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JEL Codes: D71, G33
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On the resolution of cross-liabilities*

Gabrielle Demange†

February 24, 2021

Abstract In a variety of situations, entities in a system, for example firms in the financial sector, hold liabilities on each other. The reimbursement abilities are intertwined, thereby potentially generating coordination failures and a cascade of defaults calling for interventions. Interventions can be discretionary or designed by rules such as bankruptcy laws. Bankruptcy laws, however, manage the default of a single firm towards its creditors, without considering all those that could be affected indirectly by the resolution. To account for these indirect effects, resolution rules should be defined at the system level. This paper investigates such rules, assuming that the primary goal of the resolution is to avoid defaults on external debts, say, banks' defaults on deposits. Focusing on the proportionality principle, it defines and characterizes the constrained-proportional rule, building on two approaches: the minimization of an inequality measure of the reimbursements (made and received) and the axiomatization through desirable properties.

Keywords cross-liabilities, resolution, bi-proportionality, axiomatization, entropy

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MSC 2020 90-10, 91C15

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1 Introduction

In a variety of situations, entities in a system hold liabilities on each other, for example firms in the financial sector, subsidiaries in a conglomerate, or countries in a transnational union. The reimbursement abilities are intertwined, thereby potentially generating coordination failures and cascade of defaults calling for interventions. Interventions can be discretionary or designed by rules. Rules contrast with discretionary resolution as they describe the perspective of a regulator (legal body or exchange operator) who clarifies in advance how conflicting claims will be solved in case a default arises. Bankruptcy laws are examples of such rules. They, however, manage the default of a single firm towards its creditors, leaving aside those affected indirectly by the resolution. To account for these indirect effects, resolution rules should be designed at the system level. This paper proposes to study such rules when the main objective of the regulator is to avoid default on the debts due to entities external to the system, say default by a bank on its customers' deposits. On what principle should the conflicting claims within the system be solved and reimbursements be based? In the case of a single debtor indebted to safe creditors, proportionality is a standard principle also perceived as fairness: it requires the debtor to reimburse the same amount per unit of claim to its creditors. In a system composed of multiple entities that are simultaneously debtors and creditors to each other and face bankruptcy, what proportionality means is not crystal clear as reimbursement abilities are intertwined with received payments. This paper proposes to make this precise by defining and characterizing a rule called the 'constrained-proportional' (hereafter cp) rule.

The analysis is conducted in a stylized model of financial linkages with a single liquidation date. A system of entities, called hereafter firms, have claims and liabilities between themselves, all with equal priorities. Firms also have claims and liabilities on entities external to the system.[1] The values of internal and external claims and liabilities fully describe a 'problem'. A default on external liabilities is assumed to have important consequences, as, for example, a default of a bank on its customers' deposits, and triggers bankruptcy[2] while default on its debts within the system does not. It is assumed that the primary objective of the system and its regulator is to avoid any firm to be bankrupt. In that purpose, the regulator specifies transfers from debtors to their creditors; transfers can be lower than the liabilities -allowing internal default- or larger, meaning that a debtor bails-in a creditor. Such transfers constitute a *solution* if no firm is bankrupt. The no-bankruptcy of a firm

[1]The model is similar to Eisenberg and Noe (2001) except that here firms may be indebted to entities outside the system.

[2]Bankruptcy costs and external creditors' decisions to trigger bankruptcy are not considered.

can be stated in terms of its worth, which is composed of its net external value and the net internal payments specified by the resolution. The former is equal to its external assets minus its nominal external debt (which must be fully reimbursed) and the latter is equal to the received payments from the system minus the required reimbursements to it. A firm is not bankrupt if its worth is non-negative. A problem may admit no solution, in which case bankruptcy is avoided only by injection of external cash (bail-out). The paper focuses instead on the problems admitting solutions. For such problems, solutions are typically numerous so that a resolution rule selects one of them. The cp-rule performs such a selection. It assigns to a problem the cp-solution described as follows.

The cp-solution is bi-proportional to the liabilities, reflecting proportionality in each direction, received payments and made reimbursements. Specifically, the solution is characterized by two indices per firm. Consider firm i. i's *rescue index* adjusts up each of its claim if this is necessary to avoid i's bankruptcy and i's *reimbursement index* specifies the common proportion by which i reimburses each of its adjusted liabilities. The indices are determined so as to ensure that the solution satisfies creditor's priority, 'minimal' rescue and balancedness. Creditors' priority states that a firm with positive worth transfers an amount at least equal to its liability to each creditor. Minimal rescue states that a creditor receives from a debtor a transfer strictly larger then its claim only if it needs to be rescued, in which case it receives the minimum amount to do so and its worth is null. Minimal rescue thus limits the effect of creditors' priority. Finally, the balancedness condition links together the reimbursement and rescue indices of a firm that is both a creditor and a debtor. Its reimbursement index, which determines in which proportion it reimburses its adjusted liabilities, cannot be too low or too high with respect to the scaling up of its claims computed from its rescue index. In the situation where each firm is initially solvent (precisely defined in the text), no default arises and all indices are equal to 1.

Important consequences follow from this description. First, the worth of a firm depends on the health of its debtors, as expected, *and* the health of its creditors: the firm is repaid proportionally less by its debtors in difficulty -those with a low reimbursement index- but also it reimburses proportionally more its creditors in difficulty -those with a large rescue index. In some sense, both lending and borrowing involve engagements. Second, the solution cannot be derived by considering the indebted firms separately because indices are interdependent, reflecting the interdependence in the system caused by cross-liabilities.

The cp-rule is characterized in two ways. First, the cp-solution to a problem minimizes over all solutions a 'distance' to proportional allocations as measured by an entropy index. Second, the cp-rule can be characterized by properties (also called axioms) thought to be desirable. The characterization involves considering problems with increasing complexity, starting with 'simple' problems for which there is a single debtor, then 'bipartite' problems for which firms are either debtors (short) or creditors (long) and finally general liability networks. The consistency principle, according to which 'a part of a fair solution must be fair', links the solutions to these problems. The idea of proportionality is reflected by two axioms, liabilities-invariance and claims-invariance (which are required to bipartite problems only). For example consider two simple problems with a unique debtor. Let the debtor's liabilities be, say, 20 % larger in the second problem than in the first one. If the debtor's worth is null in both problems, then the reimbursements exhaust all its resources in both cases, hence can only differ through a reshuffling. According to liabilities-invariance, such a reshuffling does not make sense because the proportions of the debtor's liabilities are identical in both problems: the solutions are identical. Creditor's priority, minimal rescue and liabilities-invariance characterize the cp-rule in simple problems. Claims-invariance and consistency on the simple reduced problems (which bear on the reimbursements made by each firm given all the others' ones) characterize the cp-rule on bipartite problems. Finally, for general networks, where a firm can be both a debtor and creditor, consistency with bipartite problems and splitting-invariance characterize the cp-rule. Splitting-invariance requires that a solution is not fundamentally changed when a firm is split into two identical firms. This property ensures that reimbursement and rescue indices are balanced.

Related literature. The paper is related to several strands of the literature.

Following O'Neill (1982), a first strand addresses the adjudication of conflicting claims when an 'estate' must be divided among claimants who do not face bankruptcy. The problem arises in a large number of situations, ranging from inheritance, default by a single firm, and tax allocation (Young 1987). The proportional rule is well defined since claimants do not face bankruptcy; other rules, such as the 'Talmud rule' suggested by the Talmud (Aumann and Maschler 1985), the Constrained equal awards rule or Constrained equal losses rule have been defined and characterized by axioms or by considering a bargaining game, as surveyed in Thomson (2003). This paper follows an axiomatic approach in a much more general setting where firms hold claims on each other and face bankruptcy constraints.

In a cross-liabilities setting, Eisenberg and Noe (2001) define a rule when a firm is required to reimburse its creditors in proportion of their claims independently of their health. Proportionality is thus restricted to reimbursements. The rule is well defined when firms' net external values are positive and Csoka and Herings (2018) provides an axiomatization. In practice, proportionality in reimbursements is not satisfied (for example, in 2011, private creditors accepted a 50 percent loss on their Greek bonds). Stutzer (2018) and Schaarsberg, Reijnierse and Borm (2018) study how to extend some of the rules defined in simple claims problems to those with cross-liabilities. Using a consistency axiom, the latter paper shows that extended solutions exist but are not unique, except for a specific structure of liabilities called 'hierarchical'. This is in line with this paper where the same consistency axiom is too weak to pin down the cp-solution.

Operation research on flow sharing problems is also concerned with the allocation of a resource among multiple agents. Flow sharing extends bipartite problems by introducing intermediate nodes between 'sources' and 'sinks'. Resource amounts available at the sources can be sent through the edges in a graph to reach the sinks so as to fulfill their demands. Motivated by transport applications, edges have capacities. The literature mainly studies the existence of flows and the computation of the maximal ones, but a few works consider equitable allocations to the sinks (see Luss (1999) for a review). Flow sharing problems extend adjudication claims, as observed by Bjørndal and Jörnsten (2010), but constraints on allocations may differ. In particular, this paper assumes away hard capacity constraints, apart that transfers are made only from a debtor to a creditor, i.e. through the directed liabilities graph. In Demange (2020), I consider alternative constraints, reflecting laws or practice such as the netting of liabilities.

A few papers take a cooperative approach to resource allocation in a network. Bjørndal and Jörnsten (2010) characterize the nucleolus in flow sharing problems as an alternative solution concept to an equitable allocation to sinks. Jackson (2005) defines a family of allocation rules for network games extending the Shapley and Myerson values to the case where the values for a subgroup of players is not defined by coalitions but by subnetworks and changes in the network structure.

A second strand of the literature evaluates the fairness of an allocation of resources or losses (in the case of taxation) by computing a measure (often called index) meant to reflect a distance to a kind of ideal. Atkison (1970) discusses the use of standard measures and advocate to derive them from a social welfare function. The use of an index applies quite generally, for example for measuring the segregation of students' assignment to schools (Frankel and Volij 2011). There are a variety of measures: the Gini index, the family of Atkinson's indices, the Mutual Information index, and the entropy one used here. Balinski and Demange (1989-a) use entropy to define proportionality in bipartite problems under various constraints and Moulin (2016) in flow sharing problems.

