Real indeterminacy and dynamics of asset price bubbles in general equilibrium

Stefano Bosi, Cuong Le Van, Ngoc-Sang Pham

Real indeterminacy and dynamics of asset price bubbles in general equilibrium*

Stefano BOSI[†] Cuong LE VAN[‡] Ngoc-Sang PHAM[§]

September 25, 2021

Abstract

We show that both real indeterminacy and asset price bubble may appear in an infinite-horizon exchange economy with infinitely lived agents and an imperfect financial market. We explain how the asset structure and heterogeneity (in terms of preferences and endowments) affect the existence and the dynamics of asset price bubbles as well as the equilibrium indeterminacy. Moreover, this paper bridges the literature on bubbles in models with infinitely lived agents and that in overlapping generations models.
Keywords: asset price bubble, real indeterminacy, borrowing constraint, intertemporal equilibrium, infinite-horizon.
JEL Classifications: D53, E44, G12.

1 Introduction

The existence and dynamics of asset price bubbles are one of the fundamental questions in economics and finance. According to the classical paper by Santos and Woodford (1997), conditions under which bubbles exist are relatively fragile. After the global financial crisis of 2007- 2009, this topic has regained momentum and different new mechanisms of bubbles have been proposed.[1] To date, the literature on rational asset price bubbles has focused on two frameworks: (1) overlapping generations models (OLG) and (2) infinite-horizon general equilibrium models with infinitely

*We thank Tomohiro Hirano, Carmen Camacho, and anonymous Reviewers for their constructive suggestions and comments.
[†]EPEE, Université Paris-Saclay. Email: stefano.bosi@universite-paris-saclay.fr
[‡]IPAG, CNRS, PSE, and TIMAS. Email: Cuong.Le-Van@univ-paris1.fr
[§]EM Normandie Business School, Métis Lab. Email: npham@em-normandie.com
[1]See Farhi and Tirole (2012), Martin and Ventura (2012), Gali (2014, 2021), Hirano and Yanagawa (2017), Miao and Wang (2012, 2018), Barbie and Hillebrand (2018) among others. The reader can also find excellent surveys in Brunnermeier and Oehmke (2012), Miao (2014) and Martin and Ventura (2018).

lived agents. Note that since the influential paper of Tirole (1985), numerous studies have privileged OLG models to study the existence of bubbles and their macroeconomic implications. Although it is also important to study infinite-horizon models of bubbles,[2] this type of framework has received relatively less attention.[3] As recognized by Kocherlakota (2008), Miao (2014) and Martin and Ventura (2018), our understanding of bubbles in infinite-horizon models is far from complete.

The present paper aims to address basic and open questions about rational asset price bubbles in intertemporal competitive equilibrium: Why do asset price bubbles exist in equilibrium? What is the connection between the existence of bubbles on the one hand and the economic agents' consumption and trade on the other? How do the existence and dynamics of the asset price bubble depend on asset structure and economic fundamentals such as endowments?

To answer these questions, we consider an infinite-horizon general equilibrium model with a finite number of agents, where there are one consumption good and one financial asset as Lucas' tree (Lucas, 1978). Our model has two key ingredients: first, agents are heterogeneous in terms of endowments and preferences; and second, there exist financial frictions in the form of short-sale constraints, i.e., the asset quantity that each agent can buy does not exceed an exogenous limit. As in Tirole (1982), Kocherlakota (1992), Santos and Woodford (1997), given an equilibrium, we say that there exists a bubble in this equilibrium if the equilibrium asset price exceeds the fundamental value of the asset, defined as the present value of dividend streams. An equilibrium with (resp., without) bubble is said to be bubbly (resp., bubbleless).

Our contribution is three-fold. First, we provide new necessary conditions for the existence of bubbles in equilibrium. The literature on bubbles in infinite-horizon models shows several conditions ruling out asset price bubbles. Kocherlakota (1992) questioned the relationship between the existence of bubbles and borrowing constraints. He pointed out that, in the presence of bubbles, the limit infimum of the differences between asset holding and borrowing limit equals zero. We go further by proving that in any equilibrium with bubbles, there exist at least two agents whose borrowing constraints bind (i.e., asset holding equals borrowing limit) at infinitely many dates and whose assets holdings fluctuate over time. Moreover, if borrowing constraints of an agent never bind, the existence of bubble requires that the asset holding of this agent must converge to the borrowing limit.

Another famous no-bubble condition, given by Theorem 3 in Santos and Woodford (1997), states that, under mild conditions, bubbles are ruled out if the present value of aggregate endowments is finite. This condition still holds in a model with debt constraints (Werner, 2014) and in a model with land and collateral constraints (Bosi et al., 2018b). In our model with short-sale constraints, we also obtain this no-bubble condition (see Corollary 3).

[2] Miao (2014) explains why we need to study infinite-horizon models of bubbles.

[3] In such models, it is difficult to characterize or compute the equilibrium. It is also not easy to provide non-trivial examples of equilibrium.

Motivated by the fact that most of the no-bubble conditions are based on endogenous variables, we contribute to the literature by providing conditions based on fundamentals. The first condition (see Corollary 2) shows the role of the borrowing limits: there is no equilibrium with bubbles if borrowing limits are high enough. The second condition (see Proposition 3) shows the role of impatience: under the assumption of uniform impatience, there is no bubble if agents strongly prefer the present. The intuition is simple: if agents strongly prefer the present, they do not buy the asset in the long run, ruling out bubbles. In particular, this is the situation in finite-horizon models in which no one buys the asset in the last period eliminating the possibility of bubbles.

Our second contribution concerns the construction of models with bubbles where we can explicitly characterize the existence and the dynamics of bubbles by using fundamentals such as agents' endowments, borrowing limits, asset dividends, and asset supply.

The above no-bubble conditions suggest us to focus on a two-agent model and characterize the equilibrium in which borrowing constraints of both agents bind infinitely many dates (more precisely, the first agent's borrowing constraint will bind, for instance, at even dates and that of the second agent at odd dates). Focusing on such equilibrium, we find that bubbles are ruled out if (1) borrowing limits of agents are high enough and (2) the value of endowments (discounted by using the interest rates of *the benchmark economy*) of the agent who buys asset vanishes in the infinity (see Proposition 4). By consequence, there cannot exist a bubble if the benchmark interest rates are high. The basic idea is that asset buyers' income must be high enough so that they are willing to buy the asset, even when the asset price exceeds the fundamental value.

Notice that this condition concerning the benchmark interest rates is based on fundamentals and cannot be obtained from the famous condition in Santos and Woodford (1997), which is based on endogenous variables. Our finding can be viewed as an extension of the no-bubble condition of Tirole (1985) in an OLG model[4] to our general equilibrium model with infinitely lived agents. In this sense, our paper is the first to create the connection between the no-bubble conditions in Tirole (1985) and those in infinite-horizon general equilibrium models. Recall that Tirole (1985), Farhi and Tirole (2012) need the convergence of interest rates of the economy without asset while we do not require such convergence.

In the existing literature, there are some examples of bubbles in general equilibrium models with infinitely lived agents.[5] Concerning the asset having zero dividends and positive supply (i.e., fiat money), Bewley (1980) (Section 13), Townsend (1980),

[4]It states that there is no bubble if the steady-state interest rate of the economy without bubble asset is higher than the population growth rate

[5]Brunnermeier and Oehmke (2012), Miao (2014), and Martin and Ventura (2018) provide excellent surveys on bubbles. Here, we focus on bubbles in general equilibrium models with infinitely lived agents.

Kocherlakota (1992) (Example 1) and Scheinkman and Weiss (1986) show in infinite-horizon general equilibrium models, that, when borrowing is not allowed, there exists an equilibrium in which fiat money's price is strictly positive. Santos and Woodford (1997) present several examples of this kind of bubbles: their examples 4.1, 4.2 study fiat money in deterministic models while their example 4.4 investigates fiat money in a stochastic model. Hirano and Yanagawa (2017) also give sufficient conditions for the existence of stochastic bubbles of an asset without dividend. There are a few examples of bubbles of assets with positive dividends. In a deterministic set-up, Example 4.3 in Santos and Woodford (1997) studies bubbles of an asset with positive dividends but with zero net supply. Like us, Example 4.5 in Santos and Woodford (1997) also investigates bubbles of the Lucas' tree, although they use a stochastic model with a single representative household.[6] Recently, Le Van and Pham (2016), Bosi et al. (2017a), Bosi et al. (2018b) show that bubbles of assets with positive dividends and positive net supply may appear even in deterministic models. Bloise and Citanna (2019) provide a sufficient condition based on trade and punishment for default for the existence of the bubble of an asset with vanishing dividends of an equilibrium whose sequence of allocations converges.

To date, no example shows how the existence and the dynamics of asset price bubbles depend on fundamentals such as endowments and the asset structure (dividends, asset supply, and borrowing limits). Our paper contributes to fill this gap. More precisely, Section 4 of the present paper provides several conditions (based on fundamentals) for the existence of bubbles. Notice that this task is not easy because studying equilibrium requires us to work with a dynamical system that has infinitely many parameters (which are our model's fundamentals and dividends). We point out that: when the benchmark economy has low interest rates and verifies the seesaw property, the following factors promote bubble in equilibrium: (1) asset supply is low, (2) borrowing limits of agents are low, (3) the level of heterogeneity (proxied by the differences between agents' fundamentals such as endowments, initial asset holdings, rates of time preferences) is high, and (4) asset dividends are low with respect to agents' endowments. Consequently, our results suggest that bubbles may appear if (i) the agents' endowments grow asymmetrically, and (ii) there is a shortage of financial assets (i.e., there is a low supply and assets provide low dividends).[7] We also prove that bubbles may not exist if one of these four conditions is not satisfied.

Let us explain the basic mechanism of asset price bubbles in our model. In each period, there is at least one agent who really needs to save by buying the asset. When the asset supply and borrowing limits are low, the asset price would be high (even higher than its fundamental value) because it is the only way allowing this agent to

[6] In this example, they introduce a sequence of non-stationary stochastic discount factors and show that bubbles may exist under a state-price process but not under another state-price process.

[7] In our model, the intertemporal utility function is time-separable. Araujo et al. (2011) consider the utility function $\sum_{t\geq 0} \zeta_{i,t} u(c_{i,t}) + \epsilon_i \inf_{t\geq 0} u_i(c_{i,t})$ and show that the parameter ϵ_i plays a key role on the existence of bubbles.

smooth consumption.[8]

To the best of our knowledge, we are the first to show that there is a continuum of bubbly equilibria (with real indeterminacy) in models with infinitely lived agents and with assets having positive supply and possibly positive dividends. With additional specifications, we can further provide a complete characterization of the set of equilibria with bubbles, and compute the bubble component as a function of fundamentals (see Proposition 6). In our models of bubbles, the asset price may converge to any value in $[0, \infty]$ or may fluctuate over time, depending on the fundamentals' properties. Furthermore, here the existence of bubbles does not violate individual transversality conditions (TVC, henceforth). Notice that individual TVC ensures the optimality of agents' choices and always holds in equilibrium while bubbles can exist or be ruled out.

Our third contribution is to clarify the relationship between the existence of bubble and real indeterminacy. The equilibrium indeterminacy in our model is global, and no local approximation is invoked to prove this indeterminacy. Our proof relies on the fact that asset prices, in some cases, can be recursively computed. Hence the sequence of prices can be computed as a function of the initial price. Therefore, at the initial date, any value can be an equilibrium price if it is low enough so that the price and the bubble component will be not too high in the future, ensuring that agents can buy them. As a result, there may be a continuum of asset prices and a continuum of equilibrium trajectories. Notice that we require neither the convergence of these trajectories nor the existence of a steady-state. So, the indeterminacy in our model is quite different from the concept of dynamic indeterminacy in macroeconomics (see Benhabib and Farmer (1999), Farmer (2019) for surveys on this issue). Our result on indeterminacy complements the findings in Kehoe and Levine (1985), Kehoe et al. (1990) who show that in a general equilibrium model with a finite number of infinitely lived consumers and with complete financial markets, equilibria are generically determinate.[9] Unlike them, we introduce financial frictions and prove that equilibria may be generically indeterminate. Moreover, the real indeterminacy in our model is associated with the existence of bubbles.

Our paper also contributes to understanding the relationship between financial assets, the existence of bubbles, and welfare. First, we prove that the equilibrium allocation in a model with bubble strictly Pareto dominates the autarkic one. The basic intuition is that the financial asset, even it contains a bubble component, provides two ways to smooth consumption: saving and borrowing. Thanks to this, agents can transfer their wealth from dates with high endowments to dates with low endowments. Second and more importantly, we show in Proposition 9 that, in our models where there are multiple equilibria, the allocation of any bubbly equilibrium strictly Pareto

[8] We can prove that, if we introduce a new asset with which agents can borrow without limit, there will be no bubble in equilibrium.

[9] More precisely, under conditions in Proposition 2 in Kehoe et al. (1990), there is a finite (odd) number of equilibria.

dominates that of the bubbleless equilibrium (notice that the bubbleless equilibrium is not necessarily the autarkic one). The idea is that both the asset prices and the bubble component increase with the initial price. Consequently, the initial asset price of the bubbleless equilibrium is lower than that of any bubbly equilibrium. When the price increases, it helps to reduce the marginal rate of substitution of agents, which in turn allows agents to smooth their consumption. Therefore, individual welfare generated by any bubbly equilibrium is higher than that generated by the bubbleless one. This point is consistent with Proposition 9 of Hirano and Yanagawa (2017). The difference is that we work under general utility functions while they only focus on the logarithmic utility function.

The rest of the paper is organized as follows. Section 2 presents the framework and provides the fundamental properties of equilibrium. Sections 3 provides no-bubble conditions. A number of models with bubbles and real indeterminacy are presented in Section 4. Finally, Section 5 concludes and mentions future works. Technical proofs are gathered in the appendices.

2 An exchange economy with short-sale constraints

Consider an infinite-horizon discrete-time model with short-sale as in Kocherlakota (1992). There are a finite number m of agents, a single consumption good and an asset. The asset structure is similar to Lucas' tree (Lucas, 1978) with exogenous dividend stream $(d_t)_t$. Denote $c_{i,t}, b_{i,t}$ the consumption and asset holding of agent i at date t while q_t is the asset price at date t. Agent i maximizes her intertemporal utility $\sum_{t=0}^{+\infty} \beta_{i,t} u_i(c_{i,t})$ subject to the following constraints:

(1) Physical constraints: $c_{i,t} \geq 0\ \forall t, \forall i$.

(2) Budget constraint: $c_{i,t} + q_t b_{i,t} \leq e_{i,t} + (q_t + d_t) b_{i,t-1}\ \forall t, \forall i$, where $e_{i,t} > 0$ is the exogenous endowment of agent i at date t and $b_{i,-1}$ is exogenously given.

(3) Borrowing constraint (or short-sale constraint): $b_{i,t} \geq -b_i^*\ \forall t, \forall i$ where $b_i^* \geq 0$ is an exogenous borrowing limit.

An equilibrium is a list of prices and allocations $(q_t, (c_{i,t}, b_{i,t})_i)_{t\geq 0}$ satisfying three conditions: (1) given price, for any i, the allocation $(c_{i,t}, b_{i,t})_i$ is a solution of the optimization problem of agent i (i.e., $\sum_{t=0}^{+\infty} \beta_{i,t} u_i(c_{i,t}) \geq \limsup_{T\to\infty} \sum_{t=0}^{T} \beta_{i,t} u_i(c'_{i,t})$ for any sequence (c'_i, b'_i) satisfying physical, budget and borrowing constraints), and (2) market clearing conditions: $\sum_i b_{i,t} = L$ and $\sum_i c_{i,t} = \sum_i e_{i,t} + L d_t\ \forall t \geq 0$, where L is the net asset supply, and (3) $q_t > 0\ \forall t \geq 0$.

Denote $W_t \equiv \sum_i e_{i,t} + L d_t$ the aggregate resource at date t. We require standard assumptions in the rest of the paper.

Assumption 1. *Assume that u_i is concave, strictly increasing, and continuously differentiable for any i. We also assume that $\beta_{i,t} > 0$, $e_{i,t} > 0$, $b_{i,-1} \geq -b_i^*$, $d_t \geq 0$, $\sum_t \beta_{i,t} u_i(W_t) < \infty$, $\lim_{t\to\infty} \beta_{i,t} = 0$, $\forall i, t$, and the net asset supply is positive ($L > 0$).*

Assumption 2. *There exists an increasing function $v(c)$ such that $u_i'(c)c \leq v(c)$ $\forall c$ and $\sum_t \beta_{i,t} v(W_t) < \infty$ $\forall i$.*

Notice that when $\sum_t \beta_{i,t} < \infty$ $\forall i$, and $u_i(c) = ln(c)$ $\forall c, \forall i$ or $u_i(0)$ is finite for any i, Assumption 2 is a direct consequence of Assumption 1.

We start our exposition with the following result which plays a fundamental role in understanding bubbles.

Proposition 1. *Let Assumption 1 be satisfied.*
*(1) If $(q, (c_i, b_i)_i)$ is an equilibrium, we have first-order conditions (**FOC**):*

$$\beta_{i,t} u_i'(c_{i,t}) = \lambda_{i,t} \tag{1a}$$

$$\lambda_{i,t} q_t = \lambda_{i,t+1}(q_{t+1} + d_{t+1}) + \eta_{i,t}, \quad \eta_{i,t}(b_{i,t} + b_i^*) = 0, \quad \eta_{i,t} \geq 0. \tag{1b}$$

for any i, t. In addition, if Assumption 2 holds, then $\lim_{t\to\infty} \lambda_{i,t} q_t (b_{i,t} + b_i^*) = 0$.

(2) If the sequences $(q, (c_i, b_i)_i)$ and (λ_i, η_i) satisfy

(a) $c_{i,t}, b_{i,t}, \lambda_{i,t}, \eta_{i,t}, \geq 0$, $q_t \geq 0$, $b_{i,t} \geq -b_i^*$, $c_{i,t} + q_t b_{i,t} = e_{i,t} + (q_t + d_t) b_{i,t-1}$ $\forall i, t$;

(b) First-order conditions (1a-1b), and market clearing conditions;

*(c) Transversality conditions (**TVC**): $\lim_{t\to\infty} \lambda_{i,t} q_t (b_{i,t} + b_i^*) = 0$ $\forall i$;*

(d) For any i, the series $\sum_{t=0}^{\infty} \beta_{i,t} u_i(c_{i,t})$ converges.

then $(q, (c_i, b_i)_i)$ is an equilibrium.

Proof. See Appendix A.1. □

Proposition 1 provides necessary and sufficient conditions under which a list of prices and allocation constitutes an equilibrium.[10] Kocherlakota (1992) considers a particular function $\sum_t \beta_i^t u_i(c_{i,t})$ and states a similar result but he requires that $u_i(c) \leq 0$ $\forall c$ or $u_i(c) \geq 0$ $\forall c$ (to ensure that the sum $\sum_t \beta_i^t u_i(c_{i,t})$ always converges). Of course, his condition is not satisfied if $u_i(c) = ln(c)$. Our result is more general and also applies to unbounded utility functions, including $u_i(c) = ln(c)$. Our result is related to Proposition 1 in Bosi et al. (2018b). The difference is that we impose exogenous borrowing limits while Bosi et al. (2018b) consider collateral constraints and the borrowing limits depends on prices of assets in the future.

Our proof of TVC is quite different from that of Kamihigashi (2002). We cannot directly apply the result in Kamihigashi (2002) because he only considers positive allocations while $b_{i,t}$ may be negative in our model. It is interesting to notice that when $u_i(0) \geq 0$ $\forall i$, the second statement of Proposition 1 still holds if we replace $\lim_{t\to\infty} \lambda_{i,t} q_t (b_{i,t} + b_i^*) = 0$ $\forall i$ by $\liminf_{t\to\infty} \lambda_{i,t} q_t (b_{i,t} + b_i^*) = 0$ $\forall i$.[11]

Following the standard literature (Tirole, 1982, 1985; Kocherlakota, 1992; Santos and Woodford, 1997), we introduce the notion of rational asset price bubbles.

[10] For the existence of equilibrium, see, among others, Bosi et al. (2018b) and references therein.
[11] See Remark 8 in Appendix A.1 for a proof.

Definition 1. *Consider an equilibrium. We define discount factors* $(R_t)_t$ *by* $R_{t+1}q_t = q_{t+1} + d_{t+1}$. *The fundamental value of the asset is* $FV_0 \equiv \sum_{t=1}^{\infty} Q_t d_t$ *where* $Q_t \equiv \frac{1}{R_1 \cdots R_t}$. *We say that there is a bubble in this equilibrium if* $q_0 > FV_0$. *In this case, this equilibrium is called bubbly. Otherwise, it is called bubbleless.*

Remark 1. One can prove that $1 = R_{t+1} \max_i \frac{\beta_{i,t+1}u_i'(c_{i,t+1})}{\beta_{i,t}u_i'(c_{i,t})}$ $\forall t \geq 0$.[12]

In our deterministic framework, the sequence of discount factors (R_t) is uniquely determined. The reader is referred to Santos and Woodford (1997), Araujo et al. (2011), Pascoa et al. (2011), Bosi et al. (2018b) among others for the notion of bubbles in stochastic economies where discount factors (and state price processes) are not necessarily uniquely determined.[13]

According to the asset pricing equation $q_t = (q_{t+1} + d_{t+1})/R_{t+1}$, we have

$$q_0 = \sum_{s=1}^{T} Q_s d_s + Q_T q_T, \quad q_t = \sum_{s=t+1}^{T} \frac{Q_s}{Q_t} d_s + \frac{Q_T}{Q_t} q_T, \ \forall T \geq t \geq 1. \tag{2}$$

So, there is a bubble if and only if $q_t > \sum_{s=t+1}^{\infty} \frac{Q_s}{Q_t} d_s$. This is also equivalent to $\lim_{t\to} Q_t q_t > 0$, i.e., the discounted value of 1 unit of the asset does not vanish in the infinity. In a particular case where $d_t = 0 \ \forall t$, the fundamental value equals zero; in this case, there is a bubble iff the asset price is strictly positive (this is the notion of bubble in Tirole (1985)).

Our main goal is to understand conditions under which rational asset price bubbles may exist (or be ruled out) in equilibrium as well as the implications of this phenomenon.

3 No-bubble conditions in general cases

This section aims to study necessary conditions for the existence of bubbles and find out new conditions under which bubbles cannot appear.

