for any candidates i and j, there is an equilibrium τ with $\gamma_i, \gamma_j > \gamma_k = 0$. For any such profile r, at least two candidates can be elected at equilibrium.*

We show how to adapt this well-known result [Myerson, 2002] to our setting.

Proof. Let r be such that the share of voters preferring a given candidate to another is strictly positive and different from 1/2 (this assumption only eliminates a set of ordinal profiles r of measure 0, i.e. this assumption holds for a generic r). Let i and j be two candidates, and consider the ballot profile τ defined by $\tau_i = r_i + r_{ki}$ and $\tau_j = 1 - \tau_i$. As $\tau_i, \tau_j > 0$, we have that $\mu_{ij} > \mu_{ik} = \mu_{kj} = \mu_{abc}$. Hence, voters' asymptotic best replies lead to the profile τ, which is thus an equilibrium.

To conclude, either there is a Condorcet loser i, in which case both the Condorcet winner j (if she runs against i or k) and the other candidate k (if she runs against i) can be elected at an equilibrium. Or there is a Condorcet cycle, in which case any candidate can be elected at an equilibrium. □

A.9. *Proof of Proposition 6*

Proposition 6. *For any $\varepsilon > 0$, there are generic preference profiles ρ, satisfying Assumption S and admitting a Condorcet winner, such that there exists an equilibrium winner i under PL with $W_i < 1/4 + \varepsilon$.*

Proof. Let F be any symmetric cumulative distribution on $(0,1)$. Let $\varepsilon > 0$. Consider any profile ρ such that $F_o = F$ for each $o \in \mathscr{O}$ (so that Assumption S is satisfied) and with an ordinal profile r such that:

$$r_i < \frac{\varepsilon}{2}, \qquad r_{ki} < \frac{\varepsilon}{2} \qquad \text{and} \qquad 1/2 < r_{ji} < 1/2 + \frac{\varepsilon}{2}.$$

Note that j is the Condorcet winner and that $W_i = r_i + \frac{1}{2}(r_{ki} + r_{ji}) < 1/4 + \varepsilon$. Moreover, we know from Duverger's law (Proposition 5) that there is an equilibrium under PL with $\gamma_i, \gamma_k > \gamma_j = 0$. In this equilibrium, since ik is the main pivot, we have $\gamma_i = r_i + r_{ji} > 1/2$, so that i is the equilibrium winner. To conclude, the profile ρ just constructed is generic. □

A.10. *Proof of Proposition 7*

Proposition 7. *Under APL, for any utility distributions $(\rho_o)_{o \in \mathscr{O}}$, the set of ordinal profiles r for which an equilibrium exists is of measure* 0.

Proof. Let τ be an equilibrium under APL. If there are i, j such that $\tau_{ij} = 0$, we have then $\tau_{ik} + \tau_{jk} = 1$. In that case, it can be checked that voters with ordinal type ji weakly prefer the ballot ij to the ballot jk for any outcome that can occur. Moreover, these voters strictly prefer ij to jk in the outcome $z_{ik} = z_{jk} = 0$, which occurs with positive probability. Hence, the best reply for voters with ordinal type ji is the ballot ij. As τ is an equilibrium with $\tau_{ij} = 0$, it must be that $r_{ji} = 0$, which implies that the ordinal profile r is not generic.

In the sequel we thus consider an equilibrium τ such that $\tau_{ab} > 0$, $\tau_{ac} > 0$ and $\tau_{bc} > 0$. Hence, we can apply Lemma 1, and we note that, since $\tau_a = \tau_b = \tau_c = 0$ under APL, we can write $\delta_{ij}(\tau) = \tau_{ij} - \sqrt{\tau_{ik}\tau_{jk}}$. We consider in turn all possible ordering of τ_{ab}, τ_{ac} and τ_{bc}.

Case 1: $\tau_{ab} > \tau_{ac} > \tau_{bc}$.

In that case, $\delta_{ab}(\tau) > 0 > \delta_{bc}(\tau)$. Thus, $\mu_{ab} = -(\sqrt{\tau_{ac}} - \sqrt{\tau_{bc}})^2 > \mu_{bc} = \mu_{abc}$.

Now, if $\delta_{ac}(\tau) \leq 0$, we have that $\mu_{ab} > \mu_{ac} = \mu_{bc} = \mu_{abc}$. Then, all voters preferring a to b cast a ballot ac (i.e. against b), while all those preferring b to a cast a ballot bc (i.e. against a). We obtain that $\tau_{ab} = 0$, a contradiction.

If $\delta_{ac}(\tau) > 0$, we obtain $\mu_{ac} = (\sqrt{\tau_{ab}} - \sqrt{\tau_{bc}})^2 > (\sqrt{\tau_{ac}} - \sqrt{\tau_{bc}})^2 = \mu_{ab} > \mu_{bc}$. Then, we obtain as before that $\tau_{ac} = 0$, a contradiction.

Case 2: $\tau_{ab} > \tau_{ac} = \tau_{bc}$.

Here, we have that $\delta_{ab}(\tau) > 0 > \delta_{ac}(\tau) = \delta_{bc}(\tau)$. We obtain as before $\mu_{ab} > \mu_{ac} = \mu_{bc} = \mu_{abc}$, and then $\tau_{ab} = 0$, a contradiction.

Case 3: $\tau_{ab} = \tau_{ac} > \tau_{bc}$.

In that case, we obtain $\mu_{ab} = \mu_{ac} > \mu_{bc}$ and $\lim_{n\to\infty} \mathbb{P}[\widetilde{\text{piv}_{ab}} \mid n\tau]/\mathbb{P}[\widetilde{\text{piv}_{ac}} \mid n\tau] = 1$. We conclude that voters ab cast a ballot ab,[51] voters ac cast a ballot ac, voters bc and cb cast a ballot bc. Moreover, voters ba cast a ballot ab if they attach a utility at least one half to a, and cast a ballot bc otherwise. Similarly, voters ca cast a ballot ac if they attach a utility at least one half to a, and cast a ballot bc otherwise.

We thus have $\tau_{ab} = r_{ab} + r_{ba}(1 - F_{ba}(1/2))$ and $\tau_{ac} = r_{ac} + r_{ca}(1 - F_{ca}(1/2))$. As such equilibrium must respect the equation $\tau_{ab} = \tau_{ac}$, it can only be encountered for non-generic ordinal profiles r.

Case 4: $\tau_{ab} = \tau_{ac} = \tau_{bc}$.

In that case, all pivots are equally likely and all voters vote for their two favorite candidates: for all (i, j), we have $\tau_{ij} = r_{ij} + r_{ji}$. Such an equilibrium can only be encountered for non-generic ordinal profiles r. □

[51]Note that these voters always benefit strictly more from voting ab in the event piv_{ac} than they benefit from voting ac in the event piv_{ab}.

APPENDIX B. CONVERGENCE OF THE UTILITY THRESHOLDS

This appendix is devoted to the proof of Theorem 1.

Theorem 1. *For any ordinal type $o \in \mathcal{O}$, for any profile $\tau \in \Delta(\mathcal{M})$, the sequence $(u_o^n(\tau))_{n\geq 0}$ converges. We denote by $u_o^\infty(\tau) \in [0,1]$ its limit.*

While proving the theorem, we also provide explicit formulas for the asymptotic utility thresholds $u_o^\infty(\tau)$. The formulas provide the theoretical foundation for the Python package "Poisson Approval".

The proof is divided in three parts. First, we provide in Section B.1 asymptotic developments of basic events defined by the difference of two Poisson variables. Second, we show in Section B.2 that these developments can be directly applied to obtain asymptotic utility thresholds $u_o^\infty(\tau)$ for profiles τ such that (at least) two consecutive ballots are absent, that we refer to as *flower profiles*.[52] Third, focusing on non-flower profiles, we show in Section B.3 that asymptotic utility thresholds $u_o^\infty(\tau)$ can be obtained from asymptotic developments of relevant pivot probabilities, which are themselves derived from asymptotic developments of probabilities of tie events.

The section concludes with an example of computation of an asymptotic utility threshold (Section B.4).

B.1. *Asymptotic Development of Some Basic Events*

In this section, Z_1 and Z_2 denote two independent Poisson variables, with $Z_i \sim \mathcal{P}(n\lambda_i)$ for each $i \in \{1,2\}$. For the probabilities of the events defined below, we derive asymptotic developments of the form $\exp\left(\mu n + \nu \log n + \xi + o(1)\right)$ when $n \to +\infty$, with $\mu, \nu, \xi \in \mathbb{R}$.

