Characterizing a Sanskrit Mathematical Commentary: An exploration of Pṛthūdaka's Vāsanābhāṣya on progressions

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Characterizing a Sanskrit Mathematical Commentary: An exploration of Pṛthūdaka's *Vāsanābhāṣya* on progressions*

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Contents

1 Introduction

1.1 South Asian Studies, Commentaries and the 'Exact Sciences'

Despite a renewed interest in the different genres of scholarly literature in South Asia- in which commentaries have pride of place[1]- comprehensive studies usually do not include an examination of commentaries dealing with astral science (*jyotiṣa*) and mathematics (*gaṇita*)[2]. However editions, translations and studies of South Asian commentaries on mathematics and astral science do exist[3]. This omission no doubt is in part due to the difficult reciprocal integration of the history of the 'exact sciences' and South Asian Studies[4]. Furthermore, could such an exclusion have historical and topical reasons as well? Was there a very specific genre of Sanskrit mathematical commentaries? Were such commentaries excluded from other scholarly disciplines? Could such a an omission explain their relative isolation in the realm of South Asian studies?

With these questions in the background, this paper attempts to situate and describe some of the peculiarities of a Sanskrit mathematical commentary, Pṛthūdaka's 9th century CE 'Commentary with Explanations' (*Vāsanābhāṣya*) on Brahmagupta's 7th century CE astronomical *Theoretical Astronomical Treatise of the True Brahmā [School]* (*Brāhmasphuṭa-siddhānta*).

1.2 A Specific Commentary: Pṛthūdaka's *Vāsanābhāṣya*

Brahmagupta's *Theoretical Astronomical Treatise of the True Brahmā [School]* (628, *Brāhmasphuṭasiddhānta* henceforth abbreviated as BSS) is famously constructed in two parts: a first part presents a system of astronomy, the second supplements the first, notably with topics

1. The genre of treatises (*śāstras*) with aphoristic sometimes versified *sūtras*, using mostly nominal sentences, having attracted much attention already from (Jacobi, 1903) to (Renou, 1963), and as a genre of treatise in Pollock, 1985.

2. See notably (Angot, 2017), which nonetheless professes a thorough encyclopedic ambition. Note also (Wilden, 2009) which claims to be on Tamil commentaries in general, but focusses without explaining why on grammatical and poetic commentaries, with a small excursion into religous commentaries. For older, general considerations on the genre, notably on the vocabulary associated to names of commentaries, and what they mean, see (Bhattacharya, 1955). For prose and verse within classical Sanskrit commentaries, (Bronkhorst, 1991). As for studies on specific commentarial genres associated with specific disciplines, one can note for religious canonical commentaries (Patton and Doniger, 1996, pp. 27–35), for grammar (Bronkhorst, 1990), and (Houben, 1997). (Preisendanz, 2008) is both a historiographical study of how philosophical commentaries have been perceived, and an implicit description of their diversity. A more classical study of philosophical commentaries can be found in (Chenet, 1998), and for a more recent analysis on early modern philosophical commentaries (Ganeri and Miri, 2010 and Ganeri, 2011, pp. 91–95, 112–116); the latter includes also general considerations on commentaries.

3. Almost all editions of Sanskrit astral texts are made with commentaries, studies of the contents of astral texts are thus sometimes on the root text, sometimes on their commentaries. For an analysis of Sanskrit mathematical commentaries and how they have been edited and studied, see (Keller, 2010).

4. Thus, with notable exceptions ranging from C. Minkowski to S. R. Sarma, often historians of astronomy and mathematics in South Asia seem to consider that the 'exact sciences' are independent technical realms separated from the rest of Sanskrit scholarly lore, save for the shape of treatises, or the inclusion of mathematical considerations in ayurvedic, musical or prosodical texts. Studies in the history of mathematics and astral science in South Asia in the past have often ignored questions raised in South Asian philology, as for instance the quest to historicise and contextualise Sanskrit scholarly texts. And when they have tried to address them, they have often done so by not integrating such questions within their analysis of the contents of texts. On the other hand, despite the fact that part of the impulse that gave birth to Sanskrit studies as an academic discipline was an interest in Sanskrit texts on mathematics and astral science, many (not all) philologists and historians of South Asia seem to consider *jyotiṣa* as belonging to the very margins of their discipline.

outside the direct realm of astronomy, such as mathematics and prosody, but also with a certain number of revisions and criticisms. Pṛthūdaka (fl. 860)'s commentary on Brahmagupta's BSS is characterized by the fact that it does not comment on the treatise in its initial order, but on the contrary re-orders the text, starting with a commentary on Chapter 21 of the treatise[5]. It further provides 'explanations' (*vāsanā*) when commenting on the versified *sūtra*s that form Brahmagupta's treatise. In doing so, he may have created a specific kind of mathematical and astronomical commentary, that of the *Vāsanābhāṣya* ('Commentary with explanations'), emulated by others[6].

Brahmagupta's astronomical theoretical treatise, the BSS, contains a chapter (12) devoted to mathematics (*gaṇitādhyāya*)[7]. In the following, we will concentrate on the explanations given by Pṛthūdaka in this chapter,[8] more specifically those devoted to the 'practices' (*vyavahāra*) of progressions (*śreḍhī*) as evoked in verses 17-18[9].

What follows is an attempt, while examining a sample of Pṛthūdaka's commentary, to delineate its singularities in relation to his commentaries on Chapter 21 of the BSS[10], in relation to other mathematical commentaries, and in relation to other South Asian commentaries. First, a description of some characteristic elements of Sanskrit mathematical commentaries will be provided, before examining, in the heart of the article, how Pṛthūdaka's *Vāsanābhāṣya* situates itself within this landscape, examining how he uses versified problems and what reasonings he exposes. Such practices will serve as the basis of a cursory comparison with other commentaries, while reflecting on the *Vāsanābhāṣya*'s aim and the context in which his commentary may have been produced.

1.3 Mathematics in relation to Astral science

Astral science (*jyotiṣa*) is together with grammar (*vyakaraṇa*) and ritual (*kalpa*) one of the six scholarly disciplines of Sanskrit lore which grew from the study of and reverence for the sacred texts of the Vedas (*vedāṅga*). By the 5th century CE this discipline, like many others, becomes rather a 'knowledge system' (*śāstra*) among many other 'sites of knowledge'

5. The only published edition of part of Pṛthūdaka's commentary is devoted to this chapter, (Ikeyama, 2003). It is thus worth noting that we still have but a very partial view of his commentary.

6. Both known commentaries by Pṛthūdaka bear this title, although his commentary on Brahmagupta's *Khaṇḍakhādyaka* (665) is sometimes also referred to as the *Khaṇḍakhādyakavivaraṇa*. Āmarāja (fl. 1200) wrote a *Vāsanābhāṣya* on the *Khaṇḍakhādyaka* which includes quotations of Pṛthūdaka's commentaries, does not follow the order of the root text and is intent on providing reasonings associated with the rules. Bhāskarācārya (b.1114) also wrote a *Vāsanābhāṣya* on his own work. A more thorough investigation of the known mathematical commentaries with this title is certainly required in order to assess its importance. Also, the processes of title naming of treatises and commentaries in *jyotiṣa* certainly need clarification as well. Some titles seem to have been given (and are glossed) by the authors themselves, while others seem to have been given by editors or owners of the texts.

7. Pṛthūdaka's commentary respects the order of the verses in this chapter, although he may reflect at times on how a given verse might seemingly not be stated in the right place. An example is discussed in (Keller and Morice-Singh, Forthcoming).

8. I am currently editing Pṛthūdaka's commentary on Chapter 12 of the *Brāhmasphuṭasiddhānta*. For a description of the manuscripts used, see the Appendix, section A.

9. Numbers have been added by editors/translators of the text and do not belong to the original. An edition and translation of these commentaries can be found in the Appendix at the end of this article in sections A.1 and A.2. Concerning the topics and subdivisions of mathematics, see section 2.4.

10. (Keller, Forthcoming).

(*vidyāsthāna*)[11], in a Sanskrit cosmopolis where scholarly knowledge is thought and shaped to exist outside of time and space, to attain a form of universality[12]. This is the time were mathematics (*gaṇita*) emerges as an autonomous discipline as well: some of the transmitted texts are devoted only to mathematics, but others appear as sub-chapters of astronomical treatises, even if the topics treated in these chapters are not all of use in mathematical astronomy[13]. Thus, the mathematical chapter of the BSS starts with a definition of the knowledge to be known by one who practices *gaṇita*, which includes its elementary or fundamental constituants (*parikarman*, 'operations'), and a set of specialised topics (*vyavahāra*, 'practices') which includes 'progressions' (*śreḍhi*)[14]. As we will see, although Pṛthūdaka links Brahmagupta's rules on progressions to other mathematical topics and to disciplines outside of it, no relation to astronomy is pointed out.

Astral science and mathematical treatises have the standard shape of other medieval treatises: they are composed of versified *sūtras*[15] (that we call 'rules'), are quite concise, and provide the gist of a procedure (*karaṇa*) to be carried out, or a characterisation/definition (*lakṣaṇa*) of an object or situation. Most probably such compositions were made to be commented upon, orally and by writing. The material display in manuscripts of commentaries in relation to the text they comment on is as varied in mathematical commentaries as it is within other South Asian commentaries. Thus, some manuscripts do not quote the root text extensively, while in others the root text is quoted in the middle of a leaf while the commentary expands above and below it[16]. For the sources considered here, as seen in Figures 4, 5 and 6, the text of the commentary which quotes entirely the commented text is not separated by any spectacular textual indentation or display from the root text. A sign of punctuation, the *daṇḍa* (|), sometimes doubled (||), and sometimes a blank space, is used to signify both the end of a paragraph and a new section in the text. It thus announces also a quotation of the root text. In such manuscripts, it is the reader, the one who uses the manuscripts, who makes apparent the inner structure of the text, by separating, underlining, highlighting words or phrases, and sometimes rubricating the manuscript in its margins.

1.4 Parts of Texts in Mathematical Commentaries

Mathematical commentaries in Sanskrit, verse commentary by verse commentary, are composed of very standardised parts. They almost always include versified problems and their resolutions. This might be what makes them different from other commentaries in Sanskrit, including commentaries on texts on astral science. Commentaries can gloss a half verse,

11. As described in (Pollock, 1985). Note that however this does not mean that astral science becomes more secular. On the fact that the opposition of sacred and secular can be an anachronistic grid to read the litterature see (Angot, 2017, note 6, p. 15).

12. (Deshpande, 1985 and Pollock, 2007).

13. (Plofker, 2009, p. 121, Keller, 2010).

14. Such a subdivision between 'operations' and 'practices' will remain standard throughout the history of mathematics in Sanskrit, although the contents behind such labels will often vary from author to author. We will come back to this list of topics as 'practices' in section 2.4. Mathematics then does not include algebra, to which another chapter (18) is devoted (*kuṭṭakādhyāya*). Sometimes Chapter 12 of the BSS is considered devoted to 'arithmetic'.

15. The rules considered here are in the very common *āryā* meter.

16. See for instance the description of manuscripts in (Morice-Singh, 2015, p. 29 note 45).

a whole verse, or several verses together. Furthermore, they all seem to follow the same scheme. First, before quoting the part of the root, an introductory sentence is provided. This sentence names the mathematical topic that will be treated in the commentary, and it often states the meter or verse used in the quoted text. Because such a sentence usually has few words, it might be understood as a title for the verse commentary at hand:[17] The root text is then quoted, and this is followed by a syntactical and semantical elucidation of the text in prose. Such a practice is standard in most Sanskrit commentaries[18]. It is in what follows that there may lie one of the peculiar features of Sanskrit mathematical commentaries. That is, such a part is generally followed by a list of solved problems. As can be seen in Pṛthūdaka's commentary, the example is first announced: 'an example' (*uddeśaka*)[19], a versified example is then given. This is followed by a 'setting' (*nyāsa*), which opens in the text of the commentary the representation of a working surface in which numbers can be noted, sometimes with abbreviations as can be seen in Pṛthūdaka's commentary[20]. Such a part of the commentary represents a translation of the text of the problem into the shape it takes to be worked upon on a working surface. This is followed by a prose 'resolution' (*karaṇa*). The resolution can further provide small windows onto the different states of the working surface while the procedure is executed. Sometimes the examples are followed by 'variations' (*udāharaṇa*): within a given problem, other numerical or quantitative data can be introduced. Although this is the overall structure of the successive lists of examples, sometimes the names of the different parts are not announced or stated, but simply assumed, as can be seen in the portions of Pṛthūdaka's commentaries translated in the Appendix. Within these lists of solved examples, sometimes an additional part is devoted to reasonings to which occasionally specific names are given (*vyākhya, upapanna, vāsanā, pratyaya*).