3.1 The role of borrowing constraints

The relationship between the existence of bubble and borrowing constraints is questioned by Kocherlakota (1992). However, he did not investigate whether borrowing constraints are binding or not in equilibrium with bubbles. The following result explores such a relationship and shows our contribution with respect to Kocherlakota

[12]Indeed, let $t \geq 0$ arbitrary, then FOCs imply that $q_t \geq (q_{t+1} + d_{t+1}) \max_i \frac{\beta_{i,t+1}u_i'(c_{i,t+1})}{\beta_{i,t}u_i'(c_{i,t})}$. Since $\sum_i b_{i,t} = L > 0$, there is an agent i_t such that $b_{i_t,t} > 0$. Hence, $\eta_{i,t} = 0$. By consequence, $q_t = (q_{t+1} + d_{t+1}) \frac{\beta_{i_t,t+1}u_{i_t}'(c_{i_t,t+1})}{\beta_{i_t,t}u_{i_t}'(c_{i_t,t})}$. Therefore, we obtain our result.

[13]See Miao and Wang (2012, 2018) for the notion of bubble on the value of firm and Becker et al. (2015), Bosi et al. (2017a) for the notion of bubble on physical capital.

(1992) as well as the connection between the existence of bubble and the trading on the asset market.

Proposition 2 (bubble existence and borrowing constraint)**.** *Let Assumption 1, 2 be satisfied. If there is a bubble in equilibrium, then we have that:*

1. *For each i, at least one of the two following statements is true: (i) there exists an infinite increasing sequence i_n of time such that $b_{i,i_n} + b_i^* = 0$, $\forall n$; (ii) $\lim_{t\to\infty}(b_{i,t} + b_i^*) = 0$.*

2. *There exist at least 2 agents i and j such that their asset holding sequences $(b_{i,t})_t$ and $(b_{j,t})_t$ do not converge. Moreover, their borrowing constraints bind infinitely often: there exist 2 infinite increasing sequences $(i_n)_n, (j_n)_n$ such that $b_{i,i_n} + b_i^* = 0$ and $b_{j,j_n} + b_j^* = 0$ for all n.*

Proof. See Appendix A.1. □

Corollary 1 (Proposition 3 in Kocherlakota (1992))**.** *Let Assumption 1, 2 be satisfied. If there is a bubble in equilibrium, then we have $\liminf_{t\to\infty}(b_{i,t} + b_i^*) = 0$, $\forall i$.*

Proposition 2 show that the existence of bubbles implies the fluctuations of asset trading of at least 2 agents. More interesting, borrowing constraints of these two agents must infinitely often bind. Our result is stronger than those in Bosi et al. (2018b), where they prove that, if bubbles exist, there is at least 1 agent whose collateral constraints bind infinitely often.

Let us provide a sketch and intuition of our proof. When the borrowing constraint of an agent, say agent i, is not binding from some date, say t_0, then $Q_t = \lambda_{i,t}\frac{Q_{t_0}}{\lambda_{i,t_0}}$ $\forall t \geq t_0, \forall i$. It means that the discount factor of any agent is proportional to that of the economy. So, the TVC which ensures the optimality of agent i's allocation implies that $\lim_{t\to\infty} Q_t q_t(b_{i,t} + b_i^*) = 0$. When bubbles exist (i.e., $\lim_{t\to\infty} Q_t q_t > 0$), we have $\lim_{t\to\infty}(b_{i,t} + b_i^*) = 0$.

Proposition 2's point 2 leads to the following result showing the role of borrowing limits (b_i^*) and dividends (d_t) on the existence of bubbles.

Corollary 2. *Let Assumption 1, 2 be satisfied. If there is a date T such that $b_i^* d_t > e_{i,t}$ $\forall i, \forall t \geq T$, then there is no equilibrium with bubble.*[14]

Notice that this result still applies for the case where borrowing limits depend on time, i.e., when the borrowing constraint of agent i at date t is $b_{i,t} + b_{i,t}^* \geq 0$ where $b_{i,t}^* \geq 0$ is exogenous.

[14] Indeed, suppose that there is an equilibrium with bubble. According to point 2 of Proposition 2, there is an agent i and an infinite sequence $(i_n)_n$ such that $b_{i,i_n} + b_i^* = 0$ $\forall n$. Let n be such that $i_n > T$. We have $c_{i,i_n+1} = e_{i,i_n+1} - d_{i_n+1}b_i^* - q_{i_n+1}(b_i^* + b_{i,i_n}) \leq e_{i,i_n+1} - d_{i_n+1}b_i^* < 0$, a contradiction.

3.2 Interest rates, impatience and bubble

A famous result in Santos and Woodford (1997) states that, under mild conditions, bubbles are ruled out if the present value of total future resources is finite (this condition was named "high implied interest rates" by Alvarez and Jermann (2000)).[15] In our model with short-sale constraints, we can also prove a similar result.

Corollary 3. *Let Assumption 1, 2 be satisfied. There is no bubble if*

$$\sum_{t\geq 0} Q_t\Big(\sum_i e_{i,t}\Big) < \infty. \tag{3}$$

Consequently, there is no bubble if $\liminf_{t\to\infty} \frac{d_t}{\sum_i e_{i,t}} > 0$.

Proof. See Appendix A.1. □

We provide here a sketch of our proof which is different from that in Santos and Woodford (1997). By using condition (3), we prove that, for any agent i, the discounted value of asset holding $Q_t q_t b_{i,t}$ converges when t tends to infinity. If there is a bubble (i.e., $\lim_{t\to\infty} Q_t q_t > 0$), then the sequence $(b_{i,t})$ converges for any i. By market clearing condition, there is at least one agent, say j, whose asset holding $b_{j,t}$ converges to a strictly positive value. So, this agent's borrowing constraint is not binding from some date on. By consequence, the TVC implies that $\lim_{t\to\infty} Q_t q_t (b_{j,t} + b^*_j) = 0$ which is a contradiction.

Corollary 3 also indicates bubbles can exist only if there is an infinite sub-sequence of times $(t_n)_{n\geq 0}$ such that the ratio $\frac{d_{t_n}}{\sum_i e_{i,t_n}}$ converges to zero, i.e., the dividend will be very low with respect to the aggregate endowment. This condition is consistent with those in Le Van and Pham (2016), Bosi et al. (2018b).

Our main goal in this subsection is to find out other conditions (based on fundamentals) under which bubbles cannot appear. To do so, we borrow the concept "uniform impatience" in the existing literature (Magill and Quinzii, 1994, 1996; Levine and Zame, 1996). Given a consumption plan $c = (c_t)_{t\geq 0}$, a date t, a vector $(\gamma, \delta) \in (0,1) \times \mathbb{R}_+$, we define another consumption plan, called $z = z(c, t, \gamma, \delta)$, by $z_s = c_s\ \forall s < t$, $z_t = c_t + \delta$, $z_s = \gamma c_s\ \forall s > t$. We also denote $U_i^T(c) = \sum_{t=0}^T \beta_{i,t} u_i(c_{i,t})$ and $U_i(c) \equiv \limsup_{T\to\infty} U_i^T(c)$.

Assumption 3 (Uniform impatience). *There exists $\gamma \in (0,1)$ such that for any consumption plan $c = (c_t)$ with $0 \leq c_t \leq W_t\ \forall t$, we have*

$$U_i\Big(z(c, t, \gamma', W_t)\Big) > U_i(c)\ \forall i, \forall t, \forall \gamma' \in [\gamma, 1).$$

[15] Theorem 6.1 in Huang and Werner (2000) provides a version of Santos and Woodford (1997)'s Theorem 3 in a model with debt constraints. Proposition 12 in Bosi et al. (2018b) shows a related result concerning the bubbles of land.

Proposition 1 in Pascoa et al. (2011) provides sufficient conditions for the uniform impatience. Notice that they only consider the case where $u_i(c) \geq 0\ \forall c, \forall i$. If $u_i(c) = \frac{c^{1-\sigma}}{1-\sigma}\ \forall i$, where $\sigma > 0$, $\sigma \neq 1$, and there exists $\gamma \in (0,1)$ such that $\beta_{i,t}\frac{2^{1-\sigma}-1}{1-\sigma}W_t^{1-\sigma} + (\gamma^{1-\sigma} - 1)\sum_{s=t+1}^{\infty}\beta_{i,s}\frac{W_s^{1-\sigma}}{1-\sigma} > 0\ \forall t$, then the uniform impatience holds. We can also consider logarithmic utility functions: If $u_i(c) = ln(c)\ \forall c, \forall i$ and there exists $\gamma \in (0,1)$ such that $\beta_{i,t} > -\frac{\ln(\gamma)}{ln(2)}\sum_{s=t+1}^{\infty}\beta_{i,s}\ \forall t$, then the uniform impatience holds.

Our main contribution in this subsection can be stated as follows.

Proposition 3. *Assume that Assumptions 1, 2, 3 hold and $e_{i,t} - d_t b_i^* > 0\ \forall i, \forall t$. There is no bubble if*

$$\lim_{T\to\infty} W_T \prod_{t=0}^{T-1} \max_i \frac{\beta_{i,t+1}u_i'(e_{i,t+1} - d_{t+1}b_i^*)}{\beta_{i,t}u_i'(\sum_i e_{i,t} + Ld_t)} = 0. \tag{4}$$

This leads to two consequences.

1. *When $u_i(c) = ln(c)$, $\beta_{i,t} = \beta^t\ \forall i, \forall t$, and $\frac{1-\beta}{\beta} > -\frac{ln(\gamma)}{ln(2)}$ with $\gamma \in (0,1)$, there is no bubble if*

$$\lim_{T\to\infty} \beta^T W_T \cdots W_1 W_0 \prod_{t=1}^{T} \max_i \frac{1}{e_{i,t+1} - d_{t+1}b_i^*} = 0. \tag{5}$$

2. *When $u_i(c) = \frac{c^{1-\sigma}}{1-\sigma}$ where $\sigma > 0$, $\beta_{i,t} = \beta^t\ \forall i, \forall t$, and there exists $\gamma \in (0,1)$ such that $\frac{2^{1-\sigma}-1}{1-\sigma}W_t^{1-\sigma} + (\gamma^{1-\sigma} - 1)\sum_{s=t+1}^{\infty}\beta^{s-t}\frac{W_s^{1-\sigma}}{1-\sigma} > 0\ \forall t$, there is no bubble if*

$$\lim_{T\to\infty} \beta^T W_0^{\sigma} W_T^{1-\sigma} \prod_{t=1}^{T} \max_i \frac{W_t^{\sigma}}{(e_{i,t} - d_t b_i^*)^{\sigma}} = 0. \tag{6}$$

Proof. See Appendix A.1. ☐

The basic intuition of Proposition 3 is that the value of bubble (i.e., $\lim_{t\to} \frac{q_t}{R_1\cdots R_t}$) must be zero if the discount factors $(R_t)_t$ are high enough. There are two key points helping us to get (4). First, we use the uniform impatience to find an upper bound of asset price q_t: $q_t \leq \frac{mW_t}{L(1-\gamma)}\ \forall t$. Second, and more importantly, by deriving an upper bound on the intertemporal marginal rate of substitution of asset holders, we can find an upper bound of $1/R_t$:

$$\frac{1}{R_{t+1}} \leq \max_i \frac{\beta_{i,t+1}u_i'(e_{i,t+1} - d_{t+1}b_i^*)}{\beta_{i,t}u_i'(\sum_i e_{i,t} + Ld_t)}\ \forall t \geq 0. \tag{7}$$

and so an upper bound of the discount factor Q_t. By consequence, we obtain (4).

Proposition 3 contributes to the literature by providing conditions (based on fundamentals) under which bubbles are ruled out. Let us mention some consequences of Proposition 3. First, when borrowing limits (b_i^*) and dividends (d_t) are not high (in the sense that $e_{i,t} > d_t b_i^* \ \forall i, \forall t$), Proposition 3 implies that bubbles do not exist if the agents prefer strongly the present (formally, $\beta_{i,t+1}/\beta_{i,t}$ is low). In a particular case, where $\beta_{i,t} = \beta^t$ with β is low enough, there is no bubble. Notice that, when there is T such that $\beta_{i,t} = 0 \ \forall i, \forall t > T$, we recover a T-horizon model where we have $q_0 = \sum_{s=1}^{T} Q_s d_s$ and $q_s = 0 \ \forall s > T$, and therefore, there is no bubble.

Second, let us look at condition (6). Assume that W_t is bounded and $\frac{W_t}{e_{i,t} - d_t b_i^*} = \frac{\sum_i e_{i,t} + L d_t}{e_{i,t} - d_t b_i^*} < m_c, \ \forall i, \forall t$. Then, condition (6) holds if $\beta m_c^\sigma < 1$. It implies that there is no bubble if σ is small enough, or, equivalently, the elasticity of intertemporal substitution $1/\sigma$ is high enough.

Third, in the case of zero dividends ($d_t = 0 \ \forall t$), conditions (4-6) do not depend on borrowing limits b_i^*. So, bubbles may be ruled out even borrowing limits are too low. This in turn suggests that financial frictions are only necessary conditions for asset price bubbles.

We end this section by mentioning that conditions (4-6) are only sufficient conditions (not necessary) to rule out bubbles. They are quite restrictive (because $\frac{u_i'(e_{i,t+1} - d_{t+1} b_i^*)}{u_i'(\sum_i e_{i,t} + L d_t)}$ may be high if the number of agent is large). However, these conditions are useful because they suggest that the rate of time preference and the elasticity of intertemporal substitution play a role on the existence of bubble.

4 Models with bubbles

4.1 A model with two types of agents

We are now interested in constructing model economies in which bubbles exist. Proposition 2 shows that such models must contain at least 2 heterogeneous agents. So, we should focus on a model with two types of agents, say 1 and 2.[16] Suggesting by Proposition 2, we should look at equilibria in which borrowing constraints of agent 1 (agent 2) bind at any even (odd) date because this is the simplest model under which bubbles may exist. Formally, we aim to find economies where there is an equilibrium such that

$$b_{1,2t} = -b_1^*, \quad b_{2,2t} = L + b_1^*, \quad b_{1,2t+1} = L + b_2^*, \quad b_{2,2t+1} = -b_2^*. \tag{8}$$

With these asset holdings, we have that

$$c_{1,0} = e_{1,0} + (q_0 + d_0) b_{1,-1} + q_0 b_1^*, \quad c_{2,0} = e_{2,0} + (q_0 + d_0) b_{2,-1} - q_0 (L + b_1^*) \tag{9a}$$
$$c_{1,2t-1} = e_{1,2t-1} - b_1^* d_{2t-1} - q_{2t-1} H, \quad c_{2,2t-1} = e_{2,2t-1} + d_{2t-1}(L + b_1^*) + q_{2t-1} H \tag{9b}$$
$$c_{1,2t} = e_{1,2t} + d_{2t}(L + b_2^*) + q_{2t} H, \quad c_{2,2t} = e_{2,2t} - d_{2t} b_2^* - q_{2t} H \tag{9c}$$

[16] Our results below can be extended to a model with n types of agents and asset allocation is given by $b_{1,nt} = -b_1^*, b_{1,nt+1} = -b_1^*, \dots b_{1,nt+n-1} = H_n - b_1^*$, $b_{2,nt} = H_n - b_2^*, b_{2,nt+1} = -b_2^*, \dots b_{2,nt+n-1} = -b_2^*$, $b_{n,nt} = -b_n^*, \dots b_{n,nt+n-2} = H_n - b_n^*, b_{n,nt+n-1} = -b_n^*$, where $H_n \equiv L + \sum_{i=1}^n b_i^*$.

where $b_{1,-1}, b_{2,-1}$ are given and $H \equiv L + b_1^* + b_2^*$.[17] The FOCs become

$$1 = \frac{\beta_{2,2t+1} u_2'(c_{2,2t+1})}{\beta_{2,2t} u_2'(c_{2,2t})} \frac{q_{2t+1} + d_{2t+1}}{q_{2t}} \geq \frac{\beta_{1,2t+1} u_1'(c_{1,2t+1})}{\beta_{1,2t} u_1'(c_{1,2t})} \frac{q_{2t+1} + d_{2t+1}}{q_{2t}} \tag{10a}$$

$$1 = \frac{\beta_{1,2t} u_1'(c_{1,2t})}{\beta_{1,2t-1} u_1'(c_{2,2t-1})} \frac{q_{2t} + d_{2t}}{q_{2t-1}} \geq \frac{\beta_{2,2t} u_2'(c_{2,2t})}{\beta_{2,2t-1} u_2'(c_{2,2t-1})} \frac{q_{2t} + d_{2t}}{q_{2t-1}} \tag{10b}$$

In the following, we will show conditions (based on fundamentals) under which such equilibrium may have a bubble. First, we provide necessary conditions. The idea is to look at the benchmark economy (i.e., the economy without assets). Let us define the exogenous sequences $(R^*_{1,t}), (R^*_{2,t}), (R^*_t)$ by

$$1 = \frac{\beta_{1,t} u_1'(e_{1,t})}{\beta_{1,t-1} u_1'(e_{1,t-1})} R^*_{1,t}, \quad 1 = \frac{\beta_{2,t} u_2'(e_{2,t})}{\beta_{2,t-1} u_2'(e_{2,t-1})} R^*_{2,t}, \text{ and } R^*_t \equiv \min(R^*_{1,t}, R^*_{2,t}). \tag{11}$$

$R^*_{1,t}$ (resp., $R^*_{2,t}$) can be interpreted as the *subjective real interest rate* of agent 1 (resp., 2) and R^*_t as *the real interest rate* between dates $(t-1)$ and t in the benchmark economy. Notice that $R_t \geq R^*_t$ $\forall t \geq 2$ which means that the interest rate of the benchmark economy is lower than that of our economy with asset.

We have the following result providing necessary conditions for the existence of (bubbly) equilibrium in this two-agent economy.

Proposition 4 (the role of interest rates of the benchmark economy)**.** *Consider a model with two agents. Assume that the sequence (q_t), asset holdings are given by (8) and agents' consumptions given by (9a-9c) constitute an equilibrium.*

1. *We have*

$$R^*_{2,2t} \geq R^*_{1,2t}, \quad R^*_{1,2t+1} \geq R^*_{2,2t+1} \ \forall t \geq 1 \quad \textit{(seesaw property)}. \tag{12}$$

2. *Moreover, there is no bubble if*

$$\lim_{t\to\infty} \frac{e_t}{R^*_1 \cdots R^*_t} = 0. \tag{13}$$

In a particular case, where $e_t = e$ and $R^*_t = R^*$ $\forall t$, there is no bubble if $R^* > 1$.

Proof. See Appendix A.2. □

The term $\frac{e_t}{R^*_1 \cdots R^*_t}$ represents the value (discounted by using the interest rates of the benchmark economy) of endowment of the agent who buys asset in the economy with asset. Proposition 4 implies that, if there is bubble, the sequence of these discounted values either diverges or converges to a strictly positive value. In the case

[17] Observe that such equilibrium exists only if $e_{1,2t-1} - b_1^* d_{2t-1} > 0$ and $e_{2,2t} - d_{2t} b_2^* > 0$ $\forall t$ (we can interpret that the borrowing limits b_1^*, b_2^* are low).

of convergence, the existence of bubble requires that $\lim_{t\to\infty} \frac{e_t}{R_1^* \cdots R_t^*} > 0$. The basic idea behind is that the income of asset buyers must be high enough so that these agents are willing to buy the asset even the asset price exceeds its fundamental value.

Condition (13) is new with respect to the literature of rational bubbles in infinite-horizon general equilibrium models. Notice that it is not implied by the well-known no-bubble condition $\sum_t Q_t(\sum_i e_{i,t}) < \infty$ (see Santos and Woodford (1997), Werner (2014), Bosi et al. (2018b)) because $R_t \geq R_t^*, \forall t$. The novelty of condition (13) is to show the importance of interest rates of the economy without asset (these interest rates are exogenous) on the existence of bubbles in the economy with assets.

Condition (13) allows us to establish the connection between the literature of bubbles in OLG models and that in infinite-horizon models. Indeed, let us compare it with the main result in the influential paper of Tirole (1985) who studies a pure bubble asset (i.e., asset pays no dividend) in an OLG model. He provides a no-bubble condition based on fundamentals: there is no bubble if the steady state interest rates of the economy without bubble asset is higher than the population growth rate. Condition (13), also based on exogenous variables, can be interpreted as a high interest rates condition (indeed, it becomes $R^* > 1$ if $e_t = e$, $R_t^* = R^*\ \forall t$). So, our result is consistent with that in Tirole (1985). The difference is that we do not require the convergence of interest rates R_t^* as in Tirole (1985) or in Farhi and Tirole (2012).

Remark 2. Condition (13) helps us to understand better a number of examples of bubbles in the literature. Indeed, in Example 1 in Kocherlakota (1992) and Example 4.2 in Santos and Woodford (1997) of fiat money, we can verify that $\lim_{t\to\infty} \frac{e_t}{R_1^* \cdots R_t^*} = \infty$, i.e., condition (13) is violated. Moreover, in examples of bubbles in Bosi et al. (2018b), we have $R_t^* = 0$, and hence condition (13) is also violated.

We are now interested in the set of (bubbly) equilibria satisfying (8) and (9a-9c). We observe that, for $x > 0$, the sequence $(q_t)_{t\geq 0}$, defined by $q_0 = x$ and the system (10a-10b), is unique. So, we denote this sequence by $(q_t(x))_t$. We also denote $\mathcal{S}_0$ the set of initial prices, i.e., the set of all values $x > 0$ such that the sequence $(q_t(x))$ is a sequence of prices of an equilibrium whose allocations are given by (8) and (9a-9c).

The following result shows interesting properties of the set $\mathcal{S}_0$.

Proposition 5 (the set of (bubbly) equilibria). *Let Assumption 1, 2 be satisfied. Assume that for $i = 1, 2$, the function $cu_i'(c)$ is increasing in c, and that $e_{1,t} - d_t b_1^* > 0, e_{2,t} - d_t b_2^* > 0\ \forall t$.*

The set $\mathcal{S}_0$ is bounded and connected (in the sense that, if $x, y \in \mathcal{S}_0$ and $x < y$, then $(x, y) \subset \mathcal{S}_0$). So, if the set $\mathcal{S}_0$ is non-empty, either it contains a unique element or it is an interval. By consequence, we have that:

1. *There is at most one bubbleless equilibrium.*

2. *If $\mathcal{S}_0$ contains at least 2 elements, there is a continuum of bubbly equilibria.*

Proof. See Appendix A.2. ∎

The key point of this result is that q_t and R_t are strictly increasing in q_0 while the fundamental value $FV_0 \equiv \sum_{t\geq 1} Q_t d_t$ is strictly decreasing in q_0.