Lemma 4. *For any $k \in \mathbb{N}$,*

- *(i) If $\lambda_1 = 0$*
  - *(a) and $k = 0$, then $\mathbb{P}[Z_1 = Z_2 + k] = \exp(-\lambda_2 n)$.*
  - *(b) and $k > 0$, then $\mathbb{P}[Z_1 = Z_2 + k] = 0$.*
- *(ii) If $\lambda_1 > 0$ and $\lambda_2 = 0$, then $\mathbb{P}[Z_1 = Z_2 + k] = \exp(-\lambda_1 n + k\log n + k\log\lambda_1 - \log k!)$.*
- *(iii) If $\lambda_1 > 0$ and $\lambda_2 > 0$, then*

$$\mathbb{P}[Z_1 = Z_2 + k] = \exp\left(-\left(\sqrt{\lambda_1} - \sqrt{\lambda_2}\right)^2 n - \frac{1}{2}\log n - \frac{1}{2}\log\left(4\pi\lambda_2^{\frac{1}{2}+k}\lambda_1^{\frac{1}{2}-k}\right) + o(1)\right).$$

[52]With the graphical representation of our game in Figure 1, flower profiles are such that relevant ballots belong to the same half-space, and are thus included in a stylized (lotus) flower.

Proof. Cases (i) and (ii) are direct applications of the definition of a Poisson distribution. Consider case (iii). As the difference between two independent Poisson distributions follows a Skellam distribution, we have:

$$\mathbb{P}[Z_1 = Z_2 + k] = e^{-(\lambda_1+\lambda_2)n} \left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{k}{2}} I_k\left(2n\sqrt{\lambda_1\lambda_2}\right),$$

where I_k denotes the modified Bessel function of the first kind of order k [Myerson, 2000]. Since $2n\sqrt{\lambda_1\lambda_2} \to +\infty$, we can use the Hankel development: $\log I_k(x) = x - \frac{1}{2}\log(2\pi x) + o(1)$ when $x \to +\infty$, hence the result. □

Lemma 5. *For any* $k \in \mathbb{N}$,

(i) *If* $\lambda_1 = 0$
  (a) *and* $k = 0$, *then* $\mathbb{P}[Z_1 \geq Z_2 + k] = \exp(-\lambda_2 n)$.
  (b) *and* $k > 0$, *then* $\mathbb{P}[Z_1 \geq Z_2 + k] = 0$.

(ii) *If* $\lambda_1 > \lambda_2 \geq 0$, *then* $\mathbb{P}[Z_1 \geq Z_2 + k] = \exp\big(o(1)\big)$.

(iii) *If* $\lambda_1 = \lambda_2 > 0$, *then* $\mathbb{P}[Z_1 \geq Z_2 + k] = \exp\big(-\log 2 + o(1)\big)$.

(iv) *If* $\lambda_2 > \lambda_1 > 0$, *then*

$$\mathbb{P}[Z_1 \geq Z_2 + k] = \exp\left(-\left(\sqrt{\lambda_1} - \sqrt{\lambda_2}\right)^2 n - \frac{1}{2}\log n - \frac{1}{2}\log\left(4\pi\lambda_2^{\frac{1}{2}+k}\lambda_1^{\frac{1}{2}-k}\right) - \log\left(1 - \sqrt{\frac{\lambda_1}{\lambda_2}}\right) + o(1)\right).$$

Proof. Case (i) is obtained as a direct application of the definition of a Poisson distribution.

Case (ii): consider n large enough so that $(\lambda_1 - \lambda_2)n - k > 0$. Applying Chebyshev's inequality, we obtain:

$$\begin{aligned}\mathbb{P}[Z_1 < Z_2 + k] &\leq \mathbb{P}\left(|Z_1 - \lambda_1 n| \geq \frac{(\lambda_1-\lambda_2)n-k}{2}\right) + \mathbb{P}\left(|Z_2 - \lambda_2 n| \geq \frac{(\lambda_1-\lambda_2)n-k}{2}\right) \\ &\leq \frac{4n\lambda_1}{((\lambda_1-\lambda_2)n-k)^2} + \frac{4n\lambda_2}{((\lambda_1-\lambda_2)n-k)^2},\end{aligned}$$

which tends to 0 since $\lambda_1 > \lambda_2$.

Case (iii): since $\mathbb{P}[Z_1 = Z_2] = o(1)$ (Lemma 4), we have by symmetry of the variables that $\mathbb{P}[Z_1 \geq Z_2] = \frac{1}{2} + o(1)$. On the other hand, we have $\mathbb{P}[Z_2 \leq Z_1 < Z_2 + k] = \sum_{k'=0}^{k-1} \mathbb{P}[Z_1 = Z_2 + k']$. Using case (iii) of Lemma 4, we have $\mathbb{P}[Z_1 = Z_2 + k'] = \frac{1}{\sqrt{4\pi\lambda_1 n}}\big(1 + o(1)\big)$, hence $\mathbb{P}[Z_2 \leq Z_1 < Z_2 + k] = \frac{k}{\sqrt{4\pi\lambda_1 n}}\big(1 + o(1)\big) = o(1)$. Finally, we deduce that $\mathbb{P}[Z_1 \geq Z_2 + k] = \mathbb{P}[Z_1 \geq Z_2] - \mathbb{P}[Z_2 \leq Z_1 < Z_2 + k] = \frac{1}{2} + o(1)$, hence the result.

Case (iv): we use the same technique as in the proof of Lemma 3 (towards the end of the proof). The event $\{Z_1 > Z_2 + k\}$ can be written as $\{Z_1 \geq Z_2 + k\} - 2$, i.e. the event for which adding a unit to the variable Z_2 creates an event in $\{Z_1 \geq Z_2 + k\}$. Applying the *Offset Theorem* [Myerson, 2000], we obtain that $\frac{\mathbb{P}(Z_1 > Z_2 + k)}{\mathbb{P}(Z_1 \geq Z_2 + k)} \sim \phi_2$, where ϕ_2 denotes the offset of 2 in the event $\{Z_1 \geq Z_2 + k\}$. As k is finite and $\lambda_2 > \lambda_1$, the offset of 2 in $\{Z_1 \geq Z_2 + k\}$ is the same as in $\{Z_1 = Z_2\}$, equal to $\phi_2 = \sqrt{\frac{\lambda_1}{\lambda_2}}$. Simple algebra yields $\frac{\mathbb{P}(Z_1 \geq Z_2+k)}{\mathbb{P}(Z_1 = Z_2+k)} \sim 1/(1-\sqrt{\frac{\lambda_1}{\lambda_2}})$. We obtain the desired formula by using the asymptotic development of $\mathbb{P}(Z_1 = Z_2 + k)$ from Lemma 4 (iii). □

B.2. *Flower profiles*

We first derive asymptotic utility thresholds for *flower profiles* τ, i.e. such that $\tau_i + \tau_{ij} = 0$ for some $i, j \in \mathcal{K}$. Lemmas 4 and 5 allow us to compute the asymptotic developments of $\mathbb{P}[\text{piv}^o_{ij} \mid n\tau]$, $\mathbb{P}[\text{piv}^o_{jk} \mid n\tau]$, $\mathbb{P}[\text{tie}^{o,1}_{abc} \mid n\tau]$, and $\mathbb{P}[\text{tie}^{o,2}_{abc} \mid n\tau]$ for any flower profile τ and ordinal type $o = ij$.

Then, using expression (1) from Proposition 1*, we obtain the asymptotic utility threshold:

$$u^\infty_o(\tau) = \lim_{n\to\infty} \frac{3\mathbb{P}[\text{piv}^o_{ij} \mid n\tau] + 2\mathbb{P}[\text{tie}^{o,1}_{abc} \mid n\tau] + \mathbb{P}[\text{tie}^{o,2}_{abc} \mid n\tau]}{3\mathbb{P}[\text{piv}^o_{ij} \mid n\tau] + 3\mathbb{P}[\text{piv}^o_{jk} \mid n\tau] + 4\mathbb{P}[\text{tie}^{o,1}_{abc} \mid n\tau] + 2\mathbb{P}[\text{tie}^{o,2}_{abc} \mid n\tau]}.$$

In particular, given the length of the asymptotic developments, we obtain that $(u^n_o(\tau))_{n\geq 1}$ always converge.

B.3. *Non-flower profiles*

We now focus on non-flower profiles τ, for which $\tau_i + \tau_{ij} > 0$ for any $i, j \in \mathcal{K}$. The section is divided in three parts. Section B.3.1 introduces the notion of pseudo-offsets and establishes some of their basic properties. This notion is used in Section B.3.2 to show that all relevant pivot probabilities can be written as equivalent to tie events. Section B.3.3 wraps up and provides formulas for the asymptotic utility thresholds, derived from the equivalents of pivot probabilities.

B.3.1. *Pseudo-offsets.* For each ballot $m \in \mathcal{M}$, we introduce the notion of *pseudo-offset* of ballot m at an event $E \subseteq \mathcal{Z}$, which extends the notion of offset to the case where $\tau_m = 0$.