This structure is by and large the same one as is found from Bhāskara's (628) commentary on the mathematical chapter of the *Āryabhaṭīya* to the later commentaries on the *Bījagaṇita* of Bhāskarācārya, as for instance in Kṛṣṇa Daivajña (ca. 1600-1625)'s *Bījapallava*. Later treatises sometimes integrate within the root texts lists of examples[21], commentaries however

17. Thus, Pṛthūdaka introduces verse 17 as follows:

> Here, in the beginning, he states an *āryā* [verse] in order to show the contents of a stack having the shape of half a Mt Meru Instrument.

but we could also probably translate less literally

> Where, in an *āryā* [verse], the contents of a stack having the shape of half a Mt Meru Instrument (*meru-yantra*) is shown.

This style then is perhaps more immediately recognisable as being that of a title. Similarly, the second rule is introduced thus:

> Now he states an *āryā* [verse] for the computation of the rank with the knowledge of the first, the increase and the overall value.

.

18. For a thorough detailed structural analysis of parts of philosophical and medical texts in Sanskrit, see (Preisendanz, 2018).

19. See section 2.2 for a discussion of the meaning of these different words.

20. For a study of the diagrams that can be found in such settings, in another mathematical commentary, see (Keller, 2005). For the tabular displays, see (Keller, 2015, Keller, Montelle, and Koolakudlu, n.d.).

21. In some cases one might question also how the edition was crafted, such as (Rangacarya, 1912), who integrated in the root text examples that are found in some commentaries and not in others, as remarked by (Morice-Singh, 2015, 73, 143 sqq, 166sqq).

still retain the same structure as they ‘complete’ the root text with the required resolutions[22].

The implications of this structure of mathematical commentaries- the different ways such problems relate to the rule they comment on, elucidating for instance the steps of a procedure that the root text just alludes to, or on the contrary focussing on a point which explains the procedure, the fact that such versified examples circulate from commentary to commentary and may reflect a wider living culture of mathematical riddles, etc.,- all of this and other topics are beyond the scope of this article. In what follows, the focus will only be on the way the problems provided in Pṛthūdaka’s commentary are used by him to specify the mathematical topics that can be related to the rules given by Brahmagupta, while more standardly, and progressively, exploring the scope of the rules.

2 Problems, Mathematical Topics and Quantities

In Pṛthūdaka’s commentary on Brahmagupta’s rules on progressions, a representation of arithmetic progressions as a stack of objects is put forward, varying what is stacked and how it is stacked. If the problems have a context that resonates both with mathematical topics and ‘real life’ situations, their treatment focusses on the numerical, not on what is quantified. In doing so, Pṛthūdaka not only specifies the kinds of quantities that can be inserted into the procedure, but also how its different components, when varied, can enable the exploration of the progression considered. In this sense his commentary explores the realm of the rule. Its limits.

2.1 Brahmagupta’s Rules for Arithmetic Progressions

Indeed, Brahmagupta’s rules for arithmetic progressions[23], like most mathematical *sūtra*s, do not provide the mathematical context in which they are to be understood. Verse 17[24] is devoted to the sum of terms of an arithmetical progression. It runs as follows:

> BSS.12.17. The rank less one, multiplied by the increase, added to the first is the last value. Half the first increased by the last value is the mean value; which, multiplied by the rank, is the total.

Three different elements of the progression are computed: first the ‘last value’ (*antya-dhana*), then ‘the mean value’ (*madhya-dhana*), finally, ‘the total’ (*gaṇita*), which is glossed by Pṛthūdaka as ‘the overall value’ (*savra-dhana*). If U is an arithmetical progression of n terms (‘rank’ *pada*), of first term (*ādi*) U_1, of common difference (or ‘increase’, *uttara*) u, then the rule states:

22. As, most canonically, in Bhāskarācārya’s autocommentary, the *Vāsanābhāṣya*.

23. Rules for diverse kinds of arithmetic progressions are given by Brahmagupta in verses 17-20 of the mathematical chapter of the BSS. We will here only concentrate on two of the verses.

24. The verse and its commentary by Pṛthūdaka are partly edited and translated in Appendix A.1.

1. 'The rank (*pada*) less one, multiplied by the increase (*uttara*), added to the first is the last value (*antya-dhana*)'
$$U_n = U_1 + u(n-1)$$

2. 'Half the first increased by the last value is the mean value (*madhya-dhana*)'
$$M = \frac{U_1 + U_n}{2}$$

3. 'which, multiplied by the rank, is the total'
$$\sum_{i=1}^{n} U_i = M \times n$$

Each new component is built from the previous one. In the end then, the sum of the terms of the progressions appears as being computed in three separate steps, each step having a mathematical meaning[25].

Verse 18[26] deals also with arithmetic progressions: knowing the sum of its terms, the rank is to be found. It runs as follows:

> BSS.12.18. Having added the square of the remainder of the subtraction of the increase with twice the first to the product of the [overall] value by eight times the increase, the rank (*gaccha*) is the square-root, less the remainder, divided by twice the increase.

In other words, with the same notation as before, taking $S = \sum_{i=1}^{n} U_i$, the rule states:

$$n = \frac{\sqrt{(2U_1 - u)^2 + (8uS)} - (2U_1 - u)}{2u}$$

In what follows, let us first see how the solved problems introduced by Pṛthūdaka help build a representation of progressions as a stack of quantified things.

2.2 Enunciated Problems and Variations

Two Sanskrit terms are used by Pṛthūdaka to name what we refer to generally as the introduced 'problems' and which we translate as 'example', *uddeśaka*, and 'variation', *udāharaṇa*. Do they have different meanings and functions in Pṛthūdaka's commentary?

25. K. Chemla has showed that this could be a way of grounding an algorithm (Chemla, 2009). Indeed, one can also argue that in many parts of the BSS Brahmagupta's statements about procedures do not only aim at describing how they should be executed, but could also have the aim of explaining, justifying, or maybe proving something. This would imply that explanations of why a procedure provides a correct answer were not necessarily confined to commentaries.

26. The verse and its commentary by Pṛthūdaka are edited and translated in Appendix A.2.

Both terms are familiar to readers of Sanskrit mathematical commentaries. Both expressions are used in Bhāskarācārya's *Līlāvatī* (1104), as synonyms it seems, to announce the statement of versified problems. One or the other or both are used in mathematical commentaries that supply such problems. Thus Bhāskara's (628) commentary on the *Āryabhaṭīya* prefers *uddeśaka* while Sūryadeva Yājvan (12th century) in his own commentary on the same text uses both.

The term *uddeśaka* is derived from the verbal root *ud-diś*, which means to 'enunciate' or 'indicate'[27], and should probably be understood at first as meaning 'enunciation [of a problem]' [28]. The word however takes on a technical meaning: since it announces the beginning of the part of the commentary where a problem will be stated and solved, it should most probably be understood as naming the whole commentary section where this takes place.

The term *udāharaṇa* derives from a similar semantic background, since the verbal root it derives from, *udāhṛ-*, signifies first 'to quote', and then to 'name' and 'illustrate'. *udāharaṇa*s as examples substantiating a reasoning are known to be standard parts of religious, philosophical, and grammatical commentaries[29]. In Pṛthūdaka's commentary on verse 17, the *udāharaṇa* can designate numerical/quantitative variations to the data of an already stated and solved problem[30]. He ends here his enumerations of 'variations' with a peculiar expression: *evaṃ ādyudāharaṇīya*, 'thus should the following be varied'[31]. Probably this is an invitation to explore the scope of the rule by varying the kind of quantities that it can deal with. It also is a way of indicating that this sub-part of the resolution of a problem is finished. In Pṛthūdaka's commentary on verse 18, an extension of the rule by varying the givens and results ends with the following remark:

> In this way, many variations (*bahudhodāhraṇīya*) aiming at proficiency (*vyutpatti*) [can be given]. It is not made explicit for fear of the weight of the composition since we have undertaken to explain (*vyākhyā*) the whole treatise (*siddhānta*).

Therefore, 'variations' of numerical examples can be used in different ways to explore the scope of a rule. They are valued as a way of mastering a topic. As such, they show how

27. The category of *uddeśa* has been famously singled out in the canonical logic commentary, the *Nyāyabhāṣa*, in its enumeration of the aims of a commentary's work as relating to the first function of a commentary 'to designate' (the two others being to define-*lakṣaṇa* and to investigate- *parīkṣā*). (Angot, 2017, 117, §20) and (Angot, 2009, Introduction, C §10). These categories are general enough to apply to Pṛthūdaka's (and others') commentary, however no serious trace of a reference to this text can be detected here.

28. As noted by (Filliozat, 2004, p. 151): 'Very often the *uddeśaka* is a stanza. That indicates that it was also an item to be learnt by heart, so that the user has in mind models to imitate in the course of his work.' He consequently (p. 154) translates the term above as: 'Enunciation [of the problem]', while dealing with examples in Sūryadeva Yājvan's 12th century commentary on the *Āryabhaṭīya*.

29. For example, (Angot, 2017, 714, §282), quotes such *udāharaṇa*s in Śabara's commentary on the *Jaiminisūtra*: in a reasoning intending to show that the meaning of a *mantra* is not what makes its utterance important in ritual, he provides 'examples' of *mantras* whose meanings have nothing to do with the object of the ritual. In Bhāskara's commentary on the *Āryabhaṭīya*, the term understood as an 'illustration' appears in a quoted stanza that specifies that a *sūtra* should not contain any. In this quoted verse, *udāharaṇa* is opposed to 'definition/characterisation' (*lakṣaṇa*).

30. A similar phenomenon can be found in Sūryadeva Yājvan's 12th century commentary on the *Āryabhaṭīya*, notably SYAB.2.3ab (Sarma, 1976)

31. This expression is also used at the end of his commentary on verse 17, just at the end of a solved example.

examples are to be understood as instruments to expand the rule: they are more than specific instantiations of it.

2.3 An Arithmetical Progression as a Stack of Quantified things

The first interpretation Pṛthūdaka gives of the rules understands arithmetical progressions as a stack of objects, epitomized by a stack of bricks[32]. This indeed is the context of the first example solved in Pṛthūdaka's commentary on BSS.12.17 and BSS.12.18.

With this interpretation, the rule given in BSS.12.17 first describes the computation of the amount of objects in the last layer, U_n. It then defines an 'amount in the middle [layer]', M[33]. Finally, the total amount of bricks in the stack is computed, S. The compound translated as 'last value' (*antyadhana*) can be understood as also meaning 'the amount in the last [layer]', both interpretations are possible . And thus, as Pṛthūdaka comments generally BSS.12.17, he specifies:

> Therefore, the number of bricks (*iṣṭaka*) in that last [layer] is the meaning (*artha*) of '**the rank less one, multiplied by the increase, added to the first is the last value**'.

Similarly, the 'mean value' can also be understood as 'the amount in the middle [layer]' (*madhyadhana*) and the expression *sarvadhana* which is Pṛthūdaka's gloss for the 'total' (*gaṇita*) can be the 'total/overall value' but also the 'overall amount [of bricks]'. Indeed, Pṛthūdaka specifies:

> That '**mean value multiplied by the rank**' is the overall value. The number of bricks of that stack is produced.

This floating meaning uses the polysemy of the word *dhana* which can mean wealth, amount, or value together with the syntactical flexibility allowed by the compound.

A numerical example provides the case for a stack of 5 layers of bricks, whose top layer would be made of 2 bricks, each layer being increased by 3 bricks. Such an example makes explicit the algorithm's different steps for integers. It also can enable a visualisation and understanding of the stack and what the different steps compute, as seen in Figure 1, although such a visualisation is not specifically alluded to here.

In the first example of the commentary on BSS.12.18, a pile of a 100 bricks whose first layer is made of 10 bricks, and whose following layers increase 5 by 5 is given. The number of layers in the pile (the 'rank', *gaccha*) is then found to be the same as in the example in the commentary on BSS.12.17, 5.

32. This appears as a standard understanding of progressions. A similar situation is notably evoked in Bhāskara's commentary on the *Āryabhaṭīya*, and in the Ł. Problems dealing with stacks of bricks are known also from Mesopotamia, although they do not seem to be dealing with progressions, (Friberg, 2001, Friberg, 2007, pp. 169–178. Friberg and Al-Rawi, 2016, pp. 339–477).

33. If n is uneven, M corresponds indeed to the amount of bricks in the middle of the stack; if n is even, M represents a 'mean' amount in a 'mean' layer.

