Direct Proofs of the Existence of Equilibrium, the Gale-Nikaido-Debreu Lemma and the Fixed Point Theorems using Sperner's Lemma

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Abstract

In this paper, we use Sperner's lemma to prove the existence of general equilibrium for a competitive economy with production or with uncertainty and financial assets. We then show that the direct use of Sperner's lemma together with Carathéodory's convexity theorem and basic properties of topology such as partition of unit, finite covering of a compact set allow us to bypass the Kakutani fixed point theorem even in establishing the Gale-Nikaido-Debreu Lemma. We also provide a new proof of the Kakutani fixed point theorem based on Sperner's lemma.
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1 Introduction

The classic proofs of the existence of general equilibrium mainly rely on Brouwer and Kakutani fixed point theorems (Brouwer, 1911; Kakutani, 1941). They make use of either Gale-Nikaido-Debreu (Debreu, 1959; Gale, 1955; Nikaido, 1956) or Gale and Mas-Colell (Gale and Mas-Colell, 1975, 1979) lemmas, the proofs of which in turn require Kakutani or Brouwer fixed point theorems.[1]

Sperner's lemma (Sperner, 1928) is a combinatorial variant of the Brouwer fixed point theorem and actually equivalent to it.[2] By enabling us to work with topological spaces in a

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[1] See, for excellent treatments of the existence of equilibrium, Debreu (1982) and Florenzano (2003).

[2] For instance, Knaster, Kuratowski, and Mazurkiewicz (1929) use the Sperner lemma to prove the Knaster-Kuratowski-Mazurkiewicz lemma which implies the Brouwer fixed point theorem. Meanwhile, Yoseloff (1974) and Park and Jeong (2003) prove the Sperner lemma by using the Brouwer fixed point theorem. The reader is referred to Park (1999) for a more complete survey of fixed point theorems and Ben-El-Mechaiekh et al. (2009) for a survey of general equilibrium and fixed point theory.

purely combinatorial way, Sperner's lemma has been proven useful in computing the fixed points of mappings, critical points of dynamical systems, and the fair division problems (Scarf and Hansen, 1973; Su, 1999). However, this intuitive yet powerful lemma has not been fully exploited in the theory of general equilibrium.

This paper highlights the role of the Sperner's lemma as an alternative, purely conbinatorial approach, to equilibrium analysis. First, following the excess demand approach, we use the Sperner lemma and elementary mathematical results, which allow us to bypass the fixed points theorems and the Gale-Nikaido-Debreu lemma, to prove the existence of an equilibrium in an economy with or without production and a two-period stochastic economy with incomplete financial markets.

The key point when applying the Sperner's lemma is to construct a labeling which is proper (i.e., it satisfies Sperner condition) and, more importantly, will generate a point corresponding to an equilibrium price. In an earlier attempt, Scarf (1982) (page 1024) also uses the Sperner's lemma to prove the existence of general equilibrium, but for a pure exchange economy. In an economy with production, thanks to the Weak Walras Law and by adapting the labeling in Scarf (1982), we can construct a proper labeling which generates an equilibrium price.

In a two-period economy with incomplete financial markets, constructing a proper labeling is more difficult because the budget sets may have empty interiors when some prices are null. To overcome this difficulty, we introduce an artificial economy where all agents except for one have an additional income ($\epsilon > 0$) in the first period so that their budget sets have a non-empty interior for any prices system in the simplex. For this artificial economy, we can construct a proper labeling and hence prove the existence of an equilibrium which depends on ϵ. Then, we let ϵ go to zero to get an equilibrium for the original economy. It should be noticed that our proof works for nominal, and numéraire assets as well. Our result leads to an important implication: in the case of numéraire asset, there is a continuum of equilibrium.

Second, we use Sperner's lemma to give a new proof of the Gale-Nikaido-Debreu lemma. It is noteworthy that the existing proofs of the several versions of the Gale-Nikaido-Debreu lemma require the use of the fixed point theorems (see Florenzano (2009) for an excellent review). For instance, Debreu (1956, 1959) and Nikaido (1956) use the Kakutani fixed point theorem while Gale (1955) uses the Knaster-Kuratowski-Mazurkiewicz lemma. According to Duppe and Weintraub (2014), Khan (2021), Debreu wanted to discuss the question whether one could dispense with a fixed point theorem in proving the lemma. We address the question of Debreu by providing a new proof of the Gale-Nikaido-Debreu lemma directly from Sperner's lemma and the basic elements of topology.