Finally, a third strand of the literature bears on bi-proportional matrices. They appear in various areas: in statistics for adjusting contingencies tables, in economics for balancing international trade accounts (Bacharach 1965), or in voting problems (Balinski and Demange 1989-a). In the latter paper, we consider simultaneously the adjustment of an initial matrix through scale factors on rows and columns and the rounding of the elements so as to approximate proportionality when the solution must assume integer values. This leads us to introduce and characterize a family of *bi-divisor methods*, each one associated to a distinct rounding method and to develop algorithms for finding bi-proportional matrices in both the real and integer cases (Balinski and Demange 1989-b). The New Apportionment Procedure adopted in various cantons in Switzerland, developed by Pukelsheim, is based on a bi-divisor method (see Simeone and Pukelsheim 2007 for a general account). In Demange (2014), I define and axiomatize a ranking method based on bi-proportionality. The method applies in particular to the 'peers' context when peers are ranked according to their judgements over each other, as for example, when journals are ranked according to the citations to each other. In that case, the direction in the network of citations matters and the method defines two indices per journal, one for its relevance for users and another one for its ability to point to relevant journals, akin to the hubs and authorities introduced by Kleinberg (1999) to rank Webpages. Although the method in this paper entirely differs due to the different interpretation of the network, the cp-solution also relies on bi-proportionality and defines two indices per institution that is both borrowing and lending. Finally, algorithms for finding the matrix bi-proportional to another one and meeting constraints on rows' and columns' totals have been studied extensively (see the survey of Censor and Zenios 1997).

Section 2 introduces the model, defines solutions, studies their existence, and characterizes the cp-rule in simple problems. Section 3 defines the cp-solution as minimizing the entropy index and compares it with the Eisenberg and Noe solution. Section 4 provides an axiomatization of the cp-rule in bipartite networks (Section 4.1) and general networks (Section 4.2). Section 5 concludes and Section 6 gathers the poofs.

2 Problems, solutions and rules

Consider a system $N = \{1, \cdots, n\}$ composed of n entities. Entities are for example firms in the same sector, members of a club, or intermediaries in a financial system. Call them firms in the sequel. Firms have claims and liabilities on each other. The analysis takes place at the liquidation date, where nominal values are known. ℓ_{ij} represents i's nominal liability to firm j, equivalently ℓ_{ij} is j's claim on i, for i and j in N. ℓ_{ij} is non-negative and ℓ_{ii} is null. Denote $\boldsymbol{\ell} = (\ell_{ij})_{i,j=1,\cdots,n}$. Firms also hold assets (stocks, loans..) on entities external to the system N, with value a_i, and have liabilities towards them (debts, deposits), with value d_i.

A firm may default on its external or internal liabilities. External liabilities have senior priority and default on them triggers bankruptcy. The primary objective of the system is to avoid any bankruptcy. In that purpose, the regulator can choose freely transfers from debtors to creditors within N, possibly involving defaults within the system.

Formally, let the *liabilities graph* $\mathcal{G}$ be the graph with node set N and edges (i,j) if $\ell_{ij} > 0$. An *allocation* is represented by $\boldsymbol{b}_\mathcal{G} = (b_{ij})_{(i,j)\in\mathcal{G}}$ where b_{ij} is non-negative. b_{ij} is referred to as i's reimbursement to j or j's payment by i; i defaults on its debt to j if $b_{ij} < \ell_{ij}$ and i bails-in j if $b_{ij} > \ell_{ij}$. It is sometimes convenient to extend $\boldsymbol{b}_\mathcal{G}$ into a (n,n)-matrix denoted $\boldsymbol{b}$ by setting the values b_{ij} to zero for (i,j) not in $\mathcal{G}$. In particular, we can write[3] $b_{Ni} = \sum_{j\in N} b_{ji}$ as the sum of the payments to i and $b_{iN} = \sum_{j\in Nj} b_{ij}$ as the sum of the reimbursements made by i.

Consider now default on external creditors. i is *bankrupt* at $\boldsymbol{b}$ if $a_i + b_{Ni} - b_{iN} < d_i$. This inequality says that the amount of resources available to i for reimbursing external creditors, the left hand side, is lower than i's external debt, the right hand side: i defaults on its external creditors. Default arises because i has no access to further resources or cannot be forced to add further funds due to limited liability. Defining *i's worth* by $W_i = a_i - d_i + b_{Ni} - b_{iN}$, i is bankrupt if its worth is negative. i's worth depends on i's external assets and liabilities only through their difference: $z_i = a_i - d_i$, called i's *net external value*. A *problem* is thus summarized by $\boldsymbol{\pi} = (\boldsymbol{z}, \boldsymbol{\ell})$ and solutions to $\boldsymbol{\pi}$ are defined as follows:

Definition 1 *A* **solution** *to problem $\boldsymbol{\pi} = (\boldsymbol{z}, \boldsymbol{\ell})$ is an allocation $\boldsymbol{b}_\mathcal{G}$ under which no firm is bankrupt:*

$$W_i = z_i + b_{Ni} - b_{iN} \geq 0 \text{ for each } i. \quad (1)$$

Firm i is said to be **solvent** *if $z_i + \ell_{Ni} - \ell_{iN} \geq 0$. The* **exact** *allocation $\boldsymbol{b} = \boldsymbol{\ell}$ is a solution if each firm is solvent.*

A solution has to be thought as a simultaneous determination of all reimbursements. Indeed, even if all firms are solvent and the exact allocation is a solution, bankruptcy can occur due to mis-coordination: a solvent firm for which $z_i - \ell_{iN} < 0$ can be made bankrupt if its internal creditors do not reimburse their loans. Requiring a firm with positive worth to repay at least its liabilities and receive at most its claims, as follows from creditor's priority and minimal rescue defined below, prevent such type of mis-coordination when all firms are solvent.[4] If some firms are insolvent, avoiding their bankruptcy implies losses to some firms in the system: insolvent i is not bankrupt if its net reimbursements, $b_{iN} - b_{Ni}$, are lower

[3] I use throughout the following notation: given matrix $\boldsymbol{x} = (x_{ij})$ and two subsets A and B of the rows and columns' indices $x_{A,B} = \sum_{i\in A, j\in B} x_{ij}$, simplified into $x_{iB} = \sum_{j\in B} x_{ij}$ for A singleton. $\boldsymbol{x}_{A\times B}$ denotes the restriction of the matrix to $A \times B$. For a vector $\boldsymbol{x} = (x_i)$, $x_A = \sum_{i\in A} x_i$ and $\boldsymbol{x}_A$ are similarly defined.

[4] Letting $A = \{i, W_i > 0\}$, the requirements imply: For each i in A, $b_{iN} \geq \ell_{iN}$ and $b_{Ni} \leq \ell_{Ni}$. Summing

than its net liabilities $\ell_{iN} - \ell_{Ni}$ (because $\ell_{iN} - \ell_{Ni} > z_i$ and $z_i \geq b_{iN} - b_{Ni}$); as a result, some firms surely receive less than their net claims, which may induce them to reimburse less than expected to avoid bankruptcy even if they are solvent. The process might propagate and result in defaults on external liabilities that could be avoided at a solution, provided one exists.

Existence of solutions The two next lemma characterize problems for which a solution exists (Lemma 1) or a positive solution, with positive transfers on $\mathcal{G}$ (Lemma 2). The characterizations depend on the liability structure. Given a subset A of N, let $D(A)$ be the set of its debtors and $C(A)$ the set of its creditors:

$$D(A) = \{i \in N \text{ s.t. } \ell_{ij} > 0 \text{ for some } j \in A\}, C(A) = \{j \in N \text{ s.t. } \ell_{ij} > 0 \text{ for some } i \in A\}.$$

$D(A)$ and $C(A)$ are alternatively defined as the sets of direct predecessors to A and direct successors of A in the liabilities graph $\mathcal{G}$.

Lemma 1 *Problem $\boldsymbol{\pi} = (\boldsymbol{z}, \boldsymbol{\ell})$ admits a solution if and only if*

$$\textit{for each subset } A \textit{ of } N \textit{ such that } D(A) \subset A : z_A \geq 0. \tag{2}$$

The conditions (2) are surely satisfied if all external values are non-negative: this is not surprising since $\boldsymbol{b} = \boldsymbol{0}$ is a solution in that case. Applying (2) to $A = N$ a necessary condition for a solution to exist is $z_N \geq 0$, which says that the system as a whole must not be indebted to outsiders. The condition is easy to explain by the conservation of aggregate worth within the system: W_N is equal to z_N at any allocation because the transfers within N cancel out. Hence the non-negativity of worth levels, $W_i \geq 0$ for each i, requires $z_N \geq 0$. A similar argument explains why (2) is necessary for a set A such that all debtors are in A: if $D(A) \subset A$, A receives payments from A only so that the net transfers from A are equal to $b_{A,N-A}$ and $W_A = z_A - b_{A,N-A}$.[5] Worth levels in A can be non-negative only if $z_A \geq 0$. The conditions (2) are shown to be sufficient by considering the graph composed of $\mathcal{G}$ and an additional node sending z_i and receiving i's worth for each i. The existence of a solution is equivalent to the existence of a circulation in that graph (with well defined capacities).

over A implies: $b_{A,N} - b_{N,A} \geq \ell_{A,N} - \ell_{N,A}$. By contradiction, let A be not the full set N: $N - A = \{i, W_i = 0\}$. Using the identities $b_{N,N-A} - b_{N-A,N} = b_{A,N} - b_{N,A}$ and $\ell_{A,N} - \ell_{N,A} = \ell_{N,N-A} - \ell_{N-A,N}$, we derive $b_{N,N-A} - b_{N-A,N} \geq \ell_{N,N-A} - \ell_{N-A,N}$ hence $W_{N-A} \geq z_{N-A} + \ell_{N,N-A} - \ell_{N-A,N}$. If all firms are 'strictly' solvent, the right hand side is positive, which implies $W_{N-A} > 0$, a contradiction: this proves that $A = N$, hence no firm defaults.

[5]The net transfers from A satisfy the following identity: $b_{A,N} - b_{N,A} = b_{A,N-A} - b_{N-A,A}$. The nullity of $b_{N-A,A}$ implies they are equal to $b_{A,N-A}$ hence $W_A = z_A - b_{A,N-A}$.

The characterization provided by Hoffman's Theorem proves the sufficiency of conditions (2).[6]

Consider now the case where an inequality in (2) holds as an equality: $z_A = 0$ for a subset A such that $D(A) \subset A$. From the above computation, if $A = N$, then $W_N = 0$: all worth levels are null at any solution; if A is a strict subset of N, then $W_A = -b_{A,N-A}$ so that both terms are null at a solution: firms in A have null worth levels and there are no transfers from A to $N - A$. If no firm in A is indebted to $N - A$, then all debtors and creditors of A belong to A, $D(A) \subset A$ and $C(A) \subset A$: $\mathcal{G}$ is said to be *decomposable* so that the allocations can be studied separately on A and $N - A$. If a firm in A is indebted to $N - A$, then no solution can be positive since reimbursements from A to $N - A$ are null. The reverse is true and Lemme 2 follows:

Lemma 2 *Problem $\boldsymbol{\pi} = (\boldsymbol{z}, \boldsymbol{\ell})$ is said to be strictly feasible if each of the inequality in (2) is satisfied strictly.*

(i) Strictly feasible $\boldsymbol{\pi}$ admits a positive solution, i.e. $\boldsymbol{b}_{\mathcal{G}}$ whose elements are all strictly positive, and the set of positive solutions has a non-empty interior.