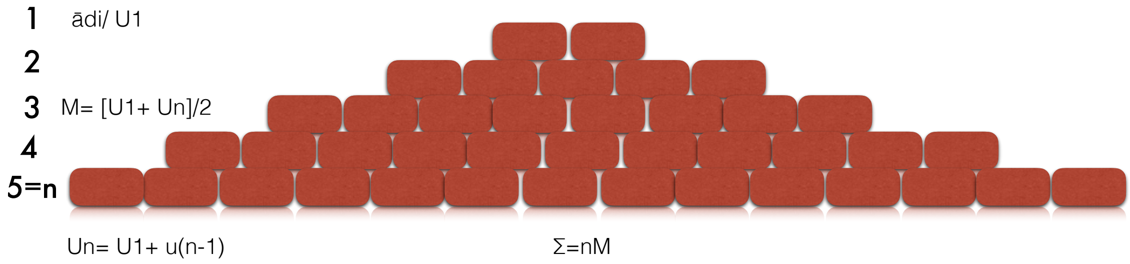

Figure 1: Sum of terms of an arithmetical progression as a stack of bricks

2.4 Extending the Context of Stacks

The next example in the commentary on verse 17 concerns increasing daily gifts. This example and the two following ones vary in terms of the 'stacks' considered. Indeed, in problems concerning gift donations the amount of given money/gold is stacked in time, not spatially. In problems of accumulated prices of conch shells, what is accumulated is not objects, but the object's value, its price (*mūlya*).

2.4.1 Practices

One might see these problems as relating the 'practice' of progressions with other 'practices'. Indeed, Pṛthūdaka gives the following enumeration of the 8 topics that make mathematics, as he comments on Brahmagupta's elusive definition of the knowledge of a mathematician (*gaṇaka*) in his commentary on BSS.12.1[34]:

> And the practices are: mixtures (*miśraka*), series (*śreḍhī*), geometry (*kṣetra*), excavations (*khāta*), stacks (*citi*), sawings (*krākacika*), piles (*rāśi*), shadows (*chāyā*), that is the meaning of 'eight'.

Now rules of 'mixtures' (*miśraka*) deal with invested capital (*dhana*) and respective shares, while 'stacks' (*citi*) deals with the cubic content of stacks, notably of stacks of bricks. Possibly, then, part of the commentarial work which provides problems that a rule can solve might also be to show that the rules belonging to 'practices' of progressions can be applied in problems which appear as belonging to the realm of 'mixtures' or 'stacks'.

2.4.2 Quantities

From objects to their values, the problems become increasingly numerical[35]. The initial stack becomes an image in which other situations can be grasped, in which the quantification is more important than what is quantified. Numerical instances, as can be seen in the tables

34. *vyavahārāś ca miśrakaḥ śreḍhī-kṣetra-khātaṃ citikrākaciko rāśiś chāyā artha caity aṣṭau|*

35. Numbers (*saṃkhyā*- probably integers) are distinguished from quantities (*rāśi*- probably a broader concept which includes forms of fractions, zero, positives and negative, and undetermined), both being actor's categories. By 'numerical' is evoked a spectrum by which values considered would be more or less considered beyond what they quantify.

of section 2.2 of the Appendix, are progressive, and increase in complexity, from simple integers to fractional quantities (of the form $a \pm \frac{b}{c}$) and even 'negative' quantities (as in example 4 of rule 17 and its variations). The latter are introduced by the commentator although operations on these entities have not been defined in the mathematical chapter.[36] As seen in Table 12, the introduction of these negative givens allows also an exploration of its consequences for the results, which thus can be equal to zero, or be themselves negative, and fractional. Although they are increasingly numerical, when considering the examples provided in the commentary on BSS.12.17, they can also each time still be interpreted in relation to what they quantify. For example, the last variation of Example 4 provides numerical values for a 'gift' of coins given over a certain number of days, but using 'negative' quantities. This arithmetic progression then has as first term -5 of 'increase' -3 , of 'rank' 8. The last term is found to be -26, the mean value is $-\frac{31}{2}$, and the sum -124. It is possible to understand these results in terms of literal 'debts' : the father-in-law would not be giving coins to his son-in-law, but borrowing money daily, obtaining in the end an accumulated debt.

These variations and explorations of the quantities that can enter as data in the procedure all seem to play with the different meanings of the word *dhana*: it can be used to designate both objects and the values given to the objects, an invested capital, or be a technical term for a 'positive' quantity vs. a 'negative' quantity. As Pṛthūdaka explores the quantitative range of Brahmagupta's rules he seems to be mapping exactly the different semantic ranges the word *dhana* can take.

2.4.3 Givens and Results

The introduction of 'variations' within solved 'examples' points not only to the exploration of the realm of the rule in relation to the quantities that can be included in it, but also to a wider grasping of the kind of problems the rule opens a door onto. Thus Pṛthūdaka, in his commentary on verse 17, adds and comments on a rule on the sum of terms of a geometrical progression. He also first alludes to the progression and then launches into a more or less systematic examination of the resolution of problems, when varying their givens and results. Thus, although Pṛthūdaka does not introduce these variations in Example 3 of his commentary on rule 17 (PBSS.12.17.Ex.3.), which evaluates the price of a 7th conch, he adds in the end:

> Similarly, one should show [the prices] of the remaining ones with an [other] assumed last one.

In other words, another variation of the same problem could be explored computing the price of the six other conchs (of which we know the price of the first). Pṛthūdaka further suggests, in his commentary on verse 17, variations of givens and results after having interpreted in general terms the progression as computing a total number of bricks, remarking:

36. Indeed, rules to operate with 'positives' (*dhana*, indeed the term translated by 'value', 'amount' or 'invested capital') and 'negative' (*ṛṇa*, lit. 'debts') are provided in chapter 18, devoted to algebra. Negatives are noted in the manuscripts here as a dot placed above a given numerical value.

And similarly, in another case too, the computation of that number should be made with a fixed given rank.

Therefore, the progressions considered are finite: the rule can only work if the number of terms is known, fixed. If the rank is not known, what should be done? In the commentary on verse 18, extensions of the procedure evoked in verse 17 are examined, by taking numerical examples which vary the givens and the results[37]. Indeed, in the rule of verse 17, the first term, the number of terms and increase are known from which the total is deduced; in the rule of verse 18, the first term, total and increase are known, and the number of terms is found. Pṛthūdaka explores varying what is given and what is not, as seen in Table 13, in section 2.2 of the Appendix, progressively diminishing the number of givens. He thus examines the effects that choosing a common difference can have on the first term, when the number of terms and the total is known. Finally, he examines the case when all the elements are given and computed: the first term, the number of terms, the common difference, the mean and total value are required to be square. In the numerical examples considered here, the givens and results are purely numerical, what the numbers represent is not at stake.[38] This is especially striking in the last condition of the results, which indeed points to their common conditions as numbers, and not to the fact that they quantify different entities.

It is within the part of the commentary that deals with solved examples that reasonings bearing specific names are exposed by Pṛthūdaka. In what follows, we will examine one called an '*upapanna*' ('proof') and another called a '*vāsanā*' ('explanation').

3 Peculiar Reasonings

3.1 A Brief Historiography of Proofs in Sanskrit Mathematical and Astral texts

Despite some late 20th century claims[39], proofs in Sanskrit mathematical commentaries have long attracted the attention of historians of mathematics and mathematicians. In 1817, H. T. Colebrooke published English translations of medieval Sanskrit mathematical texts by Bhāskarācārya and Brahmagupta. In the footnotes of his translations, he included some of the explanations of algorithms that can be found in commentaries (Colebrooke, 1817). These justifications caught the attention of mathematicians. For instance, Hermann Hankel saw them as epitomizing an 'intuitive proof' that he sought to synthetize with what he called the 'analytical' proofs of Greek mathematics (Charette, 2012; Smadja, 2015). It is true that

37. In the tabular 'settings' of such problems, the element whose value is yet to be found is announced, like the others, with an abbreviation, followed by a small circle, which indicates an empty space in the table. This small circle is the same as the one used to note zero while noting numbers with the decimal place value notation.

38. At the end of this set of variations, Pṛthūdaka adds: *evam abhinneṣu ca sarvatra yojyam*. I have translated as: 'And in this way, [the rule] should be used on non-fractions in all cases'. However, I am not sure that this is exactly what Pṛthūdaka means, since he actually gives an example, just before this sentence, in which the chosen common difference is $2 + \frac{1}{2}$, also noted $\frac{5}{2}$. What is the noun/subject modified by *yojyam* ('should be used')? Should *abhinna* be understood as qualifiying quantities ('non-fractions'), or a more abstract idea 'in unbroken' (traditions?)? In my translation I follow here Colebrooke, who translates as follows (Colebrooke, 1817, p. 293): 'This is applicable in all cases and in whole numbers.'

39. (Srinivas, 1990, Srinivas, 2005, Srinivas, 2008, Ramasubramanian, 2011).

during the first two thirds of the 20th century these explanations were not recognized as proofs *per se* (Chemla, 2012). They hence ceased to be studied, and at times were simply forgotten. There has since been a renewed interest in Sanskrit mathematical commentaries and the kinds of reasonings they contain[40].

We have inherited from the historiography of proof a certain number of questions in this respect: What kind of 'geometrical proofs' can be found in such texts? What kind of 'algebraical' analysis? Are there any arithmetical reasonings that also aim at proving in one way or another? Such questions however need to be addressed to texts as a whole, and therefore require that there be englobed within them modes of argumentation that we do not necessarily understand, or that cannot be grasped immediately within such frameworks. Indeed, the groundings that can be found in Sanskrit mathematical texts are still approached with many preconceptions. The way algorithms are justified, the values by which a rule is considered as proven, have been taken implicitly as being those that are familiar to us. Consequently, historians eager to present reasonings have often performed, as it were, anatomical sections on commentaries, extracting from them what they deemed to be the relevant explanatory portions, those that seemed to make sense, and whose modes of reasonings were familiar[41]. Indeed, certain types of reasonings seem to have been studied more often by historians than others. Thus, geometrical and algebraical explanations in commentaries have been presented as the standard mathematical toolbox for proofs, while other forms, such as the re-reading of an algorithm by another algorithm, for example, or the linking of two algorithms, have been considered but little[42].

In what follows, the aim is to analyse the processes of explaining from the point of view of actors' categories. What kind of explanations were recognized and valued in Sanskrit mathematical commentaries?

3.2 Naming Reasonings

Several words are known to have been used in Sanskrit mathematical commentaries to indicate reasonings, groundings, explanations, or justifications of the procedures that the root text evokes. The word usually translated as 'proof' is *upapatti*, and other words derived from the same verbal root *upa-pad-*, such as *upapanna*. But other terms are known to have been used for explanations that could be proofs, such as *yukti*, *vāsanā*, *vyākhyāna*, *pratyayakaraṇa* and *pradarśita*.[43] Although nuances have been suggested for the different meanings of these words, they have been understood as designating practices of algorithm justification considered as a homogeneous whole. In doing so, it is possible that what were considered different practices of grounding by a given author may have been jumbled together, while others may have been left in the dark.

What follows is an effort to understand what Pṛthūdaka could have understood when

40. See the works of Srinivas and Ramasubramanian quoted above but also (Keller, 2012a; Keller, 2012b).
41. This involves also shaping corpuses so that they can compare to the prestigious 'Greek' corpus.
42. The overlooking of the first kind of justification is evoked in (Keller, 2012b).
43. (Ramasubramanian, 2011, Keller, 2012a).

he used the word *upapanna* or the word *vāsanā*. We will adopt a fixed translation of these words but we will try to describe the reasonings followed by Pṛthūdaka in order to better approach what these words could have meant for him.

3.3 A Geometrical 'Proof' (*upapanna*)

3.3.1 A Geometrical Interpretation of the Rule Evoked in BSS.12.17

Pṛthūdaka's geometrical interpretation of the rule first arises when he comments in general terms on BSS.12.17, and more specifically after he glosses the computation of the overall amount of terms. The last amount computed is interpreted as the area of a rectangle, one side having as length the mean value (M), the other side that of the 'rank' or the 'rank number' here (n). Thus:

> The meaning is: differently, the second side of a rectangular figure whose other [side] is equal to the rank number is produced with that [rule].

Pṛthūdaka comes back to this interpretation, after PBSS.12.17ex.4.[44]:

> In this case, one side should be assumed to be equal to the 'rank' and the second reckoned as the mean value. On should show (*pradṛś*) such a rectangle.

Indeed, the final computation of $M \times n$ can be interpreted as providing the area of a rectangle of length M and of width n, as seen in Figure 2. This new reading of the result (*phala*) rests on the fact that the arithmetical operation of multiplying can also be understood geometrically as the computation of an area. This double understanding, both numerical and geometrical, of arithmetical operations was theorised by another commentator, Bhāskara, whom we have reasons to believe Pṛthūdaka has read[45]. It may have been quite a standard hermeneutical operation.

This statement is followed by the instruction to 'show' (*pradṛś*) the rectangle. The term used here may have the double meaning of 'showing' and 'teaching'. In this case, from what follows, the term might literally mean that a diagram of the rectangle should be drawn, to visually explain how it relates to another geometrical interpretation of the progressions, that of the 'progression figure' (*śreḍhīkṣetra*).