If $m = i$, we denote by ψ^E_i the *pseudo-offset* associated to i at E, defined as follows:

$$\psi^E_i = \begin{cases} \phi_i, & \text{if } \tau_i > 0, \\ \phi_{ij}\phi_{ik} & \text{otherwise.} \end{cases}$$

where each ϕ_m is the offset of ballot m at the event E. Similarly, if $m = ij$, we denote by ψ_{ij} the *pseudo-offset* associated to ij at E, defined as follows:

$$\psi^E_{ij} = \begin{cases} \phi_{ij} & \text{if } \tau_{ij} > 0, \\ \phi_i\phi_j & \text{otherwise.} \end{cases}$$

In the sequel, we simply write ψ_m for ψ_m^E when there is no ambiguity on the event E. When there is ambiguity, we write $\psi_m^{[abc]}$ for ψ_m^E when $E = \text{tie}_{abc}$ and $\psi_m^{[ij]}$ for ψ_m^E when $E = \widetilde{\text{piv}}_{ij}$ or $E = \text{piv}_{ij}$ (the pseudo-offsets being the same in these two events).

The following lemma establishes basic properties of pseudo-offsets at pivotal and tie events.

Lemma 6. *For any non-flower profile τ, for any event $E \in \{\text{piv}_{ij}, \widetilde{\text{piv}}_{ij}, \text{tie}_{abc}\}$:*

(i) The pseudo-offset of each ballot m at E is well-defined and positive: $\psi_m > 0$,
(ii) These pseudo-offsets satisfy for any $i, j, k \in \mathcal{K}$, $\psi_i \psi_{jk} = 1$ and $\psi_{ij} = \psi_i \psi_j$.

Proof. Let τ be a non-flower profile. We consider first the case for which $E = \text{tie}_{abc}$ and then $E \in \{\text{piv}_{ij}, \widetilde{\text{piv}}_{ij}\}$. We prove the validity of (i) for the ballot $m = ij$. The proofs for all other ballots m are similar (since τ is a non-flower profile), and are thus omitted.

Setting $E = \text{tie}_{abc}$, consider the pseudo-offset of $m = ij$. If $\tau_{ij} > 0$, the offset ϕ_{ij} is well-defined and $\psi_{ij} = \phi_{ij}$. If not, we must have $\tau_i > 0$ and $\tau_j > 0$. Thus, ϕ_i and ϕ_j are well-defined so that $\psi_{ij} = \phi_i \phi_j$. The *Dual Magnitude Theorem* [Myerson, 2002] implies that $\mu[E] = \mu_{abc} = \min_{x,y \in \mathbb{R}} F(x,y)$ with:

$$F(x,y) = \tau_i e^{x+y} + \tau_j e^{-x} + \tau_k e^{-y} + \tau_{ij} e^{y} + \tau_{ik} e^{x} + \tau_{jk} e^{-x} e^{-y} - 1.$$

Let x^* and $y^* \in [-\infty, +\infty]$ be respectively the optimal values of x and y. Hence, the offset-ratios of ballots i, j and ij are equal to (if well-defined): $\phi_i = e^{x^*+y^*}$, $\phi_j = e^{-x^*}$ and $\phi_{ij} = e^{y^*}$. It follows that $\phi_i \phi_j = e^{x^*+y^*} e^{-x^*} = e^{y^*} = \phi_{ij}$. This, in turn, implies that $\psi_{ij} = e^{y^*}$, independently of whether $\tau_i > 0$ or $\tau_{ij} > 0$. Thus, it suffices to show that $y^* > -\infty$ to conclude the proof of (i).

Since τ is non-flower, we have either $\tau_k > 0$ or ($\tau_{ik} > 0$ and $\tau_{jk} > 0$). If $\tau_k > 0$, then F cannot be minimized for $y = -\infty$, hence $y^* > -\infty$. If $\tau_{ik} > 0$ and $\tau_{jk} > 0$, $\tau_{ik} > 0$ implies that F cannot be minimized for $x = +\infty$, hence $x^* < +\infty$. Moreover, $\tau_{jk} > 0$ implies that $\tau_{jk} e^{-x^*} > 0$. As a consequence, F cannot be minimized for $y = -\infty$. Therefore, $y^* > -\infty$, which proves that $\psi_{ij} > 0$, concluding the proof of (i) for the case $E = \text{tie}_{abc}$.

Consider now the case for which $E \in \{\text{piv}_{ij}, \widetilde{\text{piv}}_{ij}\}$. We have

$$\mu[\text{piv}_{ij}] = \mu[\widetilde{\text{piv}}_{ij}] = \mu_{ij} = \min_{x \in \mathbb{R}, y \in \mathbb{R}^+} F(x,y),$$

with the same function F as above. We note $x^* \in [-\infty, +\infty]$ and $y^* \in [0, +\infty]$ the optimal values of x and y. As before, we obtain $\psi_{ij} = e^{y^*}$, independently of whether $\tau_i > 0$ or $\tau_{ij} > 0$. Since $y^* \geq 0$, we obtain $\psi_{ij} \geq 1 > 0$.

The formulas in (ii) simply derive from the following observations. With the previous formulas, we always have in a non-flower profile $\psi_i = e^{x^*+y^*}$ and $\psi_{jk} = e^{-x^*-y^*}$ (where x^* and y^* are the optima from the minimization program associated to the magnitude of E), so that $\psi_i\psi_{jk} = 1$. We also have $\psi_{ij} = e^{y^*}$, $\psi_i = e^{x^*+y^*}$ and $\psi_j = e^{-y^*}$, so that $\psi_{ij} = \psi_i\psi_j$, as required. □

The next lemma provides the key reason for the introduction of pseudo-offsets. It establishes an extension of the *Offset Theorem* [Myerson, 2000] to cases where some offsets ϕ_m are not defined, applicable to any non-flower profile τ.

Lemma 7. *(Pseudo-Offset Lemma)*
For any non-flower profile τ, for any event $E \in \{\text{piv}_{ij}, \widetilde{\text{piv}_{ij}}, \text{tie}_{abc}\}$, for any ballot $m \in \mathcal{M}$, for any integer q,

$$\lim_{n\to\infty} \frac{\mathbb{P}[E - qm \mid n\tau]}{\mathbb{P}[E \mid n\tau]} = (\psi_m^E)^q.$$

Proof. Take any event $E \in \{\text{piv}_{ab}, \widetilde{\text{piv}_{ab}}, \text{tie}_{abc}\}$, let q be an integer and consider first the ballot $m = ij$. If $\tau_{ij} > 0$, the offset of ij in E is well-defined, so that $\psi_{ij}^E = \phi_{ij}$, and we obtain the result by direct application of the *Offset Theorem* [Myerson, 2000].

Consider now the case where $\tau_{ij} = 0$. As τ is non-flower, we must have $\tau_i > 0$ and $\tau_j > 0$. Moreover, the set $E - q(ij)$ coincides with the set $E - q(i) - q(j)$. Hence, the *Offset Theorem* implies that:

$$\lim_{n\to\infty} \frac{\mathbb{P}[E - q(ij) \mid n\tau]}{\mathbb{P}[E \mid n\tau]} = \lim_{n\to\infty} \frac{\mathbb{P}[E - q(i) - q(j) \mid n\tau]}{\mathbb{P}[E \mid n\tau]} = (\phi_i)^q(\phi_j)^q = (\psi_{ij}^E)^q,$$

where the last equality comes from $\psi_{ij}^E = \phi_i\phi_j$, as $\tau_{ij} = 0$.