Although Proposition 5 shows important characteristics of the set of (bubbly) equilibrium, it remains to find conditions under which this set contains at least 2 elements. We will work under logarithmic utility functions, i.e., $u_i(c) = \ln(c)\ \forall i = 1, 2$. In this case, the FOCs give

$$\begin{cases} \dfrac{e_{2,1} - b_2^* d_1}{q_1 + d_1} & = \dfrac{\beta_{2,1}(e_{2,0} + d_0 b_{2,-1})}{\beta_{2,0} q_0} - \frac{\beta_{2,1}}{\beta_{2,0}}(L + b_1^* - b_{2,-1}) - H \\ \dfrac{e_{1,2t} - d_{2t} b_1^*}{q_{2t} + d_{2t}} & = \dfrac{\beta_{1,2t}(e_{1,2t-1} - b_1^* d_{2t-1})}{\beta_{1,2t-1} q_{2t-1}} - H(\frac{\beta_{1,2t}}{\beta_{1,2t-1}} + 1) \\ \dfrac{e_{2,2t+1} - d_{2t+1} b_2^*}{q_{2t+1} + d_{2t+1}} & = \dfrac{\beta_{2,2t+1}(e_{2,2t} - b_2^* d_{2t})}{\beta_{2,2t} q_{2t}} - H(\frac{\beta_{2,2t+1}}{\beta_{2,2t}} + 1). \end{cases} \tag{14}$$

where recall that $H \equiv L + b_1^* + b_2^*$.

It is not easy to study this system because there are infinitely many parameters $(e_{i,t}, \beta_{i,t}, d_t)$.

4.1.1 Asset without dividends

We begin our exposition by studying a specific case.

Example 1 (a simple example). *Assume that $u_i(c) = ln(c), \beta_{i,t} = \beta^t$ where $\beta \in (0, 1)$ and there is no dividend ($d_t = 0\ \forall t$). Assume also that $b_{1,-1} = L + b_2^*$, $b_{2,-1} = -b_2^*$, and endowments are periodic:*

$$(e_{1,t})_{t\geq 0} = (w, e, w, e, \ldots), \quad (e_{2,t})_{t\geq 0} = (e, w, e, w, \ldots), \tag{15a}$$

where $e, w > 0$ (so $e_t = e > 0, w_t = w > 0\ \forall t$).

Let us focus on equilibrium satisfying (8) and (9a-9c).

1. *If $\frac{\beta e}{w} \leq 1$ (i.e., $R^* \geq 1$), there is no bubble.*
2. *If $\frac{\beta e}{w} > 1$ (i.e., $R^* < 1$: low interest rate condition), then the initial price of any equilibrium with bubble must satisfy condition $q_0 \leq \frac{1}{H}\frac{\beta e - w}{1+\beta}$. Conversely, we have:*
   - *(a) There is a unique equilibrium with initial price $q_0 = \frac{1}{H}\frac{\beta e - w}{1+\beta}$. Moreover, we have $q_t = \frac{1}{H}\frac{\beta e - w}{1+\beta} > 0\ \forall t$.*
   - *(b) (Continuum of equilibria with bubble) For any value x in the interval $[0, \frac{1}{H}\frac{\beta e - w}{1+\beta})$, the sequence (q_t) determined by $q_0 = x$ and $\frac{1}{Hq_{t+1}} = \frac{\beta e}{w}\frac{1}{Hq_t} - \frac{1+\beta}{w}$ $\forall t \geq 0$,[18] is a system of price of an equilibrium with bubble. Moreover, (1) q_t is decreasing in t and converges to zero, (2) the interest rate $R_t \equiv q_t/q_{t-1}$ is decreasing in t and converges to $R^* = \frac{w}{\beta e} < 1$.*

[18] By convention, if $q_0 = 0$, we determine $q_t = 0\ \forall t \geq 1$.

Proof. See Appendix A.2. □

Notice that in the case of bubbles in Example 1, the seesaw property (12) holds and high interest rate condition (13) is violated (because $e_t = e$ and $R^* < 1$)

Example 1 is related to several models of bubbles in general equilibrium, for instance, Example 4.2 in Santos and Woodford (1997), Townsend (1980), Chapter 27 in Ljungqvist and Sargent (2012) (their model corresponds to the case $e = 1, w = 0$ in our model), Section 2 in Bloise and Citanna (2019). An added value of Example 1 is to show that multiple equilibria may exist and we completely characterizes the set of multiple equilibria. By the way, it complements Example 4.2 in Santos and Woodford (1997), which only examines the steady state $q_t = q > 0$, $\forall t$ (but with a general utility function).

However, Example 1 and those in the existing literature do not clearly show us how the existence of bubbles depends on the dynamics of economic fundamentals and on the asset structure (dividends and borrowing limits). This observation motivates us to study the system (14) to understand bubbles in a more general case.

We firstly focus on the case of fiat money or pure bubble asset (i.e., $d_t = 0\ \forall t$). To simplify our exposition, we introduce some notations.

$$\gamma_{i,t} = \frac{\beta_{i,t+1}}{\beta_{i,t}},\ \gamma_{2t} \equiv \gamma_{2,2t},\ \gamma_{2t+1} \equiv \gamma_{1,2t+1}, \mu_{2t} \equiv \gamma_{1,2t},\ \mu_{2t+1} \equiv \gamma_{2,2t+1}, \forall t \geq 0 \tag{16a}$$

$$e_{2t} \equiv e_{2,2t}, \quad e_{2t+1} \equiv e_{1,2t+1}, \quad w_{2t} \equiv e_{1,2t}, \quad w_{2t+1} \equiv e_{2,2t+1}, \forall t \geq 0 \tag{16b}$$

$$\Gamma_t \equiv \frac{\gamma_{t-1}e_{t-1}}{w_t} \cdots \frac{\gamma_0 e_0}{w_1} = \frac{1}{R_1^* \cdots R_t^*}, \forall t \geq 1 \tag{16c}$$

$$D_t \equiv \frac{1+\gamma_{t-1}}{w_t} + \frac{1}{R_t^*}\frac{1+\gamma_{t-2}}{w_{t-1}} + \cdots + \frac{1}{R_t^* \cdots R_2^*}\frac{1+\gamma_0\frac{L+b_1^*-b_{2,-1}}{L+b_1^*+b_2^*}}{w_1}, \forall t \geq 2 \tag{16d}$$

$$D_1 \equiv \frac{1+\gamma_0\frac{L+b_1^*-b_{2,-1}}{L+b_1^*+b_2^*}}{w_1}, \forall t \geq 1. \tag{16e}$$

The following result provides a necessary and sufficient condition under which bubbles exist in equilibrium.

Proposition 6 (continuum equilibria with bubble)**.** *Assume that $d_t = 0\ \forall t$ and $u_i(c) = ln(c)\ \forall i = 1, 2$. The sequences $(b_{i,t})$ given by (8), $(c_{i,t})$ given by (9a-9c), and $(q_t)_{t\geq 0}$ constitute an equilibrium with bubble if and only if*

$$X_t \equiv \frac{\gamma_t e_t}{w_{t+1}} - \frac{\mu_t w_t}{e_{t+1}} > 0, \forall t \geq 0 \tag{17}$$

$$\sup_{t\geq 1}\left\{\frac{\frac{1+\gamma_t}{w_{t+1}} + \frac{1+\mu_t}{e_{t+1}}}{\frac{\gamma_t e_t}{w_{t+1}} - \frac{\mu_t w_t}{e_{t+1}}} R_1^* \cdots R_t^*\right\} < \infty \tag{18}$$

$$\sup_{t\geq 2}\left\{\frac{R_1^* \cdots R_{t-1}^*}{e_{t-1}}\Big(\frac{1}{\gamma_{t-1}}+1\Big) + \cdots + \frac{1}{e_0}\Big(\frac{1}{\gamma_0} + \frac{L+b_1^*-b_{2,-1}}{L+b_1^*+b_2^*}\Big)\right\} < \infty, \tag{19}$$

and the sequence of asset prices $(q_t)_{t\geq 0}$ *is determined by*

$$q_0 \in (0, \bar{q}], \quad \frac{1}{Hq_t} = \frac{1}{Hq_0}\Gamma_t - D_t \ \forall t \geq 1, \tag{20}$$

where the upper bound $\bar{q}$ *is defined as follows:*

$$\bar{q} \equiv \frac{1}{L + b_1^* + b_2^*} \min\Big\{ \frac{\frac{\gamma_{2,0}e_{2,0}}{e_{2,1}} - \frac{\gamma_{1,0}e_{1,0}}{e_{1,1}}}{\frac{1+\gamma_{2,0}\frac{L+b_1^*-b_{2,1}}{L+b_1^*+b_2^*}}{e_{2,1}} + \frac{1+\gamma_{1,0}\frac{b_1^*+b_{1,-1}}{L+b_1^*+b_2^*}}{e_{1,1}}}, \inf_{t\geq 1} \frac{X_t\Gamma_t}{X_tD_t + Y_t} \Big\}, \tag{21}$$

where $Y_t \equiv \frac{1+\gamma_t}{w_{t+1}} + \frac{1+\mu_t}{e_{t+1}}$.

By consequence, there is a continuum of equilibria and all such equilibria are bubbly if (17-19) are satisfied.

Proof. See Appendix A.2. □

Proposition 6 is a generalized version of Example 1 for the case where endowments and ratios $\frac{\beta_{i,t+1}}{\beta_{i,t}}$ are time dependent.[19] It provides a complete characterization of all equilibria satisfying (8) and (9a-9c). Importantly, we can explicitly describe all such equilibria by using fundamentals. Conditions (17-19) are necessary and sufficient for the existence of bubble, and they are satisfied for a large class of parameters. Condition (17) implies the seesaw property (12), i.e., $R^*_{2,2t} \geq R^*_{1,2t}$ and $R^*_{1,2t+1} \geq R^*_{2,2t+1}$, $\forall t$, for the logarithmic utility $u(c) = ln(c)$. Conditions (18-19) can be interpreted as interest rates of the economy without asset (R^*_t) are low enough. It implies that, when $u_i(c) = ln(c)$, there is no bubble if $\sum_t \frac{R^*_1 \cdots R^*_t}{e_t} = \infty$. This is consistent but much stronger than condition (13) in Proposition 4 with a general utility function.

Under conditions (17-19), the sequence (q_t) is part of equilibrium with bubble if and only if $0 < q_0 \leq \bar{q}$. So, the value $\bar{q}$ can be interpreted as the maximum value of bubble. The higher the value of $\bar{q}$, the more chance to have a bubble in equilibrium. So, it is important to understand how the upper bound $\bar{q}$ depends on fundamentals. Observe that

$$\begin{aligned}
&\frac{X_tD_t + Y_t}{X_t\Gamma_t} = \frac{D_t}{\Gamma_t} + \frac{Y_t}{X_t\Gamma_t} \\
=&\frac{w_1 \cdots w_{t-1}}{e_0 \cdots e_{t-1}} \frac{1+\gamma_{t-1}}{\gamma_0 \cdots \gamma_{t-1}} + \frac{w_1 \cdots w_{t-2}}{e_0 \cdots e_{t-2}} \frac{1+\gamma_{t-2}}{\gamma_0 \cdots \gamma_{t-2}} + \cdots + \frac{1}{e_0}\Big(\frac{1}{\gamma_0} + \frac{L + b_1^* - b_{2,-1}}{L + b_1^* + b_2^*}\Big) \\
&+ \frac{1 + \gamma_t + (1+\mu_t)\frac{w_{t+1}}{e_{t+1}}}{\gamma_t\frac{e_t}{w_t} - \mu_t\frac{w_{t+1}}{e_{t+1}}} \frac{w_1 \cdots w_{t-1}}{e_0 \cdots e_{t-1}} \frac{1}{\gamma_0 \cdots \gamma_{t-1}}.
\end{aligned} \tag{22}$$

Therefore, the maximum value $\bar{q}$ of bubble defined by (21) is decreasing in the asset supply L, borrowing limits b_1^*, b_2^*, the endowment ratio $\frac{w_t}{e_t}$, the initial asset holding

[19] Indeed, when $\beta_{i,t} = \beta^t$, $e_t = e, w_t = w$, $\forall t, \forall i = 1, 2$, we recover Example 1; in this case, we can compute that $\bar{q} = \frac{\beta e}{w}$.

$b_{1,-1}$ of agent 1. Moreover, $\bar{q}$ is increasing in the rate of time preference γ_t, the initial asset holding $b_{2,-1}$ of agent 2. Therefore, the following factors contribute to promote the existence of bubble:

1. Asset supply L is low. (Asset shortage.)

2. Borrowing limits b_1^* and b_2^* are low. (Financial frictions matter.)

3. The initial asset $b_{2,-1}$ is high and/or the initial asset $b_{1,-1}$ is low. (Heterogeneity matters.)

4. The endowment ratios $\frac{e_{2,2t}}{e_{1,2t}}$ and $\frac{e_{1,2t+1}}{e_{2,2t+1}}$ are high. (Heterogeneity and seesaw property.)

5. The rates of time preference $\frac{\beta_{2,2t+1}}{\beta_{2,2t}}$ and $\frac{\beta_{1,2t}}{\beta_{1,2t-1}}$ are high. (Heterogeneity and seesaw property).[20]

Making clear the role of these factors on the existence and value of bubbles is a contribution of Proposition 6 with respect to Example 1 and the examples in the literature. For instance, Bewley (1980) (Section 13), Townsend (1980), Kocherlakota (1992) (Example 1), Scheinkman and Weiss (1986), Santos and Woodford (1997) (Example 4.2), borrowing is not allowed, which corresponds to the case $b_i^* = 0$, $\forall i$, in our model.

Proposition 6 can be viewed as a version of the classical result in Tirole (1985) (Proposition 1) for an exchange general equilibrium model with infinitely lived agents and short-sale constraints. With our specification, we explicitly compute the maximum level $\bar{q}$ of initial price bubble while it is implicit in more general models. Moreover, we do not require the convergence of interest rate as in Tirole (1985) and Farhi and Tirole (2012).

Remark 3 (Equilibrium indeterminacy and bubbles)**.** Proposition 6 and Example 1 show that, not only asset price bubbles but also real indeterminacy exist. Indeed, in equilibrium, the consumption allocation is given by

$$\begin{aligned} c_{1,0} &= e_{1,0} + q_0(b_{1,-1} + b_a^*), & c_{2,0} &= e_{2,0} + q_0(b_{2,-1} - L - b_a^*) \\ c_{1,2t} &= e_{1,2t} + q_{2t}H, & c_{2,2t} &= e_{2,2t} - q_{2t}H \\ c_{1,2t+1} &= e_{1,2t+1} - q_{2t+1}H, & c_{2,2t+1} &= e_{2,2t+1} + q_{2t+1}H. \end{aligned}$$

where the sequence of prices (q_t) is determined by (20). In our model, since there is a continuum of equilibrium price systems, there is real indeterminacy. This point is interesting because our model contains only one consumption good and a single asset. Our framework indicates that financial frictions and heterogeneity may generate real indeterminacy.

[20]When $\beta_{i,t} = \beta^t \ \forall i, t$, the existence of bubbles requires that β must be high enough (this is consistent with the finding in Proposition 3).

4.1.2 Assets with positive dividends

We now focus on the case where dividends are strictly positive (i.e., $d_t > 0\ \forall t \geq 0$). Looking back to Definition 1, the asset pricing equation $q_t = \frac{q_{t+1}+d_{t+1}}{R_{t+1}}$ implies that $q_tQ_t = q_{t+1}Q_{t+1}(1+\frac{d_{t+1}}{q_{t+1}})$. By iterating, we get that $q_0 = q_TQ_T\prod_{t=1}^{T}(1+\frac{d_t}{q_t})$. Bubbles exist (i.e., $\lim_{t\to 0} Q_tq_t > 0$) if and only if $\lim_{t\to\infty}\prod_{t=1}^{T}(1+\frac{d_t}{q_t}) < \infty$, or, equivalently, $\sum_t \frac{d_t}{q_t} < \infty$.[21] From this, we obtain a necessary condition for the existence of bubble.

Corollary 4. *Consider an equilibrium whose allocations given by (8) and (9a-9c). The existence of bubble requires that*

$$\sum_t \frac{d_{2t}}{e_{2,2t} - b_2^* d_{2t}} < \infty \text{ and } \sum_t \frac{d_{2t-1}}{e_{1,2t-1} - b_1^* d_{2t-1}} < \infty. \tag{24a}$$

This condition helps us to construct our example of bubble. Before providing conditions under which there is a continuum of bubbly equilibria, we introduce notations.

$$\begin{cases} a_1 \equiv \frac{\gamma_{2,0}(e_{2,0}+d_0b_{2,-1})}{e_{2,1}-b_2^*d_1} \\ a_{2t} \equiv \frac{\gamma_{1,2t-1}(e_{1,2t-1}-b_1^*d_{2t-1})}{e_{1,2t}-b_1^*d_{2t}} \\ a_{2t+1} \equiv \frac{\gamma_{2,2t}(e_{2,2t}-b_2^*d_{2t})}{e_{2,2t+1}-b_2^*d_{2t+1}} \end{cases} \quad \begin{cases} H_1 \equiv \frac{\gamma_{2,0}(L+b_1^*-b_{2,-1})+H}{e_{2,1}-b_2^*d_1} \\ H_{2t} \equiv \frac{H(1+\gamma_{1,2t-1})}{e_{1,2t}-b_1^*d_{2t}} \\ H_{2t+1} \equiv \frac{H(1+\gamma_{2,2t})}{e_{2,2t+1}-b_2^*d_{2t+1}} \end{cases} \quad \begin{cases} \bar{q}_0 \equiv \frac{e_{2,0}-e_{1,0}-d_0(b_{1,-1}-b_{2,-1})}{L+2b_1^*+b_{1,-1}-b_{2,-1}} \\ \bar{q}_{2t-1} \equiv \frac{e_{1,2t-1}-e_{2,2t-1}-(L+2b_1^*)d_{2t-1}}{2H} \\ \bar{q}_{2t} \equiv \frac{e_{2,2t}-e_{1,2t}-(L+2b_2^*)d_{2t}}{2H}. \end{cases} \tag{25}$$

With these notations, the FOCs (14) can be rewritten as

$$\frac{1}{q_t+d_t} = \frac{a_t}{q_{t-1}} - H_t\ \forall t \geq 1, \text{ or, equivalently, } q_t = \frac{q_{t-1}}{a_t - H_tq_{t-1}} - d_t\ \forall t \geq 1 \tag{26}$$

The main result in this section is stated as follows.

Proposition 7 (continuum of equilibria with bubbles)**.** *Let $u_i(c) = ln(c)\ \forall i = 1,2$ and $d_t > 0$, $\forall t$. Assume that $H_t > 0$, $a_{t+1}/H_{t+1} < \bar{q}_t\ \forall t$ and there are sequences $(\alpha_t)_{t\geq 1}, (\sigma_t)_{t\geq 1}$ satisfying $0 < \alpha_t < 1 < \sigma_t$ and*

$$\textit{Strong heterogeneity: } \frac{a_{t+1}H_t}{H_{t+1}} > \frac{\alpha_t}{\alpha_{t+1}(1-\alpha_t)} \tag{27a}$$

$$\textit{Low dividend condition: } \begin{cases} \frac{d_t}{d_{t+1}} > \frac{\sigma_{t+1}}{\sigma_t - 1}a_{t+1} \\ 1-(\sigma_t-1)d_tH_t > 0 \\ \text{and } \frac{\sigma_1a_1d_1}{1+d_1H_1} < \frac{\alpha_1a_1}{H_1} \end{cases} \tag{27b}$$

Then, there is a continuum of bubbly equilibria. More precisely, any sequence $(q_t)_{t\geq 0}$ determined by

$$q_0 \in (\frac{\sigma_1a_1d_1}{1+d_1H_1}, \frac{\alpha_1a_1}{H_1}) \text{ and } (q_t)_{t\geq 1} \text{ is computed by the system (14)} \tag{28}$$

[21] This condition was also proved in Montrucchio (2004), Bosi et al. (2018b).

is a system of prices of an equilibrium in which asset holdings are given by (8) and agents' consumptions are given by (9a-9c). Moreover for such equilibrium, we have

$$\frac{\sigma_t a_t d_t}{1+d_t H_t} < q_{t-1} < \frac{\alpha_t a_t}{H_t} \; \forall t \geq 1. \tag{29}$$

Proof. See Appendix A.2. □

Proposition 7 partially extends Proposition 6 to the case where dividends are time dependent. To the best of our knowledge, Proposition 7 is the first result showing the existence of multiple equilibria with bubbles of assets with positive dividends in deterministic general equilibrium models.[22] Note that dividends and endowments are time dependent.

Look at condition (27a), we observe that

$$\begin{aligned}\frac{a_{2t+1}H_{2t}}{H_{2t+1}} &= \frac{\gamma_{2,2t}(1+\gamma_{1,2t-1})}{1+\gamma_{2,2t}}\frac{e_{2,2t}-b_2^*d_{2t}}{e_{1,2t}-b_1^*d_{2t}}\\ \frac{a_{2t}H_{2t-1}}{H_{2t}} &= \frac{\gamma_{1,2t-1}(1+\gamma_{2,2t-2})}{1+\gamma_{1,2t-1}}\frac{e_{1,2t-1}-b_1^*d_{2t-1}}{e_{2,2t-1}-b_2^*d_{2t-1}}.\end{aligned}$$

Therefore, condition (27a) ensures that there are a strong heterogeneity and a seesaw property in our model. It can also be viewed as a "low interest rate condition".

It is essential to mention that there exist exogenous parameters satisfying all conditions in Proposition 7. Indeed, we can choose parameters as follows.

Step 1. We choose $\alpha_t = \alpha$, $\sigma_t = \sigma \; \forall t$.