Pṛthūdaka indeed continues:

> With the area of the progression figure (*śreḍhīkṣetra*), so many small figures in a progression figure that have been discarded are in that case gathered in front, in the rectangle. Thus the computation of the result (*phala*) has been proven (*upapanna*).

44. That the last variation dealt with 'negative' quantities does not seem to have been taken into account, so that we can assume that this part of the commentary, although situated after a solved problem, does not deal with one specific example, or kind of quantity involved.

45. (Keller, 2007, Keller, Forthcoming).

In other words, the rectangle itself can be seen as a rearrangement of the rectangles of a 'progression figure' . We can assume that as a Mt Meru Instrument (as seen in Figure 3) which Pṛthūdaka evokes at the beginning of his commentary on BSS.12.17 and to which we will come back later, such a 'progression figure' was seen as a stack of [areas of] rectangles, as illustrated in Figure 2.

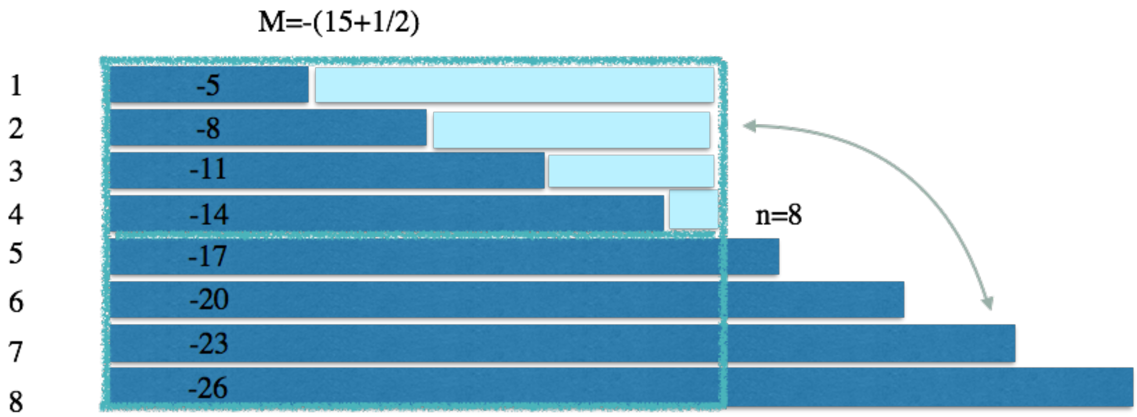

Figure 2: A progression as a stack of [areas of] rectangles

These two geometrical representations of the progression and its results thus relate this 'practice' of series (*śreḍhī*) to the 'practice' of geometry (*kṣetra*).

3.3.2 Relating Two Representations

The reference Pṛthūdaka makes to a 'Progression Figure' is noteworthy: it was previously understood that such a representation of progressions belonged to the 'Kerala School' [46]. If progressions are standardly ascribed in the literature to geometry, and if Bhāskara's (628) *Āryabhaṭīyabhāṣya* provides diagrams of progressions in the form of stacks of objects, the use of a 'progression figure' is usually attributed to Nīlakaṇṭha Somayāji (1444-1545)[47]. From the way in which it is evoked in Pṛthūdaka's commentary, it seems to be a much more standard and long-standing representation of progressions than what was previously assumed[48].

If a link between these two representations of the total value of the progression in a geometrical setting is explained, what does the term *upapanna* ('proved') mean here? Pṛthūdaka replies with an elusive expression: 'the computation of the result'. The term translated as 'result', *phala-* literally 'fruit' - is used to name both an area and the result of a computation. Pṛthūdaka uses it when presenting the rule, in the compound *phala-citi*, 'the content of a stack'. Here then, visually, is shown how the 'overall value' can be understood as the area of a figure that is made of a stack of rectangles, and then how this figure can be made into a rectangle with the same area. The equivalence of the two representations has been

46. (Mallayya, 2001, Mallayya, 2003).

47. As noted by Sho Hirose in his PhD concerning another member of the Kerala school, Parameśvara (Hirose, 2017), although Nīlakaṇṭha provides an intellectual lineage emphasizing Āryabhaṭa, he may have silently also read Brahmagupta (critical of Āryabhaṭa) and his commentators.

48. Pṛthūdaka alludes also to this figure that he attributes to Master Skandasena, in his commentary on BSS.12.2.

shown, but also that the numerical result can be read as having two precise geometrical interpretations.

Thus, Pṛthūdaka would be adding a new point of view on the rule by providing different representations of it. He would additionally be linking these different representations. Such multiple representations use the polysemy of some key words (*phala*, *dhana*) and the fact that summing, subtracting, and multiplying can be interpreted numerically and geometrically. To which of these operations does the word *upapanna* apply? The term *upapanna*, lit. 'approaching', is part of a larger family of words known in scholarly commentaries to be used to qualify a reasoning as 'legitimate' or 'sound'. In Pṛthūdaka's commentary, it obviously has a technical meaning: however, does it really designate a reasoning that intends to ground the rule as, usually, scholars understand the term *upapatti*?

As we will see, using a new topic to re-interpret a rule and make it signify differently can also be seen in Pṛthūdaka's commentary on the second rule given by Brahmagupta concerning arithmetical progressions.

3.4 An 'Algebraical Explanation' (*bījavāsanā*) of the Rule evoked in BSS.12.18

As seen above, the rule evoked in BSS.12.18 can be understood as a computation of the number of elements, n, in an arithmetical progression of common increase u, where $S = \sum_{i=1}^{n} U_i$:

$$n = \frac{\sqrt{(2U_1 - u)^2 + (8uS)} - (2U_1 - u)}{2u}$$

Like the preceding rule, this one too was standard in the treatment of progressions. It has attracted much attention by historians of mathematics who have seen in it an early proof of the knowledge of the resolutions of equations of the second degree.[49] However, there is no evidence of an understanding of this rule in relation to algebra before the passage in Pṛthūdaka's commentary that we examine here.

3.4.1 Solving a Problem

Pṛthūdaka starts with an example:

> When a stack is made of a hundred of bricks, having in its first layer (*mukha*) ten bricks increasing five by five, tell me the rank (*pada*).

We recognize here a standard understanding of progressions as a stack of bricks. Using the above procedure, and recognizing that in this example, $U_1 = 10$, $u = 5$, $S = 100$, Pṛthūdaka in a step by step computation finds that $n = \frac{\sqrt{(20-5)^2+(8\times5\times100)}-(20-5)}{10} = 5$.

Once this solution is obtained, Pṛthūdaka then states that in this case there is 'an algebraical explanation' (*bījavāsanā*)[50] using what he calls an 'elimination of the middle'.

49. Thus, for instance, this rule is treated by Datta & Singh in a section on algebra, in the context of another text (Datta and Singh, 1935, Volume 2, p. 61).

50. In one of the three manuscripts- Manuscript V_1- a reader- probably Pdt Sudhākara Dvivedin- corrects the expression to make it as follows: *madhyamāharaṇabījena vasanā*, 'an explanation using an 'algebraical elimination of

He then proceeds to make explicit what this means.

3.4.2 An 'Algebraical Explanation' with an 'Elimination of the Middle'

Since the sum of terms is known (100), he considers the 'rank' as an unknown, and proceeds then to rewrite the computation spelled out in verse 17. The algebraical interpretation involves a restatement of the givens of the problem, displaying the algebraical unknown (*avyakta*, lit. undetermined) for the 'rank' that is sought. As is standard in Sanskrit algebraical texts, the unknown of first degree is noted by a या (*yā* -an abbreviation of *yāvattāvat*, 'as much as' transcribed in our translation by '*as.*'), its square (*varga*) by या व (*yā va*-translated as '*as.* s.') and constant terms by a रू (*rū*-an abbreviation of *rūpa*, 'integer', the abbreviation has been translated as '*in.*'). Numbers that are to be subtracted are noted with an overhead dot ($\dot{1}$ standing for our -1, in the *devanagarī* transcription here above left of the written numerals.).

Each step of the computation progressively writes a polynomial, as seen in Table 1[51].

Table 1: Writing the computation of the computation in BSS.12.17 into a polynomial

General	Numerical	Transliteration	Translation
$x-1$	$x-1$	*yā 1 rū* $\dot{1}$	*as.* 1 *in.* $\dot{1}$
$u(x-1)$	$5x-5$	*yā 5 rūpa* $\dot{5}$	*as.* 5 integer $\dot{5}$
$U_n = u(x-1)+U_1$	$5x+5$	*yā 5 rūpa 5*	*as.* 5 integer 5
$U_n + U_1$	$5x+15$	*yā 5 rū 15*	*as.* 5 *in.*15
$M = \frac{U_n+U_1}{2}$	$\frac{5}{2}x+\frac{15}{2}$	*yā* $\frac{5}{2}$ *rū* $\frac{15}{2}$	*as.* $\frac{5}{2}$ *in.* $\frac{15}{2}$
$S = M \times n$	$\frac{5}{2}x^2+\frac{15}{2}x$	*yā va* $\frac{5}{2}$ *yā* $\frac{15}{2}$	*as.* s. $\frac{5}{2}$ *as.* $\frac{15}{2}$

So that finally the polynomial $\frac{5}{2}x^2+\frac{15}{2}x$ (or with Pṛthūdaka's notation: *as.* s. $\frac{5}{2}$ *as.* $\frac{15}{2}$) is obtained. The process of making it into an equation and then solving it is then presented. The equation considered by Pṛthūdaka, since the above polynomial should be according to the rule of verse BSS.12.17 equal to S, given as 100, should be: $\frac{5}{2}x^2+\frac{15}{2}x = 100$. However, the following equivalent is what is solved: $5x^2+15x = 200$.

The solution involves what Pṛthūdaka calls the 'elimination of the middle' (*madhyamāharaṇa*). This procedure is provided in Chapter 18 of the BSS[52]. We will just present here its mathematical gist. If a quadratic equation has the form $ax^2+bx = c$, a is called 'the [coefficient

the middle''. Such a correction emphasizes that there would be no 'algebraical explanation', but just an 'explanation' that uses an algebraical tool.

51. Note however that the abbreviations are not systematically used and that sometimes the text redevelops them, referring to an 'integer' (*rūpa*) rather than to '*in.*' (*rū*).

52. BSS.18.44. (Dvivedin, 1902, p. 314).

of] the square' (*varga*), c is called the 'integer' (*rūpa*), and b is called the '[coefficient of] the middle' (*madhya*), not to be confused with the middle (also *madhya*) being here the unknown, x, which is sought[53]:

> BSS.18.44. The middle is the square-root of four times [the coefficient of] the square and the integer increased by the square of the [coefficient] of the middle, decreased by the [coefficient of] the middle, divided by twice [the coefficient] of the square.

In other words:

$$x = \frac{\sqrt{4ac + b^2} - b}{2a}$$

Applying this rule to the equation above, we thus have $a = 5$, $b = 15$, $c = 200$; Pṛthūdaka thus computes $x = \frac{\sqrt{4000+225}-15}{10}$. He thus finds again $x = 5$.

We can recognize here precisely the computation carried out when solving the example, using the procedure evoked in BSS. 12. 18.

Therefore, this *vāsanā* is a process which produces the same result as the one found in the first solution of the problem, but from what at first seems to be a different path. This is done by reading with a new mathematical tool the procedure alluded to in BSS.12.17. By changing the nature of the quantities involved in the process, Pṛthūdaka changes the meaning of the operations to be carried out on them: from a procedure with numbers which produces a number as a result, it becomes a procedure dealing with an unknown quantity and producing a polynomial. The problem then becomes an equation. The process of using this polynomial to make it into an equation, and then solving it, using yet other algebraical tools, further strengthens the links between the two rules. Finally, as we reach the final computation, what seemed to be deriving the result from a different perspective, from a different rule with a specific mathematical tool, actually re-invests the computation of BSS.12.18 with a new meaning: it demonstrates indeed that this rule which involves the same computational steps- simply taking $b = 2U_1 - u$, $a = u$ and $c = 2S$- is a formulation of a solution of a second degree equation, or of an 'elimination of the middle'. From being an arbitrary set of operations to be carried out, the process evoked in BSS.12.18 is now grasped in its deep algebraical relation with the preceeding rule. Indeed, Pṛthūdaka is very careful to show how each step that involves the use of an algebraical procedure provides simultaneously a step of the rule given in BSS.12.17 and in BSS.12.18. Thus, when constructing the polynomial from BSS.12.17, he is careful at every step to recognize in each polynomial what it represents in terms of progressions. He thus specifies: ' *as.* 5 integer 5, this is the last term' , and then

> '...*as.* $\frac{5}{2}$ *in.* $\frac{15}{2}$. This is the mean value; multiplied by the rank, produces *as.* s. $\frac{5}{2}$ *as.* $\frac{15}{2}$, its overall value.'