(ii) Conversely, if $\mathcal{G}$ is non-decomposable, then $\boldsymbol{\pi}$ admits a positive solution only if it is strictly feasible.

(iii) If $\boldsymbol{\pi}$ is strictly feasible, at any solution, at least one firm has a positive worth.

(i) and (ii) follow from the comments following Lemma 1. (iii) follows from the conservation of aggregate worth, $W_N = z_N$, and the fact that $z_N > 0$ at a strictly feasible problem.

From now on, the paper restricts to strictly feasible problems, and, without loss of generality, exclude those with a decomposable graph.

Definition 2 *Let $\mathcal{F}$ denote the set of strictly feasible problems with a non-decomposable graph. A rule F is a continuous function defined on $\mathcal{F}$, or on a subset of $\mathcal{F}$, that assigns to each problem in $\mathcal{F}$ a positive solution.*

We consider two subsets of $\mathcal{F}$: those constituted of simple (claims) problems where there is a single debtor, as studied in the next paragraph, and those constituted of bipartite problems, where each firm is either a creditor or a debtor but not both, studied in Section 4.1.

[6]Observe that conditions (2) depend on the external values $\boldsymbol{z}$ and the liabilities graph but not on the values assumed by liabilities. Such independence is due to the fact that reimbursements are not bounded by liabilities and payments by claims. In Demange (2020), I study the case where allocations must satisfy such bounds as well as some other constraints.

Simple (claims) problems. In a simple problem, a single firm, say 1, is indebted. Denote by T the set of its creditors and $(\ell_j)_{j\in T}$ their claims on 1. A positive solution exists if (and only if) $z_1 + \sum_{\{j,z_j<0\}} z_j > 0$, which says that 1 can cover the outside debts. The adjudication of claims introduced by O'Neill (1982) is a well studied simple problem where creditors' external values are non-negative and their total claims exceed the debtor's resources.[7] In that case, the proportional solution simply requires the debtor to use all its resources to repay the same amount per unit of claim.

When the external values of some creditors are negative, the proportional solution can make them bankrupt. In that case, the constrained-proportional solution distorts the proportional one in a minimal way to meet the bankruptcy constraints. Formally, *the cp-solution* to a strictly feasible simple problem $\boldsymbol{\pi} = (\boldsymbol{z}, (\ell_j)_{j\in T})$ is a solution $(b_j)_{j\in T}$ for which there are δ and $(\mu)_{j\in T}$ with $0 \leq \delta \leq 1$ and $\mu_j \geq 1$ for each $j \in T$ such that

$$\delta = 1 \text{ if } W_1 > 0 \text{ and for each } j \in T : b_j = \delta \ell_j \mu_j \text{ with } \mu_j = 1 \text{ if } W_j > 0. \tag{3}$$

According to (3), creditors receive the proportion δ of their claims, except those needing more to avoid bankruptcy. Call μ_j j's *rescue index.* The existence and uniqueness of the cp-solution is easily proved (see the proof of Proposition 1). The cp-rule on the set $\mathcal{S}$ of strictly feasible simple problems assigns to each problem its cp-solution. Although the cp-rule almost needs no justification, the next proposition provides an axiomatization to illustrate the approach. Observe that the cp-rule satisfies the following properties.

Creditors' priority. *$W_1 > 0$ implies $b_j \geq \ell_j$ for each j in T.*

The debtor reimburses at least its liabilities to its creditors if its worth is positive. This is a strong form of creditors' priority, which makes sense when a liability represents an engagement to the creditor. If reimbursements were restricted to be bounded by liabilities, a weaker form would obtain requiring a firm with positive net worth to reimburse exactly each of its liabilities.

Minimal rescue. *$b_j > \ell_j$ implies $W_j = 0$.*

Minimal rescue limits the effect of creditors' priority: a creditor can receive strictly more than its claim only to avoid its bankruptcy, in which case it receives the minimum amount to do so. Minimal rescue has two consequences on solutions. First, the reimbursements to

[7] The subsequent literature does not consider minimal bounds on the payments (i.e. implicitly assumes $z_j \geq 0$) except Balinski and Young (1982). Their problem is to allocate a total number of seats T in a parliament to districts, given the population numbers $(c_1, \cdots, c_p)$ in the districts and a minimum m_j for each j. Since the seats are not divisible, the allocation must be integer-valued. Considering the cp-solution to be the ideal one, Balinski and Young study how to transform it into integers.

the insolvent firms just cover their loss: $b_j = -z_j$ for insolvent j because $z_j + \ell_j < 0$ implies that j receives more than its claim to avoid bankruptcy, $b_j > \ell_j$, so that minimal rescue requires $W_j = z_j + b_j$ to be null. Second, the reimbursements to the solvent firms are at most equal to their claims: $b_j \leq \ell_j$ for solvent j because $z_j + \ell_j \geq 0$ and $b_j > \ell_j$ would imply $W_j > 0$, in contradiction with minimal rescue.

The next property reflects a very weak form of proportionality.

Liabilities-invariance. *F is* **liabilities-invariant** *on $\mathcal{S}$ if the following holds: consider two problems that differ by the scale of the liabilities: $\boldsymbol{\pi} = (\boldsymbol{z}, (\ell_j)_{j\in T})$ and $\boldsymbol{\pi}' = (\boldsymbol{z}, (\lambda\ell_j)_{j\in T})$ for some $\lambda > 0$. Let $\boldsymbol{b} = F(\boldsymbol{\pi})$ and $\boldsymbol{b}' = F(\boldsymbol{\pi}')$ be the solutions assigned by F to these problems. If 1's worth is null at both $\boldsymbol{b}$ and $\boldsymbol{b}'$ then $\boldsymbol{b}' = \boldsymbol{b}$.*

The justification is the following one. 1's worth is null when it repays a total equal to its resources. If 1's worth is null at both $\boldsymbol{b}$ and $\boldsymbol{b}'$, the payments received by the creditors add up to the same value, z_1, in both problems so that they can only differ through a reshuffling. Such a reshuffling does not make sense when the proportion of their claims are identical: this is the requirement of liabilities-invariance.

It is easy to check that the three above properties are independent. The next proposition states that the cp-rule is the unique rule that satisfies the three above properties.

Proposition 1 *The cp-rule on $\mathcal{S}$ is the unique rule that is liabilities-invariant and assigns solutions satisfying creditor's priority and minimal rescue.*

3 Constrained-proportional solutions

Defining cp-solutions in general problems is less straightforward than in simple problems and several definitions can be contemplated. Here, cp-solutions are built by considering their distance to the exact allocation measured by the entropy index defined by

$$f(\boldsymbol{b}_{\mathcal{G}}) = \sum_{(i,j)\in\mathcal{G}} b_{ij}\left[log\left(\frac{b_{ij}}{\ell_{ij}}\right) - 1\right]. \tag{4}$$

Following the optimization approach, given a problem $\boldsymbol{\pi}$ in $\mathcal{F}$, one searches for the solutions minimizing the entropy objective:

$$\mathcal{P}: \text{ minimize } f(\boldsymbol{b}_{\mathcal{G}}) \text{ over the solutions } \boldsymbol{b}_{\mathcal{G}} \text{ of } \boldsymbol{\pi}.$$

Observe first that the exact allocation solves $\mathcal{P}$ if all firms are solvent: $\boldsymbol{b} = \boldsymbol{\ell}$ is a solution of $\boldsymbol{\pi}$ and reaches the global minimum of f (because the global minimum of $b[log(\frac{b}{\ell}) - 1]$ is reached at $b = \ell$). The objective f measures the deviation to proportionality in both directions of reimbursements and payments. To see this, consider first a simple problem (Section 2). f writes $\sum_{j\in T} b_j[log(\frac{b_j}{\ell_j}) - 1]$. When all external values are non-negative, bankruptcy constraints never bind; entropy is minimized at the exact allocation if possible, i.e. if $z_1 \geq \ell_T$, or at the proportional one allocating z_1 if $z_1 < \ell_T$. With multiple indebted firms, write $f(\boldsymbol{b}_{\mathcal{G}})$ as the sum over i of $\sum_{j,(i,j)\in\mathcal{G}} b_{ij}[log(\frac{b_{ij}}{\ell_{ij}}) - 1]$. f thus aggregates the measures of how much each debtor's reimbursements depart from proportionality. Solving $\mathcal{P}$ however does not reduce to minimizing each of these measures separately: first, the resources available to indebted i for reimbursing its creditors are not known in advance if i is also a creditor, as they depend on the payments i receive, second the minimal amount that i must give to creditor j depends on what j receives from its other debtors though j's bankruptcy constraint. Simply put, reimbursements and payments cannot be determined by considering a collection of simple problems. Similarly, exchanging summation order, the entropy objective measures inequality in how much a firm is repaid per unit of claim by its debtors: writing $f(\boldsymbol{b}_{\mathcal{G}})$ as the sum over all i of $\sum_{j,(j,i)\in\mathcal{G}} b_{ji}[log(\frac{b_{ji}}{\ell_{ji}}) - 1]$, $f(\boldsymbol{b})$ reflects that the ideal payments to i from its borrowers are proportional to i's claims. The solution to $\mathcal{P}$ is unique and easily characterized by the first order conditions on the Lagrangian.

Proposition 2 *Consider $\boldsymbol{\pi}$ in $\mathcal{F}$, $D(N) = \{i, \ell_{iN} > 0\}$ the set of debtors and $C(N) = \{i, \ell_{Ni} > 0\}$ the set of creditors. By the non-decomposability of $\mathcal{G}$, $N = D(N) \cup C(N)$.*

The cp-solution to $\boldsymbol{\pi}$ is the unique solution $\boldsymbol{b}_{\mathcal{G}}$ for which there are positive scalars $(\delta_i)_{i\in D(N)}$ and $(\mu_i)_{i\in C(N)}$ that satisfy

$$\text{for each } (i,j)\in\mathcal{G}: \quad b_{ij} = \delta_i \ell_{ij} \mu_j \quad \text{(bi-proportionality)} \tag{5}$$

$$\text{for each } j\in C(N): \quad \mu_j \geq 1 \text{ with } \mu_j = 1 \text{ if } W_j > 0 \quad \text{(rescue conditions)} \tag{6}$$

$$\text{for each } i\in D(N): \quad \delta_i \leq 1 \text{ with } \delta_i = 1 \text{ if } W_i > 0 \quad \text{(creditors' priority)} \tag{7}$$

$$\text{for each } i\in D(N)\cap C(N): \quad \delta_i \mu_i = 1 \quad \text{(balancedness)} \tag{8}$$

$(\delta_i)_{i\in D(N)}$ and $(\mu_i)_{i\in C(N)}$ are said to support $\boldsymbol{b}_{\mathcal{G}}$. They are unique. Call δ_i the reimbursement index of debtor i and μ_j the rescue index of creditor j. The cp-rule assigns to each problem in $\mathcal{F}$ its cp-solution.