Consider now the ballot $m = i$. If $\tau_i > 0$, the *Offset Theorem* directly implies the result. If $\tau_i = 0$, we must have both $\tau_{ij} > 0$ and $\tau_{ik} > 0$ since τ is non-flower. Then, observe that $E - q(i)$ coincides with the set $E - q(ij) - q(ik)$, so that

$$\lim_{n\to\infty} \frac{\mathbb{P}[E - q(i) \mid n\tau]}{\mathbb{P}[E \mid n\tau]} = \lim_{n\to\infty} \frac{\mathbb{P}[E - q(ij) - q(ik) \mid n\tau]}{\mathbb{P}[E \mid n\tau]} = (\phi_{ij})^q(\phi_{ik})^q = (\psi_i^E)^q.$$

□

B.3.2. *Equivalents of Pivot Probabilities.*

Lemma 8. *For any non-flower profile τ and any ordinal type $o = ij$, we have for $n \to \infty$:*

$$\mathbb{P}[\text{tie}_{abc}^{o,1} \mid n\tau] \sim \psi_i^{[abc]}\mathbb{P}[\text{tie}_{abc} \mid n\tau], \qquad \mathbb{P}[\text{tie}_{abc}^{o,2} \mid n\tau] \sim \psi_{ij}^{[abc]}\mathbb{P}[\text{tie}_{abc} \mid n\tau],$$

and

$$\mathbb{P}[\text{piv}_{ij}^o \mid n\tau] \sim (1+\psi_{ik}^{[ij]})\mathbb{P}[\widetilde{\text{piv}_{ij}} \mid n\tau], \qquad \mathbb{P}[\text{piv}_{jk}^o \mid n\tau] \sim (\psi_i^{[jk]})^2(1+\psi_j^{[jk]})\mathbb{P}[\widetilde{\text{piv}_{jk}} \mid n\tau].$$

Proof. By application of the *Pseudo-Offset Lemma* (Lemma 7), we have when $n \to \infty$:

$$\mathbb{P}[\text{tie}^{o,1}_{abc} \mid n\tau] = \mathbb{P}[\text{tie}_{abc} - i \mid n\tau] \sim \psi_i \mathbb{P}[\text{tie}_{abc} \mid n\tau] \,,$$
$$\mathbb{P}[\text{tie}^{o,2}_{abc} \mid n\tau] = \mathbb{P}[\text{tie}_{abc} - ij \mid n\tau] \sim \psi_{ij} \mathbb{P}[\text{tie}_{abc} \mid n\tau],$$

where the pseudo-offsets are defined at the event tie_{abc}.

Recall that $\widetilde{\text{piv}_{ij}} = \text{piv}_{ij} \cup \text{tie}_{abc}$. Then, using the notation ψ_m for the pseudo-offset of m at piv_{ij}, we obtain:

$$\mathbb{P}[\widetilde{\text{piv}_{ij}} \mid n\tau] = \mathbb{P}[\text{piv}_{ij} - ij \mid n\tau] \sim \psi_{ij} \mathbb{P}[\text{piv}_{ij} \mid n\tau].$$

It follows that:

$$\begin{aligned}\mathbb{P}[\text{piv}^o_{ij} \mid n\tau] = \mathbb{P}[\text{piv}_{ij} - i \mid n\tau] + \mathbb{P}[\text{piv}_{ij} - ij \mid n\tau] &\sim \mathbb{P}[\text{piv}_{ij} \mid n\tau](\psi_i + \psi_{ij}) \\ &\sim \mathbb{P}[\widetilde{\text{piv}_{ij}} \mid n\tau] \frac{\psi_i + \psi_{ij}}{\psi_{ij}} \\ &\sim \mathbb{P}[\widetilde{\text{piv}_{ij}} \mid n\tau](1 + \psi_{ik}),\end{aligned}$$

where we use the identities $\psi_i/\psi_{ij} = \psi_i\psi_k = \psi_{ik}$ (Lemma 6). Similarly, using now the notation ψ_m for the pseudo-offset of m at piv_{jk}, we have

$$\mathbb{P}[\text{piv}^o_{jk} \mid n\tau] \sim \mathbb{P}[\text{piv}_{jk} \mid n\tau](\psi_i + \psi_{ij}) \sim \mathbb{P}[\widetilde{\text{piv}_{jk}} \mid n\tau] \frac{\psi_i + \psi_{ij}}{\psi_{jk}} = \mathbb{P}[\widetilde{\text{piv}_{jk}} \mid n\tau](\psi_i)^2(1+\psi_j),$$

where we use the identities $1/\psi_{jk} = \psi_i$ and $\psi_{ij} = \psi_i\psi_j$ (Lemma 6). □

The following result goes beyond Lemma 1 by providing precise estimates of relevant pivot probabilities, rather than magnitudes. In each case, pivot probabilities are proportional to some tie event, whose asymptotic development can then be easily computed.

Lemma 9. *For any non-flower profile τ, noting ψ_k for $\psi_k^{[abc]}$, we have:*

(i) If $\delta_{ij}(\tau) > 0$, then $\psi_k > 1$. We have $\mathbb{P}[\widetilde{\text{piv}_{ij}} \mid n\tau] \sim \mathbb{P}[\text{tie}_{ij} \mid n\tau]$ and $\frac{\mathbb{P}[\text{tie}_{abc} \mid n\tau]}{\mathbb{P}[\widetilde{\text{piv}_{ij}} \mid n\tau]} \to 0$.

(ii) If $\delta_{ij}(\tau) = 0$, then $\psi_k = 1$. We have $\mathbb{P}[\widetilde{\text{piv}_{ij}} \mid n\tau] \sim \frac{1}{2}\mathbb{P}[\text{tie}_{ij} \mid n\tau]$ and $\frac{\mathbb{P}[\text{tie}_{abc} \mid n\tau]}{\mathbb{P}[\widetilde{\text{piv}_{ij}} \mid n\tau]} \to 0$.

(iii) If $\delta_{ij}(\tau) < 0$, then $0 < \psi_k < 1$. We have $\mathbb{P}[\widetilde{\text{piv}_{ij}} \mid n\tau] \sim \frac{1}{1-\psi_k}\mathbb{P}[\text{tie}_{abc} \mid n\tau]$.

Proof. Case (i): $\delta_{ij}(\tau) > 0$.

The *Dual Magnitude Theorem* [Myerson, 2002] implies that $\mu_{abc} = \min_{x,y \in \mathbb{R}} F(x,y)$ with:

$$F(x,y) = \tau_i e^{x+y} + \tau_j e^{-x} + \tau_k e^{-y} + \tau_{ij} e^{y} + \tau_{ik} e^{x} + \tau_{jk} e^{-x} e^{-y} - 1.$$

Let x^* and $y^* \in [-\infty, +\infty]$ be respectively the optimal values of x and y in μ_{abc}.

The offset-ratios of ballots k, ik and ij are equal to:

$$\phi_k = e^{-y^*}, \quad \phi_{ik} = e^{x^*} \qquad \text{and} \quad \phi_{ij} = e^{-x^*} e^{-y^*}.$$

It follows that $\psi_k = e^{-y^*}$, independently of whether $\tau_k > 0$ or not.

As $\delta_{ij}(\tau) > 0$, Lemma 1 implies that $\mu_{ij} > \mu_{abc}$. Since $\mu_{ij} = \min_{x\in\mathbb{R}, y\geq 0} F(x,y)$, it follows that $y^* < 0$. Therefore, $\psi_k = e^{-y^*} > 1$, as wanted.

The probability of the event tie_{ij} may be written as:

$$\mathbb{P}[\text{tie}_{ij} \mid n\tau] = \mathbb{P}[s_i = s_j \geq s_k \mid n\tau] + \mathbb{P}[s_i = s_j < s_k \mid n\tau] = \mathbb{P}[\widetilde{\text{piv}}_{ij} \mid n\tau] + \mathbb{P}[s_i = s_j < s_k \mid n\tau].$$

Applying the *Dual Magnitude Theorem*, we have that $\mu[\text{tie}_{ij}] = \min_{x\in\mathbb{R}} F(x,0)$ and $\mu[\{s_i = s_j < s_k\}] = \min_{x\in\mathbb{R}, y\leq 0} F(x,y)$. As $y^* < 0$, this last magnitude is equal to μ_{abc} and is strictly lower than $\mu[\text{tie}_{ij}]$. Therefore, $\frac{\mathbb{P}[s_i = s_j < s_k \mid n\tau]}{\mathbb{P}[\text{tie}_{ij} \mid n\tau]} \to 0$, and we thus obtain $\mathbb{P}[\text{tie}_{ij} \mid n\tau] \sim \mathbb{P}[\widetilde{\text{piv}}_{ij} \mid n\tau]$ as wanted.

Finally, as $\mu_{ij} > \mu_{abc}$, we have that $\frac{\mathbb{P}[\text{tie}_{abc} \mid n\tau]}{\mathbb{P}[\widetilde{\text{piv}}_{ij} \mid n\tau]} \to 0$.

Case (ii): $\delta_{ij}(\tau) = 0$.

Following the proof of Lemma 1, $\delta_{ij}(\tau) = 0$ corresponds to the case where $y^* = 0$, so that $\psi_k = 1$. To show that the event tie_{abc} is negligible w.r.t. $\widetilde{\text{piv}}_{ij}$, we apply the *Pseudo-offset Lemma* (Lemma 7). As $\mathbb{P}[\widetilde{\text{piv}}_{ij} \mid n\tau] = \mathbb{P}[s_i = s_j \geq s_k \mid n\tau]$, for any fixed integer Q, we have:

$$\begin{aligned}
\mathbb{P}[\widetilde{\text{piv}}_{ij} \mid n\tau] &\geq \sum_{q=0}^{Q} \mathbb{P}[s_i = s_j = s_k + q \mid n\tau] \\
&\geq \left(\sum_{q=0}^{Q} (\psi_k)^q\right) \mathbb{P}[\text{tie}_{abc} \mid n\tau]\big(1 + o(1)\big) \geq Q\,\mathbb{P}[\text{tie}_{abc} \mid n\tau]\big(1 + o(1)\big).
\end{aligned}$$

We thus obtain $\frac{\mathbb{P}[\text{tie}_{abc} \mid n\tau]}{\mathbb{P}[\widetilde{\text{piv}}_{ij} \mid n\tau]} \to 0$.