And after having applied to the equation the rule of 'elimination of the middle', he also specifies:

53. See also (Colebrooke, 1817, p. 346, Datta and Singh, 1935, Volume 2, p. 63, Hayashi, 2009, pp. 123–124). *vargacaturguṇitānāṃ rūpāṇāṃ madhyavargasahitānām/*
mūlaṃ madhyenonaṃ vargadviguṇocchṛtaṃ madhyaḥ//

> ...the middle is produced. The meaning is: the rank. Since here the '*as.*' was the rank.

Thus, as before, a new perspective is given on the rule by approaching it through a new context. In addition, a link has been made with another rule. Finally, here, indeed algebra seems to be a tool which further 'proves' the validity of the rule. For the rule evoked in BSS.12.18 would be a previously known rule, BSS.12.17, to which another known rule 'the elimination of the middle' is applied. The algebraical reading may have been considered by Pṛthūdaka as a tool outside of the realm of mathematics (*gaṇita*), since it seems to have been understood by Brahmagupta as forming a distinct topic of the realm of theoretical astronomy. But to what part of the following reason should *vāsanā* be understood as referring?

4 Conclusion

4.1 Diversifying and Unifying

Pṛthūdaka's commentary on BSS.12.17-18 can be caracterized by the way it provides different mathematical topics in which Brahmaputa's rules can have a meaning: progressions as stacks of bricks, problems of donations, areas of rectangles, algebra. Each of these contexts connects the rule to a specific field of mathematics, to one of the 'practices' (*vyavahāra*) that define mathematics (*gaṇita*). In some cases, this context provides an illustration, a visual representation for the topic of the rule, in others it enables the reader to understand the rule in a different light, with a new representation of what it computes, and to understand its different steps.

In creating such contexts, Pṛthūdaka resorts to topics that are not necessarily of the realm of mathematics. In his commentary on BSS.12.17, he thus provides a rule to compute the sum of terms of a geometrical progression. The rule itself is 'shown' (*darśita*) to be prosody (*chaṃdas*) as 'a kind of substitute for an increasing stack' (*cityuttarādeśaprakāra*). As he clarifies (*vyākhyā*) the execution of the procedure alluded to in his own verses, he describes the generation of a column built from top to bottom and then computed from bottom to top, whose last cell provides the sum of the progression. The table that he describes seems indeed to be inspired from the metrical rules as we know them from Hemacandra's *Chando'nuśāna* (ca. 1150) commentary on Piṅgala's *Chandaḥsūtras* (ca 200 BCE)[54]. However, the direct relation of the table described in this verse commentary with those known to us in prosody would still need to be elaborated.

Pṛthūdaka's exploration of the rule is thus anchored first in the solved examples and variations he provides. As such, these examples are not to be seen as simple specific illustrations of the rule. They say something general about the rule and its contexts of application. Further, Pṛthūdaka's reasonings, his 'explanations' and 'proofs' are anchored

54. Note that Brahmagupta's BSS contains a still unstudied chapter devoted to this topic.

in solved examples: this shows how important they were, that they must have been thought of as having a general scope. Why and how these reasonings are linked to examples still needs to be understood though.

Within this diversity it is striking that Pṛthūdaka does not use one mode of explanation consistently. He does not follow up with a geometrical representation of arithmetical progressions, nor does he systematically work with algebra to link one rule to another. However, he may have conceived the Mount Meru instrument as a sort of unifying mnemonic token when dealing with arithmetic progressions. Thus in his introductory sentence for the rule of verse 17, Pṛthūdaka explains:

> In this case, in the beginning, he states an *āryā* [verse] in order to show the contents of a stack having the shape of half a Mt Meru Instrument (*meru-yantra*).

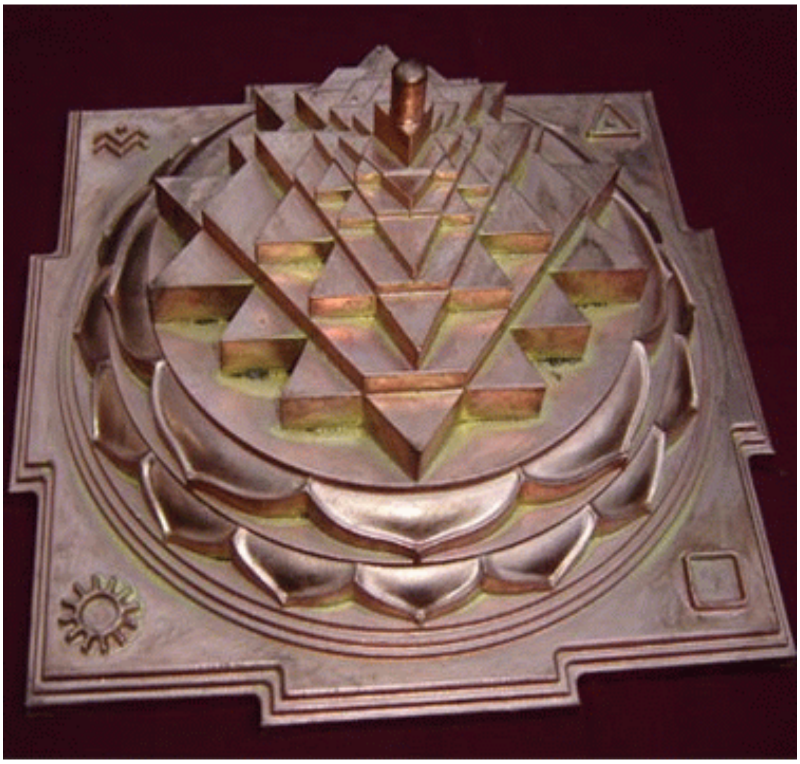

Figure 3: A Mt Meru Instrument (*Meru-yantra*)

The devotional object called *meru-yantra* could have been thought of as a synthesizing representation allowing one to imagine stacks, a geometrical arrangement of rectangles providing an alternative reading of arithmetical operations, and finally being a kind of prosody table (the prosody table called the *meru-prastāra*- which is a kind of Pascal's table- is maybe understood here as a generic prosody table), from which one can derive yet another sum, that of the sums of the terms of a geometrical progression.

Pṛthūdaka, although exploring the many different possible understandings of the rule, may more deeply be trying to unify not only the 'practices' of mathematics, but also mathematical astronomy as it appears in Brahmagupta treatise, as a discipline containing elements of prosody and algebra. Only a more thorough exploration of his commentary on the mathematical chapter will tell us if this hypothesis holds true.

4.2 Situating Pṛthūdaka's commentary

Pṛthūdaka's commentaries on progressions are striking in the way they involve all sorts of topics included by Brahmagupta in the realm of astral science: this is indeed the singularity of his commentary in this case. Indeed, the *Vāsanābhāṣya* does not use grammatical arguments nor quote canonical law treatises as Bhāskara does in his *Āryabhaṭīyabhāṣya*[55], although we have reasons to believe that Pṛthūdaka was familiar with his commentary. Unlike the anonymous and undated commentary on the *Pāṭīgaṇita*, Pṛthūdaka's commentary is not intent on reconstructing all the details of a procedure execution, nor does he, like Kṛṣṇa commenting on the *Bījagaṇita*, systematically try to show the validity of a procedure using an algebraical reading, nor does he, like Nīlakaṇṭha, sometimes resort to philosophical arguments[56].

Further, Pṛthūdaka's commentary on these verses is not shaped like his commentary on the verses given in chapter 21 of the BSS. Thus, Pṛthūdaka's commentary on Chapter 21 does not provide any versified problems to illustrate computational rules given by Brahmagupta. Further, if it is still much too early to make a general assessment of his use of the term *vāsanā* in his commentary on verse 21.19 which provides the values of some of Brahmgupta's Sine table, Pṛthūdaka's *vāsanā* is closer to the literal meaning of the word. The term *vāsanā*, which derives from the word that names the smell of perfume, literally means 'the impression left on one's mind'. In PBSS.21.19 it consists of recalling rules from Āryabhaṭa's treatise and Bhāskara's commentary on it which relate (in ways that he explains) to the rule he comments on. So in a *vāsanā* what should be considered more important, the fact that it 'recalls' or that it searches for a 'link'? In many ways, Pṛthūdaka's *Vāsanābhāṣya* seems also a standard Sanskrit scholarly commentary. Like all commentators on Sanskrit texts, while he resorts to glosses and syntactical clarification, Pṛthūdaka is not attempting to clarify Brahmagupta's intention, but rather to make sense of the variety of possible understandings of the text, taken in itself. Such an attitude has often been noted in other scholarly disciplines of South Asia as well[57].

In other words, the diverse explanations found in Pṛthūdaka's commentary provide different contexts in which the rule is meaningful, in which the different steps of the procedure can be represented or the final result considered. All further weave relations between procedures and topics in the treatise. One word seems to punctuate Pṛthūdaka's endeavor

55. Pṛthūdaka's syntactical commentaries clearly aim at making the literal meaning of the verse clear, they do not comment particularly on the choice of Brahmagupta's words, contrary to *Āryabhaṭīyabhāṣya*.

56. We may note here that if links have been drawn between early South Asian 'logic' (*nyāya*) and medecine (*ayurveda*), (Preisendanz, 2009), no such ties have come to light in studied commentaries in astral or mathematics texts in Sanskrit, before the 15th century. The use of a logical reasoning in relation to the Rule of Three reminiscent of Mimāṃsa logics in the 16th century *Krīyakramakarī* commentary on the *Līlāvatī* is evoked in (Hayashi, 2000, Section 3.2, 212–215), which also mentions a similar passage in the 15th century commentary of Nīlakaṇṭha on the *Āryabhaṭīya*.

57. (Angot, 2017, p. 69).

throughout his commentary: the term *artha*, which together names a meaning and an aim, a purpose. Pṛthūdaka's unrelenting task seems to be to bring out the *artha* of a rule. It is striking that the word is always in the singular, but that the meanings brought out are always plural. How specific to mathematics is this aim? Of course, presenting the literal meaning of the root text was required of all commentaries[58]. This at times indicates that these texts were meant for readers of different levels[59]. But should this search for purpose and meaning be extended to the weaving of connections, showing the overall unity of the treatise? The explanations of Pṛthūdaka's *Vāsanābhāṣya*, then, may aim at leaving durable impressions concerning the rule[60], a durable impression also of the dexterity and scholarship of the commentator. But such impressions might refer to what a study of the rule itself might bring about. The explanations of the commentator might aim at exhausting the different possible fragrances a given rule contains.

4.3 The Elusive Context

We do not know in which contexts Sanskrit astronomical treatises and their commentaries were produced, used, and copied. It is indeed possible to understand the structure and contents of the BSS as an advanced curriculum in mathematical astronomy, but there is no hard evidence that the text was used or composed with this in mind[61]. In standard mathematical commentary vocabulary, the rule of a root text shows/teaches (the verbal root is the same *pradṛṣ-*) something. The commentator expands the meaning and aim of the rule (*artha*). So in terms of the words whether the endeavor of a commentary is as a tool in a curriculum, or as a tool in a mathematical research, cannot be distinguished. Pṛthūdaka's virtuosity in multiplying contexts for rules displays this very ambiguity: such multiple contexts could be just as much those needed to teach the rule, as those needed to open the door onto yet unexplored possibilities.

58. (Angot, 2017, p. 66).

59. For the way in which Mallinātha Sūris (1350-1450)'s commentaries on Kālidāsa's (4th or 5th century) poems would thus have been devised for 'students' of different levels see (ibid., pp. 67, 73).

60. As suggested in (Ramasubramanian, 2011).

61. (Keller, 2014). How little we know of the way in which astral sciences was taught can be contrasted with the fact that we know that it was much studied, first because it was part of the duty of scholarly Brahmans to study the six *vedāṅgas*, and also because of the unceasing popularity of astrology.

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A Edition and Translation of Pṛthūdaka's commentary on BSS.12.17-18

This edition was made with the only three known manuscripts of the commentary for this part of the text[62]. Colebrooke's dates from the end of the 18th century or beginning of the 19th century, and probably the two others are later than this one.

1. Manuscript I_1: Paper manuscript IO 2769 belonging to Colebrooke (C) with many marginal notes by him; as presented in Figure 4, now in the British Library.