Last, but certainly not least, we provide a new proof of the Kakutani fixed point theorem by using the Sperner lemma. There have been earlier attempts to use the Sperner lemma to prove the Kakutani fixed point theorem. For example, Sondjaja (2008) uses the Sperner lemma but she also requires to make use of von Neumann (1937)'s approximation lemma. Shmalo (2018) proves the so-called *hyperplane labeling* lemma, generalizing Sperner's lemma, and uses it together with the approximate minimax theorem to prove the Kakutani fixed point theorem. In comparison, it seems that our method provides a more straightforward and direct proof of the theorem as it only uses the core notions of topology.

Note that the Sperner lemma and the mathematical tools that we have used to prove the existence of general equilibrium and the Gale-Nikaido-Debreu lemma dates back to 1928. In this respect, our proofs suggest retrospectively that the existence of general equilibrium could have been proved almost two decades earlier before the seminal papers of Arrow and Debreu (1954) and Debreu (1959).[3]

The paper proceeds as follows. In Section 2, we review some basic concepts such as the notions of subsimplex, simplicial subdivision, and Sperner's lemma. In Section 3, we use the Sperner's lemmma to prove the existence of general equilibrium (in two models - one with production and the other with incomplete financial markets), and the Gale-Nikaido-Debreu lemma. Finally, Section 4 concludes the paper.

2 Preliminaries

In this section, we introduce basic terminologies and necessary background for our work. First, we present definitions from combinatorial topology based on which we state the Sperner's lemma. After that, we provide a brief overview of correspondences and the maximum theorem which are extensively used for proving the existence of a general equilibrium.

2.1 On the Sperner lemma

Consider the Euclidean space $\mathbb{R}^n$. Let $e^1 = (1, 0, 0, \dots, 0), e^2 = (0, 1, 0, \dots, 0), \dots,$ and $e^n = (0, 0, \dots, 0, 1)$ denote the n unit vectors of $\mathbb{R}^n$. The unit-simplex Δ of $\mathbb{R}^n$ is the convex hull of $\{e^1, e^2, \dots, e^n\}$. A simplex of Δ, denoted by $[[x^1, x^2, \dots, x^n]]$, is the convex hull of $\{x^1, x^2, \dots, x^n\}$ where $x^i \in \Delta$ for any $i = 1, \dots, n$, and the vectors $(x^1 - x^2, x^1 - x^3, \dots, x^1 - x^n)$ are linearly independent, or equivalently, the vectors $(x^1, x^2, \dots, x^n)$ are affinely independent (i.e., if $\sum_{i=1}^n \lambda_i x_i = 0$ and $\sum_{i=1}^n \lambda_i = 0$ imply that $\lambda_i = 0\ \forall i$).

Given a simplex $[[x^1, x^2, \dots, x^n]]$, a face of this simplex is the convex hull $[[x^{i_1}, x^{i_2}, \dots, x^{i_m}]]$ with $m < n$, and $\{i_1, i_2, \dots, i_m\} \subset \{1, 2, \dots, n\}$.

We now define the notions of simplicial subdivision (or triangulation) and labeling (see Border (1985), Su (1999) or Chapter 23 in Maschler et al. (2013) for a general treatment) before stating the Sperner's lemma.

Definition 1. *T is a simplicial subdivision of Δ if it is a finite collection of simplices and their faces $\Delta_i,\ i = 1, \dots, p$ such that*

- $\Delta = \cup_{i=1}^p \Delta_i$,
- $ri(\Delta_i) \cap ri(\Delta_j) = \emptyset, \forall i \neq j$.

Recall that if $\Delta_i = [[x^{i_1}, x^{i_2}, \dots, x^{i_m}]]$, then $ri(\Delta_i) \equiv \{x \mid x = \sum_{k=1}^m \alpha_k x^k(i); \sum_k \alpha_k = 1;$ and $\forall k : \alpha(k) > 0\}$.

Simplicial subdivision simply partitions an n-dimensional simplex into small simplices such that any two simplices are either disjoint or share a full face of a certain dimension.

[3] Recall that Gérard Debreu was awarded the Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel in 1983 for having incorporated new analytical methods into economic theory and for his rigorous reformulation of the theory of general equilibrium.

Remark 1. *For any positive integer K, there is a simplicial subdivision $T^K = \{\Delta_1^K, \ldots, \Delta_{p(K)}^K\}$ of Δ such that $Mesh(T^K) \equiv \max_{i\in\{1,\ldots,p(K)\}} \sup_{x,y}\{\|x-y\| : x, y \in \Delta_i^K\} < 1/K$. For example, we can take equilateral subdivisions or barycentric subdivisions.*

We focus on the labeling of these subdivisions with certain restrictions.