Step 2. We choose $\gamma_{i,t} = \beta \in (0,1) \; \forall i, \forall t$. So, condition $\frac{a_{t+1}}{H_{t+1}} < \bar{q}_t$, $\forall t$, becomes

$$\frac{\beta(e_{2,0}+d_0 b_{2,-1})}{\beta(L+b_1^*-b_{2,-1})+H} < \frac{e_{2,0}-e_{1,0}-d_0(b_{1,-1}-b_{2,-1})}{L+2b_1^*+b_{1,-1}-b_{2,-1}} \tag{30a}$$

$$\frac{\beta(e_{1,2t-1}-b_1^*d_{2t-1})}{(1+\beta)H} < \frac{e_{1,2t-1}-e_{2,2t-1}-(L+2b_1^*)d_{2t-1}}{2H} \tag{30b}$$

$$\frac{\beta(e_{2,2t}-b_2^*d_{2t})}{(1+\beta)H} < \frac{e_{2,2t}-e_{1,2t}-(L+2b_2^*)d_{2t}}{2H}. \tag{30c}$$

Step 3. We choose $e_{2,2t+1}, e_{1,2t}$ such that $H_t = h > 0 \; \forall t$. Hence, $\frac{H_{t+1}}{H_t} = 1$.

Step 4. Given that (d_t) is low, we can choose $e_{2,2t}, e_{1,2t+1}$ sufficiently high so that $(1-\alpha)a_{t+1} > 1$ and (30a-30c) hold. (This is a low interest rates condition.)

[22] Le Van and Pham (2016) (Section 6.1) and Bosi et al. (2017a) provide examples of bubbles of the Lucas' tree, where the asset price may be multiple (due to the portfolio effect) but the consumption is not affected by the existence of bubbles. Our added-value is that the equilibrium indeterminacy in our model is real (in the sense that different equilibria have different consumption allocations) and the asset price affects agents' consumptions.

Step 5. Choose (d_t) and $\frac{d_{t+1}}{d_t}$ low enough such that (27a), (27b) are satisfied and $\frac{\sigma a_1 d_1}{1+d_1 H_1} < \frac{\alpha a_1}{H_1}$. (This is a low dividend condition.)

Although Proposition 7 provides a general sufficient condition under which there is a continuum of equilibria with bubbles, it would be useful to give examples with explicit mechanisms. We firstly focus on parameters satisfying the following assumption.

Assumption 4. *Assume that $\gamma_{i,t} = \beta \in (0,1)$ (i.e., $\beta_{i,t} = \beta^t$) and endowments are*

$$e_{1,2t-1} = b_1^* d_{2t-1} + e, \quad e_{1,2t} = b_1^* d_{2t} + w, \quad e_{2,2t-1} = b_2^* d_{2t-1} + w, \quad e_{2,2t} = b_2^* d_{2t} + e$$

where $e, w > 0$.

Under this specification, $(a_t), (H_t)$ defined by (25) become $a_t = a = \frac{\beta e}{w}$, $H_t = h \equiv \frac{H(\beta+1)}{w}$ $\forall t$, and the system of price (q_t) satisfies

$$\frac{1}{q_t + d_t} = \frac{a}{q_{t-1}} - h \ \forall t \geq 1, \text{ or equivalently } q_t = \frac{q_{t-1}}{a - hq_{t-1}} - d_t \ \forall t \geq 1 \tag{32}$$

In this case, we have the following result which helps us to identify all possible outcomes of equilibrium.

Proposition 8. *Let $u_i(c) = ln(c)$ $\forall i = 1, 2$ and Assumption 4 be satisfied. Assume that (q_t) is the price of an equilibrium in which asset holdings are given by (8) and agents' consumptions are given by (9a-9c).*

1. *If $\frac{\beta e}{w} < 1$ (i.e., $R^* > 1$: the interest rate of the benchmark economy is high), then there is no bubble.*
2. *If $\frac{\beta e}{w} > 1$ (i.e., $R^* < 1$: the interest rate of the benchmark economy is low), then there are only three cases:*
   - *(a) There is no bubble.*
   - *(b) The equilibrium is bubbly and q_t converges to zero.*
   - *(c) The equilibrium is bubbly, $q_t > \frac{\frac{\beta e}{w} - 1}{\frac{H(\beta+1)}{w}}$ $\forall t$, and q_t converges to $\frac{\frac{\beta e}{w} - 1}{\frac{H(\beta+1)}{w}}$.*

   Moreover, when $\frac{\beta e}{w} > 1$, there is almost one equilibrium satisfying conditions (8), (9a-9c) and $q_t > \frac{\frac{\beta e}{w} - 1}{\frac{H(\beta+1)}{w}}$ $\forall t$.

Proof. See Appendix A.2. ∎

According to Proposition 8, in equilibrium with bubbles, the asset price q_t converges either to zero or to $\frac{\frac{\beta e}{w}-1}{\frac{H(\beta+1)}{w}}$.[23]

We now complement the general results of Proposition 8 by providing examples of bubbles for each case. We start by the case where q_t converges to zero or to a positive value.

Example 2 (continuum of equilibria with bubble and $q_t \to 0$)**.** Let $u_i(c) = ln(c)$ $\forall i = 1, 2$ and Assumption 4 be satisfied. Assume that there exists σ such that $1 < \sigma$ and

$$\text{Low interest rate condition: } \frac{\beta e}{w} > 1 \tag{33a}$$

$$\text{Low dividend condition: } \begin{cases} \frac{\sigma-1}{\sigma}\frac{d_t}{d_{t+1}} > \frac{\beta e}{w} \\ d_t < \frac{w}{(\sigma-1)(\beta+1)H} \\ d_t < \frac{\frac{1-\beta}{1+\beta}e-w}{H} \\ \frac{\sigma a d_1}{1+d_1\frac{H(\beta+1)}{w}} < \frac{\beta e-w}{H(\beta+1)} \end{cases} \tag{33b}$$

$$\text{and } \frac{\beta(e_{2,0}+d_0b_{2,-1})}{\beta(L+b_1^*-b_{2,-1})+H} < \frac{e_{2,0}-e_{1,0}-d_0(b_{1,-1}-b_{2,-1})}{L+2b_1^*+b_{1,-1}-b_{2,-1}} \tag{33c}$$

Then, any sequence $(q_t)_{t\geq 0}$ determined by the system (26) and

$$q_0 \in \Big(\frac{\sigma a d_1}{1+d_1 h}, \frac{a-1}{h}\Big]$$

is a system of prices of an equilibrium at which asset holdings are given by (8) and agents' consumptions are given by (9a-9c). Moreover, Proposition 5 implies that there is a continuum of bubbly equilibria. For any equilibrium with $q_0 \leq \frac{a-1}{h}$ (including bubbly equilibrium), the asset price q_t decreasingly converges to zero when t tends to infinity.

Example 3 (an equilibrium with bubble and $q_t \to q > 0$)**.** Let $u_i(c) = ln(c)$ $\forall i = 1, 2$ and Assumption 4 be satisfied. Assume also that $a > 1$. Let $x > 0$ such that $\frac{x+1}{x} > a > 1$ and define the sequence (d_t) by

$$\frac{1}{d_t} = \Big(\frac{x+1}{xa}\Big)^t\Big(\frac{1}{d_0} - \frac{hx(x+1)}{1-(a-1)x}\Big) + \frac{hx(x+1)}{1-(a-1)x} \tag{34a}$$

$$0 < d_0 < \frac{1-(a-1)x}{hx(x+1)}, \quad d_0 < \frac{\frac{1-\beta}{1+\beta}e-w}{H} \tag{34b}$$

[23] This result is related to Propositions 2 and 3 in Bosi et al. (2018a). The difference is that Bosi et al. (2018a) consider an OLG model with descending altruism while we study a general equilibrium model with infinitely lived agents.

Observe that $0 < hxd_t < 1\ \forall t$ and $xd_t + d_t = \frac{axd_{t-1}}{1-hxd_{t-1}}$. Moreover, $\sum_t d_t < \infty$.

Define the sequence (q_t) by $q_t = \frac{a-1}{h} + xd_t\ \forall t \geq 0$. Then (q_t) is a system of prices of an equilibrium at which asset holdings are given by (8) and agents' consumptions are given by (9a-9c). Moreover, q_t decreasingly converges to $(a-1)/h$.

In this equilibrium, we have $\sum_t (d_t/q_t) = \sum_t (\frac{d_t}{\frac{a-1}{h}+xd_t}) < \sum_t d_t \frac{h}{a-1} < \infty$. So, this equilibrium experiences a bubble.

Proof. See Appendix A.2. □

Let us explain the basic intuition of our Examples 2, 3. By definition (25) of a_t, condition $a = \beta e/w > 1$ (i.e., the interest rate R^* of the benchmark economy is low) is equivalent to

$$\frac{\beta(e_{1,2t-1} - b_1^* d_{2t-1})}{e_{1,2t} - b_1^* d_{2t}} > 1 \text{ and } \frac{\beta(e_{2,2t} - b_2^* d_{2t})}{e_{2,2t+1} - b_2^* d_{2t+1}} > 1\ \forall t \tag{35}$$

We can interpret that agent 1 is richer than agent 2 at date $2t-1$ but agent 2 is richer than agent 1 at date $2t$; note that this is consistent with the seesaw property (12). Hence, agent 1 (resp., agent 2) may accept to buy the financial asset at date $2t-1$ (resp., date $2t$) even the asset price is higher than the fundamental value (i.e., there is a bubble). In both Examples 2, 3, we design that the sequence of dividends is low enough in order to ensure that, for any i, t, the asset value $q_t b_{i,t}$ is lower than the resource of agent i at date t so that agent i can buy the financial asset.

In Example 2, there is a continuum of equilibrium prices but any sequence of price converges to zero. In Example 3, we have $q_0 = (a-1)/h$ and the sequence of prices converges to $(a-1)/h > 0$ (notice that, according to Proposition 8, this is the unique bubbly equilibrium such that q_t converges to a strictly positive value.)

In Examples 2 and 3, the aggregate endowment is uniformly bounded and the sequence of dividends converges to zero. The following result shows that, in an economy with unbounded and asymmetric growth, bubbles may exist and the asset price may go to infinity.

Example 4 (asymmetric growth and multiple equilibria with $q_t \to \infty$)**.** Let $u_i(c) = ln(c)\ \forall i = 1, 2$, $\gamma_{i,t} = \beta \in (0,1)$ (i.e., $\beta_{i,t} = \beta^t$). Assume that $d_t = d > 0\ \forall t$ and endowments are

$$\begin{aligned} e_{1,2t-1} &= b_1^* d_{2t-1} + e_{2t-1}, \quad & e_{1,2t} &= b_1^* d_{2t} + w_{2t} \\ e_{2,2t-1} &= b_2^* d_{2t-1} + w_{2t-1}, \quad & e_{2,2t} &= b_2^* d_{2t} + e_{2t} \end{aligned}$$

Let α and σ be such that $0 < \alpha < 1 < \sigma$. Assume that, for any t,

$$\frac{e_{2,0} - e_{1,0} - d_0(b_{1,-1} - b_{2,-1})}{L + 2b_1^* + b_{1,-1} - b_{2,-1}} > \frac{\beta(e_{2,0} + d_0 b_{2,-1})}{\beta(L + b_1^* - b_{2,-1}) + H}$$

$$\frac{1-\beta}{1+\beta}e_t - w_t > Hd$$

$$w_{t+1} > \frac{\sigma}{\sigma - 1}\beta e_t, \quad e_t > \frac{1}{\beta(1-\alpha)}w_t,$$

$$w_t > (\sigma - 1)H(\beta + 1)d.$$

Notice that the two first conditions ensure that $\frac{a_{t+1}}{H_{t+1}} < \bar{q}_t \ \forall t$.

According to Proposition 7, any sequence $(q_t)_{t\geq 0}$ determined by the system (26) and $q_0 \in (\frac{\sigma a_1 d_1}{1+d_1 H_1}, \frac{\alpha a_1}{H_1})$, is a system of prices of an equilibrium in which asset holdings are given by (8) and agents' consumptions are given by (9a-9c). By consequence, Proposition 5 implies that there is a continuum of bubbly equilibria.

In this example, endowments of both agents go to infinity. However, there is an asymmetric growth: $\frac{e_t}{w_t} > \frac{1}{\beta(1-\alpha)} > 1$, or equivalently $\frac{e_{1,2t-1} - b_1^* d_{2t-1}}{e_{2,2t-1} - b_2^* d_{2t-1}} > \frac{1}{\beta(1-\alpha)}$ and $\frac{e_{2,2t} - b_2^* d_{2t}}{e_{1,2t} - b_1^* d_{2t}} > \frac{1}{\beta(1-\alpha)} \ \forall t$. The basic intuition of bubble in this example is consistent with that in Examples 2 and 3. Indeed, agent 1 is richer than agent 2 at date $2t-1$ but agent 2 is richer than agent 1 at date $2t$. Hence, agent 1 (resp., agent 2) accepts to buy the financial asset at date $2t-1$ (resp., date $2t$) even the asset price contains a bubble. Moreover, the price q_t goes to infinity when t tends to infinity because endowments of both agents grow without bound.

Example 5 (an equilibrium with bubbles and q_t may fluctuate over time)**.** Consider a particular case where $\beta_{i,t} = \beta^t \ \forall i, \forall t$ where $\beta \in (0,1)$, $b_1^* = b_2^* = 0$ (no short-sales) and $e_{2,2t+1} = e_{1,2t} = 0$. In this case, $\gamma_{2,2t} = \gamma_{1,2t-1} = \beta$, $H = L$ and there is a unique equilibrium satisfying condition (8)

$$q_{2t} = \frac{\beta}{(1+\beta)L}e_{2,2t} \text{ and } q_{2t-1} = \frac{\beta}{(1+\beta)L}e_{1,2t-1}. \tag{38}$$

In other words, the set $\mathcal{S}_0$ contains a unique element. This equilibrium experiences a bubble iff $\sum_t d_t/q_t < \infty$ which now becomes $\sum_t \frac{d_{2t}}{e_{2,2t}} + \sum_t \frac{d_{2t-1}}{e_{1,2t-1}} < \infty$. So, we recover (24a).[24]

It is interesting to notice that, in our example there is no causal connection between the monotonicity of the sequence of price (q_t) and the existence of bubble. The fact that the price q_t increases or decreases in time does not depend on the existence of bubble but depends on the dynamics of endowments.

[24]This corresponds to the key condition for bubbles in Section 5.1.1 in Bosi et al. (2018b)

We now look at the consumption in our example.

$$c_{1,0} = e_{1,0} + (q_0 + d_0)b_{1,-1}, \quad c_{2,0} = e_{2,0} + (q_0 + d_0)b_{2,-1} - q_0 L \tag{39a}$$
$$c_{1,2t-1} = e_{1,2t-1} - q_{2t-1}L, \quad c_{2,2t-1} = e_{2,2t-1} + d_{2t-1}L + q_{2t-1}L \tag{39b}$$
$$c_{1,2t} = e_{1,2t} + d_{2t}L + q_{2t}L, \quad c_{2,2t} = e_{2,2t} - q_{2t}L \tag{39c}$$

Since $Lq_{2t} = \frac{\beta}{1+\beta}e_{2,2t}$ and $Lq_{2t-1} = \frac{\beta}{1+\beta}e_{1,2t-1}$, we see that $c_{1,2t-1}$ and $c_{2,2t}$ do not depend on $(d_t)_t$ but $c_{1,2t}$ (resp., $c_{2,2t-1}$) is strictly increasing in d_{2t} (resp., d_{2t-1}). So, when dividends decrease, bubbles will be more likely to exist but the individual welfares will be lower.

4.2 Example of bubbles in a model with three types of agents

Section 4.1 presents several models of bubble in a two-agent model. A natural question arises: In models with more than 2 agents, how do examples of bubble look like? According to Proposition 2, a model with bubble must have at least two agents whose asset holdings fluctuate over time, and $\liminf_{t\to\infty}(b_{i,t}+b_i^*) = 0\ \forall i$. Moreover, for other agents, if their borrowing constraints never bind, we must have that $\lim_{t\to\infty}(b_{i,t}+b_i^*) = 0\ \forall i$. So, in this section we focus on an economy with three types of agents (n identical agents of type 1, n identical agents of type 2 and N identical agents of type 3) and we are interested in equilibrium with the following asset allocation:

$$b_{1,2t} = -b_1^*,\ b_{2,2t} = \frac{L + nb_1^* + Nb_3^*}{n},\ b_{1,2t+1} = \frac{L + nb_2^* + Nb_3^*}{n},\ b_{2,2t+1} = -b_2^*,\ b_{3,t} = -b_3^*. \tag{40}$$

Assume that $u_i(c) = ln(c)$, $\forall i = 1, 2, 3$ and $d_t = 0$, $\forall t$. Assume also that the agents of types 1 and 2 have the same characteristics as mentioned in Proposition 6, and

$$1 \geq \frac{\beta_{3,t+1}}{\beta_{3,t}} \frac{e_{3,t}}{e_{3,t+1}} \frac{e_{t+1}}{\mu_t w_t}, \tag{41}$$

By using the same argument of Proposition 6's proof, we have that: there exists an equilibrium with bubble and the asset allocation given by (40) if and only if conditions (17-19) holds (see Appendix A.2 for a proof). More importantly, the set of bubbly equilibrium prices is determined by

$$q_0 \in (0, q^*], \quad \frac{1}{H_3 q_t} = \frac{1}{H_3 q_0}\Gamma_t - D'_t\ \forall t \geq 1, \tag{42}$$

where $H_3 \equiv \frac{L+Nb_3^*}{n} + b_1^* + b_2^*$ and the sequences (D'_t) is defined as (D_t) but we replace L by $L_3 \equiv \frac{L+Nb_3^*}{n}$. The maximum value q^* of bubble is equal to

$$q^* \equiv \frac{1}{H_3} \min\Big\{ \frac{\frac{\gamma_{2,0}e_{2,0}}{e_{2,1}} - \frac{\gamma_{1,0}e_{1,0}}{e_{1,1}}}{\frac{1+\gamma_{2,0}\frac{L_3+b_1^*-b_{2,1}}{H_3}}{e_{2,1}} + \frac{1+\gamma_{1,0}\frac{b_1^*+b_{1,-1}}{H_3}}{e_{1,1}}}, \inf_{t\geq 1} \frac{X_t\Gamma_t}{X_t D'_t + Y_t} \Big\}. \tag{43}$$

As in discussion after Proposition 6, we have that q^* is a decreasing function of L_3. So, q^* is increasing in n and decreasing in Nb_3^*. The higher the borrowing limit b_3^* and the number of agent of type 3 (who borrows by short-selling at every date), the lower the maximum value q^* of bubble, the less chance we have a bubble in equilibrium.

Remark 4. *By using the same technique, we can obtain a result similar to Proposition 7, which provides a sufficient condition for bubble in an economy with three types of agents and positive dividends.*

4.3 Welfare analysis

It would be important to compare the individual welfares generated by different equilibria. The following result allows us to do so.

Proposition 9. *Consider equilibria satisfying conditions (8) and (9a-9c). Since equilibrium outcomes can be uniquely computed from the initial price q_0, the individual welfare of agent i is a function of q_0, and so denoted by $W_i(q_0)$.*

Assume that the utility function u_i is differentiable and strictly concave ($u_i'' < 0$) for any $i = 1, 2$. Then, we have that:

1. *For any $i = 1, 2$, the individual welfare $W_i(q_0)$ is strictly increasing in the initial price q_0.*
2. *By consequence, in the case of multiple equilibria (for example, in Proposition 6, Proposition 7, Examples 1-4), the allocation of a bubbly equilibrium strictly Pareto dominates that of the bubbleless equilibrium.*

Proof. See Appendix A.2. □

Let us provide a sketch of our proof. Recall that the agent i's welfare is $\sum_{t\geq 0}\beta_{i,t}u_i(c_{i,t})$. By using the FOCs and the concavity of utility functions, we can show that, for any t, $\beta_{1,2t-1}u_1(c_{1,2t-1})+\beta_{1,2t}u_1(c_{1,2t})$ and $\beta_{2,2t}u_2(c_{2,2t})+\beta_{2,2t+1}u_2(c_{2,2t+1})$ are strictly positive and strictly increasing in q_0. By consequence, we can prove that $\sum_{t=0}^{T}\beta_{i,t}\big(u_i(c_{i,t}) - u_i(c'_{i,t})\big)$ converge to a positive number when T tends to infinity, for any two equilibria $(q_t, (c_{i,t})_i)$ and $(q'_t, (c'_{i,t})_i)$ with $q_0 > q'_0$.

To understand the intuition behind our result, let us look at, for example $\beta_{1,2t-1}u_1(c_{1,2t-1})+\beta_{1,2t}u_1(c_{1,2t})$ which equals to

$$\beta_{1,2t-1}u_1(e_{1,2t-1} - b_1^*d_{2t-1} - q_{2t-1}H) + \beta_{1,2t}u_1(e_{1,2t} + d_{2t}(L + b_2^*) + q_{2t}H) \quad (44)$$

Recall that the rate of substitution $\frac{\beta_{1,2t}u_1'(e_{1,2t})}{\beta_{1,2t-1}u_1'(e_{1,2t-1})}$ is high (see conditions (12) and (13)) which implies that $e_{1,2t-1}$ is relatively high with respect to $e_{1,2t}$. When q_0 increases, both q_{2t} and q_{2t-1} increase. This implies that $c_{1,2t-1}$ decreases and $c_{1,2t}$ increases, and then helps agent 1 to better smooth her consumption. So, increasing q_0 is welfare-improving.

Remark 5. *Our welfare analysis is consistent with that in Proposition 9 of Hirano and Yanagawa (2017). The different is that we work with general utility functions while they only focus on the logarithmic utility function.*

Remark 6. *Notice that increasing q_0 is not necessarily strictly welfare-improving if the utility function u_i of some agent i is linear. Indeed, assume that $u_i(c) = u_i c \ \forall c$ where $u_i > 0$. We can check that $W_i(q_0) = W_i(q_0')$ for any two equilibria satisfying conditions (8) and (9a-9c) (because $\beta_{1,2t-1}u_1(c_{1,2t-1}) + \beta_{1,2t}u_1(c_{1,2t})$ and $\beta_{2,2t}u_2(c_{2,2t}) + \beta_{2,2t+1}u_2(c_{2,2t+1})$ do not depend on the initial equilibrium price q_0).*[25]

Remark 7. *In a particular case when assets have no dividend, the consumption allocation of the bubbleless equilibrium coincides with that of the autarkic equilibrium. Since the utility function is strictly concave, we can easily prove that $U_i(c_i) > U_i(e_i)$ $\forall i$. So, its allocation is strictly Pareto dominated by that of bubbly equilibrium. This argument has been used in many papers in the literature. However, it can no longer be applied for the case of positive dividend because the consumption allocation of the bubbleless equilibrium is different from that of the autarkic equilibrium. By the way, our proof of Proposition 9 is new and so part of our contribution.*

5 Conclusion and discussion

In general equilibrium models with infinitely lived agents, we have provided new conditions under which assets (with or without dividend) do not generate price bubbles. We have provided several mechanisms where bubbles and real indeterminacy exist in equilibrium. The existence of bubbles is not a matter of a single factor but the result of an interaction between heterogeneous agents in an imperfect financial market.