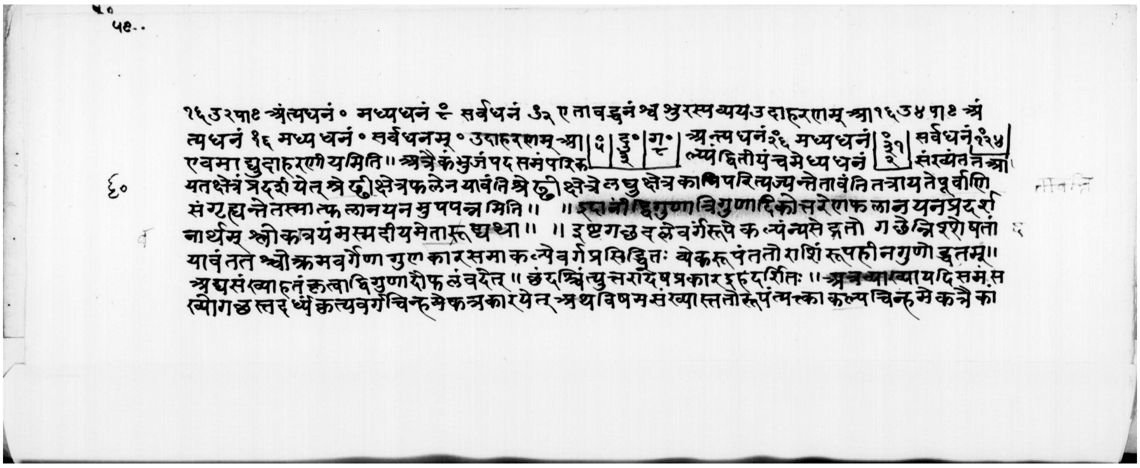

Figure 4: Manuscript I_1 belonging to Colebrooke

2. Manuscript I_2: Paper manuscript IO 2770; a copy of I_1, easier to read, in A4 format, as presented in Figure 5, now in the British Library. Maybe written with the intention to have an edition of the commentary published.

3. Manuscript V_1: Paper manuscript belonging to Dvivedin (D) with many marginal notes and corrections by him, possibly a copy of I_1 or of the manuscript from which I_1 was copied, as presented in Figure 6, now in Varanasī, at the Sarasvatī Bhavana.

This edition and translation also uses Dvivedin's edition and commentary of the BSS (Dvivedin, 1902), which quotes parts of PBSS and Colebrooke's translation of BSS and parts of PBSS (Colebrooke, 1817).

In the translation, boldface indicates the versified treatise (and quotations of it). It is thus contrasted with the prose parts of the commentary. The versified examples of the commentary are given in bold italics. In the translation, square braquets [] indicate words added to the translation from the Sanskrit[63]. Parenthesis () provide the transliteration of crucial Sanskrit terms and sometimes additional information. Sentences between ? ? indicate parts

62. I have chosen here not to provide a critical edition of the text but just an edition of the best text with a translation.

63. I have not included between brackets elements not stated in the Sanskrit but clearly part of the meaning of the sentence, and thus required for the English translation. Brackets thus underline my own interpretation of some parts of the text.

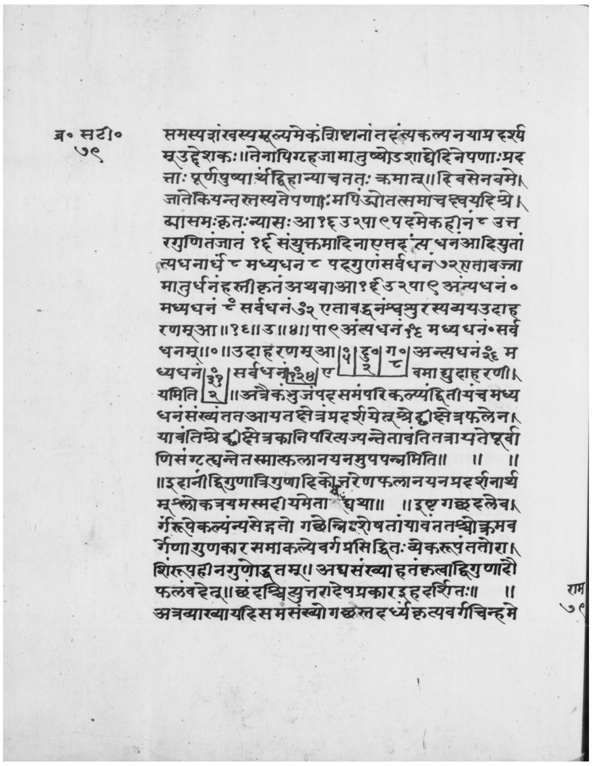

Figure 5: Manuscript I_2, a copy of I_1easier to read

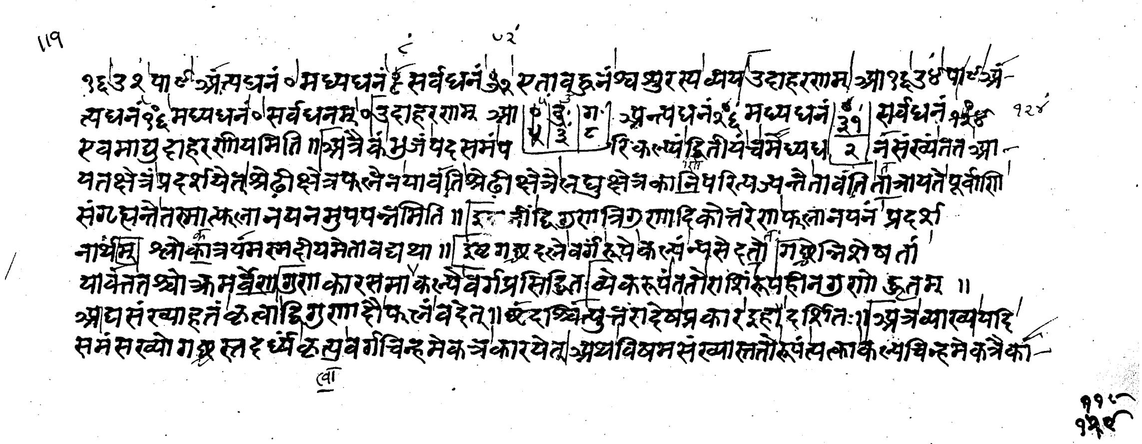

Figure 6: Manuscript V_1 belonging to Dvivedin, now in Benares

of the text that have been displaced: the editor places them here while the manuscripts place it where a ¿ can be found.

Finally, an important part of the commentary on PBSS.12.17, in which Pṛthūdaka adds his own rules and examples dealing with geometric progressions has been left out. The text transmitted by the manuscripts in PBSS.12.18 is quite corrupt, the edition here is still quite tentative and might be amended differently in further publications.

A.1 Text 1: Edition and Translation of PBSS.12.17

इदनीं श्रेढीव्यवहार आराभ्यते । तत्रादौ फलप्रदर्शनार्थम
मेरुयन्त्रार्धाकारायाश्चितेः फलप्रदर्शनार्थमार्यामाह ।

Now, the practice of progressions (*śreḍhī-vyavahāra*) is undertaken. Here, in the beginning, he states an *āryā* [verse] in order to show (*pradarśana*) the contents (*phala*) of a stack (*citi*) having the shape of half a Mt Meru Instrument (*meru-yantra*).

BSS.2.17.

पदमेकहीनमुत्तरगुणितं संयुक्तमादिनाऽन्त्यधनम ।
आदियुतान्त्यधनार्धं मध्यधनं पदगुणं गणितम ।। [64]

The rank (*pada*) less one, multiplied by the increase (*uttara*), added to the first is the last value (*antya-dhana*). Half the first increased by the last value is the mean value (*madhya-dhana*);[65] which, multiplied by the rank, is the total (*gaṇita*, e.g., value of all terms taken together or overall value).

64. An edition of this verse is given in [Dvivedin 1902, 177].

65. The Sanskrit and English expressions are ambiguous here. The mean value is: 'half [the first increased by the last value]' ($M = \frac{U_1+U_n}{2}$), and not 'half the first, increased by the last value' ($M = \frac{U_1}{2} + U_n$). But this last reading springs from the English translation. The Sanskrit, on the other hand, includes the first reading and 'the first increased by half the last value' ($M = U_1 + \frac{U_n}{2}$). Another possible reading of the Sanskrit compound could also have been: half the value of the first increased by the last. The commentator will make the reading of the compound clear. It is his reading that we have translated into the verse.

चितिसंख्यानं पदमुच्यते । उत्तरं चयत्प्रतिवर्धत आदिश्चारम्भचितिसंख्या । तेन पदमेकहीनमुत्तरगुणितं संयुक्तमादिनाऽन्त्यधनम भवति अन्त्यस्य चेष्टकानं इयं संख्या भवतीत्यः । एवमन्यस्यापि चयस्येष्टपदकल्पनया तत्संख्यानयनं कार्यम । अतः तस्यादियुतस्यान्त्यधनस्य तदर्धं तन्मध्यधनं भवति । पदसंख्या तुल्यान्यायाता[66] क्षेत्रस्य द्वितीयभुजः पृथक्तेन भवतीतयर्थः । तन्मध्यधनं पदगुणंर् सर्वधनं भवति । तस्याश्चितेरिष्टकासंख्या भवतीति ।

उद्देशकः

PBSS.12.17ex1.

मुखे द्वे इष्टके यत्र तिस्रस्तिस्रोऽधिकाश्चये ।

पञ्चहारचितिर् दृष्टावद तत्रेष्टकाफलम ।।

न्यासः

आदिः २ उ ३ प ५

The rank is said to be the stack's counting (*citisaṃkhyāna*). And the increase (*uttara*) is that which increases (*prati-vṛdh-*) the first [layer] and what initiates the stack's count (*citi-saṃkhyā*). Therefore, the number of bricks of that last [layer] is the meaning (*artha*) of '**the rank less one, multiplied by the increase, added to the first is the value of the last.**' And similarly, in another case too, the computation of that number should be made with a fixed given rank. Then, **half** the first increased by that last value is the **mean value**. The meaning is: differently, the second side of a rectangular figure whose other [side] is equal to the rank number is produced with that [rule]. That '**mean value multiplied by the rank**' is the overall amount (*sarvadhanaṃ* paraphrasing the *gaṇitaṃ* 'computation' of the treatise). The number of bricks of that stack is produced.

An example:

A five layer (hāra) stack is seen with two bricks at the beginning, increasing three by three.
Tell in this case the contents (phala) of bricks.

Setting:

The first is 2, i (abbreviation of 'increase') 3, r (abbreviation of 'rank') 5.

66. I have amended the Sanskrit text to attempt to make some sense here. The manuscripts read *padasaṃkhyā-tulyanyāyatatkṣetrasya*, lit. 'of a figure of that computation equal to the rank number', which makes no sense to me.

एकहीनं ४ उत्तरेणानेन ३ गुणितं १२ आदिना २ संयुक्तं १४ एतदन्तन्त्यधनं पुनरादिना युतम् १६ अस्यार्धम ८ एतन्मध्यधनां पदेनानेन ५ गुणितं ४० एतत्सर्वधनम ।

Subtracted by one, 4; multiplied by the common difference 3, 12; added to the first 2, 14. This is the amount in the last [layer]. The previous increased by the first 16; its half 8; this is the amount in the middle; multiplied by the number of terms 5, 40; this is the overall amount .

उद्देशकः

An example:

PBSS.12.17ex2.

अध्यार्धमादौ किल पादवृद्ध्या दत्तं द्विजेभ्यः सततं नृपेन हेमः त्रिरात्रं नवभगयुक्तं मध्यान्त्यसर्वाख्यधनानि कानि ।

Gold was given without interruption to twice born in three days and a ninth by a king, beginning indeed with one and a half [bhāras] increased [daily] by a quarter [of a bhāra]. What is the amount of those called the 'mean', the 'last' and the 'total' ?

न्यासः

Setting:

आ

१	०	३
१	१	१
२	४	९

f (abbreviation of 'first')

1	0	3
1	1	1
2	4	9

पदं $\begin{array}{|c|}\hline \text{२८} \\ \text{९} \\ \hline\end{array}$ एकहीनं $\begin{array}{|c|}\hline \text{१९} \\ \text{९} \\ \hline\end{array}$ उत्तरगुणितम् $\begin{array}{|c|}\hline \text{१९} \\ \text{३६} \\ \hline\end{array}$ संयुक्तमादिना $\begin{array}{|c|}\hline \text{७३} \\ \text{३६} \\ \hline\end{array}$ एतदन्तयधनं ।

The rank is $\begin{array}{|c|}\hline 28 \\ 9 \\ \hline\end{array}$, less one, $\begin{array}{|c|}\hline 19 \\ 9 \\ \hline\end{array}$ multiplied by the increase $\begin{array}{|c|}\hline 19 \\ 36 \\ \hline\end{array}$ added to the first $\begin{array}{|c|}\hline 73 \\ 36 \\ \hline\end{array}$, this is the amount of the last [offering].