As expected, if all firms are solvent, the exact allocation is the cp-solution: $\boldsymbol{b} = \boldsymbol{\ell}$ satisfies all conditions supported by indices all equal to 1. Let us interpret the conditions on a cp-solution. According to (5), each claim of creditor j is scaled by the rescue index μ_j and each debtor i reimburses the same amount δ_i per unit of its scaled liabilities. An alternative formulation is that the matrix $\boldsymbol{b}$ derived from $\boldsymbol{b}_{\mathcal{G}}$ is bi-proportional to $\boldsymbol{\ell}$: set indices δ_i for i not in $D(N)$ and μ_j for j not in $C(N)$ to 1 (other values can work for firms with null worth);

since the elements of $\boldsymbol{b}$ for (i,j) not in $\mathcal{G}$ are null as those of $\boldsymbol{\ell}$, $\boldsymbol{b}$ is obtained from $\boldsymbol{\ell}$ by multiplying i's row by δ_i and j's column by μ_j. According to (6), a firm with a rescue index equal to 1, in particular a firm with positive net worth, receives from each of its debtors their minimum reimbursement per unit, δ_i for i, whereas a firm with a rescue index strictly larger than 1 would be bankrupt if it received those minima. Minimal rescue is thus satisfied: only firms with a null worth can receive a payment exceeding the corresponding claim (because the δ_is are less than or equal to 1). Combining (6) and (7), a firm with reimbursement index δ_i equal to 1, in particular a firm with positive net worth, transfers an amount at least equal to its liability to each creditor: creditors' priority is satisfied. The balancedness condition (8) links together the reimbursement and rescue indices of a firm that is both a creditor and a debtor. For a firm with positive worth, balancedness is automatically satisfied since its indices are both equal to 1. For a firm with null worth, its reimbursement index, which determines in which proportion it reimburses its adjusted liabilities, cannot be too low or too high with respect to the scaling up of its claims computed from its rescue index.

Observe that a reimbursement index can be equal to 1 only for a solvent debtor: $\delta_i = 1$ implies that i's reimbursements satisfy $b_{iN} \geq \ell_{iN}$ and if i is also a debtor $b_{Ni} \leq \ell_{Ni}$ (since $\mu_i = 1$ by balancedness (8). Therefore $z_i + \ell_{Ni} - \ell_{iN} \geq W_i \geq 0$. Similarly only solvent creditors can have a rescue index equal to 1. The converse is not true: a solvent firm may need to be rescued because of failing debtors or creditors in need to be rescued (as in the next simple example).

Clearing vectors Eisenberg and Noe (2001) It is instructive to compare the cp-solutions with the clearing vectors defined by Eisenberg and Noe (2001) for problems with positive external values. The main difference is that clearing vectors require strict proportionality in the reimbursements without accounting for creditors' health. Specifically, let each firm reimburse the same fraction of its claims to each of its creditors, τ_i for i with $0 \leq \tau_i \leq 1$. The vector $(\tau_i)_{i=1,\cdots,n}$ is said to be clearing if no firm is bankrupt and creditors' priority is satisfied:

$$\text{for each } i: \ \tau_i \ell_{iN} - \sum_j \tau_j \ell_{ji} \leq z_i \text{ with } \tau_i = 1 \text{ if the inequality is strict.}$$

Assuming $z_i > 0$ for each i, a clearing vector exists and is unique, which defines a rule by setting $b_{ij} = \tau_i \ell_{ij}$ where $(\tau_i)_{i=1,\cdots,n}$ is the clearing vector for $\boldsymbol{\pi}$.

For problems with positive external values, both the Eisenberg-Noe and the cp-solution exist. To compare them, observe that they coincide if (and only if) creditors' rescue indices are all equal to 1. For a firm that is both a creditor and debtor, this holds only if the firm is solvent. In a complete network, the condition is very strong as it implies that all firms are solvent, in which case both solutions coincide with the exact one. In a bipartite problem (see next section), the condition is less strong because rescue indices are defined only for the long firms. As they are surely equal to 1 for a long firm with non-negative external value, the Eisenberg-Noe and the cp-solution coincide in bipartite problems when all external values are positive.

For problems with negative external values, a cp-solution may exist whereas a clearing vector does not, meaning that reimbursements' proportionality leads to bankruptcy. This is illustrated in the following simple claims problem with two debtors: $z_1 = 1$, $z_2 = -0.7$, $z_3 = 2$ and $\ell_2 = \ell_3 = 1$. Solutions exist since $z_1 + z_2 > 0$. Creditor's priority implies that 1 exhausts its resources since z_1 is less than its liabilities' total. At a clearing vector, if any, 1 reimburses the same amount per unit of liability, which implies $\tau_1 = 0.5$, hence 2 and 3 receive 0.5 and 2's net worth is -0.2: 2, who is solvent, is bankrupt by 'contagion'. At the cp-solution, bankruptcy is avoided as firm 2 is rescued: $b_2 = 0.7$ and $b_3 = 0.3$, supported by $\delta = 0.3$, $\mu_2 = 7/3$ and $\mu_3 = 1$. The rescue is supported indirectly by firm 3, which receives less than half of z_1.

4 Axiomatization

Let us start by characterizing problems with bipartite graphs and then proceed with general ones.

4.1 Bipartite problems

In a bipartite problem, a firm is either a debtor (short) or a creditor (long) but not both: $D(N)$ and $C(N)$ are disjoint; denote them respectively by S and T. The graph $\mathcal{G}$ is bipartite with links from S to T. A simple claims problem is a bipartite one with a unique debtor. Denote a bipartite problem by $(\boldsymbol{z}, \boldsymbol{\ell}_{S\times T})$ where $\boldsymbol{\ell}_{S\times T}$ describes the liabilities from S to T. Distinguishing the short firms from the long ones, the problem is strictly feasible if

$$\text{for each } i \in S: \quad z_i > 0 \tag{9}$$

$$\text{for each } J \subset T \text{ s.t. } z_j < 0 \text{ for each } j \in T: \quad \textstyle\sum_{j\in J} z_j + \sum_{i\in D(J)} z_i > 0. \tag{10}$$

These conditions are obtained by applying (2) respectively to short firms ($D(A)$ is empty for $A \subset S$) or to $A = J \cup D(J)$ composed of a subset J with negative external values and its debtors. One may restrict to subsets A of this form since adding to A a long firm with non-negative z_j or a short firm (for which $z_j > 0$ by (9)) can only weaken the condition.

(10) says that the debtors inside the system to a set of long firms can cover their external debt.

Let $\mathcal{B}$ denote the subset of $\mathcal{F}$ composed of bipartite problems. The cp-rule satisfies the following properties on $\mathcal{B}$.

Claims-invariance. Consider two problems that differ only by the scale of the claims of a long firm. Claims-invariance requires that if that firm has a null worth at the solutions assigned by F to both problems, then the solutions are identical. Formally:

F is **claims-invariant** *on $\mathcal{B}$ if the following holds: Let two problems in $\mathcal{B}$ that differ only by the scale $\lambda > 0$ of the claims of a firm j in T: $\boldsymbol{\pi} = (\boldsymbol{z}, \boldsymbol{\ell}_{S\times T})$ and $\boldsymbol{\pi}' = (\boldsymbol{z}, \boldsymbol{\ell}'_{S\times T})$ where for each i in S: $\ell'_{ij} = \lambda \ell_{ij}$ and $\ell'_{ik} = \ell_{ik}$ $k \neq j$. Let $\boldsymbol{b} = F(\boldsymbol{\pi})$ and $\boldsymbol{b}' = F(\boldsymbol{\pi}')$ the solutions assigned by F. If j's worth is null at both $\boldsymbol{b}$ and $\boldsymbol{b}'$, then $\boldsymbol{b} = \boldsymbol{b}'$.*

The justification is the following one: j's net worth is null if j receives total payments from S equal to $-z_j$. Hence, under scaling, as long as j's worth stays null, j's payments can only change through a reshuffling. Such a reshuffling does not make sense when the proportion of j's claims does not change. Since the solution for j is not changed, there is no reason to change the solution for firms other than j: this is the requirement of claims-invariance.[8]

1-consistency. 1-consistency applies the consistency principle to a solution $\boldsymbol{b}$ to evaluate the proportionality of the reimbursements made by each single firm. For that, imagine the reimbursements made by all short firms other than i fixed to those recommended by $\boldsymbol{b}$. This leaves a simple claims problem where i is the unique debtor with external value z_i and liabilities ℓ_{ij} towards j in T and j's external value is equal to $z_j + b_{S-i,j}$, composed of j's original external value plus the payments j receives from the short firms other than i. Denote this reduced problem by $(\boldsymbol{z}, \boldsymbol{\ell}_{\{i\}\times T})_{|\boldsymbol{b}}$. 1-consistency requires i's reimbursements at $\boldsymbol{b}$ to be allocated according to the cp-rule in this simple problem.

Let $\boldsymbol{b}$ a solution to $\boldsymbol{\pi}$ in $\mathcal{B}$. $\boldsymbol{b}$ is **1-consistent** *if for each i in S, i's reimbursements, $(b_{ij})_{j\in T}$, coincide with the cp-solution to the reduced simple claims problem $(\boldsymbol{z}, \boldsymbol{\ell}_{\{i\}\times T})_{|\boldsymbol{b}}$.*

It is easy to check that the cp-rule on $\mathcal{B}$ is claims-invariant and assigns 1-consistent solutions. The next proposition states the converse:

[8] Claims-invariance is similar to liabilities-invariance defined for simple problems in Section 2 but differ on two aspects. First it considers scaling claims instead of liabilities; second invariance is extended to all firms: the fact that the payments to firm j are unchanged extends to other long firms; this is a very natural requirement since their claims are unchanged.

Proposition 3 *The cp-rule is the unique rule on $\mathcal{B}$ that is claims-invariant and assigns 1-consistent solutions.*

Applying 1-consistency to each short firm provides its reimbursement index and a rescue index for each long firm.[9] The rescue indices may however vary across the short firms, as illustrated in the next example; claims-invariance ensures that they are equalized.