We end our paper by mentioning some avenues of research in the future. First, it would be interesting to extend our results to an economy with uncertainty and with a continuum of agents. To address this question, it is essential to understand how agents' decision depends on the economy's fundamentals, the degree of incompleteness of the financial market and the degree of uncertainty. Notice also that it is important (and not trivial) to prove the existence of equilibrium in such an economy. Another direction is to introduce a production sector in order to study the interplay between the production sector, the financial market, and bubbles as well as the interaction between borrowing limits, equilibrium stability, and dynamics of bubbles.

A Appendices

A.1 Equilibrium properties and no-bubble conditions

Proof of Proposition 1. Part 1. It is easy to see that $q_t > 0 \ \forall t$. Indeed, if $q_t = 0$

[25] See (B.49) and (B.50) in Appendix.

for some t, we can increase $b_{i,t}$ and obtain a higher income in $t+1$ and increase $c_{i,t+1}$: a contradiction.

To prove the FOCs, it suffices to prove that $q_t\beta_{i,t}u_i'(c_{i,t}) \geq \beta_{i,t+1}u_i'(c_{i,t+1})(q_{t+1}+d_{t+1})$ and we have equality if $b_{i,t}+b_i^*>0$. Fix $t\geq 0$ and consider another allocation $(c'_{i,s},b'_{i,s})_s$ given by $(c'_{i,s},b'_{i,s})=(c_{i,s},b_{i,s})$ $\forall s\notin\{t,t+1\}$ and $(c'_{i,s},b'_{i,s})_{s=t,t+1}$ determined by

$$\underbrace{c_{i,t}-\epsilon}_{c'_{i,t}}+q_t(b_{i,t}+\frac{\epsilon}{q_t}) = e_{i,t}+(q_t+d_t)b_{i,t-1}$$

$$\underbrace{c_{i,t+1}+(q_{t+1}+d_{t+1})\frac{\epsilon}{q_t}}_{c'_{i,t+1}}+q_{t+1}b_{i,t+1} = e_{i,t+1}+(q_{t+1}+d_{t+1})(b_{i,t}+\frac{\epsilon}{q_t}).$$

where $\epsilon>0$ is low enough so that $c_{i,t}-\epsilon>0$.

By the optimality $(c_{i,t},b_{i,t})_t$, we have

$$\beta_{i,t}u_i(c_{i,t})+\beta_{i,t+1}u_i(c_{i,t+1}) \geq \beta_{i,t}u_i(c'_{i,t})+\beta_{i,t+1}u_i(c'_{i,t+1}), \text{ and hence}$$
$$\beta_{i,t}\frac{u_i(c_{i,t})-u_i(c_{i,t}-\epsilon)}{\epsilon} \geq \beta_{i,t+1}\frac{u_i\Big(c_{i,t+1}+(q_{t+1}+d_{t+1})\frac{\epsilon}{q_t}\Big)-u_i(c_{i,t+1})}{\epsilon\frac{q_{t+1}+d_{t+1}}{q_t}}\frac{q_{t+1}+d_{t+1}}{q_t}.$$

Let ϵ tend to zero, we get that $q_t\beta_{i,t}u_i'(c_{i,t}) \geq \beta_{i,t+1}u_i'(c_{i,t+1})(q_{t+1}+d_{t+1})$.

If $b_{i,t}+b_i^*>0$, we can do as above but with $\epsilon<0$ and get that $q_t\beta_{i,t}u_i'(c_{i,t}) \leq \beta_{i,t+1}u_i'(c_{i,t+1})(q_{t+1}+d_{t+1})$. Therefore, we have the equality.

We finally define $\lambda_{i,t}\equiv\beta_{i,t}u_i'(c_{i,t})$ and $\eta_{i,t}\equiv\lambda_{i,t}q_t-\lambda_{i,t+1}(q_{t+1}+d_{t+1})$. We have just proved the FOCs.

We now prove the TVCs. Note that our proof is different from those of Kamihigashi (2002) and Bosi et al. (2018b). The FOCs imply that the sequence $(\lambda_{i,t}q_t)_t$ is decreasing in t. Moreover, we have

$$\lambda_{i,t}q_tb_{i,t} = \Big(\lambda_{i,t+1}(q_{t+1}+d_{t+1})+\eta_{i,t}\Big)b_{i,t} = \lambda_{i,t+1}(q_{t+1}+d_{t+1})b_{i,t}-\eta_{i,t}b_i^*.$$

We rewrite the budget constraint of agent i at date t as follows

$$\lambda_{i,t}(c_{i,t}-e_{i,t}) = \lambda_{i,t}(q_t+d_t)b_{i,t-1}-\lambda_{i,t}q_tb_{i,t}.$$

By taking the sum of this constraint from $t=0$ until T and using (1b), we get

$$\begin{aligned}\sum_{t=0}^{T}\lambda_{i,t}(c_{i,t}-e_{i,t}) &= \sum_{t=0}^{T}\Big(\lambda_{i,t}(q_t+d_t)b_{i,t-1}-\lambda_{i,t}q_tb_{i,t}\Big)\\ &= \lambda_{i,0}(q_0+d_0)b_{i,-1}-\lambda_{i,T}q_Tb_{i,T}+\sum_{t=1}^{T}\eta_{i,t}b_i^*,\end{aligned}$$

and hence $\lambda_{i,0}(q_0+d_0)b_{i,-1}+\sum_{t=0}^{T}\lambda_{i,t}e_{i,t}+\sum_{t=1}^{T}\eta_{i,t}b_i^* = \lambda_{i,T}q_Tb_{i,T}+\sum_{t=0}^{T}\lambda_{i,t}c_{i,t}$.

We will prove that $\lim_{T\to+\infty} \lambda_{i,T} q_T (b_{i,T} + b^*)$ exists in $\mathbb{R}^+$. Recall that the sequence $(\lambda_{i,t} q_t)_t$ is positive and decreasing in t. So, $\lim_{t\to+\infty} \lambda_{i,t} q_t$ exists and is in $\mathbb{R}^+$. We have $-b_i^* \le b_{i,t} = L - \sum_{j\neq i} b_{j,t} \le L + \sum_i b_i^* \forall t$, and hence

$$-\infty < \liminf_{T\to+\infty} \lambda_{i,T} q_T b_{i,T} \le \limsup_{T\to+\infty} \lambda_{i,T} q_T b_{i,T} < \infty.$$

Under our assumptions, we have $\sum_t \lambda_{i,t} c_{i,t} < \infty \ \forall i$. Indeed, we have $\sum_t \lambda_{i,t} c_{i,t} = \sum_t \beta_{i,t} u_i'(c_{i,t}) c_{i,t} \le \sum_t \beta_{i,t} v(c_{i,t}) \le \sum_t \beta_{i,t} v(\sum_i e_{i,t} + L d_t) < \infty$. Thus, we obtain that $\sum_t \lambda_{i,t} c_{i,t} < \infty \ \forall i$.

Since $\sum_t \lambda_{i,t} c_{i,t} < \infty$, both series $\sum_t \lambda_{i,t} e_{i,t}$ and $\sum_t \eta_{i,t} b_i^*$ converge. By consequence, $\lim_{T\to+\infty} \lambda_{i,T} q_T b_{i,T}$ exists in $\mathbb{R}$. Therefore $\lambda_{i,T} q_T (b_{i,T} + b^*)$ converges and

$$\lim_{T\to+\infty} \lambda_{i,T} q_T (b_{i,T} + b^*) = \lim_{T\to+\infty} \lambda_{i,T} q_T b_{i,T} + \lim_{T\to+\infty} \lambda_{i,T} q_T b_i^* \in \mathbb{R}.$$

There are two cases:

- Case (a): If $\liminf_{t\to+\infty}(b_{i,t} + b_i^*) = 0$, then $\lim_{t\to+\infty} \lambda_{i,t} q_t (b_{i,t} + b^*) = 0$ because $\lambda_{i,t} q_t \le \lambda_{i,0} q_0 \ \forall t$.

- Case (b): If $\liminf_{t\to+\infty}(b_{i,t} + b_i^*) > 0$, then there exist $\alpha > 0$ and T such that $b_{i,t} + b_i^* > \alpha \ \forall t \ge T$. In this case $\eta_{i,t} = 0 \forall t \ge T$. For simplicity of the proof, assume $T = 0$. We know that $\lim_{t\to+\infty} \lambda_{i,t} q_t$ exists. Let $\zeta \equiv \lim_{t\to+\infty} \lambda_{i,t} q_t$. We claim that $\zeta = 0$. Assume the contrary: $\zeta > 0$. In this case $\zeta = \lim_{\tau\to+\infty} \lambda_{i,T+\tau+1} q_{T+\tau+1} \le \lambda_{i,T} q_T \ \forall T$. Construct a sequence $(c'_{i,t}, b'_{i,t})$ as follows:

  $$c'_{i,0} = c_{i,0} + \frac{\zeta\alpha}{\lambda_{i,0}}, \ c'_{i,t} = c_{i,t}, \forall t \ge 1, \quad b'_{i,t} = b_{i,t} - \frac{\zeta\alpha}{q_t \lambda_{i,t}}, \forall t \ge 0.$$

  Since $b'_{i,t} \ge -b_i^* + \alpha - \frac{\zeta\alpha}{q_t\lambda_{i,t}} = -b_i^* + \alpha(1 - \frac{\zeta}{q_t\lambda_{i,t}}) \ge -b_i^*, \forall t$, the sequence $(c'_{i,t}, b'_{i,t})$ satisfies physical, budget and borrowing constraints. However $\sum_{t=0}^{+\infty} \beta_{i,t} u_i(c'_{i,t}) > \sum_{t=0}^{+\infty} \beta_{i,t} u_i(c_{i,t})$ which is a contradiction. Hence $\zeta = 0$, i.e. $\lim_{t\to\infty} q_t \lambda_{i,t} = 0$. Since $b_{i,t} + b_i^* = L - \sum_{j\neq i} b_{j,t} + b_i^* \le L + \sum_i b_i^* \forall t$, we get

  $$\lambda_{i,t} q_t (L + \sum_i b_i^*) \ge \lambda_{i,t} q_t (b_{i,t} + b_t^*) \ge \lambda_{i,t} q_t \alpha.$$

  This implies that $\lim_{t\to\infty} \lambda_{i,t} q_t (b_{i,t} + b_t^*) = 0$.

By combining the two cases (a) and (b), we obtain that $\lim_{t\to\infty} \lambda_{i,t} q_t (b_{i,t} + b_t^*) = 0$.

Part 2 (sufficient condition). It suffices to prove the optimality of the allocation (c_i, b_i). Consider another sequence (c'_i, b'_i) satisfying physical, budget and borrowing constraints.

We have, for any T,

$$\sum_{t=0}^{T}\lambda_{i,t}(c_{i,t}-c'_{i,t}) \geq \sum_{t=0}^{T}\lambda_{i,t}\Big(e_{i,t}+(q_t+d_t)b_{i,t-1}-q_tb_{i,t}-e_{i,t}-(q_t+d_t)b'_{i,t-1}+q_tb'_{i,t}\Big)$$
$$=\sum_{t=0}^{T-1}\lambda_{i,t+1}(q_{t+1}+d_{t+1})(b_{i,t}-b'_{i,t})-\sum_{t=0}^{T-1}\lambda_{i,t}q_t(b_{i,t}-b'_{i,t})-q_T\lambda_{i,T}(b_{i,T}-b'_{i,T})$$
$$=-q_T\lambda_{i,T}(b_{i,T}+b^*_i-(b'_{i,T}+b^*_i))+\sum_{t=0}^{T-1}\eta_{i,t}(b'_{i,t}+b^*_i-(b_{i,t}+b^*_i))$$
$$\geq -q_T\lambda_{i,T}(b_{i,T}+b^*_i)+\sum_{t=0}^{T-1}\eta_{i,t}(b'_{i,t}+b^*_i)\geq -q_T\lambda_{i,T}(b_{i,T}+b^*_i).$$

Therefore, we have

$$\sum_{t=0}^{T}\Big(\beta_{i,t}u(c_{i,t})-\beta_{i,t}u(c'_{i,t})\Big)\geq \sum_{t=0}^{T}\lambda_{i,t}(c_{i,t}-c'_{i,t})\geq -q_T\lambda_{i,T}(b_{i,T}+b^*_T).$$

Denote $U_T \equiv \sum_{t=0}^{T}\beta_{i,t}u(c_{i,t})$ and $U'_T \equiv \sum_{t=0}^{T}\beta_{i,t}u(c'_{i,t})$. Observe that the sequence U_T converges when T tends to infinity.

If $\lim_{T\to\infty} q_T\lambda_{i,T}(b_{i,T}+b^*_T)=0$, then $\limsup_{T\to\infty} U'_T \leq \lim_{T\to\infty} U_T$; we have finished our proof.

Remark 8. *If $u_i(0)\geq 0$, then the series $\sum_{t=0}^{\infty}\lambda_{i,t}u_i(c_{i,t})$ always converges. By consequence, conditions $U_T\geq U'_T-q_T\lambda_{i,T}(b_{i,T}+b^*_T)$ $\forall T$ and $\liminf_{T\to\infty} q_T\lambda_{i,T}(b_{i,T}+b^*_T)=0$ imply that $\lim_{T\to\infty}U_T\geq \lim_{T\to\infty}U'_T=\limsup_{T\to\infty}U'_T$.*

□

Proof of Proposition 2. We mainly use Proposition 1.

1. Let $i\in\{1,2,\ldots,m\}$. If there does not exist any infinite increasing sequence i_n of time such that $b_{i,i_n}+b^*_i=0$, then there exists T such that $b_{i,t}+b^*_i>0$, $\forall t\geq T$. So, $\frac{\lambda_{i,t+1}}{\lambda_{i,t}}=\frac{q_t}{q_{t+1}+d_{t+1}}=\frac{1}{R_{t+1}}$, $\forall t\geq T$. This implies that $Q_t=Q_T\frac{\lambda_{i,t}}{\lambda_{i,T}}$, $\forall t\geq T$. By combining with the TVC, we get that $\lim_{t\to\infty}Q_tq_t(b_{i,t}+b^*_i)=0$ $\forall i$. Since $\lim_{\to\infty}Q_tq_t>0$, we get that $\lim_{t\to\infty}b_{i,t}+b^*_i=0$.

2. Let us prove the first part: there exist at least 2 agents, say i and j, such that whose asset holdings $(b_{i,t}),(b_{j,t})$ do not converge. Suppose, by contradiction, that asset holdings of at least $m-1$ agents converge. By market clearing conditions, the asset holding of all agents converges. So, there exists an agent i such that $\lim_{t\to\infty}b_{i,t}>0$ (because the asset supply is strictly positive). According to point 1, this is impossible.

To prove the second part, it suffices to prove that: if the sequence of asset holding $(b_{i,t})$ does not converge, then there exists an infinite increasing sequences $(i_n)_n$ such that $b_{i,i_n}+b^*_i=0$. Suppose, by contradiction, that there does not exist any infinite increasing sequences $(i_n)_n$ such that $b_{i,i_n}+b^*_i=0$. According to point 1, we have $\lim_{t\to\infty}b_{i,t}+b^*_i=0$. It means that $b_{i,t}$ converges, a contradiction.

□

Proof of Corollary 3. Suppose that $\sum_{t\geq 0} Q_t e_{i,t} < \infty$ $\forall i$. Budget constraint of agent i implies that $Q_t c_{i,t} + Q_t q_t b_{i,t} = Q_t e_{i,t} + Q_t(q_t + d_t) b_{i,t-1}$. By summing this equation over t and noticing that $Q_t q_t = Q_{t+1}(q_{t+1} + d_{t+1})$, we have

$$\sum_{t=0}^{T} Q_t c_{i,t} + Q_T q_T b_{i,T} = \sum_{t=0}^{T} Q_t e_{i,t} + (q_0 + d_0) b_{i,-1} \ \forall t.$$

Since $\sum_{t\geq 0} Q_t e_{i,t} < \infty$ and $(Q_T q_T b_{i,T})$ is bounded (because $b_{i,T}$ and $Q_T q_T$ are bounded), the series $\sum_{t\geq 0} Q_t c_{i,t}$ converges, and so does the sequence $(Q_T q_T b_{i,T})_T$. If there is a bubble, we have $\lim_{t\to\infty} Q_t q_t > 0$. By consequence, $(b_{i,t})$ converges for any i. Market clearing conditions imply that there is an agent i such that $b_i \equiv \lim_{t\to\infty} b_{i,t} > 0$. So, borrowing constraints of agent i do not bind from some date on, say T. Hence, $\frac{\lambda_{i,t+1}}{\lambda_{i,t}} = \frac{q_t}{q_{t+1}+d_{t+1}} = \frac{1}{R_{t+1}}$ $\forall t \geq T$. This implies that $Q_t = Q_T \frac{\lambda_{i,t}}{\lambda_{i,T}}$ $\forall t \geq T$. By combining with the TVC, we get that $\lim_{t\to\infty} Q_t q_t (b_{i,t} + b_i^*) = 0$ $\forall i$. This is impossible because $\lim_{t\to\infty}(b_{i,t} + b_i^*) > 0$ and $\lim_t Q_t q_t > 0$.

If $\liminf_t \frac{d_t}{\sum_i e_{i,t}} > 0$, there exist a date $t_0 \geq 1$ and a positive constant x such that $x d_t \geq \sum_i e_{i,t}$ $\forall t \geq t_0$. Therefore, we have

$$\begin{aligned}\sum_{t\geq 0} Q_t\big(\sum_i e_{i,t}\big) &= \sum_{t=0}^{t_0-1} Q_t\big(\sum_i e_{i,t}\big) + \sum_{t\geq t_0} Q_t\big(\sum_i e_{i,t}\big) \\ &\leq \sum_{t=0}^{t_0-1} Q_t\big(\sum_i e_{i,t}\big) + x\sum_{t\geq t_0} Q_t d_t \leq \sum_{t=0}^{t_0-1} Q_t\big(\sum_i e_{i,t}\big) + x q_0 < \infty.\end{aligned}$$

Hence $\sum_{t\geq 0} Q_t\big(\sum_i e_{i,t}\big) < \infty$, and so there does not exist bubble.

□

Proof of Proposition 3. We need intermediate results (Lemmas 1, 2, 3).

Lemma 1. *At each date t, there exists i such that $b_{i,t} \geq b_{i,t+1}$ and borrowing constraint is not binding (i.e., $b_{i,t} + b_i^* > 0$).*

Proof. Define i_0 such that $b_{i_0,t} - b_{i_0,t+1} = \max_i \{b_{i,t} - b_{i,t+1}\}$. Then, we have $b_{i_0,t} - b_{i_0,t+1} \geq 0$. We consider two cases.

Case 1: If $b_{i_0,t} - b_{i_0,t+1} > 0$, then $b_{i_0,t} + b_i^* > b_{i_0,t+1} + b_i^* \geq 0$.

Case 2: If $b_{i_0,t} - b_{i_0,t+1} = 0$, then $b_{i,t} - b_{i,t+1} \leq 0$ $\forall i$. Since $\sum_i (b_{i,t} - b_{i,t+1}) = 0$, we get that $b_{i,t} - b_{i,t+1} = 0$ for every i. Since $\sum_i b_{i,t} > 0$, we can choose i_1 such that $b_{i_1,t} > 0$. So, we have $b_{i_1,t} = b_{i_1,t+1}$ and $b_{i_1,t} + b_i^* > 0$. □

Lemma 2. *Assume that $e_{i,t} - d_t b_i^* > 0$ $\forall i, \forall t$, then we have*

$$\frac{1}{R_{t+1}} \leq \max_i \frac{\beta_{i,t+1} u_i'(e_{i,t+1} - d_{t+1} b_i^*)}{\beta_{i,t} u_i'(\sum_i e_{i,t} + L d_t)} \ \forall t \geq 0. \tag{A.1}$$

Proof. Recall that we have $1 = R_{t+1} \max_i \frac{\beta_{i,t+1} u_i'(c_{i,t+1})}{\beta_{i,t} u_i'(c_{i,t})}$ $\forall t \geq 0$. Let $t \geq 0$, Lemma 1 implies that there exists an agent $i = i(t)$ (depending on t) such that $b_{i(t),t} \geq b_{i(t),t+1}$ and $b_{i(t),t} + b^*_{i(t)} > 0$. Then, we have $\eta_{i(t),t} = 0$ and hence

$$1 = R_{t+1} \frac{\beta_{i(t),t+1} u'_{i(t)}(c_{i(t),t+1})}{\beta_{i(t),t} u'_{i(t)}(c_{i(t),t})}.$$

We observe that $c_{i(t),t+1} = e_{i(t),t+1} + (q_{t+1} + d_{t+1}) b_{i(t),t} - q_{t+1} b_{i(t),t+1} \geq e_{i(t),t+1} - d_{t+1} b^*_{i(t)}$ and $c_{i(t),t} \leq W_t \equiv \sum_i e_{i,t} + L d_t$. By consequence, we get that

$$\begin{aligned}
\frac{1}{R_{t+1}} &= \frac{\beta_{i(t),t+1} u'_{i(t)}(c_{i(t),t+1})}{\beta_{i(t),t} u'_{i(t)}(c_{i(t),t})} \\
&\leq \frac{\beta_{i(t),t+1} u'_{i(t)}(e_{i(t),t+1} - d_{t+1} b^*_{i(t)})}{\beta_{i(t),t} u'_{i(t)}(\sum_i e_{i,t} + L d_t)} \leq \max_i \frac{\beta_{i,t+1} u'_i(e_{i,t+1} - d_{t+1} b^*_i)}{\beta_{i,t} u'_i(\sum_i e_{i,t} + L d_t)}.
\end{aligned}$$

□

Lemma 3. *Consider an equilibrium. Take γ in Assumption 3, we have that $(1-\gamma) q_t b_{i,t} \leq W_t$ $\forall i, \forall t$.*

Proof. Suppose, by contradiction, that there exist i and t such that $(1-\gamma) q_t b_{i,t} > W_t$. Let us consider a new allocation of agent i: $z_i := z\big(c_i, t, \gamma, (1-\gamma) q_t b_{i,t}\big)$. We check that this allocation is in the budget set of agent i because

$$\begin{aligned}
\big(c_{i,t} + (1-\gamma) q_t b_{i,t}\big) + q_t\big(\gamma b_{i,t}\big) &\leq e_{i,t} + (q_t + d_t) b_{i,t-1} \\
\gamma c_{i,s} + q_s\big(\gamma b_{i,s}\big) = \gamma e_{i,s} + (q_t + d_t)\big(\gamma b_{i,s-1}\big) &\leq e_{i,s} + (q_t + d_t)\big(\gamma b_{i,s-1}\big) \ \forall s \geq t+1.
\end{aligned}$$

According to Assumption 3, we have

$$U_i(c_i) < U_i\Big(z(c_i, t, \gamma, W_t\Big) < U_i\Big(z\big(c_i, t, \gamma, (1-\gamma) q_t b_{i,t}\big)\Big). \tag{A.2}$$

This is in contradiction to the optimality of (c_i, b_i). □

We now prove Proposition 3. Since points 1 and 2 are direct consequences of (4), let us prove (4). According to Lemma 3, we have $(1-\gamma) q_t b_{i,t} \leq W_t$ $\forall i, \forall t$. Taking the sum over i, we get $(1-\gamma) q_t L \leq m W_t$ $\forall t$. Since $L(1-\gamma) > 0$, we get that

$$q_t \leq \frac{m W_t}{L(1-\gamma)} \ \forall t. \tag{A.3}$$

According to Lemma 2, we have

$$\frac{1}{R_{t+1}} \leq \max_i \frac{\beta_{i,t+1} u'_i(e_{i,t+1} - d_{t+1} b^*_i)}{\beta_{i,t} u'_i(\sum_i e_{i,t} + L d_t)} \ \forall t \geq 0. \tag{A.4}$$

Recall that there is no bubble iff $\lim_{t\to\infty} Q_t q_t = 0$. By combining these above arguments, there is no bubble if condition (4) is satisfied.