आदियुतान्तधनोऽर्ध $\begin{array}{|c|}\hline \text{१२७} \\ \text{७२} \\ \hline\end{array}$ एतन्मध्यधनं ।

Half the first increased by the last (*anta*) value, $\begin{array}{|c|}\hline 127 \\ 72 \\ \hline\end{array}$, this is the mean amount.

पदगुणं जातम्सर्वधनं
८८९
१६२

।

[This] multiplied by the rank produces the overall amount [offered]

889
162

छेदभक्ते भार ५ शते

९
६०
८०

When dividing by the denominator, *bhāras* 5, in hundreds [of *palas*][67]

9
60
80

उद्देशकः

An example:

PBSS.12.17ex3

पुरा शङ्खो यदा शङ्खिः शेषाशचैव पणोत्तरः ।
सप्तमस्य तु शङ्खस्य मूल्यं ब्रूहि तदा कियत् ।।

The first conch [costs] six [paṇas], and the [prices of the] remaining ones increase by exactly a paṇa. Say then what is the price (mūlya) of that seventh conch.

न्यासः

Setting:

आदिः ६ उ १ प ७

The first is 6, *i* 1, *r* 7.

पदमेकहीनं ६ उत्तरगुणितं जातं ६ संयुक्तमादिना १२ एतत्सप्तमस्य शंखस्य मूल्यम ।

The rank less one, 6, multiplied by the increase produces 6, added to the first, 12. This is the price of that seventh conch.

एवंशिष्टानां तदन्त्यकल्पनया प्रदर्श्यम् ।

Similarly, one should show [the prices] of the remaining ones with an [other] assumed last one.

न्यासः

An example:

PBSS.12.17ex4[68]

67. These are standard measures of weights: Although the statement of the problem does not contain any measuring unit, the statement of the result shows that Pṛthūdaka assumed that the gold given by the king was weighted in *bhāras*. The following computation can then be reconstructed, knowing that 1 *bhāra* contains 2000 *palas*.: $\frac{889}{162} = 5 + \frac{79}{162}$. The quotient provides the 5 *bhāras* the text puts forth. A conversion to *palas* of the remaining fraction would then have been carried out: $2000 \times \frac{79}{162} = \frac{79000}{81} = 975 + \frac{25}{81}$. This would have been approximated to 975. Since $975 = (9 + \frac{3}{4}) \times 100$ and $\frac{3}{4} = \frac{60}{80}$, then 975 can be written in 'hundreds' as $9 + \frac{60}{80}$. It is possible that this expression of the result points to standard denominations: there may have been denominations for 100 of *palas* and for multiples of $\frac{1}{80}$th of hundreds of *palas* which would explain precisely why this is the way the result was finally expressed.

68. This example is edited in (Dvivedin, 1902, p. 186).

केनापि गृहजामातुः शोडशाऽऽद्ये दिने पणाः । प्रदत्ताः पुन्यपुष्यार्थम द्विऽहान्या च ततः क्रमात ।। दिवसे नवमे जाते कियन्तस्तस्य ते पंणाः । सम्पीदयैतत्समाचक्षव यदि श्रेध्याम श्रमः ।।

Somebody aiming at being very virtuous gave his son-in-law sixteen paṇas on the first day and decreased it by two per day in due order. How many paṇas were provided on the ninth day, report [the answer] if you have worked on progressions.

न्यासः आ १६ उ ˙२[69] प ९

Setting: f 16 i $\dot{2}$ r 9.

पदम एकहीनं ८ उत्तरगुणितं जातं ˙१६ सयुक्तमादिना ० एतदन्त्यधनं आदियुतान्त्यधनार्धं ८ मध्यधनं ८ पदगुणं सर्वधनं ७२ एतावज्जामातुर्धनं हस्तीकृतम ।

The rank less one, 8, multiplied by the increase produces $\dot{1}6$. Increased by the first [term], 0, this is the amount of the last. Half the sum of the amount of the first and the last value, 8, is the mean amount; multiplied by the rank is the whole amount, 72. As much is placed in the hands of the brother in law.

अथवा

Or else

आ ˙१६ उ २ प ९

f $\dot{1}6$ i 2 r 9

अन्त्यधनं ० मध्यधनं ˙८ सर्वधनं ˙७२ एतावद्धनं श्वशुरस्य व्ययः

The last value is 0, the mean value is $\dot{8}$ the whole value is $\dot{7}2$ so much is the disbursement amount for the father-in-law.

उदाहरणम्

A variation (*udāharaṇa*)

आ १६ उ ˙४ प ९

f 16 i $\dot{4}$ r 9

अन्त्यधनं ˙ मध्यधनम् ० सर्वधनम् ०

The last value is $\dot{1}6$, the mean value is 0, the whole value is 0.

उदाहरंणम्

A variation:

आ ˙५ उ ˙ग ८

f $\dot{5}$ i $\dot{3}$ r 8

69. I haven't figured out how to allow the dot to appear immediately above the *devanagarī* numbers, and so I have written it on their left, as they appear sometimes as well.

अन्त्यधनं ˙२६ मध्यधनं ˙३१/२ सर्वधनं ˙१२४
एवमाद्युदाहरणीयमिति ।।

The last value is $\dot{2}6$, the mean value is $\frac{\dot{3}1}{2}$, the whole value is $1\dot{2}4$; thus should the following be varied.

अत्रैकं भजं पदसमं परिकल्प्यं द्वितीयं च मध्यधनं संख्यं एततायतक्षेत्रं प्रदर्शयेत्श्रेधीक्षेत्रफलेन यावन्ति श्रेधीक्षेत्रे लघुक्षेत्रेकानि परित्यज्यन्ते तावन्ति तत्रायते पूर्वाणि संगृह्यन्ते तस्मात्फलानयनमुपन्नमिति ।।

In this case, one side should be assumed to be equal to the 'rank' and the second reckoned as the mean value. On should show this rectangle. With the area of the progression figure (*śreḍhīkṣetraphala*), so many small figures in a progression figure (*śreḍhīkṣetra*) that have been discarded are in that case gathered in front, in the rectangle. Thus the computation of the result has been proven.

(...)

A.2 Text 2: Edition and translation of PBSS.12.18

इदानिमाद्युत्तरसर्वधनानां परिज्ञानेन गच्छानयनामार्यामाह-

Now he states an *āryā* [verse] for the computation of the rank (*gaccha*) with the knowledge of the first (*ādi*), the increase (*uttara*) and the overall value (*sarvadhana*):

BSS.12.18

उत्तरहिनद्विगुणादिशेषवर्गं धनोत्तराष्टवधे ।
प्रक्षिप्य पदं शेषानं द्विगुणोत्तरहृतं गच्छः ।।

Having added the square of the remainder of the subtraction of the increase with twice the first to the product of the [overall] value by eight times the increase, the rank (*gaccha*) is the square-root, less the remainder, divided by twice the increase.

द्विगुणश्चादिश्च द्विगुणादिः तस्मादुत्तरहीनाद् द्विगुणादेर्यः शेषः स उत्तरहीनादिशेषः तस्यानष्टस्य वर्गः कार्यः । तं प्रक्षिप्य के ? इत्याह धनोत्तराष्टवधे धनं सर्वधनमुच्यते । तत्र शेषवर्गं प्रक्षिप्य मूलं ग्राह्यं । ततस्तत्पदं शेषेनानष्टेनोनं कृत्वा द्विगुणोत्तारेण विभजेत् फलं गच्छो भवति ।

That which is the remainder of the subtraction of the increase with twice the first- which is [a *karmadhāraya* compound to be understood as] that which is twice and the first is '**twice the first**'- one should make the square of that undestroyed remainder of the '**subtraction of the increase with twice the first**'. '**Having added**' that [and] what? he states: '**the product of the [overall] value by eight times the increase**',[with the term] 'value' (*dhana*), the overall value is stated. That which is the product of that increase and eight [is 'eight times the increase']. In that case, having added the square of the remainder, the square-root should be taken. Then, having decreased its square-root by the undestroyed remainder, one should divide by twice the increase, the rank is produced.

उद्देशकः

An example:

PBSS.12.18ex1[70]

मुखेदशेष्टका यत्र पञ्चपञ्चाधिकाश्चये । इष्टकानां शतं लग्नं चेतौ तत्र पदं वद ।।

When a pile is made of a hundred adhering bricks, whose first (mukha) [layer] has ten bricks-[each subsequent layer] increasing five by five-tell me the pile's rank (pada) in this case.

न्यासः

Setting:

आ १० उ ५ ग ० सर्वधनं १

f 10, i 5, r 0, overall amount 100

करणं

Procedure:

70. This example is quoted in (Dvivedin, 1902, p. 180).

द्विगुणादिः २० उत्तरेणानेन ५ हीनः १५ अस्य वर्गो यं २२५ सर्वधनं १०० अष्टो ८ उत्तरं ५ एतेषां वधः ४००० । अत्र शेषवर्गं २२५ प्रक्षिप्य जातो राशिः ४२२५ । अतः पदं ६५ शेषेणानेन १५ उनं जातं ५० द्विगुणोत्तेणानेन १० भक्त्वा लब्धं ५ एषो गच्छः । एवमन्यत्रापि ।

Twice the first, 20, decreased by this increase, 5, is 15; its square, 225; that which is the overall value, 100, eight, 8, and the increase, 5, their product, 4000. In this case, having added the remaining square 225, the quantity obtained is 4225. Then, decreasing the square root, 65, by 15, what is produced is 50; having divided [it] by twice this increase 10, what is obtained is 5. This is the rank. It is just like that in other cases also.

अत्र मध्यमाहरणबीजवसना ।

In this case, an algebraical explanation with an 'elimination of the middle'.

तद्यथा आ १० उ ५ ग या १

It is as follows: *f* 10, *i* 5, *as* (abbreviation of 'as much as') 1.

पदं या १ एकहीनं जातं या १ रू ˙ १ उत्तरगुणितं या ५ रूप ˙५ संयुक्तमादिना जातं या ५ रूप ५ एतादनत्यधनमादियुतं या ५ रू १५ अस्यार्धं या $\frac{५}{२}$ रू $\frac{१५}{२}$ एतन्मध्यधनं पदगुणं जातं या व $\frac{५}{२}$ या $\frac{१५}{२}$ तत्सर्वधनं ।

The rank, *as.* 1, decreased by one, produces *as.* 1 *in.* (first letters of 'integer') $\dot{1}$; which multiplied by the increase is *as.* 5 integer $\dot{5}$; and, added to the initial term, produces *as.* 5 integer 5, this is the last term. Added to the first term, *as.* 5 *in.* 15; its half is *as.* $\frac{5}{2}$ *in.* $\frac{15}{2}$. This is the mean value; multiplied by the rank, what is produced is *as. s.* (abbreviation for square) $\frac{5}{2}$ *as.* $\frac{15}{2}$, its overall value.

शतेन सममिति समीकरणे द्विको गुणको युज्यते यो य मध्यस्थछेधः प्राक्प्रदर्शितराशेस्तथा च कृते पक्षद्वयस्य दर्शनं रू २०० । अत्र वर्गाव्यक्ताः शोध्या यस्माद्रूपाणि तदधस्ताद (BSS.18.43) इति कृते प्रथमपक्षे प्रदर्शनं या व ५ या १५ द्वितीयपक्षस्य रु २०० ।

Since it is equal to a hundred, when equalling, two, which is the denominator standing in the middle is harnessed as multiplier to a hundred, and thus when it is made, what is seen for the second side is *in.* 200. In this case, according to '**Having subtracted the [coefficients of the] square of the unknowns [and the unknowns] from which the integers are below.**' (BSS.18.43), when made, what is shown on the first side is *as. s.* 5 *as.* 15, for the second side, *in.* 200.

अत्र वर्गचतुर्गुणानामित्यादि कृते रूपराशिदर्शनं ४००० ।

In this case, when applying '**of four times [the coefficient of] the square**' (BSS.18.44) and so forth, the apparent integer quantity is 4000.[71]

एषो धनोत्तराष्टवधः

This is **the product of eight times the [overall] amount and the increase (BSS.12.18)**

यतो यत्र सवर्ण्यने गूनकः २ अनेनावश्यं गुणनीयो रूपराशिश्चतुर्भिश्चातो ऽष्टभिर्गुणानि वधः

because the multiplier is 2 when homogenizing (*savarṇyane*), therefore, necessarily (*avaśya*), the integer quantity should be multiplied by four times that, which is thus a multiplication by eight.[72]

?तदुत्तरगुणितम् ?

? that is multiplied by the increase ?[73]

(...)[74]

...यावकाच्च यावद्रूपं विशोध्यते तावदृणगतं भवति

...and from whatever the integer should be subtracted, as much becomes a 'subtractive'.[75]

¿

उत्तरतुल्यरूपाण्यृणगताणि भवन्ति ।

the units (*rūpa*) equal to the increase become negative

71. As cursorily described in section 3.4.2, the computation here follows rule BSS.18.44, that is computes $\frac{\sqrt{4ac+b^2}-b}{2a}$. Pṛthūdaka further underlines how at each step, the computation can simultaneously be understood as being what is computed according to PBSS.12.18, $\frac{\sqrt{(2U_0-u)^2+(8uS)}-(2U_0-u)}{2u}$.