Example Consider a bipartite problem with 6 firms where 1 and 2 are short and 3, 4, 5, 6 are long. Let $z_1 = z_2 = 3$, $z_3 = z_4 = -1.7$, $z_5 = z_6 > 0$, $\ell_{13} = \ell_{14} = 1$, $\ell_{15} = 2$, $\ell_{16} = 0$ and $\ell_{23} = \ell_{24} = 1, \ell_{25} = 0, \ell_{26} = 2$. The cp-solution[10] is $b_{13} = b_{14} = \frac{1.7}{2}, b_{15} = 1.3$ and $b_{23} = b_{24} = \frac{1.7}{2}, b_{26} = 1.3$ supported by $\delta_1 = \delta_2 = \frac{1.3}{2}$, $\mu_3 = \mu_4 = \frac{1.7}{1.3}$, $\mu_5 = \mu_6 = 1$.

There are other solutions satisfying 1-consistency. Let for example $\boldsymbol{b}_{\mathcal{G}}$: $b_{13} = b_{14} = 0.9, b_{15} = 1.2$ and $b_{23} = b_{24} = 0.8, b_{26} = 1.4$. Consider 1's reimbursements first. Given the amount 0.8 reimbursed by 2, firms 3 and 4 must each receive at least 0.9 from 1 to avoid bankruptcy; it follows that 1's reimbursements constitute the cp-solution to the reduced problem supported by $\delta_1 = 0.6$ and rescue indices $\mu_3^1 = \mu_4^1 = \frac{3}{2}$. Similarly, given the amount 0.9 reimbursed by 1, 2's reimbursements constitute the cp-solution supported by $\delta_2 = 0.7$ and rescue indices $\mu_3^2 = \mu_4^2 = \frac{8}{7}$. This shows that $\boldsymbol{b}_{\mathcal{G}}$ is 1-consistent, but the reimbursements by 1 and 2 are supported by different rescue indices. Due to the fact that the constraints on 1's reimbursements depend on 2's and vice et versa, a range of 1-consistent solutions can be build similarly.

4.2 General network

The characterization of the cp-rule relies on two properties. The first one is the consistency property applied to sub-problems that are bipartite, and the second one considers the effect of splitting a firm into two identical firms.

Consistency on bipartite problems. Let S and T two disjoint sets of N and $\boldsymbol{b}$ an allocation. The reduced problem on $S \times T$ is the bipartite problem obtained when the allocation outside $S \times T$ is fixed to $\boldsymbol{b}$. Denote it by $(\boldsymbol{z}, \boldsymbol{\ell})_{S\times T,\boldsymbol{b}}$. S are the short firms, T the

[9] There is a caveat here: the reduced problem associated to a short firm is not strictly feasible if its worth as well as those of its creditors are null; in that case, the indices are not well defined. The proof relies on a perturbation argument.

[10] Surely $\mu_5 = \mu_6 = 1$ so that reimbursements made by $i = 1, 2$ satisfy: $b_{i3} = \delta_i\mu_3$, $b_{i4} = \delta_i\mu_4$, $b_{i5} = 2\delta_i$ and $b_{i6} = 2\delta_i$ with a total $\delta_i(\mu_3 + \mu_4 + 2)$ summing to 3. This implies $\delta_1 = \delta_2 = \delta$, hence $b_{13} = b_{23} = \delta\mu_3$ and $b_{14} = b_{24} = \delta\mu_4$. Since W_3 and W_4 are null, we must have $-1.7 + 2\delta\mu_3 = 0$ and $-1.7 + 2\delta\mu_4 = 0$. We thus obtain $\mu_3 = \mu_4 = \mu$ and $\delta\mu = 1.7/2$. 1 and 2's total reimbursements are thus given by $\delta(2\mu + 2) = 1.7 + 2\delta$; since they are equal to 3 we obtain $\delta = \frac{1.3}{2}$ and the cp- solution.

long ones, the liabilities from S to T are unchanged and the external values are:

$$\hat{z}_i \quad = z_i + b_{Ni} - \sum_{k \notin T} b_{ik} \quad i \in S \tag{11}$$

$$\hat{z}_j \quad = z_j - b_{jN} + \sum_{k \notin S} b_{kj} \quad j \in T \tag{12}$$

(11) uses that i in S receives b_{ki} from each other firm and pays b_{ik} to the k not in T, and (12) that j in T reimburses b_{jk} to each other firm and receives b_{kj} from the k not in S. By construction, $\boldsymbol{b}_{S\times T}$ is a positive solution to the reduced problem, and, furthermore, any other solution leads to a solution to the original problem. Consistency requires $\boldsymbol{b}_{S\times T}$ to be the cp-solution to the reduced problem.

Let $\boldsymbol{b}$ *be a solution to* $(\boldsymbol{z}, \boldsymbol{\ell})$. $\boldsymbol{b}$ *is* **bipartite-consistent** *if, for each pair of disjoint subsets* S *and* T *of* N*, the reimbursements from* S *to* T*,* $\boldsymbol{b}_{S\times T}$*, coincide with the cp-solution to the bipartite reduced problem* $(\boldsymbol{z}, \boldsymbol{\ell})_{S\times T, \boldsymbol{b}}$.

A bipartite-consistent solution is 1-consistent: since a firm is not indebted to itself, it suffices to choose $S = \{i\}$ and $T = N - \{i\}$ for each indebted i.

Splitting-invariance. Splitting-invariance says that splitting a firm into two half firms does not affect the solution in a fundamental way. Specifically, firm i is said to be split if it is replaced by two identical firms, i' and i'', whose external values, liabilities and claims to other firms are equal to half those of i and liabilities between them are equal:

$$\text{for } k = i', i'', z_k = \frac{z_i}{2} \text{ and } \ell_{kj} = \frac{\ell_{ij}}{2}, \ \ell_{jk} = \frac{\ell_{ji}}{2} \text{ for each } j \neq i$$

$$\ell_{i'i''} = \ell_{i''i}.$$

Let F *be defined on* $\mathcal{F}$. F *is* **splitting-invariant** *if the following holds for each problem* $\boldsymbol{\pi}$ *in* $\mathcal{F}$ *and* i *in* N. *Let* $\boldsymbol{\pi}'$ *be the problem that results from the splitting of* i. *Let* $\boldsymbol{b} = F(\boldsymbol{\pi})$ *and* $\boldsymbol{b}' = F(\boldsymbol{\pi}')$. $\boldsymbol{b}'$ *is derived from* $\boldsymbol{b}$ *as follows:*

$$\boldsymbol{b}'_{N-\{i\}\times N-\{i\}} = \boldsymbol{b}_{N-\{i\}\times N-\{i\}}$$

$$\text{for each } k \neq i : b'_{i'k} = b'_{i''k} = \tfrac{b_{ik}}{2} \text{ and } b'_{ki'} = b'_{ki''} = \tfrac{b_{ki}}{2}$$

$$b'_{i'i''} = b'_{i''i'} = \ell_{i'i''}$$

These equations say that the transfers $\boldsymbol{b}'$ within $N - \{i\}$ are equal to those of $\boldsymbol{b}$, the reimbursements from i' and i'' to $N - \{i\}$ are equal to half those from i in $\boldsymbol{b}$ and the reimbursements from $N - \{i\}$ to them are half those to i in $\boldsymbol{b}$, and finally, the reimbursements between i' and i'' are equal to their common liability; since any equal transfer between them does not affect the worth of any firm, there is no reason to deviate from exact reimbursements.

Proposition 4 *The cp-rule is the unique rule on $\mathcal{F}$ that is splitting-invariant and assigns bipartite-consistent solutions.*

Bipartite-consistency alone is not enough to guarantee the bi-proportionality of matrix $\boldsymbol{b}$ to $\boldsymbol{\ell}$. Consider for example two firms indebted to all the other ones, and these others not to be indebted. Let non-null liabilities be equal to 1. A solution $\boldsymbol{b}$ is characterized by the transfers from 1 and 2 to the other firms. Consider

$$\boldsymbol{b} = \begin{pmatrix} 0 & \mu_2 & \mu_3 & \cdots & \mu_4 & \cdots . \\ \delta & 0 & \delta c \mu_3 & \cdots & \delta c \mu_4 & \cdots \end{pmatrix}$$

where c is positive and choose the values for $\boldsymbol{z}$ such that 1's worth level is positive and all others are null. $\boldsymbol{b}$ is bipartite-consistent: for that it suffices to consider $S = \{1\}$ or $S = \{2\}$ (i.e. check 1-consistency) and $S = \{1, 2\}$. 1-consistency is met for 1 and 2 with rescue indices for $j \geq 3$ equal respectively to μ_j and $c\mu_j$. Consistency on $S = \{1, 2\}$ and $\{3, .., n\}$ is satisfied since the reimbursement of 1 and 2 to T are proportional between each other. $\boldsymbol{b}$ is not bi-proportional to $\boldsymbol{\ell}$ except if $c = 1$.

5 Concluding remarks

The paper proposes the cp-rule for liquidating cross-liabilities in a system. The rule assigns transfers bi-proportional to liabilities, characterized by two indices per firm representing their ability to reimburse and need to be rescued. Various developments are worth investigating. Let us mention a few.

First, although relatively simple, cp-solutions cannot be computed by hand. Standard algorithms iteratively scaling rows and columns compute the matrix that is bi-proportional to another one and has specified rows' and columns' totals. These algorithms need to be adapted to account for two specificities of our problems: first, the network is not bipartite, second the values for the rows and columns totals, which correspond to reimbursements and repayments totals, are not fixed as they depend on the no-bankruptcy constraints. An algorithm adjusting iteratively the reimbursement and rescue indices is easy to define for a bipartite network, and its convergence is likely to follow from similar arguments as in the standard fixed totals' problem. The case of a general network needs to be investigated.

Second, the paper assumes that the system's primary objective is to avoid any bankruptcy. Introducing a bankruptcy cost would allow to analyze situations where the system chooses not to rescue some firms even if their bankruptcy is avoidable (see Rogers and Veraart 2013 for such an approach in the Eisenberg and Noe framework).

Third, the analysis takes as given the values for the internal and external liabilities. Rules presumably impact ex-ante firms' decisions that will determine these values; analyzing how different rules impact firms' decisions is challenging but worthwhile. On this respect, the worth of a firm at the cp-solution is affected by the capacity of its debtors to reimburse their debts and the need to rescue its lenders, thereby providing incentives to screen both borrowers and lenders.

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6 Proofs

Proof of Proposition 1.

Let us first prove the existence and uniqueness of a solution satisfying (3). The condition on the rescue index μ_j is equivalent to $\delta\ell_j\mu_j = \max(\delta\ell_j, -z_j)$. So, considering the function ϕ defined by $\phi(\delta) = \sum_{j\in T}\max(\delta\ell_j, -z_j)$, a cp-solution is associated to δ in $[0, 1]$ that satisfies: either $\delta < 1$ and $\phi(\delta) - z_1 = 0$ or $\delta = 1$ and $\phi(1) \leq z_1$. Consider two cases.