We now check the uniform impatience with two specific utility functions. Before doing so, we observe that, for $T > t$,

$$U_i^T\Big(z(c_i, t, \gamma', W_t)\Big) = \sum_{s=0}^{t-1} \beta_{i,s} u_i(c_{i,s}) + \beta_{i,t} u_i\big(c_{i,t} + W_t\big) + \sum_{s=t+1}^{T} \beta_{i,s} u_i(\gamma' c_{i,s}).$$

1. If $u_i(c) = ln(c)$, we have

$$U_i^T\Big(z(c_i, t, \gamma', W_t)\Big) - U_i^T(c_i) = \beta_{i,t}\Big(u_i\big(c_{i,t} + W_t\big) - u_i(c_{i,t})\Big) + \sum_{s=t+1}^{T} \beta_{i,s}\Big(u_i(\gamma' c_{i,s}) - u_i(c_{i,s})\Big)$$

$$= \beta_{i,t} ln\Big(1 + \frac{W_t}{c_{i,t}}\Big) + \ln(\gamma') \sum_{s=t+1}^{T} \beta_{i,s} \geq \beta_{i,t} ln(2) + \ln(\gamma) \sum_{s=t+1}^{T} \beta_{i,s} \quad \forall \gamma' \geq \gamma.$$

So, we have the uniform impatience if $\beta_{i,t} > -\frac{\ln(\gamma)}{ln(2)} \sum_{s=t+1}^{\infty} \beta_{i,s}$ $\forall t$.
2. If $u_i(c) = \frac{c^{1-\sigma}}{1-\sigma}$, we have

$$U_i^T\big(z(c_i, t, \gamma', W_t)\big) - U_i^T(c_i) = \beta_{i,t}\Big(\frac{\big(c_{i,t} + W_t\big)^{1-\sigma}}{1-\sigma} - \frac{\big(c_{i,t}\big)^{1-\sigma}}{1-\sigma}\Big) + (\gamma^{1-\sigma} - 1) \sum_{s=t+1}^{T} \beta_{i,s} \frac{\big(c_{i,s}\big)^{1-\sigma}}{1-\sigma}$$

$$\geq \beta_{i,t} \frac{2^{1-\sigma} - 1}{1-\sigma} W_t^{1-\sigma} + (\gamma^{1-\sigma} - 1) \sum_{s=t+1}^{T} \beta_{i,s} \frac{W_s^{1-\sigma}}{1-\sigma}$$

where the last inequality is come from $c_{i,t} \leq W_t$ $\forall i, \forall t$, the function $u_i(c + W_t) - u_i(c)$ is decreasing in c, and $\gamma < 1$. So, the uniform impatience holds if $\beta_{i,t} \frac{2^{1-\sigma}-1}{1-\sigma} W_t^{1-\sigma} + (\gamma^{1-\sigma} - 1) \sum_{s=t+1}^{\infty} \beta_{i,s} \frac{W_s^{1-\sigma}}{1-\sigma} > 0$ $\forall t$. □

A.2 Proofs for Section 4

We firstly state and prove a condition under which a sequence is a system of prices. This condition is very useful for the next proofs.

Lemma 4. *Let Assumption 1, 2 be satisfied. (1) The sequence $(q_t)_{t \geq 0}$, asset holdings given by (8) and agents' consumptions given by (9a-9c) constitute an equilibrium if and only if* *consumptions are strictly positive and the following conditions hold*

$$1 = \gamma_{2,0}\frac{u_2'(e_{2,1} + d_1(L + b_1^*) + q_1 H)}{u_2'(e_{2,0} + (q_0 + d_0)b_{2,-1} - q_0(L + b_1^*))}\frac{q_1 + d_1}{q_0} \tag{B.1a}$$

$$1 = \gamma_{1,2t-1}\frac{u_1'(e_{1,2t} + d_{2t}(L + b_2^*) + q_{2t}H)}{u_1'(e_{1,2t-1} - b_1^* d_{2t-1} - q_{2t-1}H)}\frac{q_{2t} + d_{2t}}{q_{2t-1}} \tag{B.1b}$$

$$1 = \gamma_{2,2t}\frac{u_2'(e_{2,2t+1} + d_{2t+1}(L + b_1^*) + q_{2t+1}H)}{u_2'(e_{2,2t} - b_2^* d_{2t} - q_{2t}H)}\frac{q_{2t+1} + d_{2t+1}}{q_{2t}} \tag{B.1c}$$

$$\gamma_{1,2t-1}\frac{u_1'(e_{1,2t} + d_{2t}(L + b_2^*) + q_{2t}H)}{u_1'(e_{1,2t-1} - b_1^* d_{2t-1} - q_{2t-1}H)} \geq \frac{\gamma_{2,2t-1}u_2'(e_{2,2t} - d_{2t}b_2^* - q_{2t}H)}{u_2'(e_{2,2t-1} + d_{2t-1}(L + b_1^*) + q_{2t-1}H)} \tag{B.1d}$$

$$\gamma_{2,2t}\frac{u_2'(e_{2,2t+1} + d_{2t+1}(L + b_1^*) + q_{2t+1}H)}{u_2'(e_{2,2t} - d_{2t}b_2^* - q_{2t}H)} \geq \gamma_{1,2t}\frac{u_1'(e_{1,2t+1} - b_1^* d_{2t+1} - q_{2t+1}H)}{u_1'(e_{1,2t} + d_{2t}(L + b_2^*) + q_{2t}H)} \tag{B.1e}$$

$$\gamma_{2,0}\frac{u_2'(e_{2,1} + d_1(L + b_1^*) + q_1 H)}{u_2'(e_{2,0} + (q_0 + d_0)b_{2,-1} - q_0(L + b_1^*))} \geq \gamma_{1,0}\frac{u_1'(e_{1,1} - b_1^* d_1 - q_1 H)}{u_1'(e_{1,0} + (q_0 + d_0)b_{1,-1} + q_0 b_1^*)} \tag{B.1f}$$

and

$$\lim_{t\to\infty} \beta_{1,2t+1}u_1'(e_{1,2t+1} - b_1^* d_{2t+1} - q_{2t+1}H)q_{2t+1}H = 0 \tag{B.2}$$

$$\lim_{t\to\infty} \beta_{2,2t}u_2'(e_{2,2t} - d_{2t}b_2^* - q_{2t}H)q_{2t}H = 0. \tag{B.3}$$

(2) When $u_i(c) = ln(c)$ $\forall i = 1, 2$ and $d_t = 0$, $\forall t \geq 0$, the sequence $(q_t)_t$, asset holdings given by (8) and agents' consumptions given by (9a-9c) constitute an equilibrium with $q_t > 0$, $\forall t \geq 0$, if and only if

$$\frac{1}{Hq_1} = \frac{\gamma_{2,0}e_{2,0}}{e_{2,1}}\frac{1}{Hq_0} - \frac{1 + \gamma_{2,0}\frac{L+b_1^*-b_{2,-1}}{L+b_1^*+b_2^*}}{e_{2,1}} \geq \frac{\gamma_{1,0}e_{1,0}}{e_{1,1}}\frac{1}{Hq_0} + \frac{1 + \gamma_{1,0}\frac{b_1^*+b_{1,-1}}{L+b_1^*+b_2^*}}{e_{1,1}} \tag{B.4}$$

$$\frac{1}{Hq_{t+1}} = \frac{\gamma_t e_t}{w_{t+1}}\frac{1}{Hq_t} - \frac{1 + \gamma_t}{w_{t+1}} \geq \frac{\mu_t w_t}{e_{t+1}}\frac{1}{Hq_t} + \frac{1 + \mu_t}{e_{t+1}}, \forall t \geq 1, \tag{B.5}$$

where $\gamma_t, \mu_t, e_t, w_t$ are defined by (16a-16b), and

$$\lim_{t\to\infty} \beta_{1,2t+1}\frac{q_{2t+1}H}{e_{1,2t+1} - q_{2t+1}H} = \lim_{t\to\infty} \beta_{2,2t}\frac{q_{2t}H}{e_{2,2t} - q_{2t}H} = 0. \tag{B.6}$$

(3) When $u_i(c) = ln(c)$ $\forall i = 1, 2$, the sequence $(q_t)_t$, asset holdings given by (8) and agents' consumptions given by (9a-9c) constitute an equilibrium if, for any $t \geq 1$, $\gamma_{1,2t-1} \geq \gamma_{2,2t-1}$, $\gamma_{2,2t} \geq \gamma_{1,2t}$

$$e_{1,2t-1} \geq e_{2,2t-1} + (L + 2b_1^*)d_{2t-1} + 2Hq_{2t-1} \tag{B.7a}$$

$$e_{2,2t} \geq e_{1,2t} + (L + 2b_2^*)d_{2t} + 2Hq_{2t} \tag{B.7b}$$

$$e_{2,0} \geq e_{1,0} + d_0(b_{1,-1} - b_{2,-1}) + q_0(L + 2b_1^* + b_{1,-1} - b_{2,-1}) \tag{B.7c}$$

and

$$q_0 = (q_1 + d_1)\frac{\gamma_{2,0}(e_{2,0} + (q_0 + d_0)b_{2,-1} - q_0 b_{2,0})}{e_{2,1} + d_1(L + b_1^*) + q_1 H} \tag{B.8a}$$

$$q_{2t-1} = (q_{2t} + d_{2t})\frac{\gamma_{1,2t-1}(e_{1,2t-1} - b_1^* d_{2t-1} - q_{2t-1}H)}{e_{1,2t} + d_{2t}(L + b_2^*) + q_{2t}H} \tag{B.8b}$$

$$q_{2t} = (q_{2t+1} + d_{2t+1})\frac{\gamma_{2,2t}(e_{2,2t} - b_2^* d_{2t} - q_{2t}H)}{e_{2,2t+1} + d_{2t+1}(L + b_1^*) + q_{2t+1}H}. \tag{B.8c}$$

Proof of Lemma 4. The part 1 of Lemma 4 is a direct consequence of Proposition 1. Let us prove points 2 and 3.

Point 2. When $u_i(c) = ln(c)$ $\forall i = 1, 2$ and $d_t = 0$, $\forall t \geq 0$, the FOCs become

$$\begin{aligned}
1 &= \frac{\gamma_{1,2t-1}(e_{1,2t-1} - q_{2t-1}H)}{e_{1,2t} + q_{2t}H}\frac{q_{2t}}{q_{2t-1}} \geq \gamma_{2,2t-1}\frac{e_{2,2t-1} + q_{2t-1}H}{e_{2,2t} - q_{2t}H}\frac{q_{2t}}{q_{2t-1}} \\
1 &= \frac{\gamma_{2,2t}(e_{2,2t} - q_{2t}H)}{e_{2,2t+1} + q_{2t+1}H}\frac{q_{2t+1}}{q_{2t}} \geq \gamma_{1,2t}\frac{e_{1,2t} + q_{2t}H}{e_{1,2t+1} - q_{2t+1}H}\frac{q_{2t+1}}{q_{2t}} \\
1 &= \gamma_{2,0}\frac{e_{2,0} - q_0(L + b_1^* - b_{2,-1})}{e_{2,1} + q_1 H}\frac{q_1}{q_0} \geq \gamma_{1,0}\frac{e_{1,0} + q_0(b_1^* + b_{1,-1})}{e_{1,1} - q_1 H}\frac{q_1}{q_0}.
\end{aligned}$$

With notations (16a-16e), these conditions are equivalent to (B.4-B.5). Conditions (B.6) are indeed the TVCs (B.2-B.3) for our special case.

Point 3. When $u_i(c) = ln(c)$ $\forall i$, the FOCs now become

$$\begin{aligned}
1 &= \frac{\gamma_{1,2t-1}u_1'(c_{1,2t})}{u_1'(c_{1,2t-1})}\frac{q_{2t} + d_{2t}}{q_{2t-1}} = \frac{\gamma_{1,2t-1}(e_{1,2t-1} - b_1^* d_{2t-1} - q_{2t-1}H)}{e_{1,2t} + d_{2t}(L + b_2^*) + q_{2t}H}\frac{q_{2t} + d_{2t}}{q_{2t-1}} \\
1 &= \frac{\gamma_{2,2t}u_2'(c_{2,2t+1})}{u_2'(c_{2,2t})}\frac{q_{2t+1} + d_{2t+1}}{q_{2t}} = \frac{\gamma_{2,2t}(e_{2,2t} - d_{2t}b_2^* - q_{2t}H)}{e_{2,2t+1} + d_{2t+1}(L + b_1^*) + q_{2t+1}H}\frac{q_{2t+1} + d_{2t+1}}{q_{2t}} \\
&\frac{\gamma_{1,2t-1}c_{1,2t-1}}{c_{1,2t}} \geq \frac{\gamma_{2,2t-1}c_{2,2t-1}}{c_{2,2t}}, \quad \frac{\gamma_{2,2t}c_{2,2t}}{c_{2,2t+1}} \geq \frac{\gamma_{1,2t}c_{1,2t}}{c_{1,2t+1}}.
\end{aligned}$$

The two last inequalities of the FOCs become

$$\gamma_{1,2t-1}\frac{e_{1,2t-1} - b_1^* d_{2t-1} - q_{2t-1}H}{e_{1,2t} + d_{2t}(L + b_2^*) + q_{2t}H} \geq \gamma_{2,2t-1}\frac{e_{2,2t-1} + d_{2t-1}(L + b_1^*) + q_{2t-1}H}{e_{2,2t} - d_{2t}b_2^* - q_{2t}H} \tag{B.11a}$$

$$\gamma_{2,2t}\frac{e_{2,2t} - d_{2t}b_2^* - q_{2t}H}{e_{2,2t+1} + d_{2t+1}(L + b_1^*) + q_{2t+1}H} \geq \gamma_{1,2t}\frac{e_{1,2t} + d_{2t}(L + b_2^*) + q_{2t}H}{e_{1,2t+1} - b_1^* d_{2t+1} - q_{2t+1}H}. \tag{B.11b}$$

Notice that at the period 0, FOCs are

$$1 = \frac{\gamma_{2,0}u_2'(c_{2,1})}{u_2'(c_{2,0})}\frac{q_1 + d_1}{q_0} = \gamma_{2,0}\frac{e_{2,0} + (q_0 + d_0)b_{2,-1} - q_0 b_{2,0}}{e_{2,1} + d_1(L + b_1^*) + q_1 H}\frac{q_1 + d_1}{q_0}$$

$$\text{and } \frac{\gamma_{2,0}u_2'(c_{2,1})}{u_2'(c_{2,0})} \geq \frac{\gamma_{1,0}u_1'(c_{1,1})}{u_1'(c_{1,0})} \quad \Leftrightarrow \gamma_{2,0}\frac{c_{2,0}}{c_{2,1}} \geq \gamma_{1,0}\frac{c_{1,0}}{c_{1,1}}$$

$$\Leftrightarrow \gamma_{2,0}\frac{e_{2,0} + (q_0 + d_0)b_{2,-1} - q_0 b_{2,0}}{e_{2,1} + d_1(L + b_1^*) + q_1 H} \geq \gamma_{1,0}\frac{e_{1,0} + (q_0 + d_0)b_{1,-1} - q_0 b_{1,0}}{e_{1,1} - b_1^* d_1 - q_1 H}.$$

The above inequalities are guaranteed by conditions (B.7a-B.7c) while the above equalities are ensured by conditions (B.8a-B.8c).

According to Proposition 1, it remains to prove the transversality conditions:

$$\lim_{t\to\infty} q_{2t}(b_{1,2t}+b_1^*)\lambda_{1,2t}=0, \quad \lim_{t\to\infty} q_{2t+1}(b_{1,2t+1}+b_1^*)\lambda_{1,2t+1}=0$$
$$\lim_{t\to\infty} q_{2t}(b_{2,2t}+b_2^*)\lambda_{2,2t}=0, \quad \lim_{t\to\infty} q_{2t+1}(b_{2,2t+1}+b_2^*)\lambda_{2,2t+1}=0.$$

Since $b_{1,2t}=-b_1^*$ and $b_{2,2t-1}=-b_2^*$, they becomes

$$\lim_{t\to\infty} q_{2t+1}(b_{1,2t+1}+b_1^*)\lambda_{1,2t+1}=0 \text{ and } \lim_{t\to\infty} q_{2t}(b_{2,2t}+b_2^*)\lambda_{2,2t}=0$$
$$\text{or, equivalently, } \lim_{t\to\infty} q_{2t+1}H\beta_{1,2t+1}\frac{1}{c_{1,2t+1}}=0 \text{ and } \lim_{t\to\infty} q_{2t}H\beta_{2,2t}\frac{1}{c_{2,2t}}=0. \quad \text{(B.14)}$$

These conditions are satisfied thank to (B.7a-B.7c).

□

Proof of Proposition 4. Since the dividends and asset prices are non-negative, the FOCs (B.1b-B.1e) imply that

$$R^*_{2t}=R^*_{1,2t}\leq R^*_{2,2t}, R^*_{2t+1}=R^*_{2,2t+1}\leq R^*_{1,2t+1}\ \forall t\geq 1.$$

The value of asset price bubble is $b_0=q_0-FV_0=\lim_{t\to\infty}\frac{q_t}{R_1\cdots R_t}$. Since the function u_i' is decreasing, we have

$$\frac{q_t}{R_1\cdots R_t}\leq \frac{q_t}{R_1^*\cdots R_t^*}\frac{u_2'(c_{2,1})}{u_2'(c_{2,0})}\frac{u_2'(e_{2,0})}{u_2'(e_{2,1})}\ \forall t\geq 2.$$

The positivity of the consumptions implies that $Hq_t\leq e_t$, where we denote $e_{2t}\equiv e_{2,2t}$ and $e_{2t+1}\equiv e_{1,2t+1}$. So, there is no bubble if $\lim_{t\to\infty}\frac{e_t}{R_1^*\cdots R_t^*}=0$.

□

Proof of Proposition 5. First, we observe that $u_1'\big(e_{1,2t}+d_{2t}(L+b_2^*)+q_{2t}H\big)=u_1'\big(e_{1,2t}-d_{2t}b_1^*+(q_{2t}+d_{2t})H\big)$ and $u_2'\big(e_{2,2t+1}+d_{2t+1}(L+b_1^*)+q_{2t+1}H\big)=u_2'\big(e_{2,2t+1}-d_{2t+1}b_1^*+(q_{2t+1}+d_{2t+1})H\big)$. Since $e_{1,t}-d_tb_1^*, e_{2,t}-d_tb_2^*$ are strictly positive and the function $cu_i'(c)$ is increasing in c, the three numerators in FOCs (B.1a), (B.1b), (B.1c) are increasing in q_1, q_{2t}, q_{2t+1} respectively. Moreover, the three denominators in FOCs (B.1a), (B.1b), (B.1c) are increasing in q_0, q_{2t-1}, q_{2t} respectively. By consequence, q_{t+1} is increasing in q_t for any t. This in turns implies that $\frac{q_{t+1}+d_{t+1}}{q_t}$ is increasing in q_t for any t. By consequence, q_t is strictly increasing in q_0 and $\frac{1}{R_t}=\frac{q_{t-1}}{q_t+d_t}$ is strictly decreasing in q_0. Thus, the fundamental value $FV_0=\sum_{t\geq 1}Q_td_t$ is strictly decreasing in q_0. Hence, the asset price bubble component $B_0\equiv q_0-FV_0$ is strictly increasing in q_0.

Second, we prove that $\mathcal{S}_0$ is connected. Let $x,y\in\mathcal{S}_0$ with $x<y$, and let $z\in(x,y)$. Since $q_t(\cdot)$ is a strictly increasing function, we have $q_t(y)>q_t(z)>q_t(x)$. So, the sequence $(q_t(z))_t$ is strictly positive and individual consumptions generated by this sequence are strictly positive. To prove that the sequence $(q_t(z))$ is a sequence of equilibrium prices, we have to now verify FOCs and TVCs. FOCs (B.1a), (B.1b), (B.1c) are obviously satisfied.