As $\mathbb{P}[\widetilde{\text{piv}}_{ij} \mid n\tau] = \mathbb{P}[s_i = s_j \geq s_k \mid n\tau]$, we may write:

$$\begin{aligned}
\mathbb{P}[\widetilde{\text{piv}}_{ij} \mid n\tau] &= \mathbb{P}[\text{tie}_{ij} \mid n\tau] \times \mathbb{P}[s_i \geq s_k \mid \text{tie}_{ij}, n\tau] \\
&= \mathbb{P}[\text{tie}_{ij} \mid n\tau] \times \mathbb{P}[Z_i - Z_{jk} \geq Z_k - Z_{ij} \mid Z_i - Z_{jk} = Z_j - Z_{ik}, n\tau] \\
&= \mathbb{P}[\text{tie}_{ij} \mid n\tau] \times \mathbb{P}[X^n \geq Y^n \mid n\tau],
\end{aligned}$$

where the (independent) variables X^n and Y^n are defined by:

- $X^n = (Z_i - Z_{jk} \mid Z_i - Z_{jk} = Z_j - Z_{ik})$,
- $Y^n = Z_k - Z_{ij}$.

Since $(Z_m)_{m\in\mathcal{M}}$ are independent Poisson variables, we may apply Proposition 1 from Durand and de Panafieu [2021]. If we note $E_X = \tau_i \sqrt{\frac{\tau_j + \tau_{jk}}{\tau_i + \tau_{ik}}} - \tau_{jk} \sqrt{\frac{\tau_i + \tau_{ik}}{\tau_j + \tau_{jk}}}$, we obtain that the variable $\frac{X^n - nE_X}{\sqrt{n}}$ is asymptotically gaussian and centered, unless $\tau_i = \tau_{jk}$ or $\tau_j = \tau_{jk}$ or $\tau_j = \tau_{ik}$ or $\tau_i = \tau_{ik}$, in which case it follows a Dirac measure centered in 0.

Let us note $E_Y = \tau_k - \tau_{ij}$. The variable Y^n can be seen as the sum of n independent random variables following the Skellam distribution of parameters τ_k and τ_{ij}. If $\tau_k + \tau_{ij} > 0$, by application of the central limit theorem, we obtain that the variable $\frac{Y^n - nE_Y}{\sqrt{n}}$ is asymptotically gaussian and centered. If $\tau_k + \tau_{ij} = 0$, $\frac{Y^n - nE_Y}{\sqrt{n}}$ simply follows a Dirac measure centered in 0.

Note that we have by assumption $0 = \delta_{ij}(\tau) = E_X - E_Y$, hence $E_X = E_Y$. We obtain:

$$\mathbb{P}[X^n \geq Y^n \mid n\tau] = \mathbb{P}\left[\frac{X^n - nE_X}{\sqrt{n}} \geq \frac{Y^n - nE_Y}{\sqrt{n}} \mid n\tau\right].$$

As τ is a non-flower profile, the variables $\frac{X^n - nE_X}{\sqrt{n}}$ and $\frac{Y^n - nE_Y}{\sqrt{n}}$ cannot simultaneously follow the Dirac measure centered in 0 (the conditions are incompatible). Therefore, one variable is asymptotically gaussian and centered, while the other variable either is asymptotically gaussian and centered or follows the Dirac measure centered in 0. As a result, the probability that one variable is higher or equal to the other tends to one half. We thus obtain:

$$\mathbb{P}[\widetilde{\text{piv}}_{ij} \mid n\tau] = \mathbb{P}[\text{tie}_{ij} \mid n\tau] \times \mathbb{P}[X^n \geq Y^n \mid n\tau] \sim \frac{1}{2}\mathbb{P}[\text{tie}_{ij} \mid n\tau].$$

Case (iii): $\delta_{ij}(\tau) < 0$.

Following the same logic as in case (i), we obtain that $y^* > 0$ and $\psi_k < 1$. Moreover, we know from Lemma 6 that $\psi_k > 0$. As in the proof of Lemma 3 (towards the end), we have $y^* > 0$, so that the pseudo-offset of ballot k is the same in events tie_{abc} and $\widetilde{\text{piv}}_{ij}$. Following that proof, we may apply the *Pseudo-offset Lemma* (Lemma 7) by observing that $\text{piv}_{ij} = \widetilde{\text{piv}}_{ij} - k$, from which we finally obtain:

$$\frac{\mathbb{P}[\text{tie}_{abc} \mid n\tau]}{\mathbb{P}[\widetilde{\text{piv}}_{ij} \mid n\tau]} \sim 1 - \psi_k.$$

□

B.3.3. *Asymptotic utility thresholds.* We study the decision problem for a voter with ordinal type $o = ab$ so that the relevant pivot events involve pairs ab and bc. We consider three cases:

- If $\delta_{ab}(\tau) \geq 0$ and $\delta_{bc}(\tau) \geq 0$, then Lemma 8 implies that:

$$\mathbb{P}[\text{piv}^o_{ab} \mid n\tau] \sim (1 + \psi^{[ab]}_{ac})\mathbb{P}[\widetilde{\text{piv}}_{ab} \mid n\tau].$$

Moreover, since $\delta_{ab}(\tau) \geq 0$, Lemma 9 implies that:

$$\mathbb{P}[\widetilde{\text{piv}}_{ab} \mid n\tau] \sim \frac{1 + \mathbb{1}_{\{\delta_{ab}(\tau) > 0\}}}{2}\mathbb{P}[\text{tie}_{ab} \mid n\tau].$$

Thus, $\mathbb{P}[\text{piv}^o_{ab} \mid n\tau]$ is equivalent (up to a constant) to $\mathbb{P}[\text{tie}_{ab} \mid n\tau]$. Similarly, since $\delta_{bc}(\tau) \geq 0$, we obtain:

$$\mathbb{P}[\text{piv}^o_{bc} \mid n\tau] \sim (\psi^{[bc]}_a)^2(1+\psi^{[bc]}_b)\frac{1+\mathbb{1}_{\{\delta_{bc}(\tau)>0\}}}{2}\mathbb{P}[\text{tie}_{bc} \mid n\tau].$$

Since $\lim_{n\to\infty} \frac{\mathbb{P}[\text{tie}_{abc}|n\tau]}{\mathbb{P}[\widetilde{\text{piv}_{ab}}|n\tau]} = 0$ by Lemma 9, Lemma 8 implies that $\mathbb{P}[\text{tie}^{o,1}_{abc} \mid n\tau]$ and $\mathbb{P}[\text{tie}^{o,2}_{abc} \mid n\tau]$ are also negligible w.r.t. $\mathbb{P}[\widetilde{\text{piv}_{ab}} \mid n\tau]$ when $n \to \infty$. Hence, applying expression (1) from Proposition 1*, we obtain:

$$u^n_o(\tau) \sim \frac{1}{1 + \frac{(\psi^{[bc]}_a)^2(1+\psi^{[bc]}_b)(1+\mathbb{1}_{\{\delta_{bc}(\tau)>0\}})}{(1+\psi^{[ab]}_{ac})(1+\mathbb{1}_{\{\delta_{ab}(\tau)>0\}})} \times \frac{\mathbb{P}[\text{tie}_{bc} \mid n\tau]}{\mathbb{P}[\text{tie}_{ab} \mid n\tau]}}. \tag{3}$$

Finally, we can compute the limit $u^\infty_o(\tau)$ thanks to the asymptotic developments of $\mathbb{P}[\text{tie}_{ab} \mid n\tau]$ and $\mathbb{P}[\text{tie}_{bc} \mid n\tau]$, obtained from Lemma 4.[53]

- If $\delta_{ab}(\tau) \geq 0 > \delta_{bc}(\tau)$ or $\delta_{bc}(\tau) \geq 0 > \delta_{ab}(\tau)$. Assume first that $\delta_{ab}(\tau) \geq 0$ and $\delta_{bc}(\tau) < 0$. Then, combining Lemma 8 and Lemma 9, we obtain that $\mathbb{P}[\text{piv}^o_{bc} \mid n\tau]$, $\mathbb{P}[\text{tie}^{o,1}_{abc} \mid n\tau]$ and $\mathbb{P}[\text{tie}^{o,2}_{abc} \mid n\tau]$ are all equivalent (up to a constant) to $\mathbb{P}[\text{tie}_{abc} \mid n\tau]$, and thus negligible with respect to $\mathbb{P}[\text{piv}^o_{ab} \mid n\tau] \sim (1+\psi^{[ab]}_{ac})\mathbb{P}[\widetilde{\text{piv}_{ab}} \mid n\tau]$. We thus have $u^\infty_o(\tau) = 1$, by application of (1).