In the first case, all creditors are insolvent or at the borderline: $\ell_j + z_j \leq 0$ for each j in T. ϕ is constant over $[0, 1]$ with value $-\sum_j z_j$. Surely $z_j < 0$ for each j in T, so that the strict feasibility of the problem writes $z_1 + \sum_j z_j > 0$, which implies $\phi(\delta) < z_1$ for each δ: the cp-solution is $b_j = -z_j$, supported by $\delta = 1$. In words, each creditor needs to be rescued and 1 rescues them at the minimum level.

In the second case, there is k in T for which $\ell_k + z_k > 0$. Letting $\underline{\delta}$ be the upper bound to the non-negative δ such that $\delta \ell_j + z_j < 0$ for each j in T, surely $\underline{\delta} < 1$. ϕ is constant over $[0, \underline{\delta}]$, assuming the value $-\sum_{\{j, z_j < 0\}} z_j$, and strictly increasing over $[\underline{\delta}, 1]$. The strict feasibility of the problem implies $\phi(\underline{\delta}) < z_1$. Hence if $\phi(1) \leq z_1$, the cp-solution is supported by $\delta = 1$, described by $b_j = -z_j$ for the insolvent firms and $b_j = \ell_j$ for the solvent ones, and if $\phi(1) > z_1$, there is a unique $\delta < 1$ such that $\phi(\delta) - z_1 = 0$. This proves the existence and uniqueness of a cp-solution.

Let F be liabilities-invariant and assign solutions satisfying creditor's priority and minimal rescue. Let us prove that $\boldsymbol{b} = F(\boldsymbol{\pi})$ is the cp-solution to $\boldsymbol{\pi}$. Let K be the set of solvent creditors: $K = \{k \in T, z_k + \ell_k \geq 0\}$. We know that minimal rescue determines the payments to the insolvent firms: $b_j = -z_j$ and bounds those to the solvent ones: $b_k \leq \ell_k$. Thus letting $\overline{z}_1 = -z_{T-K} + \ell_K$, $\overline{z}_1$ is an upper-bound on 1's total reimbursements. Furthermore, if $W_1 > 0$, then creditor's priority implies $b_k \geq \ell_k$ hence solvent firms receive exactly their claim: $b_k = \ell_k$ on K and 1's payments are equal to the upper bound $\overline{z}_1$. Two cases follow.

Case 1: $z_1 > \overline{z}_1$. 1's worth is strictly positive when it pays off their claims to K. From the above arguments, necessarily $b_k = \ell_k$ on K: $\boldsymbol{b}$ coincides with the cp-solution associated to $\delta = 1$ and $W_1 = z_1 - \overline{z}_1$, which is positive.

Case 2: $z_1 < \overline{z}_1$. Surely $W_1 = 0$ since otherwise 1 would have to pay off a total equal to $\overline{z}_1$, ending up with negative worth $z_1 - \overline{z}_1$. Letting λ, $\lambda \leq 1$, be a scale for 1's liabilities, consider $\boldsymbol{b}(\lambda)$ the associated solution and $\boldsymbol{W}(\lambda)$ the worth levels. Let $\Lambda = \{\lambda, W_1(\lambda) = 0\}$ be the set of scales for which 1's worth is null. Λ is non-empty as it contains 1. Liabilities-invariance implies that $\boldsymbol{b}(\lambda) = \boldsymbol{b}$ for λ in Λ. We show that Λ has a positive lower bound. Choose λ small enough such that $z_1 > -z_{T-K} + \sum_{k \in K} \max(-z_k, \lambda \ell_k)$; such values exist: $z_j < 0$ for j insolvent so that strict feasibility implies $z_1 > -z_{T-K}$. Define $K' = \{k, z_k + \lambda \ell_k \geq 0\}$. The problem $(\boldsymbol{z}, (\lambda \ell_j)_{j \in T})$ satisfies Case 1's condition so that its solution is: $b_j(\lambda) = \lambda \ell_j$ for each $j \in K'$ and $b_j(\lambda) = -z_j$ for each $j \in T - K'$; $\boldsymbol{b}(\lambda)$ is thus the cp-solution associated to $\delta = \lambda$. Since λ does not belong to Λ and 1 belongs to Λ, there is λ^* adherent to Λ and its complement in $[0, 1]$. By continuity of F, $b_j(\lambda^*)$ is the cp-solution associated to $\delta = \lambda^*$ and $\boldsymbol{b}(\lambda^*) = \boldsymbol{b}$. This implies that $\boldsymbol{b}$ is the cp-solution associated to $\delta = \lambda^*$. ■

Proof of Proposition 2. The objective function f of $\mathcal{P}$ is separable and strictly convex. The feasible set of $\mathcal{P}$ defined by the linear inequalities (1) has a non-empty interior for a problem $\boldsymbol{\pi}$ in $\mathcal{F}$ by Lemma 2. It follows that the solution to $\mathcal{P}$ is unique and characterized by the first order conditions on the Lagrangian. Denoting by α_i the Kuhn-Tucker multiplier to i's positivity worth constraint (1), the Lagrangian writes:

$$\mathcal{L}(\boldsymbol{b}_{\mathcal{G}}) = f(\boldsymbol{b}_{\mathcal{G}}) + \sum_{i \in N} \alpha_i (b_{iN} - b_{Ni} - z_i).$$

The first order conditions with respect to b_{ij} for $(i,j) \in \mathcal{G}$ and the complementarity conditions are

$$\text{for each } (i,j) \in \mathcal{G} \quad : \quad \frac{\partial \mathcal{L}}{\partial b_{ij}} = \log \frac{b_{ij}}{\ell_{ij}} + \alpha_i - \alpha_j = 0 \tag{13}$$

$$\text{for each } i \quad : \quad \alpha_i \geq 0 \text{ and } \alpha_i (b_{iN} - b_{Ni} - z_i) = 0. \tag{14}$$

Taking exponential, (13) is equivalent to

$$\text{For each } (i,j) \in \mathcal{G} : \qquad b_{ij} = \delta_i \mu_j \ell_{ij} \tag{15}$$

$$\text{where for each } i : \quad \delta_i = \exp(-\alpha_i) \text{ and } \mu_i = \exp \alpha_i \tag{16}$$

(15) proves the bi-proportionality conditions (5). Conditions (16) imply (6), (7) and (8). Conversely, a solution $\boldsymbol{b}$ satisfying (5) to (8) solves $\mathcal{P}$: for such $\boldsymbol{b}$, defining $\alpha_i = log(\mu_i)$ one checks that all the sufficient conditions for optimality are satisfied. Since $\mathcal{P}$ has a unique solution, this proves that a solution satisfying (5) to (8) is unique.

Let us prove the uniqueness of indices supporting a cp-solution $\boldsymbol{b}$. Let us say that supporting indices are determined on a subset A of N if μ_j is unique on $A \cap C$ and δ_j is unique on $A \cap D(N)$. We prove that supporting indices are determined on N in three steps.

Step 1- Indices are determined on the non-empty set $\{j \in N, W_j > 0\}$.

This is immediate: there are firms with positive worth by strict feasibility (Lemma 2) and indices for them are necessarily equal to 1.

Step 2- If indices are determined on A, then they are determined on $A \cup C(A) \cup D(A)$. This follows from the following points:

(a) μ_j is determined for each j in $C(A)$. If j belongs to $C(A)$, j is creditor to some i in A. Thus $\ell_{ij} > 0$ where i belongs to $A \cap D(N)$ (since i belongs to $D(N)$). Hence δ_i is unique and the equation (5) $b_{ij} = \delta_i \ell_{ij} \mu_j$ determines μ_j.

(b) δ_k is determined on $D(A)$. If k belongs to $D(A)$, k is debtor to some j in A: $\ell_{kj} > 0$ where j belongs to $A \cap C(N)$. Hence μ_j is unique and the equation $b_{kj} = \delta_k \ell_{kj} \mu_j$ determines δ_k.

(c) It remains to show that μ_j is unique on $D(A) \cap C(N)$ and δ_j is unique on $C(A) \cap D(N)$. Let j belong to $D(A) \cap C(N)$. Since j belongs to $D(A)$, δ_j is unique (by (b)); as j is also a creditor, the balancedness relation (8), $\delta_j \mu_j = 1$, determines μ_j. Similarly, for j in $C(A) \cap D(N)$, μ_j is unique (by (a)); as j is also a debtor, the balancedness relation determines δ_j.

Step 3- Since N is finite, Step 1 and 2 imply that there is A such that $\{j \in N, W_j > 0\} \subset A$ and $A \cup C(A) \cup D(A) = A$. A contains all its creditors and debtors. If A was not the whole set N, the network would be decomposable. Thus $A = N$, which proves that the indices supporting $\boldsymbol{b}_{\mathcal{G}}$ are unique. ∎

Proof of Proposition 3. Let F be claims-invariant and assign 1-consistent solutions on $\mathcal{B}$. Given $\boldsymbol{\pi} = (\boldsymbol{z}, \boldsymbol{\ell}_{S\times T})$ a problem in $\mathcal{B}$, let $\boldsymbol{b} = F(\boldsymbol{\pi})$ and $\boldsymbol{W} = (W_i)$ the associated worth levels. To show that $\boldsymbol{b}$ is the cp-solution to $\boldsymbol{\pi}$, let us first consider problems where matrix $\boldsymbol{\ell}_{S\times T}$ has all its elements positive.

1- $\boldsymbol{\ell}_{S\times T}$ has all its elements positive.

Let $J = \{j \in T, W_j = 0\}$; hence $T - J = \{j \in T, W_j > 0\}$. 1-consistency applied to i in S implies that the worth of each creditor is W_j at the cp-solution of the reduced problem. Hence, if $J = \emptyset$, by (3), there are $\delta_i \leq 1$ such that

$$\text{for each } j \in T : b_{ij} = \delta_i \ell_{ij} \text{ with } \delta_i = 1 \text{ if } W_i > 0.$$

Thus $\boldsymbol{b}$ is the cp-solution supported by the values $\boldsymbol{\delta}_S$ and the μ_j all equal to 1.

Let us now consider the case where $J \neq \emptyset$. Let $\boldsymbol{\lambda} = (\lambda_j)_{j\in J}$, $\lambda_j \geq 1$ for each $j \in J$ be a vector scaling up the claims of firms in J. Denote by $\boldsymbol{b}(\boldsymbol{\lambda})$ the associated solutions and by $\boldsymbol{W}(\boldsymbol{\lambda})$ the worth levels. Consider $\Lambda = \{\boldsymbol{\lambda},\ W_j(\boldsymbol{\lambda}) = 0 \text{ for each } j \in J\}$ the set of scales such that the worth of firms in J are null. Λ is non-empty as it contains the vector of ones. We show that a maximal value of Λ exists and constitutes rescue indices for $\boldsymbol{b}$.

Step 1 Λ *is bounded.* Let $\lambda \in \Lambda$. Claims-invariance implies that $\boldsymbol{b}(\boldsymbol{\lambda}) = \boldsymbol{b}$. Consider two cases.