72. While Pṛthūdaka explains how $4c = 8S$, he details that $c = 2 \times 100$ because when the fractional coefficients of the equation were made into integers, its constant term, 'on the second side', called 'the integer' was doubled. Therefore $4c = 8 \times 100 = 8S$. Note that in this part of the text the term 'integer' (*rūpa*) can also apply to any term which is not an 'undetermined quantity', eg. an algebraical unknown.

73. In all manuscripts, this [part of a] sentence is stated after *ṛṇa-gataṃ bhavati*, 'becomes a subtractive'.

74. The manuscripts seem to have transmitted a damaged part of the text here. They give the impression that a part has been omitted: from the doubling of the integer of the equation (c) we slip to the increase u possibly because there had been a sentence on the doubling of the increase $2u = 2a$, when drawing parallels between the process described in BSS.18.44 and BSS.12.18. Furthermore, another part of a sentence (the one translated above) seems to have 'slipped down' into a portion of the text it has no concern with. The text here is like a fragmented pot whose pieces i have tentatively glued back together.

75. Possibly what is considered here could be the computation of $2U_0 - u$ in which $-u$ is computed. This makes sense with the computation detailed bellow. But from the point of view of of the generality of the fragmentary statement here, this sentence could equally concern, in the process described in BSS.18.44, the computation of $-b$, which corresponds to $-(2U_0 - u)$ in the process described in BSS.12.18. Here the term *rūpa* translated as 'integer' is understood as referring to any quantity which isn't an algebraical unknown.

तानि च सयुक्तान्यादिना यावत्क्रियन्ते यावच्च पुनरप्यादियुतानि तावदादिर्द्विगुणो धनराशिस्

and these are united to the first [which has been doubled] and these [units] are increased again by the first [which is doubled] as much as the first multiplied by two is a ‘positive’ quantity[76]

तेन च सहोतरमेव धनर्णयोर्योगे युज्यते शेषश्च स एवेह यावकराशिः स एव तेन मध्यवर्गसहितानामिति कृते दर्शनम ४२२५ ।

and one should sum that [doubled first term] with the increase just as when the sum of the ‘positive’ and ‘negative’ [is taken] and that precisely is the **the remainder** (BSS.12.18) which is also the [coefficient of the] *as.* quantity with that [which was previously computed], when **increased by the square of the middle** (BSS.18.44) is performed, 4225 appears.

अतः पदं १५ मध्योनं १० द्विगुणितवर्गैनहृतं मध्यो भवति

Then, the square-root, 65, decreased by 10, is divided by twice the [coefficient of the] square; the middle is produced.

गच्छः इत्यर्थः यतो यावको ऽत्रगच्छ इति।

The meaning is: the rank is produced, since here the *as* was the rank.

अथ यत्रादिर्नज्ञायते उत्तरगच्छसर्वधनानि र्नज्ञायन्ते तत्र सर्वधनं गच्छेन विभज्य मध्यधनं भवति । तद्द्वगुणितं कृत्वा तस्माद्रूपेन गच्छोत्तरवधं विशोध्य ये शेषस्यार्धमादिर्भवति ।

Now, when the first is not known, and the increase, rank and overall value are known then having divided the overall value by the rank, the mean value is produced. Having multiplied it by two, having subtracted the product of the rank [decreased] by one and the increase, half of that remainder produces the first term.

76. The corruption of the text makes the interpretation uncertain. Here, the computation might be focussed on the units (*rūpa* in plural) which accumulated provide the values involved in the computation of the process evoked in BSS.12.18: the ‘increase’ (*uttara*) and twice the ‘first’ [term] (*ādi*). First, while computing $2U_0 - u$, $-u$ would be evoked as made of several units, then $-u + 2U_0$ would be carried out. The idea that an integer is made of several units, gives the feeling that Pṛthūdaka could be evoking a computational technique not unlike what is known of cauri shell computations...

तद्यथा आ ० उ ३ ग ५ सर्वधनं ४० एतद्गच्छेन विभक्तं ८ मध्यधनं द्विगुणमेतत् १६ व्येकपदं ४ उत्तरं ३ अनयोर्वधः १२ एतं विशोध्य शेषार्धं २ अयमादिः । एवमन्यात्रापि योज्यमिति ।

For instance, f 0 i 3 r 5 overall value 40. This, divided by the rank, 8, is the mean amount. This is doubled, 16. The product of the rank minus one, 4, and the increase, 3, is 12. Having subtracted this [from the doubled value], half of the remainder is 2. This is the first. This to be applied in other cases also.

यत्र तूत्तरं नज्ञायते तत्रापि सर्वधनं गच्छभक्तं मध्यधनं भवति । तद्द्विगुणं कृत्वा तस्माद्द्विगुणमादिं ४ विशोधयेत्ततः शेषाद्व्येकेन पदेन यद् लब्धं तदुत्तरं भवति । तद्यथा आ २ उ ० प ७ सर्वधनं ७७ सर्वधनाद्गच्छेन मध्यधनं मानिय तद्द्विगुणं कृत्वा ...[77]

When indeed the increase is not known, then also the overall value divided by the rank is the mean amount/middle value. Having doubled it, one should subtract from it twice the first, 4. Then what is obtained from the remainder [divided] by the rank less one is the increase. For instance, f 2 i 0 r 7 the overall value 77. One should measure the mean measure from the overall value [divided] by the rank, having doubled it...

...[यत्र चादिः चोत्तरं नज्ञायते तत्रापि सर्वधनं गच्छभक्तं मध्यधनं भवति । द्विगुणं कृत्वा]नष्टं स्थपयेत्तत इष्टमुत्तरं परिकल्प्यं तेन व्येकं गच्छं गुणायित्वानष्टाद् विशोधयेच्छेषस्यार्धम आदिर्भवति । उत्तरं च यदिकल्पितं तदेवभवति । तद्यथा आ ० उ ० ग ९ सर्वधनम् ५७६ अतो गच्छेन लब्धं मध्यधनं ६४ एतद्द्विगुणमनष्टसंज्ञां १२८ अत एकं रूपमुत्तरं परिकल्प्य तद्गुणं व्येकं पदं ८ अनाष्टाद्विशोध्य शेषं अस्यार्धमादिः ६० उ १ ग ९ ।

...[when the first and increase are not known, then also the overall value divided by the rank is the mean amount. Having doubled it], one should place the [result which is] undestroyed. Then, one should choose a desired increase. Having multiplied the increase by that [rank] decreased by one, one should subtract [it] from the undestroyed. Half the remainder is the first, and the increase is just as it was chosen. For instance, f 0 i 0 r 9 overall value 576. Now what is obtained [from the division of this] by the rank is the mean value 64. Its double is what is called the undestroyed 128. Now having chosen the integer one as increase, the product of that [with] the rank less one is 8, having subtracted [it] from the undestroyed, half the remainder is the first 60, i 1 r 9.

77. All the manuscripts are corrupt here and seem to ommit some parts of the original text.

अथवा द्विगुणमुत्तरं परिकल्प्य तद्गुणं व्येकं पदं १६ अनष्टराशेर्विशोध्य शेषं ११२ अस्यार्धमादिः ५६ उ २ ग ९ सर्वधनं ५७५ । सार्धं रूपद्वयमुत्तरं परिकल्प्य तेन व्येकं पदं गुणितं २० एतदानषटाद विशोध्य शेषार्धमादिः ५४ उ $\frac{५}{२}$ ग ९ सर्वधनं ५७६ ।

Or else, having chosen a doubled increase (e.g. the increase is two), the product of that [with] the rank less one is 16, having subtracted [it] from the undestroyed quantity, the remainder is 112, its half is the first 56, i 2, r 9, overall value 576. Having chosen an increase which is two integers and a half, the rank less one, multiplied by that is 20. Having subtracted this from the undestroyed, half the remainder is the first 54, i $\frac{5}{2}$ r 9, the total amount is 576.

एवमभिन्नेषु च सर्वत्र योज्यमिति यदा पुनराद्युत्तरगच्छास्त्रयो ऽपि न ज्ञायन्ते तदा गच्छमिष्टं परिकल्प्य तदेवानतरप्रदर्शि क्रमं कृत्वासर्वं योजयेत् ।

And in this way, [the rule] is used on non-fractions (*abhinneṣu*) in all cases. However, when all three also, the first, the increase and the rank are not known, then when having chosen a desired rank, having executed afterwards, in due order, just as shown, all should be used.

अथ यदा उत्तरगच्छमध्यमसर्वधनानि वर्गगतानीष्यन्ते तदा कमपि वर्गराशिं गच्छं परिकल्पयेत । गच्छश्च षोडशहता उत्तरं भवति । द्व्यूनगच्छस्य वर्ग आदिर्भवत्येतैर्मध्यसर्वधनानि प्राग्वत् । तद्यथा वर्गराशिः ९ एषो गच्छः षोडशहतो जातो १४४ एतदुत्तरं गच्छे द्व्यूनः ७ अस्य वर्गः ४९ आदिरयं आ ४९ उ १४४ ग ९ मध्यधनं ५६२५ पञ्चैवैतानि वर्गौ ।

Now, when the increase, the rank, the mean and total amounts are required to be squares, then one should choose any square quantity as the rank. The increase is sixteen times the rank. The first is the square of the rank decreased by two. With these, as before, the mean and total amount [are obtained]. It is as follows: the square quantity is 9, this rank is multiplied by sixteen, what is produced is 144. This is the increase. When the rank is decreased by two, 7, its square is 49. This is the first. f 49 i 144 ga 9 the mean amount is 625 and the overall value is 5625. These five indeed are squares.

एवम्व्युत्पत्त्यर्थं बहुधोहारणीयं।
अस्मभिर्ग्रन्थगौरवभयान्नोदाहृतं यतः सकलसिद्धान्तो ऽस्माभिर्व्याख्यातुमारब्ध इति ॥ ॥

In this way, many variations aiming at proficiency (*vyutpatti*) [can be given]. It is not made explicit for fear of the weight of the composition since we have undertaken to explain/comment the whole treatise/astronomical system (*siddhānta*).

B Givens and Results in Solved Examples

2.2

Table 12: Givens and Results in Solved Examples of PBSS.12.17

Example	Givens	Results
PBSS.12.17ex1	$U_1 = 2,\ u = 3,\ n = 5$	$U_n = 14,\ M = 8,\ S = 40$
PBSS.12.17ex1	$U_1 = 1 + \frac{1}{2},\ u = \frac{1}{4},\ n = 3 + \frac{1}{9}$	$U_n = \frac{73}{36},\ M = \frac{127}{72},\ S = \frac{889}{162}$[78]
PBSS.12.17ex3	$U_1 = 1,\ u = 6,\ n = 7$	$U_n = 12$
PBSS.12.17ex4	$U_1 = 16,\ u = -2,\ n = 9$	$U_n = 0,\ M = 8,\ S = 72$
'or else'	$U_1 = -16,\ u = 2,\ n = 9$	$U_n = 0,\ M = -8,\ S = -72$
'a variation'	$U_1 = 16,\ u = -4,\ n = 9$	$U_n = -16,\ M = 0,\ S = 0$
'a variation'	$U_1 = -5,\ u = -3,\ n = 8$	$U_n = -26,\ M = \frac{31}{2},\ S = -124$
PBSS.12.17ex5	$U_1 = 6,\ q = 3,\ n = 10$	$S = 177144$
PBSS.12.17ex6	$U_1 = \frac{7}{2},\ q = \frac{5}{2},\ n = 3$	$S = \frac{273}{8}$

Table 13: Givens and Results in Solved Examples of PBSS.12.18

Example	Givens	Results
PBSS.12.18ex1	$U_1 = 10,\ u = 5,\ S = 100$	$n = 5$
'For instance'	$u = 3,\ n = 5\ S = 40$	$U_1 = 2$
'For instance'	$U_1 = 2,\ n = 7,\ S = 77$	$[u = 3]$[79]
'For instance'	$n = 9,\ S = 576$	$M = 64$, choosing $u = 1,\ U_1 = 60$
'or else'	$n = 9,\ S = 576$	$M = 64$, choosing $u = 2,\ U_1 = 56$
	$n = 9,\ S = 576$	$M = 64$, choosing $u = 2 + \frac{1}{2},\ U_1 = 54$
	$n = 9$	$u = 144$, $U_1 = 49,\ M = 625,\ S = 5625$ all are squares

78. S is converted subsequently into measures of weight.
79. The manuscripts are deficient here, so this value is interpolated by me.