  If on the other hand $\delta_{bc}(\tau) \geq 0 > \delta_{ab}(\tau)$, then all relevant pivots become negligible with respect to $\mathbb{P}[\text{piv}^o_{bc} \mid n\tau]$, and we obtain $u^\infty_o(\tau) = 0$.
- If $\delta_{ab}(\tau), \delta_{bc}(\tau) < 0$. Then, we have $\psi_a, \psi_c < 1$ (Lemma 9). As the pseudo-offsets are the same in the the events $\text{tie}_{abc}, \text{piv}_{ab}$ and piv_{bc}, we may write, by application of Lemma 8 and Lemma 9:

$$\mathbb{P}[\text{piv}^o_{ab} \mid n\tau] \sim \frac{1+\psi_{ac}}{1-\psi_c}\mathbb{P}[\text{tie}_{abc} \mid n\tau], \qquad \mathbb{P}[\text{piv}^o_{bc} \mid n\tau] \sim \frac{(\psi_a)^2(1+\psi_b)}{1-\psi_a}\mathbb{P}[\text{tie}_{abc} \mid n\tau].$$

  Combining with the equivalents of tie events in Lemma 8, we obtain by application of (1):[54]

$$u^\infty_o(\tau) = \frac{3\frac{1+\psi_{ac}}{1-\psi_c} + 2\psi_a + \psi_{ab}}{3\frac{1+\psi_{ac}}{1-\psi_c} + 3\frac{\psi_a^2(1+\psi_b)}{1-\psi_a} + 4\psi_a + 2\psi_{ab}}. \tag{4}$$

Finally, we have shown that for each profile $\tau \in \Delta(\mathscr{M})$ and for each ordinal type $o \in \mathscr{O}$, the sequence $(u^n_o(\tau))_{n\geq 1}$ converges, and we have provided a formula for computing $u^\infty_o(\tau)$ in each case. This concludes the proof of Theorem 1.

[53]The pseudo-offsets $\psi^{[bc]}_a$, $\psi^{[bc]}_b$ and $\psi^{[ab]}_{ac}$ coincide with the corresponding pseudo-offsets in the tie events tie_{bc} and tie_{ab} (respectively). These quantities can be computed analytically, and the Python package "Poisson Approval" thus uses their exact formulas.

[54]In the Python package "Poisson Approval", the pseudo-offsets are analytically computed whenever possible. Otherwise, the pseudo-offsets are obtained by a numeric optimization of the program defined by the *Dual Magnitude Theorem* [Myerson, 2002].

B.4. *A (difficult) example*

In this section, we provide an example of computation of the asymptotic utility threshold.

Consider the following profile: $\tau_a = 3/20$, $\tau_b = 9/20$, $\tau_{ac} = 1/20$ and $\tau_{bc} = 7/20$. We have $\mu_{ab} = \mu_{bc} = -1/5$ and $\mu_{ac} = \mu_{abc} = \frac{\sqrt{21}-7}{10} \approx -0.242 < -1/5$. We thus have $\delta_{ab}(\tau) > 0$ and $\delta_{bc}(\tau) > 0$ (Lemma 1).

We consider the decision problem of a voter with ordinal type $o = ab$. Numerically, we obtain the relevant pseudo-offsets: $\psi_{ac}^{[ab]} = 2$ in the event $\widetilde{\text{piv}}_{ab}$, while $\psi_a^{[bc]} = 1$ and $\psi_b^{[bc]} = 1/3$ in the event $\widetilde{\text{piv}}_{bc}$. By application of (3), we obtain:

$$u_o^n(\tau) \sim \frac{1}{1 + \frac{4}{9}\frac{\mathbb{P}[\text{tie}_{bc} \mid n\tau]}{\mathbb{P}[\text{tie}_{ab} \mid n\tau]}}.$$

Then, we may write $\mathbb{P}[\text{tie}_{ab} \mid n\tau] = \mathbb{P}[Z_a + Z_{ab} = Z_b + Z_{bc}]$, where $Z_a + Z_{ac} \sim \mathscr{P}(\frac{n}{5})$ and $Z_b + Z_{bc} \sim \mathscr{P}(\frac{4n}{5})$ are independent. Applying Lemma 4 (iii), we obtain:

$$\begin{aligned}
\mathbb{P}[\text{tie}_{ab} \mid n\tau] &= \exp\left(-\left(\sqrt{\frac{n}{5}} - \sqrt{\frac{4n}{5}}\right)^2 - \frac{1}{2}\log(n) - \frac{1}{2}\log\left(4\pi\sqrt{\frac{n \times 4n}{5 \times 5}}\right) + o(1)\right) \\
&= e^{-\frac{n}{5} - \frac{1}{2}\log(n) - \frac{1}{2}\log(\frac{8\pi}{5}) + o(1)}.
\end{aligned}$$

We obtain similarly that $\mathbb{P}[\text{tie}_{bc} \mid n\tau] = e^{-\frac{n}{5} - \frac{1}{2}\log(n) - \frac{1}{2}\log(\frac{3\pi}{5}) + o(1)}$. To conclude, we obtain:

$$u_o^n(\tau) \sim \frac{1}{1 + \frac{4}{9} \times \frac{e^{-\frac{n}{5} - \frac{1}{2}\log(n) - \frac{1}{2}\log(\frac{3\pi}{5}) + o(1)}}{e^{-\frac{n}{5} - \frac{1}{2}\log(n) - \frac{1}{2}\log(\frac{8\pi}{5}) + o(1)}}} \sim \frac{1}{1 + \frac{4}{9} \times e^{\frac{1}{2}\log(\frac{8}{3}) + o(1)}}.$$

Simple algebra yields $u_o^\infty(\tau) = \dfrac{9\sqrt{3}}{9\sqrt{3} + 8\sqrt{2}}$.

Note that, in this example, using magnitudes and the *Offset theorem* (or equivalently, the *Pseudo-Offset Lemma*) alone would not be sufficient to obtain $u_o^\infty(\tau)$. With these tools, one obtains that both events piv_{ab}^o and piv_{bc}^o are infinitely more likely than tie_{abc}, but it remains *a priori* difficult to compare the probabilities of these two events. Yet, the methods are extremely useful to relate $u_o^\infty(\tau)$ to the probabilities of tie events, whose asymptotic developments can then be computed.

APPENDIX C. NUMERICAL APPENDIX

This appendix is devoted to robustness checks for the numerical results obtained under the adaptive procedure. It also contains some examples and figures that complement the analysis.

C.1. *Monte-Carlo simulations on ordinal equilibria*

In this section, we explore the robustness of the finding that ordinal equilibria under AV not electing the Condorcet winner are rare (Section 4.1). We run Monte-Carlo simulations for each possible number $T \in \{2, \ldots 6\}$ of ordinal types present in a profile. Specifically, for each T, we generate 10,000 ordinal profiles, drawn independently from the uniform distribution on $\Delta_T(\mathcal{O}) := \{r \in \Delta(\mathcal{O}) \mid \#\{o \in \mathcal{O} \mid r_o > 0\} = T\}$.

T	$\mathbb{P}_T[\exists$ ord eq $\mid \exists$ CW$]$	$\mathbb{P}_T[\exists$ ord eq, s.t. CW not elected $\mid \exists$ CW$]$	$\mathbb{P}_T[\exists$ CW$]$
2	40.8%	0.0%	100.0%
3	24.1%	3.8%	97.2%
4	27.4%	1.8%	94.9%
5	26.9%	0.3%	94.0%
6	29.4%	0.1%	93.7%

TABLE 3. Estimated frequencies of ordinal equilibrium existence and of the election of a non-Condorcet winner at an ordinal equilibrium, conditionally on the existence of a Condorcet winner, for any given number T of ordinal types.

We note that the rare cases for which a "bad" ordinal equilibrium exists under AV are slightly more likely with few ordinal types, as they occur in 1.8% of the simulations when $T = 4$ and in 3.8% when $T = 3$.