(a) $J \neq T$. Both J and $T - J$ are non-empty: there are k with $W_k > 0$ and j with $W_j = 0$. Since $\boldsymbol{b}(\boldsymbol{\lambda}) = \boldsymbol{b}$, the reimbursement ratios of i in S to k and j are respectively b_{ik}/ℓ_{ik} and $b_{ij}/(\lambda_j \ell_{ij})$. 1-consistency applied to i in S implies that i reimburses per unit firm j at least as much as k (i is indebted to both by assumption) hence $b_{ij}/(\lambda_j \ell_{ij}) \geq b_{ik}/\ell_{ik}$. This implies that λ_j is bounded above for each j in J: Λ is a bounded set.

(b) $J = T$. All firms in T have a null worth, so that there is i in S with $W_i > 0$ (Lemma 2). Applying 1-consistency to i, $\delta_i = 1$ and i reimburses its liability at least to each firm in T. Since $\boldsymbol{b}(\boldsymbol{\lambda}) = \boldsymbol{b}$, this implies $b_{ij}/(\lambda_j \ell_{ij}) \geq 1$ for each j in T, which proves that λ_j is bounded above for each j in T, hence Λ is a bounded set. ∎

Λ is closed since F is continuous. As Λ is bounded, there is $\boldsymbol{\lambda}^*$ in Λ that is maximal in the following sense: if $\boldsymbol{\lambda}$ belongs to Λ and $\lambda_j \geq \lambda_j^*$ for each j in J, then $\boldsymbol{\lambda} = \boldsymbol{\lambda}^*$.

Step 2 *Let $\boldsymbol{\lambda}^*$ be maximal in Λ. For j in J let $\boldsymbol{\lambda}$ coincide with $\boldsymbol{\lambda}^*$ except for λ_j. If λ_j is strictly larger than λ_j^*, then $W_j(\boldsymbol{\lambda}) > 0$.*

By definition of $\boldsymbol{\lambda}^*$, $\boldsymbol{\lambda}$ does not belong to Λ so that surely $\boldsymbol{b}(\boldsymbol{\lambda}) \neq \boldsymbol{b}$. $\boldsymbol{b}(\boldsymbol{\lambda})$ and $\boldsymbol{b}(\boldsymbol{\lambda}^*)$ are solutions to identical problems except that j's claims are proportional with the factor λ_j/λ_j^*. We have $W_j(\boldsymbol{\lambda}^*) = 0$. Applying claims-invariance, if $W_j(\boldsymbol{\lambda})$ was null, then the solution $\boldsymbol{b}(\boldsymbol{\lambda})$ would be equal to $\boldsymbol{b}(\boldsymbol{\lambda}^*)$, hence equal to $\boldsymbol{b}$. This proves $W_j(\boldsymbol{\lambda}) > 0$.

Step 3 ***b*** *is the cp-solution supported by rescue indices given by* $\boldsymbol{\lambda}^*$ *for firms in* J*.*

Choose j in J and consider values of $\boldsymbol{\lambda}$ as in Step 2. Apply 1-consistency to i in S. The supporting indices are unique since j's worth level is positive[11] (by Step 2). Furthermore, for λ_j close enough to λ_j^*, $W_k(\boldsymbol{\lambda}) > 0$ for each k in $T - J$ by continuity of F. So there is a unique δ_j such that and

$$\delta_i = \frac{b_{ij}(\boldsymbol{\lambda})}{\lambda_j \ell_{ij}} \leq \frac{b_{ik}(\boldsymbol{\lambda})}{\lambda_k^* \ell_{ik}} \text{ for each } k \neq j, k \in J \text{ and } \delta_i = \frac{b_{ik}(\boldsymbol{\lambda})}{\ell_{ik}} \text{ for each } k \in T - J$$

where $\delta_i \leq 1$ with an equality if $W_i(\boldsymbol{\lambda}) > 0$. When λ_j tends to λ_j^*, $\boldsymbol{b}(\boldsymbol{\lambda})$ tends to $\boldsymbol{b}(\boldsymbol{\lambda}^*) = \boldsymbol{b}$, hence the δ_is converge to some δ_i^* such that

$$\delta_i^* = \frac{b_{ij}(\boldsymbol{\lambda}^*)}{\lambda_j^* \ell_{ij}} \leq \frac{b_{ik}(\boldsymbol{\lambda}^*)}{\lambda_k^* \ell_{ik}} \text{ for each } k \in J \text{ and } \delta_i^* = \frac{b_{ik}(\boldsymbol{\lambda}^*)}{\ell_{ik}} \text{ for each } k \in T - J$$

where $\delta_i^* \leq 1$ with an equality if $W_i = W_i(\boldsymbol{\lambda}^*) > 0$. Exchanging the role of j and k in J, the inequality $\frac{b_{ij}(\boldsymbol{\lambda}^*)}{\lambda_j^* \ell_{ij}} \leq \frac{b_{ik}(\boldsymbol{\lambda}^*)}{\lambda_k^* \ell_{ik}}$ is reversed hence must hold as an equality. So, setting $\lambda_k^* = 1$ for each $k \in T - J$:

$$\text{for each } i \in S: \ \delta_i^* = \frac{b_{ij}}{\lambda_j^* \ell_{ij}} = \frac{b_{ik}}{\lambda_k^* \ell_{ik}} \leq 1 \text{ for each } j, k \in T, \delta_i^* \leq 1 \text{ with } \delta_i^* = 1 \text{ if } W_i > 0.$$

This proves that defining rescue indices equal to $\boldsymbol{\lambda}_T^*$, $\boldsymbol{\delta}_S^*$ and $\boldsymbol{\lambda}_T^*$ support $\boldsymbol{b}$: $\boldsymbol{b}$ is the cp-solution.

2-$\boldsymbol{\ell}_{S\times T}$ has null elements. Let us perturb the liabilities matrix by setting each null ℓ_{ij} with $i \neq j$ to some positive ϵ. The perturbed problem is strictly feasible since $\boldsymbol{z}$ is unchanged and the graph is expended. Consider the associated sequences of solutions $\boldsymbol{b}(\epsilon)$, worth levels $\boldsymbol{W}(\epsilon) = (W_k(\epsilon))_{k \in S\cup T}$, indices $\boldsymbol{\delta}_S(\epsilon)$ and $\boldsymbol{\mu}_T(\epsilon)$. All but the rescue indices are in a bounded set: the solutions are non-negative and satisfy for each i in S: $W_i(\epsilon) = z_i - \sum_{j\in T} b_{ij}(\epsilon) \geq 0$, worth levels are non-negative and satisfy $\sum_k W_k(\epsilon) = \sum_k z_k$, and the reimbursement indices $\delta_i(\epsilon)$ belong to $[0, 1]$. The rescue indices are not a priori bounded, so let J denote the subset of T for which $\mu_j(\epsilon_p)$ is not bounded. There is a sequence (ϵ_p) converging to 0 when p tends to ∞ such that the sequences $\boldsymbol{b}(\epsilon_p)$, $\boldsymbol{W}(\epsilon_p)$,

[11]If W_i is null and $T = J$, the reduced problem to i is not strictly feasible. In that case the indices supporting the cp-solution of i's reimbursements are not unique. Such a problem does not arise at $\boldsymbol{\lambda}$, since $W_j(\boldsymbol{\lambda}) > 0$. The proof shows that the limit of the indices supporting $\boldsymbol{b}(\boldsymbol{\lambda})$ support $\boldsymbol{b}(\boldsymbol{\lambda}^*) = \boldsymbol{b}$.

$\boldsymbol{\delta}_S(\epsilon_p)$ and $\boldsymbol{\mu}_{T-J}(\epsilon_p)$ converge. Denote the limits respectively by $\boldsymbol{b}^*$, $\boldsymbol{W}^*$, $\boldsymbol{\delta}_S^*$ and $\boldsymbol{\mu}_{T-J}^*$. Denote by I the subset of S for which δ_i^* is null. The following properties hold:

$$\text{for } i \in I \qquad (a): W_i^* = 0 \qquad (b): b_{ij}^* = 0 \text{ for each } j \in T-J \tag{17}$$

$$\text{for } i \in S-I \quad (a): \ell_{ij} = 0 \text{ for each } j \in J \quad (b): b_{ij}^* = \delta_i^* \ell_{ij} \mu_j^* \text{ for each } j \in T-J \tag{18}$$

(17)-(a) follows from $\delta_i^* = 0$ for $i \in I$, which implies $\delta_i(\epsilon_p) < 1$ for p large enough hence $W_i(\epsilon_p) = 0$. Other properties follow from the fact that $b_{ij}(\epsilon_p)$ converges to b_{ij}^*, $b_{ij}(\epsilon_p) = \delta_i(\epsilon_p)\ell_{ij}(\epsilon_p)\mu_j(\epsilon_p)$ with $\delta_i(\epsilon_p)$ converging to δ_i^* and $\ell_{ij}(\epsilon_p)$ to ℓ_{ij}. Since for $j \in T-J$, $\mu_j(\epsilon_p)$ converges, we obtain $b_{ij}^* = \delta_i^* \ell_{ij} \mu_j^*$, which proves (17)-(b) (since $\delta_i^* = 0$) and (18)-(b). Finally to prove (18)-(a), use that for $j \in J$ and $i \in I$: $\mu_j(\epsilon_p)$ is unbounded and δ_i^* is positive so $b_{ij}(\epsilon_p)$ can converge to the finite value b_{ij}^* only if $\ell_{ij} = 0$.

Assume J non-empty. The rescue indices of j in J are surely larger than 1 hence j's worth is null. (18)-(a) implies that the debtors of J are in I: $D(J) \subset I$. Thus surely I is non-empty as well (because, by non-decomposability, any element in T has at least a debtor). Since J receives transfers from I only, $W_I^* + W_J^* = z_I + z_J - b_{I,T-J}^*$. From (17)-(b), $b_{I,T-J}^*$ is null so finally $W_I^* + W_J^* = z_I + z_J$. Since worths are null for each i in I and j in J, we obtain $z_I + z_J = 0$, which contradicts the strict feasibility of the problem since $D(J) \subset I$. This proves that both I and J are empty. Hence by (18)-(b), $\boldsymbol{b}^*$ is associated with finite indices for which all the conditions required at a cp-solution are satisfied. This ends the proof. ∎

Proof of Proposition 4. Let F be splitting-invariant and assign bipartite-consistent solutions. Let $\boldsymbol{\pi} = (\boldsymbol{z}, \boldsymbol{\ell})$ be a problem in $\mathcal{F}$, $\boldsymbol{b} = F(\boldsymbol{z}, \boldsymbol{\ell})$ and $\boldsymbol{W}$ the worths achieved at $\boldsymbol{b}$. The proof first assumes the network to be complete.