We present a proof of (B.1d) (conditions (B.1e), (B.1f) can be proved by applying the same method). Indeed, we have

$$\begin{aligned}\frac{\gamma_{1,2t-1}u_1'(e_{1,2t}+d_{2t}(L+b_2^*)+q_{2t}(z)H)}{\gamma_{2,2t-1}u_2'(e_{2,2t}-d_{2t}b_2^*-q_{2t}(z)H)} &\geq \frac{\gamma_{1,2t-1}u_1'(e_{1,2t}+d_{2t}(L+b_2^*)+q_{2t}(y)H)}{\gamma_{2,2t-1}u_2'(e_{2,2t}-d_{2t}b_2^*-q_{2t}(y)H)}\\ &\geq \frac{u_1'(e_{1,2t-1}-b_1^*d_{2t-1}-q_{2t-1}(y)H)}{u_2'(e_{2,2t-1}+d_{2t-1}(L+b_1^*)+q_{2t-1}(y)H)}\\ &\geq \frac{u_1'(e_{1,2t-1}-b_1^*d_{2t-1}-q_{2t-1}(z)H)}{u_2'(e_{2,2t-1}+d_{2t-1}(L+b_1^*)+q_{2t-1}(z)H)}.\end{aligned}$$

because $q_t(y) > q_t(z)\ \forall t$. So, we obtain (B.1d).

TVCs (B.2), (B.3) are satisfied because $q_t(y) > q_t(z)\ \forall t$, $u_1'(e_{1,2t+1}-b_1^*d_{2t+1}-q_{2t+1}H)q_{2t+1}$ is increasing in q_{2t+1}, and $u_2'(e_{2,2t}-d_{2t}b_2^*-q_{2t}H)q_{2t}$ is increasing in q_{2t}.

The two last points of Proposition 5 are a direct consequence of the fact that $B_0 \equiv q_0 - FV_0$ is strictly increasing in q_0.

□

Proof of Example 1. Assume that there exists a bubble, then we have $q_t > 0\ \forall t$. According to FOCs (B.8a-B.8c) we obtain that $\frac{1}{Hq_{t+1}} = \frac{\beta e}{w}\frac{1}{Hq_t} - \frac{1+\beta}{w}\ \forall t \geq 0$. From this, by iterating, we get that

$$\frac{1}{Hq_t} = \frac{1}{Hq_0}\Gamma_t - D_t = \frac{1}{Hq_0}\Big(\frac{\beta e}{w}\Big)^t - \frac{1+\beta}{w}\Big(1+\frac{\beta e}{w}+\cdots+\Big(\frac{\beta e}{w}\Big)^{t-1}\Big)\ \forall t \geq 1. \tag{B.15}$$

1. If $\frac{\beta e}{w} \leq 1$ (i.e., $R^* \geq 1$), then the right hand side of (B.15) is negative if t is high enough while the left hand side is strictly positive, a contradiction. Therefore, there is no bubble in this case.
2. If $\frac{\beta e}{w} > 1$ (i.e., $R^* < 1$). In this case, we have

$$\frac{1}{Hq_t} = \frac{\big(\frac{\beta e}{w}\big)^t}{Hq_0} - \frac{1+\beta}{w}\frac{\big(\frac{\beta e}{w}\big)^t - 1}{\frac{\beta e}{w}-1} = \frac{\big(\frac{\beta e}{w}\big)^t}{Hq_0}\Big(1 - Hq_0\frac{1+\beta}{\beta e - w}\big(1-\big(\frac{w}{\beta e}\big)^t\big)\Big)\ \forall t \geq 1. \tag{B.16}$$

2.a. If $q_0 > \frac{1}{H}\frac{\beta e - w}{1+\beta}$, then $1 - Hq_0\frac{1+\beta}{\beta e - w} < 0$. By consequence, the right hand side is strictly negative when t is high enough, a contradiction. In this case, there is no bubble.

2.b. If $q_0 = \frac{1}{H}\frac{\beta e - w}{1+\beta}$, then $1 - Hq_0\frac{1+\beta}{\beta e - w} = 0$. By consequence, we have $q_t = q > 0$ $\forall t \geq 1$. To verify that this is an equilibrium price, we must check conditions (B.7a-B.7c) which now become $e - w > 2H\frac{1}{H}\frac{\beta e - w}{1+\beta}$. This is satisfied because $\beta \in (0,1)$.

2.c. If $0 < q_0 < \frac{1}{H}\frac{\beta e - w}{1+\beta}$, then $1 - Hq_0\frac{1+\beta}{\gamma e - w} > 0$. In this case, we see that the sequence q_t determined by $(B.16)$ is positive and decreasing in t, and $\lim_{t\to\infty} q_t = 0$. Conditions (B.7a-B.7c) which now become $e - w > 2Hq_t\ \forall t$. Since $q_0 < \frac{1}{H}\frac{\beta e - w}{1+\beta}$, we have $2Hq_t \leq 2Hq_0 < e - w$ $\forall t$. So, conditions (B.7a-B.7c) are satisfied. Therefore, the sequence $(q_t)_t$, determined by two conditions $q_0 < \frac{1}{H}\frac{\beta e - w}{1+\beta}$ and (B.15), constitutes a system of equilibrium price with bubble.

□

Proof of Proposition 6. Part 1. We will prove (17-19) and $q_0 \leq \bar{q}$. Recall that bubble exists iff $q_t > 0\ \forall t$. According to point 2 of Lemma 4, FOCs (in form of equality) are equivalent to

$$\frac{1}{Hq_1} = \frac{\gamma_{2,0}e_{2,0}}{e_{2,1}}\frac{1}{Hq_0} - \frac{1+\gamma_{2,0}\frac{L+b_1^*-b_{2,-1}}{L+b_1^*+b_2^*}}{e_{2,1}}, \quad \frac{1}{Hq_{t+1}} = \frac{\gamma_t e_t}{w_{t+1}}\frac{1}{Hq_t} - \frac{1+\gamma_t}{w_{t+1}}, \forall t \geq 1. \tag{B.17}$$

From this, we can compute that, for any $t \geq 2$,

$$\begin{aligned}
\frac{1}{Hq_t} &= \frac{\gamma_{t-1}e_{t-1}}{w_t}\Big(\frac{\gamma_{t-2}e_{t-2}}{w_{t-1}}\frac{1}{Hq_{t-2}} - \frac{1+\gamma_{t-2}}{w_{t-1}}\Big) - \frac{1+\gamma_{t-1}}{w_t} \\
&= \frac{\gamma_{t-1}e_{t-1}}{w_t}\frac{\gamma_{t-2}e_{t-2}}{w_{t-1}}\frac{1}{Hq_{t-2}} - \frac{\gamma_{t-1}e_{t-1}}{w_t}\frac{1+\gamma_{t-2}}{w_{t-1}} - \frac{1+\gamma_{t-1}}{w_t} = \cdots \\
&= \frac{\gamma_{t-1}e_{t-1}}{w_t}\cdots\frac{\gamma_1 e_1}{w_2}\frac{1}{Hq_1} - \Big(\frac{1+\gamma_{t-1}}{w_t} + \frac{\gamma_{t-1}e_{t-1}}{w_t}\frac{1+\gamma_{t-2}}{w_{t-1}} + \cdots + \frac{\gamma_{t-1}e_{t-1}}{w_t}\cdots\frac{\gamma_2 e_2}{w_3}\frac{(1+\gamma_1)}{w_2}\Big) \\
&= \frac{\gamma_{t-1}e_{t-1}}{w_t}\cdots\frac{\gamma_0 e_0}{w_1}\frac{1}{Hq_0} \\
&\qquad - \Big(\frac{1+\gamma_{t-1}}{w_t} + \frac{\gamma_{t-1}e_{t-1}}{w_t}\frac{1+\gamma_{t-2}}{w_{t-1}} + \cdots + \frac{\gamma_{t-1}e_{t-1}}{w_t}\cdots\frac{\gamma_1 e_1}{w_2}\frac{1+\gamma_{2,0}\frac{L+b_1^*-b_{2,-1}}{L+b_1^*+b_2^*}}{w_1}\Big) \\
&= \frac{1}{Hq_0}\Gamma_t - D_t.
\end{aligned}$$

FOCs (in form of inequality) become

$$\frac{1}{Hq_1} \geq \frac{\gamma_{1,0}e_{1,0}}{e_{1,1}}\frac{1}{Hq_0} + \frac{1+\gamma_{1,0}\frac{b_1^*+b_{1,-1}}{L+b_1^*+b_2^*}}{e_{1,1}}, \qquad \frac{1}{Hq_{t+1}} \quad \geq \frac{\mu_t w_t}{e_{t+1}}\frac{1}{Hq_t} + \frac{1+\mu_t}{e_{t+1}}, \forall t \geq 1. \tag{B.19}$$

By combining with (B.17), conditions (B.19) become

$$\frac{\gamma_{2,0}e_{2,0}}{e_{2,1}}\frac{1}{Hq_0} - \frac{1}{e_{2,1}}\Big(1+\gamma_{2,0}\frac{L+b_1^*-b_{2,-1}}{L+b_1^*+b_2^*}\Big) \geq \frac{\gamma_{1,0}e_{1,0}}{e_{1,1}}\frac{1}{Hq_0} + \frac{1+\gamma_{1,0}\frac{b_1^*+b_{1,-1}}{L+b_1^*+b_2^*}}{e_{1,1}} \tag{B.20}$$

$$\frac{\gamma_t e_t}{w_{t+1}}\frac{1}{Hq_t} - \frac{1+\gamma_t}{w_{t+1}} \geq \frac{\mu_t w_t}{e_{t+1}}\frac{1}{Hq_t} + \frac{1+\mu_t}{e_{t+1}}. \tag{B.21}$$

Combining with the fact that $q_t > 0$, $\forall t$, we obtain that X_t defined by (17) is strictly positive.

By using the fact that $\frac{1}{Hq_t} = \frac{1}{Hq_0}\Gamma_t - D_t$, inequalities (B.20-B.21) are equivalent to

$$q_0 \leq \frac{1}{H}\frac{\frac{\gamma_{2,0}e_{2,0}}{e_{2,1}} - \frac{\gamma_{1,0}e_{1,0}}{e_{1,1}}}{\frac{1}{e_{2,1}}\big(1+\gamma_{2,0}\frac{L+b_1^*-b_{2,1}}{H}\big) + \frac{1}{e_{1,1}}\big(1+\gamma_{1,0}\frac{b_1^*+b_{1,-1}}{H}\big)} \tag{B.22}$$

$$q_0 \leq \frac{1}{H}\frac{X_t\Gamma_t}{X_t D_t + Y_t}, \forall t \geq 1. \tag{B.23}$$

By consequence, we have $q_0 \leq \bar{q}$.

Condition (B.23) implies that $\sup_{t\geq 1} \frac{X_tD_t+Y_t}{X_t\Gamma_t} \leq 1/(Hq_0) < \infty$. Hence, $\sup_{t\geq 1} D_t/\Gamma_t < \infty$ and $\sup_{t\geq 1} Y_t/(X_t\Gamma_t) < \infty$. Condition $\sup_{t\geq 1} Y_t/(X_t\Gamma_t) < \infty$ is indeed (18) while $\sup_{t\geq 1} D_t/\Gamma_t < \infty$ is equivalent to (19) (note that $D_1/\Gamma < \infty$).

Part 2 (sufficient condition). Now, let (17-19) be satisfied and $q_0 \leq \bar{q}$. It suffices to check that prices and consumptions are strictly positive, and all conditions in point 2 of Lemma 4 are satisfied.

First of all, our construction $\frac{1}{Hq_t} = \frac{1}{Hq_0}\Gamma_t - D_t \ \forall t \geq 1$ ensures FOCs (B.17). FOCs (in form of inequality), i.e., conditions (B.19) are satisfied thanks to (B.22-B.23) and $X_t > 0$.

Second, condition (B.20-B.21) and $X_t > 0$ imply that $q_t > 0$, $\forall t$. Again, by our construction, we have $\frac{\gamma_t e_t}{w_{t+1}}\frac{1}{Hq_t} - \frac{1+\gamma_t}{w_{t+1}} = \frac{1}{Hq_{t+1}} > 0$. Hence, $Hq_t < \frac{\gamma_t}{(1+\gamma_t)}e_t < e_t$. By consequence, consumptions $c_{2,2t} = e_{2,2t} - q_{2t}H$ and $c_{1,2t+1} = e_{1,2t+1} - q_{2t+1}H$ are strictly positive. It is obvious that $c_{2,2t+1}$ and $c_{1,2t}$ are strictly positive.

Last, we check the TVCs. As just mentioned, we have $Hq_t(1+\gamma_t) < \gamma_t e_t$, or, equivalently, $\frac{Hq_t}{e_t - Hq_t} < \gamma_t$. By combining with $\lim_{t\to\infty} \beta_{i,t} = 0$, we obtain the TVCs: $\lim_{t\to\infty} \beta_{1,2t+1}\frac{q_{2t+1}H}{e_{1,2t+1}-q_{2t+1}H} = \lim_{t\to\infty} \beta_{2,2t}\frac{q_{2t}H}{e_{2,2t}-q_{2t}H} = 0$.

$\square$

Proof of Proposition 7. We will prove, by induction, condition (29), i.e.,

$$\frac{\sigma_s a_s d_s}{1 + d_s H_s} < q_{s-1} < \frac{\alpha_s a_s}{H_s} \ \forall s \geq 1. \tag{B.24}$$

This is satisfied for $t = 1$ because we choose $q_0 \in (\frac{\sigma_1 a_1 d_1}{1+d_1H_1}, \frac{\alpha_1 a_1}{H_1})$. Assume that it holds for $s = t$. Let us prove it for $s = t + 1$. According to (26) and $q_{t-1} < \frac{\alpha_t a_t}{H_t}$, we have

$$q_t = \frac{(1 + d_tH_t)q_{t-1} - a_td_t}{a_t - H_tq_{t-1}} < \frac{(1 + d_tH_t)\frac{\alpha_t a_t}{H_t} - a_td_t}{a_t - H_t\frac{\alpha_t a_t}{H_t}} = \frac{\frac{\alpha_t}{H_t} - (1 - \alpha_t)d_t}{1 - \alpha_t} \tag{B.25}$$

$$< \frac{\alpha_t}{(1 - \alpha_t)H_t} < \frac{\alpha_{t+1}a_{t+1}}{H_{t+1}} \tag{B.26}$$

where the last inequality is from (27a).

The system (26) and condition $\frac{\sigma_t a_t d_t}{1+d_tH_t} < q_{t-1}$ imply that

$$q_t = \frac{(1 + d_tH_t)q_{t-1} - a_td_t}{a_t - H_tq_{t-1}} > \frac{(1 + d_tH_t)\frac{\sigma_t a_t d_t}{1+d_tH_t} - a_td_t}{a_t - H_t\frac{\sigma_t a_t d_t}{1+d_tH_t}} = \frac{(\sigma_t - 1)d_t}{1 - \frac{\sigma_t d_t H_t}{1+d_tH_t}}. \tag{B.27}$$

According to condition $1 - (\sigma_t - 1)d_tH_t > 0$, we have $1 - \frac{\sigma_t d_t H_t}{1+d_tH_t} > 0$ which in turn implies that $q_t > (\sigma_t - 1)d_t > \sigma_{t+1}a_{t+1}d_{t+1}$, where the last inequality is from the first condition in (27b). Finally, we get that $q_t > \frac{\sigma_{t+1}a_{t+1}d_{t+1}}{1+H_{t+1}d_{t+1}}$. Therefore, we have just proved (29).

To prove that (q_t) is a price sequence of an equilibrium, we check that all conditions in Lemma 4 are satisfied. First, since $0 < \alpha_t < 1 < \sigma_t$, condition (29) ensures that $q_t > 0$ $\forall t \geq 0$.

Second, observe that condition (29) implies that $q_t < \frac{a_{t+1}}{H_{t+1}}$ and hence $q_t < \bar{q}_t$. By definition of $\bar{q}_t$, the inequalities (B.7a-B.7c) can be rewritten as $q_t \leq \bar{q}_t \ \forall t$. So, conditions (B.7a-B.7c) are satisfied. It also ensures that consumptions are strictly positive.

Last, FOCs (B.8a-B.8c) are ensured by the system (26).

Since there is a continuum of equilibria, Proposition 5 implies that there is a continuum of bubbly equilibrium. □

Proof of Proposition 8. Part 1. According to Remark **??**, there is no bubble if $a < 1$. Now, consider the case $a > 1$. Suppose that there is a bubble. We must have $\sum_t d_t < \infty$. There are only two cases.

Case 1. If there is t_0 such that $q_{t_0} \leq \frac{a-1}{h}$, then we have

$$q_{t_0+1} - q_{t_0} = \frac{q_{t_0}(hq_{t_0} - (a-1))}{a - hq_{t_0}} - d_t < 0 \tag{B.28}$$

By induction, we have that $q_t < q_{t-1} < (a-1)/h \ \forall t > t_0$. By consequence, the sequence q_t decreasingly converges to a value $q \geq 0$. Observe that

$$(q_t + d_t)(a - hq_{t-1}) = q_{t-1}, \text{ and hence } q(a - hq) = q, \tag{B.29}$$

So, either $q = 0$ or $q = (a-1)/h$. Since $q_t < q_{t-1} < (a-1)/h \ \forall t > t_0$, the value q must be strictly lower than $(a-1)/h$. As a result, q_t converges to zero.

Case 2. $q_t > \frac{a-1}{h} \ \forall t \geq 0$. Observe that

$$(q_t + d_0 + \cdots + d_t) - (q_{t-1} + d_0 + \cdots + d_{t-1}) = q_t + d_t - q_{t-1} = \frac{q_{t-1}(hq_{t-1} - (a-1))}{a - hq_{t-1}} > 0.$$

So the sequence $(q_t + d_0 + \cdots + d_t)$ is strictly increasing. Since $\sum_t d_t < \infty$ and $q_t < \frac{a}{h}$, this sequence is bounded, and hence converges. As a result, the sequence (q_t) converges. So, it must converge to $\frac{a-1}{h}$.

Part 2. We now prove that there is almost one equilibrium satisfying $q_t > \frac{a-1}{h} \ \forall t$, in which asset holdings are given by (8) and agents' consumptions are given by (9a-9c). Let (q_t) and (q'_t) be two systems of equilibrium prices. We must have $q_t < a/h$ and $q'_t < a/h$.

Define $x_t = q_t - \frac{a-1}{h}, x'_t = q'_t - \frac{a-1}{h}$, then we have $0 < x_t, x'_t < 1/h$ and

$$q_t + d_t = \frac{q_{t-1}}{a - hq_{t-1}} \Leftrightarrow x_t + \frac{a-1}{h} + d_t = \frac{x_{t-1} + \frac{a-1}{h}}{a - h(x_{t-1} + \frac{a-1}{h})} \Leftrightarrow x_t + d_t = \frac{ax_{t-1}}{1 - hx_{t-1}}$$

Similarly, we have $x'_t + d_t = \frac{ax'_{t-1}}{1-hx'_{t-1}}$. Therefore, we get that

$$x_t - x'_t = \frac{a(x_{t-1} - x'_{t-1})}{(1 - hx_{t-1})(1 - hx'_{t-1})} \ \forall t \geq 1. \tag{B.30}$$

We will prove that $x_0 = x'_0$ (which implies that $q_t = q'_t \ \forall t$). Without loss of generality, suppose that $x_0 > x'_0$. According to (B.30), we have $x_t - x'_t > a(x_{t-1} - x'_{t-1}) \ \forall t \geq 1$. Therefore, we have $x_t - x'_t > a^t(x_0 - x'_0) \ \forall t \geq 1$. Since $a > 1$, $a^t(x_0 - x'_0)$ converges to infinity. So, $x_t - x'_t$ also converges to infinity. However, this cannot happen because both x_t and x'_t belong the interval $(0, 1/h)$.

□

Proof of Example 2. First, we see that

$$a_{2t} = a_{2t+1} = a \equiv \frac{\beta e}{w}, \quad H_{2t} = H_{2t+1} = h \equiv \frac{H(\beta+1)}{w}$$
$$2H\bar{q}_{2t-1} \equiv e - w - Hd_{2t-1}, \quad 2H\bar{q}_{2t} \equiv e - w - Hd_{2t}$$

So, condition $\frac{a_{t+1}}{H_{t+1}} < \bar{q}_t \ \forall t$ becomes

$$\frac{\beta(e_{2,0} + d_0 b_{2,-1})}{\beta(L + b_1^* - b_{2,-1}) + H} < \frac{e_{2,0} - e_{1,0} - d_0(b_{1,-1} - b_{2,-1})}{L + 2b_1^* + b_{1,-1} - b_{2,-1}} \tag{B.31}$$
$$\frac{2\beta e}{1+\beta} < e - w - Hd_{2t-1}, \quad \frac{2\beta e}{1+\beta} < e - w - Hd_{2t}. \tag{B.32}$$

These conditions and condition $\bar{q}_t > 0$ are satisfied because we assume that $d_t < \frac{\frac{1-\beta}{1+\beta}e - w}{H}$.

Second, observe that condition $\frac{\sigma a d_1}{1+d_1\frac{H(\beta+1)}{w}} < \frac{\beta e - w}{H(\beta+1)}$ ensures that $\frac{\sigma a d_1}{1+d_1 h} < \frac{a-1}{h}$. So, the interval $(\frac{\sigma a d_1}{1+d_1 h}, \frac{a-1}{h}]$ is well defined.

We next prove that $q_s \in (\frac{\sigma a d_s}{1+d_s h}, \frac{a-1}{h}] \ \forall s \geq 0$. This holds for $s = 0$ because $q_0 \in (\frac{\sigma a d_1}{1+d_1 h}, \frac{a-1}{h}]$. Assume that it holds for $t-1$, we will prove this for t. Indeed, condition $d_t < \frac{w}{(\sigma-1)(\beta+1)H}$ is equivalent to $1 - \frac{\sigma d_t h}{1+d_t h} > 0$. By combining this with $\frac{\sigma-1}{\sigma}\frac{d_t}{d_{t+1}} > \frac{\beta e}{w}$, we get that

$$q_t = \frac{(1+d_t h)q_{t-1} - a d_t}{a - h q_{t-1}} > \frac{(1+d_t h)\frac{\sigma a d_t}{1+d_t h} - a d_t}{a_t - h\frac{\sigma a d_t}{1+d_t h}} = \frac{(\sigma-1)d_t}{1 - \frac{\sigma d_t h}{1+d_t h}} \tag{B.33}$$
$$> (\sigma - 1)d_t > \sigma a_{t+1} d_{t+1} > \frac{\sigma a_{t+1} d_{t+1}}{1 + H_{t+1} d_{t+1}} \tag{B.34}$$

We also have

$$q_t - q_{t-1} = \frac{q_{t-1}(h q_{t-1} - (a-1))}{a - h q_{t-1}} - d_t < 0 \tag{B.35}$$

because $hq_{t-1} < a - 1$. So, we have $q_t < q_{t-1} < (a-1)/h \ \forall t$. This in turn implies that $q_t < (a-1)/h \ \forall t$. By consequence, the sequence q_t decreasingly converges, and hence it cannot converge to $(a-1)/h$. As a result, it converges to zero.