To better understand the circumstances under which such examples can be found, we compute the conditional probability of a "bad" ordinal equilibrium, for each possible subset of ordinal types $O \subseteq \mathcal{O}$, with Monte-Carlo simulations performed for a sample of 10,000 draws (ordinal profiles r are drawn uniformly over the simplex $\Delta(O)$). We obtain that, perhaps surprisingly, the only domain where the conditional probability is above 5% is the single-peaked domain.[55] Indeed, an ordinal equilibrium where the Condorcet winner is not elected exists in 5.9% of all single-peaked profiles and in 13.0% of single-peaked profiles with three ordinal

[55]The single-peaked domain is defined as $\mathcal{D}_{SP} = \{r \in \Delta(\mathcal{O}) \mid \exists i, j \in \mathcal{K}, r_{ij} + r_{ji} = 0\}$. Informally, one candidate (k) is never ranked last. On this domain, a Condorcet winner always exists.

types such that each of them has a different candidate ranked first.[56]

As we observe that single-peaked profiles are more likely than others to admit an ordinal equilibrium electing a non-Condorcet winner under AV, we also report the winning frequency of the Condorcet winner under the adaptive procedure (as in Section 5.2.2) over this particular domain.

	PL	APL	AV
Single-Peaked	66.1%	74.7%	99.2%

TABLE 4. Percentage of observations for which the Condorcet winner is elected.

The main conclusion of Section 5.2.2 remains valid on the domain of single-peaked profiles. Although AV may admit some equilibria on this domain for which the Condorcet winner is not elected, these are almost never reached by the adaptive procedure, as the Condorcet winner is elected in 99.2% of the simulations.

C.2. *Robustness to the parameters of the adaptive procedure*

In this section, we consider alternative choices for the parameters of the adaptive procedure. In Table 5, we report the convergence rate (computed for a maximum of $P = 1,000$ periods) and the frequency with which the Condorcet winner is elected when it exists for several values of α^p, the parameter governing voters' belief updating, while keeping $\beta^p = \frac{1}{\log(p+1)}$. The first row displays $\alpha^p = 1$, the case for which voters only take the latest poll into account. The third row corresponds to $\alpha^p = \frac{1}{\log(p+1)}$ as in the article. The last row displays $\alpha^p = \frac{1}{p}$ as in classical fictitious play.

	Convergence rate			Condorcet Consistency		
α^p	AV	PL	APL	AV	PL	APL
1	93.9%	100.0%	0.0%	99.94%	66.2%	54.7%
0.5	95.0%	100.0%	0.0%	99.98%	67.2%	52.8%
$\frac{1}{\log(p+1)}$	95.2%	100.0%	0.0%	99.96%	66.1%	49.5%
$\frac{1}{\sqrt{p}}$	94.7%	100.0%	0.0%	100.0%	66.0%	46.2%
$\frac{1}{p}$	0.0%	0.0%	0.0%	99.95%	66.5%	44.7%

TABLE 5. Robustness to the belief updating parameter α^p ($P = 1,000$).

We observe that the convergence rate of the procedure for each voting rule is

[56]We can show that, on the single-peaked domain, only the central candidate (say b) can be elected at an ordinal equilibrium. The region of parameter values for which this equilibrium exists but b is not the Condorcet winner is (i) convex, (ii) contiguous to the region where b is the Condorcet winner and (iii) such that at least $\frac{3-\sqrt{5}}{2} \approx 38\%$ of voters prefer b to the Condorcet winner.

robust to different choices of α^p, but in the case $\alpha^p = \frac{1}{p}$, for which it becomes null for all rules. This is not surprising: the adjustment of beliefs is then too slow for the procedure to meet our stringent convergence test before a number of $P = 1,000$ periods. Yet, we observe that for all choices of the parameter α^p, including the one with no convergence, the frequency with which the Condorcet winner is elected under each voting rule appears consistent. While this frequency slightly decreases for APL from the first to the lowest rows of the table, the election of the Condorcet winner always remains more likely under AV than under any other rule.

In Table 6, we report the convergence rate (computed for a maximum of $P = 1,000$ periods) and the frequency with which the Condorcet winner is elected when it exists for several values of β^p, the share of updating voters at period p, while keeping $\alpha^p = \frac{1}{\log(p+1)}$. The first row ($\beta^p = 1$) corresponds to the case for which all voters play a best-reply at each period.

	Convergence rate			Condorcet Consistency		
β^p	AV	PL	APL	AV	PL	APL
1	93.9%	100.0%	0.0%	99.8%	66.7%	10.3%
0.5	94.6%	100.0%	0.0%	99.96%	67.0%	26.8%
$\frac{1}{\log(p+1)}$	95.2%	100.0%	0.0%	99.96%	66.1%	49.5%
$\frac{1}{\sqrt{p}}$	95.0%	100.0%	0.0%	99.98%	67.0%	57.1%
$\frac{1}{p}$	5.6%	0.0%	0.0%	99.97%	65.7%	60.3%

TABLE 6. Robustness to the per-period share of updating voters β^p ($P = 1,000$).

The convergence rates under AV and PL are both high and consistent, but in the case $\beta^p = \frac{1}{p}$, for which they significantly drop. The Condorcet winner becomes less likely to be selected under APL as the per-period share of updating voter increases, but in any case, it is under AV that she is most likely to be elected.

C.3. *Robustness to the distribution of preference intensities*

In this section, we consider the robustness of our results to the distributions of preference intensities. This is important: we have seen in Proposition 3 that if utility distributions satisfy Assumption S, i.e. if they are identical across ordinal types and symmetric, then no other candidate than the Condorcet winner can be elected at a cardinal equilibrium under AV. In the article, simulations are conducted with $\rho_o \sim \mathscr{U}((0,1))$ for all o, so that Assumption S is satisfied. We thus consider two alternative utility distributions: $\rho_o \sim \mathscr{U}([0.09, 0.11])$ for all o, so that utilities are concentrated around 0.1; and the polar case $\rho_o \sim \mathscr{U}([0.89, 0.91])$ for all o, so that utilities are concentrated around 0.9. In both cases, Assumption S is violated (yet as indicated in Footnote 18, the result of Proposition 3 still holds in the latter case where the property of *weak cardinal support for the Condorcet winner* is satisfied). We report on Table 7 the frequency with which the Condorcet winner is elected when it exists, and the average utilitarian welfare loss, for each set of distributions.

	Condorcet Consistency			Average Welfare Loss		
ρ_o	AV	PL	APL	AV	PL	APL
$\mathscr{U}((0,1))$	99.96%	66.1%	49.5%	0.007	0.060	0.099
$\mathscr{U}([0.09,0.11])$	98.5%	66.5%	49.3%	0.011	0.086	0.115
$\mathscr{U}([0.89,0.91])$	98.5%	66.0%	49.0%	0.033	0.062	0.113

TABLE 7. Robustness to the distribution of preference intensities.

We observe that our results are mostly robust, as AV always dominates the two other rules on both criteria, although it becomes slightly less efficient for each criterion in absolute terms.

C.4. *Robustness to the initial poll*

We now consider the robustness of our results to the distribution from which the initial poll τ^0 is drawn in the adaptive procedure. While we assume that τ^0 is drawn uniformly on $\Delta(\mathscr{M})$ in the article, we consider here two alternative specifications. In the first one, we consider the region of ballot profiles τ that are obtained when voters use undominated strategies, and we draw τ^0 uniformly on this region. In the second one, we assume that τ^0 coincides with the ballot profile obtained when all voters are (model 1-) expressive. We report on Table 8 the frequency with which the Condorcet winner is elected when it exists, and the average utilitarian welfare loss, for each distribution of the initial poll τ^0.

	Condorcet Consistency			Average Welfare Loss		
Distribution of τ^0	AV	PL	APL	AV	PL	APL
uniform	99.96%	66.1%	49.5%	0.007	0.060	0.099
uniform undominated	99.997%	81.7%	48.9%	0.007	0.031	0.099
(model 1-) expressive	99.999%	96.8%	48.9%	0.007	0.009	0.099

TABLE 8. Robustness to the initial poll.

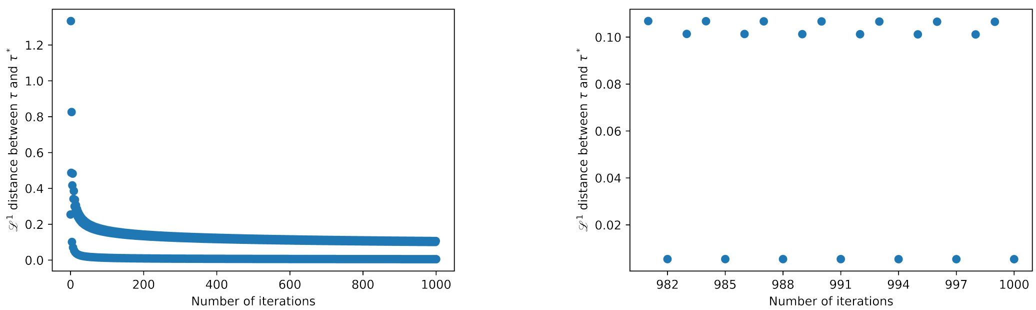
We observe that the results for AV and APL are robust to the draw of the initial poll. The main difference appears for PL, which improves going from the first to the second row, and from the second to the third one, both in terms of Condorcet consistency and average welfare. Yet, even in the hypothetical case for which the initial poll is expressive, AV still slightly dominates PL.