1. Complete liabilities network: $\ell_{ij} > 0$ for each $i \neq j$.

Step 1 *Let S and $T = N - S$ be a partition of N. The reduced bipartite problem is strictly feasible.*

Proof: The reduced problem is non-decomposable since liabilities are strictly positive. Furthermore it admits $b_{S\times T})$ as a positive solution. Hence the result follows from (i) of Lemma 2. ∎

Step 2 $\boldsymbol{b}$ *is bi-proportional to* $\boldsymbol{\ell}$ *supported by indices satisfying (6) and (7).*

Proof: Choose $S = \{i, j\}$ and $T = N - S$. The reduced problem is strictly feasible (step 1) and complete so by bipartite-consistency, there are positive scalars (δ_i^S) and (δ_j^S) and $(\mu_k^S)_{k\in T}$ satisfying (6) and (7). such that

$$\text{for each } k \in T: b_{ik} = \delta_i^S \ell_{ik} \mu_k^S \text{ and } b_{jk} = \delta_j^S \ell_{jk} \mu_k^S \tag{19}$$

There is a firm, say 1, with $W_1 > 0$. Necessarily $\delta_1^S = 1$ for each $S = \{1, j\}$, thus (19) implies $b_{1k} = \ell_{1k}\mu_k^S$ hence μ_k^S is independent of $S = \{1, j\}$; denoting its value by μ_k,

$$\text{for each } k \neq 1 : b_{1k} = \ell_{1k}\mu_k \text{ and } b_{jk} = \delta_j \ell_{jk}\mu_k \text{ where } \delta_j = \delta_j^{\{1,j\}}. \tag{20}$$

Now apply (19) to $S = \{2, j\}$ for each j. Since $\mu_1^{\{2,j\}} = 1$ whatever j, (19) writes $b_{21} = \delta_2^{\{2,j\}}\ell_{21}$, which implies $\delta_2^{\{2,j\}}$ is independent of j; let $\delta_2' = \delta_2^{\{2,j\}}$ this common value. Now (19) writes for $k \neq j$ $k \neq 2$: $b_{2k} = \delta_2' \ell_{2k}\mu_k^{\{2,j\}}$, which implies that $\mu_k^{\{2,j\}}$ is also independent of j. Letting $\mu_k' = \mu_k^{\{2,j\}}$ this common value, it follows that

$$\text{for each } j \neq 1 : b_{j1} = \delta_j' \ell_{k1} \text{ and } b_{jk} = \delta_j' \ell_{ik}\mu_k' \text{ for each } k \neq 1 \text{ where } \delta_j' = \delta_j^{\{2,j\}}. \tag{21}$$

Comparing (19) and (21), we obtain that for each $k, j \neq 1 : \delta_j \ell_{jk}\mu_k = \delta_j' \ell_{jk}\mu_k'$. This implies that δ_j'/δ_j is independent of j, equal to some scalar c, and the matrix $\boldsymbol{b}$ is of the form

$$\boldsymbol{b} = \begin{pmatrix} 0 & \ell_{12}\mu_2 & \cdots & \ell_{1k}\mu_k & \cdots . \\ c\delta_2\ell_{21} & 0 & \cdots & \delta_2\ell_{2k}\mu_k & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ c\delta_i\ell_{i1} & \cdots & \cdots & \delta_i\ell_{ik}\mu_k & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

To prove that $\boldsymbol{b}$ is bi-proportional to $\boldsymbol{\ell}$, it remains to show $c = 1$. Consider two cases.

Case 1: $W_i > 0$ for some $i > 1$ for example for 2. By 1-consistency applied to 2, 2 reimburses fully 1 so that $c\delta_2 = 1$. By bi-consistency applied to $\{1, 2\}$, δ_2 (which is defined as $\delta_2^{\{1,2\}}$ by (19)) is equal to 1. This proves $c = 1$.

Case 2: $W_i = 0$ for each $i > 1$. Let us split 1 with a the liabilities between them. Splitting invariance implies that $1'$ and $1''$ have positive worths (half that of 1) at the solution $\boldsymbol{b}'$ in the split model and displaying only the reimbursements of $1'$ and $1''$ and 2, $\boldsymbol{b}'$ is of the form

$$\boldsymbol{b}' = \begin{pmatrix} 0 & a & \frac{1}{2}\ell_{12}\mu_2 & \cdots & \frac{1}{2}\ell_{1k}\mu_k & \cdots \\ a & 0 & \frac{1}{2}\ell_{12}\mu_2 & \cdots & \frac{1}{2}\ell_{1k}\mu_k & \cdots \\ \frac{1}{2}c\delta_2\ell_{21} & \frac{1}{2}c\delta_2\ell_{21} & 0 & \cdots & \delta_2\ell_{2k}\mu_k & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

Bipartite-consistency applied to $S = \{1', 2\}$ implies that the reimbursements ratio of $1'$ and 2 to $\{1'', 3, ...\}$ must be proportional. The reimbursement ratios of $1'$ to $1''$ and k for $k > 2$ are respectively 1 and μ_k and those of 2 are $c\delta_2$ and $\delta_2\mu_k$. Hence proportionality implies $c = 1$.

Step 3 *Balancedness: $\delta_i\mu_i = 1$ for each i*

Proof: Let us split i and denote by $\boldsymbol{b}'$ the solution in the split model and the supporting indices by ' (which exist from Step 3). We first show

$$\begin{aligned} &\text{for each } j \in N - \{i\} : \delta_j = \delta'_j \text{ and } \mu_j = \mu'_j \\ &\delta_i = \frac{(\delta'_{i'} + \delta'_{i''})}{2} \text{ and } \mu_i = \frac{(\mu'_{i'} + \mu'_{i''})}{2} \end{aligned} \tag{22}$$

Splitting invariance implies that adding the reimbursements in $\boldsymbol{b}'$ made by i' and i'' and the payments to them yields solution $\boldsymbol{b}$. Hence

$$\begin{aligned} &\text{for each } j \text{ and } k \in N - \{i\} : \quad b_{jk} = \delta'_j \ell_{jk} \mu'_k \\ &\text{for each } k \in N - \{i\} : \quad b_{ik} = \tfrac{(\delta'_{i'} + \delta'_{i''})}{2} \ell_{ik} \mu'_k \text{ and } b_{ki} = \delta'_k \tfrac{(\mu'_{i'} + \mu'_{i''})}{2} \ell_{ki} \end{aligned}$$

The worths of $k \neq i$ are identical at $\boldsymbol{b}'$ and $\boldsymbol{b}$ and those of i' and i'' are half those of i. Thus their nullity or positivity are identical in the two allocations. This implies that the indices $(\delta'_j)_{j\neq i}, \frac{(\delta'_{i'}+\delta'_{i''})}{2}$ and $(\mu'_j)_{j\neq i}, \frac{(\mu'_{i'}+\mu'_{i''})}{2}$ support the allocation $\boldsymbol{b}$. The uniqueness of indices implies (22).

We now prove $\delta'_{i'} = \delta'_{i''} = \delta_i$ and $\mu'_{i'} = \mu'_{i''} = \mu_i$. By splitting invariance, $b'_{i'k} = \frac{b_{ik}}{2}$, which writes $\delta'_{i'} \frac{\ell_{ik}}{2} \mu_k = \delta_i \frac{\ell_{ik}}{2} \mu_k$ hence $\delta'_{i'} = \delta_i$. Similarly the identity $b'_{ki'} = \frac{b_{ki}}{2}$ writes $\delta_k \frac{\ell_{ki}}{2} \mu'_{i'} = \delta_k \frac{\ell_{ki}}{2} \mu_k$ hence $\mu'_{i'} = \mu_i$. Since the same argument applies to i'', we finally obtain $\delta'_{i'} = \delta'_{i''} = \delta_i$ and $\mu'_{i'} = \mu'_{i''} = \mu_i$, as desired.

We end the proof: as the reimbursements between i' and i'' are equal to their common liability, $b'_{i'i''} = \ell_{i'i''}$, we must have $\delta'_{i'}\mu'_{i''} = 1$ hence $\delta_i\mu_i = 1$. ∎

2. General liabilities network. If the network is not complete, change each null ℓ_{ij}, $i \neq j$ into ϵ positive. The problem is still strictly feasible. Consider the associated worth levels $W_k(\epsilon), k \in N$ and indices $\boldsymbol{\delta}(\epsilon), \boldsymbol{\mu}(\epsilon)$. Arguing as in the proof of Proposition 3, the worth levels and reimbursement indices are in a bounded set, so there are converging for subsequence (ϵ_p) converging to 0. Denote their limits respectively by (W^*_k) and $\boldsymbol{\delta}^*$. We cannot exclude that $\boldsymbol{b}$ and $\boldsymbol{\mu}$ are unbounded. Let I be the set for which δ^*_i is null. We can take the sequence ϵ_p so that $\delta_i(\epsilon_p) < 1$ for each i in I. The identity $\delta_i(\epsilon)\mu_i(\epsilon) = 1$ implies that $\mu^*_i(\epsilon_p)$ tends to ∞ for i in I and tends to $1/\delta^*_i$ for i not in I. The following equations follow:

$$i \in I : \quad (a)\, W_i(\epsilon_p) = 0 \quad (b) \lim_{p\to\infty} b_{ik}(\epsilon_p) = 0 \text{ for } k \in N - I \tag{23}$$

$$j \in N - I : \quad (a) \lim_{p\to\infty} b_{jk}(\epsilon_p) = \infty\, k \in I \text{ with } \ell_{jk} > 0 \quad (b) \lim_{p\to\infty} b_{jk}(\epsilon_p) = \frac{\delta^*_i}{\delta^*_k} \ell_{jk}, k \in N - I \tag{24}$$

From (24)-(b), if I is empty, the sequence $(\boldsymbol{b}(\epsilon_p))$ converges and the limit satisfies all the conditions required on a cp-solution. It thus suffices to show that I is empty to conclude the proof.

Assume by contradiction $I \neq \emptyset$. Let us prove $D(I) \subset I$. Let j not in I. From (23)-(b) and (24)-(b), j's received payments are bounded; hence $W_j \geq 0$ implies that j's reimbursements are bounded as well. From equation (24)-(a), b_{jk} is bounded for $k \in I$ only if $\ell_{jk} = 0$, i.e. j is not indebted to any k in I. This proves $D(I) \subset I$. We now show $z_I \leq 0$. From (23)-(a), $W_I(\epsilon_p) = 0$, which writes $z_I + b_{N-I,I}(\epsilon_p) - b_{I,N-I}(\epsilon_p) = 0$. Since $b_{I,N-I}(\epsilon_p)$ tends to 0 from (23)-(b), $z_I + b_{N,I}(\epsilon_p)$ tends to zero, hence surely $z_I \leq 0$, which contradicts the strict feasibility of the problem. ∎