It remains to prove that (q_t) is a price sequence of an equilibrium. To do so, we verify that all conditions in Lemma 4 are satisfied. First, it is easy to see that $q_t > 0 \ \forall t \geq 0$. Second, according to $\frac{a_{t+1}}{H_{t+1}} < \bar{q}_t$, we have $q_t < \frac{a_{t+1}}{H_{t+1}} < \bar{q}_t$. This shows that conditions (B.7a-B.7c) are satisfied. It also ensures that consumptions are strictly positive. Last, FOCs (B.8a-B.8c) are ensured by the system (26).

□

Proof of Example 3. We see that $1 - (a-1)x > 0$ and $hxd_0 < 1$. So, we can check that $0 < hxd_t < 1$ and $xd_t + d_t = \frac{axd_{t-1}}{1-hxd_{t-1}} \ \forall t \geq 0$. According to the proof of Proposition 8, the sequence (q_t), defined by $q_t = \frac{a-1}{h} + xd_t \ \forall t$, satisfies: $q_t \in (\frac{a-1}{h}, \frac{a}{h})$ and $q_t + d_t = \frac{q_{t-1}}{a - hq_{t-1}} \ \forall t$. In order to prove that (q_t) is a system of prices of an equilibrium at which asset holdings are given by (8) and agents' consumptions are given by (9a-9c), we verify all conditions in Lemma 4.

As in the proof of Example 2, condition $d_0 < \frac{\frac{1-\beta}{1+\beta}e-w}{H}$ ensures that $a/h < \bar{q}_t\ \forall t$. Thus, $q_t < a/h < \bar{q}_t\ \forall t$. This shows that conditions (B.7a-B.7c) are satisfied. It also ensures that consumptions are strictly positive. Last, FOCs (B.8a-B.8c) are ensured by the system $q_t + d_t = \frac{q_{t-1}}{a-hq_{t-1}}\ \forall t$.

□

Proof of result in Section 4.2. We check all FOCs, TVCs and market clearing conditions in Proposition 1. Firstly, with the asset allocation given by (40), we have $c_{3,t} = e_{3,t}$, $\forall t$. Moreover

$$c_{1,0} = e_{1,0} + q_0(b_{1,-1} - b_{1,0}) = e_{1,0} + q_0(b_{1,-1} + b_1^*) \tag{B.36}$$

$$c_{1,2t+1} = e_{1,2t+1} + q_{2t+1}(b_{1,2t} - b_{1,2t+1}) = e_{1,2t+1} + q_{2t+1}(-b_1^* - \frac{L + nb_2^* + Nb_3^*}{n}) \tag{B.37}$$

$$= e_{1,2t+1} - H_3 q_{2t+1} \tag{B.38}$$

$$c_{1,2t} = e_{1,2t} + q_{2t}(b_{1,2t-1} - b_{1,2t}) = e_{1,2t} + q_{2t}(\frac{L + nb_2^* + Nb_3^*}{n} + b_1^*) \tag{B.39}$$

$$= e_{1,2t} + H_3 q_{2t} \tag{B.40}$$

and

$$c_{2,0} = e_{2,0} + q_0(b_{2,-1} - b_{2,0}) = e_{2,0} + q_0(b_{2,-1} - \frac{L + nb_1^* + Nb_3^*}{n}) \tag{B.41}$$

$$= e_{2,0} - q_0(b_1^* + L_3 - b_{2,-1}) \tag{B.42}$$

$$c_{2,2t+1} = e_{2,2t+1} + q_{2t+1}(b_{2,2t} - b_{2,2t+1}) = e_{2,2t+1} + H_3 q_{2t+1} \tag{B.43}$$

$$c_{2,2t} = e_{2,2t} + q_{2t}(b_{2,2t-1} - b_{2,2t}) = e_{2,2t} - H_3 q_{2t} \tag{B.44}$$

The next step is to look at all FOCs, TVCs in Proposition 1. As in part 2 of of Lemma 4, the FOCs becomes

$$1 = \frac{\gamma_{1,2t-1}(e_{1,2t-1} - q_{2t-1}H_3)}{e_{1,2t} + q_{2t}H_3}\frac{q_{2t}}{q_{2t-1}} \geq \gamma_{2,2t-1}\frac{e_{2,2t-1} + q_{2t-1}H_3}{e_{2,2t} - q_{2t}H_3}\frac{q_{2t}}{q_{2t-1}}$$
$$1 = \frac{\gamma_{2,2t}(e_{2,2t} - q_{2t}H_3)}{e_{2,2t+1} + q_{2t+1}H_3}\frac{q_{2t+1}}{q_{2t}} \geq \gamma_{1,2t}\frac{e_{1,2t} + q_{2t}H_3}{e_{1,2t+1} - q_{2t+1}H_3}\frac{q_{2t+1}}{q_{2t}}$$
$$1 = \gamma_{2,0}\frac{e_{2,0} - q_0(L_3 + b_1^* - b_{2,-1})}{e_{2,1} + q_1H_3}\frac{q_1}{q_0} \geq \gamma_{1,0}\frac{e_{1,0} + q_0(b_1^* + b_{1,-1})}{e_{1,1} - q_1H_3}\frac{q_1}{q_0}.$$

or, equivalently,

$$\frac{1}{H_3q_1} = \frac{\gamma_{2,0}e_{2,0}}{e_{2,1}}\frac{1}{H_3q_0} - \frac{1 + \gamma_{2,0}\frac{L_3+b_1^*-b_{2,-1}}{L_3+b_1^*+b_2^*}}{e_{2,1}} \geq \frac{\gamma_{1,0}e_{1,0}}{e_{1,1}}\frac{1}{H_3q_0} + \frac{1 + \gamma_{1,0}\frac{b_1^*+b_{1,-1}}{L_3+b_1^*+b_2^*}}{e_{1,1}} \tag{B.46}$$

$$\frac{1}{H_3q_{t+1}} = \frac{\gamma_t e_t}{w_{t+1}}\frac{1}{H_3q_t} - \frac{1e + \gamma_t}{w_{t+1}} \geq \frac{\mu_t w_t}{e_{t+1}}\frac{1}{H_3q_t} + \frac{1 + \mu_t}{e_{t+1}}, \forall t \geq 1, \tag{B.47}$$

Then, we can use the same argument in the proof of Proposition 6 to show that the set of bubbly equilibria is described by (42). Note that condition (B.47) implies that

$$\frac{1}{H_3 q_{t+1}} > \frac{\mu_t w_t}{e_{t+1}} \frac{1}{H_3 q_t} \Rightarrow R_{t+1} = \frac{q_{t+1}}{q_t} < \frac{e_{t+1}}{\mu_t w_t}$$

By combining this with (41), we have that

$$1 \geq \frac{\beta_{3,t+1}}{\beta_{3,t}} \frac{e_{3,t}}{e_{3,t+1}} \frac{e_{t+1}}{\mu_t w_t} > \frac{\beta_{3,t+1}}{\beta_{3,t}} \frac{e_{3,t}}{e_{3,t+1}} \frac{q_{t+1}}{q_t} = \frac{\beta_{3,t+1} u_3'(c_{3,t+1})}{\beta_{3,t} u_3'(c_{3,t})} \frac{q_{t+1}}{q_t} \tag{B.48}$$

It means that the FOCs of agent 3 are satisfied. We have finished our proof. □

Proof of Proposition 9. We need an intermediate step.

Lemma 5. *Assume that u_i is strictly concave and $u_i'' < 0$. Then, $\beta_{1,2t-1}u_1(c_{1,2t-1}) + \beta_{1,2t}u_1(c_{1,2t})$ and $\beta_{2,2t}u_2(c_{2,2t}) + \beta_{2,2t+1}u_2(c_{2,2t+1})$ are strictly positive and strictly increasing in q_0.*

Proof of Lemma 5. The FOCs in Proposition 1 imply that

$$\begin{cases} \gamma_{1,2t-1} \dfrac{u_1'(e_{1,2t} + d_{2t}(L + b_2^*) + q_{2t}H)}{u_1'(e_{1,2t-1} - b_1^* d_{2t-1} - q_{2t-1}H)} \geq \gamma_{2,2t-1} \dfrac{u_2'(e_{2,2t} - d_{2t}b_2^* - q_{2t}H)}{u_2'(e_{2,2t-1} + d_{2t-1}(L + b_1^*) + q_{2t-1}H)} \\ \gamma_{2,2t} \dfrac{u_2'(e_{2,2t+1} + d_{2t+1}(L + b_1^*) + q_{2t+1}H)}{u_2'(e_{2,2t} - d_{2t}b_2^* - q_{2t}H)} \geq \gamma_{1,2t} \dfrac{u_1'(e_{1,2t+1} - b_1^* d_{2t+1} - q_{2t+1}H)}{u_1'(e_{1,2t} + d_{2t}(L + b_2^*) + q_{2t}H)} \end{cases}$$

and

$$\beta_{1,2t-1} q_{2t-1} u_1'\big(e_{1,2t-1} - b_1^* d_{2t-1} - q_{2t-1}H\big) = \beta_{1,2t} u_1'\big(e_{1,2t} + d_{2t}(L + b_2^*) + q_{2t}H\big)(q_{2t} + d_{2t})$$
$$\beta_{2,2t} q_{2t} u_2'\big(e_{2,2t} - b_2^* d_{2t} - q_{2t}H\big) = \beta_{2,2t+1}(q_{2t+1} + d_{2t+1}) u_2'\big(e_{2,2t+1} + d_{2t+1}(L + b_1^*) + q_{2t+1}H\big)$$

Taking the derivatives of both sides of equalities with respect to q_0 and combining with FOCs, we have

$$\begin{aligned} \beta_{1,2t-1} q_{2t-1}'(q_0) u_1'(c_{1,2t-1}) &\leq \beta_{1,2t-1} q_{2t-1}'(q_0) \Big(u_1'(c_{1,2t-1}) - q_{2t-1} H u_1''(c_{1,2t-1}) \Big) \\ &= \beta_{1,2t} q_{2t}'(q_0) \Big(u_1'(c_{1,2t}) + H u_1''(c_{1,2t})(q_{2t} + d_{2t}) \Big) \\ &< \beta_{1,2t} q_{2t}'(q_0) u_1'(c_{1,2t}) \end{aligned} \tag{B.49}$$

$$\begin{aligned} \beta_{2,2t} q_{2t}'(q_0) u_2'(c_{2,2t}) &\leq \beta_{2,2t} q_{2t}'(q_0) \Big(u_2'(c_{2,2t}) - q_{2t} H u_2''(c_{2,2t}) \Big) \\ &= \beta_{2,2t+1} q_{2t+1}'(q_0) \Big(u_2'(c_{1,2t+1}) + H u_2''(c_{2,2t+1})(q_{2t+1} + d_{2t+1}) \Big) \\ &< \beta_{2,2t+1} q_{2t+1}'(q_0) u_2'(c_{2,2t+1}). \end{aligned} \tag{B.50}$$

where we also use $u_i'' < 0$ (the function u_i is strictly concave) for $i = 1, 2$.

By consequence, we have that

$$\frac{\partial}{\partial q_0}\Big(\beta_{1,2t-1}u_1(c_{1,2t-1}) + \beta_{1,2t}u_1(c_{1,2t})\Big) = -H\beta_{1,2t-1}u_a'(c_{1,2t-1})q_{2t-1}'(q_0) + H\beta_{1,2t}u_a'(c_{1,2t})q_{2t}'(q_0) > 0$$
$$\frac{\partial}{\partial q_0}\Big(\beta_{2,2t}u_2(c_{2,2t}) + \beta_{2,2t+1}u_2(c_{2,2t+1})\Big) = -H\beta_{2,2t}u_b'(c_{2,2t})q_{2t}'(q_0) + H\beta_{2,2t+1}u_b'(c_{2,2t+1})q_{2t+1}'(q_0) > 0.$$

□

We now prove Proposition 9. According to Lemma 5, the sequences $\beta_{1,2t-1}u_1(c_{1,2t-1})+\beta_{1,2t}u_1(c_{1,2t})$ and $\beta_{2,2t}u_2(c_{2,2t})+\beta_{2,2t+1}u_2(c_{2,2t+1})$ are strictly positive and strictly increasing in q_0.

We now prove that the function $W_i(q)$ is increasing in q. Notice that we cannot directly prove this by looking at $\sum_{t\geq 0}\frac{\partial}{\partial q_0}\big(\beta_{i,t}u_i(c_{i,t})\big)$ for $i=1,2,$, because it is unclear that this series converge. So, we will prove our result as follows. Let q_0 and q'_0 be the two initial prices of two equilibria $(q_t,(c_{i,t})_i)$ and $(q'_t,(c'_{i,t})_i)$. Assume that $q_0>q'_0$. Consider agent 1 and denote $A_T\equiv\sum_{t=0}^{T}\beta_{1,t}\Big(u_1(c_{1,t})-u_1(c'_{1,t})\Big)$. We will prove that A_T converges to a strictly positive number when T tends to infinity. Indeed, we see that $A_{2T}=\sum_{t=0}^{2T}\beta_{1,t}\Big(u_1(c_{1,t})-u_1(c'_{1,t})\Big)$ is strictly positive and increasing in T (because $\beta_{1,2t-1}u_1(c_{1,2t-1})+\beta_{1,2t}u_1(c_{1,2t})$ is strictly increasing in q_0 et $c_{1,0}$ is increasing in q_0). So, it converges to a strictly positive value.

We now observe that

$$A_{2T+1}=\sum_{t=0}^{2T}\beta_{1,t}\Big(u_1(c_{1,t})-u_1(c'_{1,t})\Big)+\beta_{1,2T+1}\Big(u_1(c_{1,2T+1})-u_1(c'_{1,2T+1})\Big) \tag{B.51}$$

It is easy to see that $\beta_{1,2T+1}\big(u_1(c_{1,2T+1})-u_1(c'_{1,2T+1})\big)$ converges to zero because both $\beta_{1,2T+1}u_1(c_{1,2T+1})$ and $\beta_{1,2T+1}u_1(c'_{1,2T+1})$ converge to zero. By consequence, we have A_{2T} and A_{2T+1} converge to the same value. So, A_t converges to a strictly positive value when t tends to infinity.

By using the same method, we can prove that $\sum_{t=0}^{T}\beta_{2,t}\big(u_2(c_{2,t})-u_2(c'_{2,t})\big)$ converges to a strictly positive number when T tends to infinity.

□

References

Araujo, A., Novinski, R., Pascoa, M.R., 2011. General equilibrium, wariness and efficient bubbles. *Journal of Economic Theory* 46, p. 785-811.

Araujo, A., Pascoa, M.R., Torres-Martinez, J.P., 2011. *Long-lived collateralized assets and bubbles.* Journal of Mathematical Economics 47, 260-271.

Alvarez, F., Jermann, U.J., 2000. Efficiency, equilibrium, and asset pricing with risk of default. *Econometric* 68(4), p. 775-797.

Becker, R., Bosi, S., Le Van, C., Seegmuller, T., 2015. *On existence and bubbles of Ramsey equilibrium with borrowing constraints.* Economic Theory 58, 329-353.

Benhabib, J., Farmer R.E.A., 1999. Indeterminacy and sunspots in macroeconomics. In: Taylor, J.B., Woodford, M., (Eds.) *Handbook of Macroeconomics*, vol. 1, issue: part A, p. 387-448, Elsevier Science BV.

Barbie, M., Hillebrand, M., 2018. Bubbly Markov equilibria, *Economic Theory*, vol. 66, p. 627-679.

Bewley T., 1980. The optimal quantity of money. In: Kareken, J.H., Wallace, N., (Eds.) *Models of Monetary Economics*, Federal Reserve Bank of Minneapolis, p. 169–210.

Bloise G, Citanna A., 2019. Asset shortages, liquidity and speculative bubbles, *Journal of Economic Theory*, vol. 183, p. 952-990.

Bosi, S., Ha-Huy, T., Le Van, C., Pham, C.T., Pham, N.-S., 2018a. Financial bubbles and capital accumulation in altruistic economies, *Journal of Mathematical Economics*, vol. 75, p. 125-139.

Bosi, S., Le Van, C., Pham, N.-S., 2017a. Asset bubbles and efficiency in a generalized two-sector model, *Mathematical Social Sciences*, vol. 88, p. 37-48.

Bosi, S., Le Van, C., Pham, N.-S., 2017b. Intertemporal equilibrium with heterogeneous agents, endogenous dividends and collateral constraints, *CES working paper series*, No. 2015.67R.

Bosi, S., Le Van, C., Pham, N.-S., 2018b. Intertemporal equilibrium with heterogeneous agents, endogenous dividends and collateral constraints, *Journal of Mathematical Economics*, vol. 76, p. 1-20.

Brunnermeier, M.K., Oehmke, M., 2012. Bubbles, financial crises, and systemic risk. *Handbook of the Economics of Finance*, vol. 2.

Doblas-Madrid A., 2012. A robust model of bubbles with multidimensional uncertainty, *Econometrica* 80, p. 1845-1893.

Farhi, E., Tirole, J., 2012. Bubbly liquidity. *Review of Economic Studies* 79, p. 678-706.

Farmer R.E.A., 2019. The Indeterminacy Agenda in Macroeconomics, *NBER Working Paper*, No. 25879.

Gali, J., 2014. Monetary Policy and Rational Asset Price Bubbles. *American Economic Review* 104, p. 721–752.

Gali, J., 2021. Monetary Policy and Bubbles in a New Keynesian Model with Overlapping Generations. *American Economic Journal: Macroeconomics*, 13 (2): 121-167.

Huang, K. X. D., Werner, J., 2000. Asset price bubbles in Arrow-Debreu and sequential equilibrium. *Economic Theory* 15, p. 253-278.

Hirano, T., Yanagawa, N., 2017. Asset Bubbles, Endogenous Growth, and Financial Frictions. *The Review of Economic Studies*, vol. 84, p. 406-443.

Kamihigashi, T., 2002. *A simple proof of the necessity of the transversality condition.* Economic Theory 20, 427-433.

Kehoe, T.J., Levine, D.K., 1985. *Comparative statics and perfect foresight in infinite horizon economies.* Econometrica 53, No. 2, p. 433-453.

Kehoe, T.J., Levine, D.K., Romer, P.M., 1990. *Determinacy of equilibria in dynamic models with finitely many consumers.* Journal of Economic Theory 50, p. 1-21.

Kiyotaki, N., Moore, J., 1997. Credit cycles. *Journal of Political Economy*, 105(2), p. 211-248.

Kocherlakota, N. R., 1992. Bubbles and constraints on debt accumulation. *Journal of Economic Theory 57*, p. 245 - 256.

Kocherlakota, N. R., 2008. Injecting rational bubbles. *Journal of Economic Theory 142*, p. 218 - 232.

Le Van, C., Pham, N.-S., 2016. Intertemporal equilibrium with financial asset and physical capital. *Economic Theory* 62, p. 155-199.

Levine, D. K., Zame, W. R., 1996. Debt constraints and equilibrium in infinite horizon economies with incomplete markets. *Journal of Mathematical Economics* 26, p. 103-131.

Ljungqvist, L., Sargent, T. J. 2012. *Recursive Macroeconomic Theory*, third edition, The MIT Press

Lucas, R. E., 1978. Asset prices in an exchange economy. *Econometrica* 46, p. 1429-1445.

Magill, M., Quinzii, M., 1994. Infinite Horizon Incomplete Markets. *Econometrica* 62, p. 853-880.

Magill, M., Quinzii, M., 1996. Incomplete markets over an infinite horizon: Long-lived securities and speculative bubbles. *Journal of Mathematical Economics* 26, p. 133-170.

Martin, A., Ventura, J., 2012. Economic growth with bubbles. *American Economic Review* 102, p. 3033 - 3058.

Miao, J., 2014. *Introduction to economic theory of bubbles. Journal of Mathematical Economics* 53, p. 130-136.

Martin, A., Ventura, J., 2018. The Macroeconomics of Rational Bubbles: A User's Guide. *Annual Review of Economics*, 10(1), 505–539.

Miao, J., 2014. *Introduction to economic theory of bubbles. Journal of Mathematical Economics* 53, p. 130-136.

Miao, J., Wang, P., 2012. Bubbles and total factor productivity. *American Economic Review: Papers and Proceedings* 102, p. 82-87.

Miao, J., Wang, P., 2018. Asset bubbles and credit constraints. *American Economic Review* 108, p. 2590-2628.

Montrucchio, L., 2004. Cass transversality condition and sequential asset bubbles. *Economic Theory* 24, p. 645-663.

Pascoa, M.R., Petrassi, M., Torres-Martinez, J.P., 2011. Fiat money and the value of binding portfolio constraints. *Economic Theory* 46, p. 189-209.

Santos, M. S., Woodford, M., 1997. Rational asset pricing bubbles. *Econometrica* 65, p. 19-57.

Scheinkman J. A., Weiss L., 1986. Borrowing constraints and aggregate economic activity. *Econometrica* 54 (1), p. 23–45.

Tirole, J., 1982. On the possibility of speculation under rational expectations. *Econometrica* 50, p. 1163-1181.

Tirole, J., 1985. Asset bubbles and overlapping generations. *Econometrica* 53, p. 1499-1528.

Townsend, R., 1980. Models of Money with Spatially Separated Agents. In: Kareken, J.H., Wallace, N., (Eds.) *Models of Monetary Economics*, Federal Reserve Bank of Minneapolis, p. 265-303.

Werner, J., 2014. Rational asset pricing bubbles and debt constraints. *Journal of Mathematical Economics*, vol. 53, p. 145-152.