C.5. *APL in the Bad Apple example*

Myerson [2002] introduces an example of divided preferences with $r_{ab} = r_{ba} = 50\%$, while $\rho_{ab} = \rho_{ba}$ is the Dirac measure concentrated on 2/3. The third candidate, c, is interpreted as the *bad apple*, disliked by all voters. Myerson proves that a *large equilibrium* τ^* exists for this profile under APL, it is such that $\tau^*_{ab} = \tau^*_{ac} = \tau^*_{bc} = 1/3$. This large equilibrium is supported by a sequence of profiles $(\tau^n)_{n\geq 1}$, where τ^n is an equilibrium for an electorate of expected size n, namely $\tau^n_{ab} \approx 1/3 + \frac{0.628}{\sqrt{n}}$ and $\tau^n_{ac} = \tau^n_{bc} \approx 1/3 - \frac{0.314}{\sqrt{n}}$. Myerson shows that, along this sequence, the worst candidate c is elected with a small but positive probability, estimated at 4.4% for $n = 9{,}000{,}000$. As a first remark, this example illustrate that the construction of large equilibria may hinge on small details in beliefs, vanishing in the limit, a behavioral assumption that seems rather implausible.

We run the adaptive procedure at this profile under APL. For each of the assumptions on the distribution of the initial poll τ^0 discussed in Section C.4, we estimate that the worst candidate c gets elected with a large frequency of around 70% (between 68.6% and 71.7%). In the long run, we observe that the perceived profile $\hat{\tau}^p$ converges to τ^*. Thus, the large equilibrium τ^* has relevant predictive power for the adaptive procedure, in terms of long-run *average behavior*. Yet, the actual profile τ^p often deviates from τ^*, even in the long run, as we observe frequent and recurrent jumps along the trajectory. This electoral instability under APL is captured by the absence of equilibrium established in Proposition 7.

We illustrate the non-convergence of τ^p in this example for one (typical) run of simulation on Figure 8.



(A) Over all periods ($1 \leq p \leq 1000$) (B) Over last 20 periods ($981 \leq p \leq 1000$)

FIGURE 8. $\mathscr{L}^1$-distance between τ^p and τ^*.

C.6. *Absolute Welfare*

In this section, we focus on absolute (utilitarian) welfare levels rather than welfare losses, as done in Section 5.2.3. Figure 9 reports the empirical cumulative distributions of utilitarian welfare.

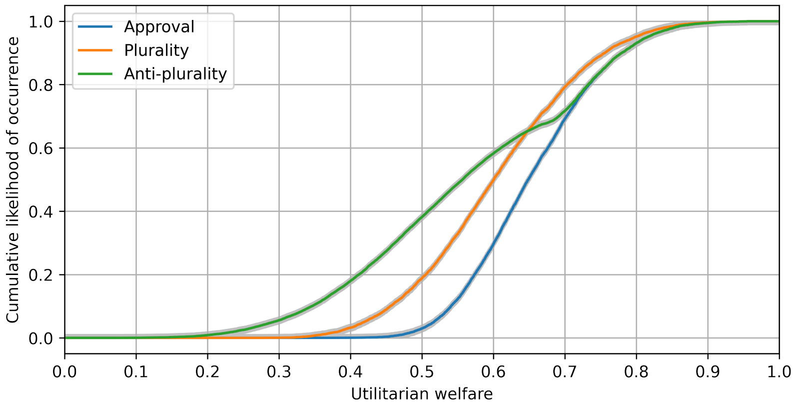
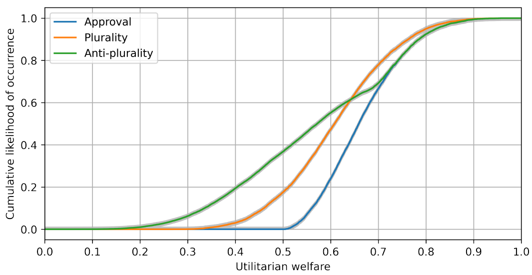
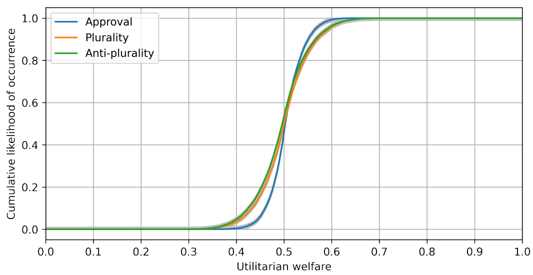


FIGURE 9. Cumulative distributions of utilitarian welfare levels.

We see that AV still robustly dominates both PL and APL, while the comparison between APL and PL is less clear-cut than with welfare losses. We further distinguish on Figure 10 below between cases where a Condorcet winner exists (left panel) and cases where it does not (right panel).



(A) With a Condorcet Winner



(B) Without a Condorcet winner

FIGURE 10. Cumulative distributions of utilitarian welfare levels.

We see that the dominance of AV over other voting rules is driven by profiles for which a Condorcet winner exists, which are exactly the profiles for which we provided equilibrium welfare guarantees (Proposition 4). We indeed confirm that for such profiles, the welfare of elected candidates under AV is always higher than 1/2, while there is a significant chance (approximately 20%) to elect a candidate yielding welfare below 1/2 under PL. By contrast, all rules perform similarly when there is a Condorcet cycle.

C.7. *An example where the challenger is approved by a majority under AV*

We describe one example of a preference profile for which the adaptive procedure sometimes reaches an election outcome such that both the winner (ranked first) and the challenger (ranked second) obtain a majority of approvals. The ordinal profile r is defined by $r_{ab} = 33\%$, $r_{ac} = 1\%$, $r_{ba} = 8\%$, $r_{bc} = 8\%$, $r_{ca} = 2\%$ and $r_{cb} = 48\%$. The distribution ρ is such that $\rho_o \sim \mathscr{U}((0,1))$ for all $o \in \mathscr{O}$. Note that c is the Condorcet winner in this example.

We ran the adaptive procedure for multiple draws of τ^0 from the uniform distribution on $\Delta(\mathscr{M})$. For some draws, the procedure converged to the ordinal equilibrium τ such that $\tau_a = 33\%$, $\tau_{ac} = 3\%$, $\tau_b = 16\%$ and $\tau_{bc} = 48\%$. For this particular profile, the non-Condorcet winner b is elected and the scores are given by $\gamma_a = 36\%$, $\gamma_b = 64\%$ and $\gamma_c = 51\%$. Hence, the challenger c is approved by a majority of voters. Finally, observe that this example corresponds to one of the very rare cases (estimated at $100-99.96 = 0,04\%$ in Table 2) for which the profile ρ and the initial poll τ^0 are such that Condorcet winner is not elected in the long-run outcome of the procedure under AV.

C.8. *Confidence Intervals*

Within the article, we do not specify confidence intervals in the tables for the sake of readability. As all estimates are computed for 10,000 draws, we report here the 95% confidence interval associated to all possible point estimates of a frequency for this number of draws. That is, we report on Figure 11 the 95% confidence interval for the parameter $f \in (0,1)$, for each possible realization $\theta \in \{0,\dots,10000\}$ of a variable drawn from the binomial distribution with parameters 10,000 and f.

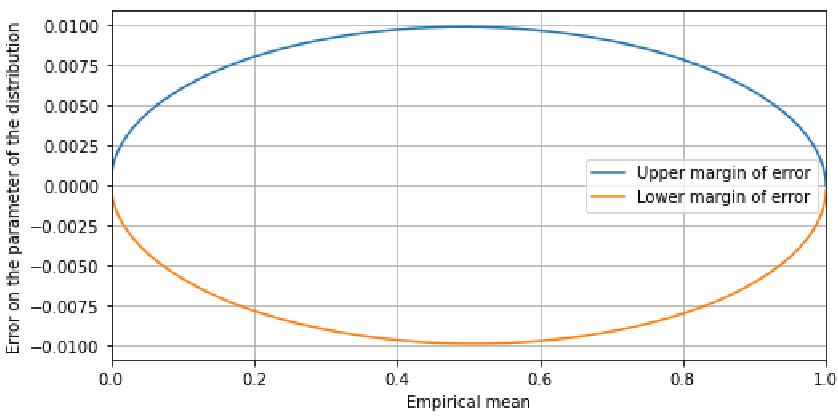


FIGURE 11. 95% confidence interval for 10,000 draws.

We observe that the boundaries of the 95% confidence interval are always within one percentage point of the point estimate for the frequency f. The confidence interval becomes asymmetric and narrower for very low or very high estimated frequencies.