Definition 2. *Consider a simplicial subdivision of Δ. Let V denote the set of vertices of all the subsimplices of Δ. A labeling R is a function from V into $\{1, 2, \ldots, n\}$. A labeling R is said to be proper if it satisfies the* ***Sperner condition****:*

$$x \in \ ri[[e^{i_1}, e^{i_2}, \ldots, e^{i_m}]] \Rightarrow R(x) \in \{i_1, i_2, \ldots, i_m\}.$$

In particular, $R(e^i) = i, \forall i$.

Note that the Sperner condition implies that all vertices of the simplex are labeled distinctly. Moreover, the label of any vertex on the edge between the vertices of the original simplex matches with another label of these vertices. With these in mind, we can now state the Sperner's lemma.

Lemma 1. *(Sperner) Let $T = \{\Delta_1, \ldots, \Delta_p\}$ be a simplicial subdivision of Δ. Let R be a labeling which satisfies the Sperner condition. Then there exists a subsimplex $\Delta_i \in T$ which is completely labeled, i.e. $\Delta_i = [[x^1(i), \ldots, x^n(i)]]$ with $R(x^l(i)) = l, \forall l = 1, \ldots, n$.*

The Sperner's lemma guarantees the existence of a completely labeled subsimplex for any simplicially subdivided simplex in accordance with the Sperner condition. A proof of this lemma can be found in several textbooks (Berge, 1959; Scarf and Hansen, 1973; Border, 1985; Maschler et al., 2013) or papers (Sperner, 1928; Le Van, 1982). In particular, the original proof uses an inductive argument based on a complete enumeration of all completely labeled simplices for a series of lower dimensional problems. Meanwhile, proofs using constructive arguments date back to Cohen (1967) and Kuhn (1968) (see Scarf (1982) for a demonstration of the constructive proof).

2.2 On correspondences

Let $X \subset \mathbb{R}^l, Y \subset \mathbb{R}^m$. A correspondence Γ from X into Y is a mapping from X into the set of subsets of Y. The graph of Γ is the set graph$\Gamma = \{(x, y) \in X \times Y : y \in \Gamma(x)\}$. A correspondence $\Gamma : X \to Y$ is *closed* if its graph is closed.

Definition 3. *A correspondence $\Gamma : X \to Y$ is upper semicontinuous at point x if for every open set V of Y for which $\Gamma(x) \subset V$, there exists a neighborhood U of x such that $\Gamma(x) \in V$ $\forall x \in U$. Γ is said to be upper semicontinuous on X if it is upper continuous at every point of X.*

Notice that if X is compact then Γ is upper semicontinuous if and only if Γ is closed. It is also clear that if Γ is upper semicontinuous and $K \subset X$ is compact, then $\Gamma(K)$ is compact. Recall that if Γ is single-valued, the notions of continuity, upper semicontinuity, and the lower semicontinuity turn out to be equivalent.

3 Main results

3.1 Using Sperner's lemma to prove the existence of general equilibrium

In this section, we show that Sperner lemma can be used as a direct tool to prove the existence of general equilibrium in competitive economies. Our proofs are novel as they only make use of the Sperner lemma and elementary mathematical results.

3.1.1 Equilibrium existence in an economy with production

Consider an economy with L consumption goods, K input goods which may be capital or labor, I consumers, and J firms.[4] Each consumer i has an initial endowment of consumption goods $\omega^i \in \mathbb{R}^L_+$, an initial endowment of inputs $y_0^i \in \mathbb{R}^K_+$, and a utility function u^i depending on her/his consumptions $x^i \in \mathbb{R}^L_+$. The firms produce consumption goods. Firm j has production functions $F^j = (F^j_1, \ldots, F^j_L)$ and uses a vector of inputs $(y^j_1, \ldots, y^j_K) \in \mathbb{R}^K_+$. The production functions satisfy $F^j_l \geq 0$, and $F^j \neq 0$. We do not exclude the case that $F^j_l = 0$ for some l (i.e., firm j does not produce good l).

We adopt the following set of standard assumptions concerning the specifications of an economy with production.

Assumption 1. *(i) Each utility function is strictly concave, continuous, and strictly increasing.*
(ii) The endowments of consumption goods satisfy: $\omega^i \gg 0$ *(i.e.,* $\omega \in \mathbb{R}^L_{++}$*)* $\forall i$.
(iii) The endowments of inputs satisfy: $y_0^i \gg 0$ *(i.e.,* $y_0^i \in \mathbb{R}^K_{++}$*)* $\forall i$.
(iv) For any l, $F^j_l(0) = 0$, *and if* $F^j_l \neq 0$ *then it is strictly concave and strictly increasing.*
(v) The firms distribute their profits among consumers. The share coefficients θ^{ij}, $i = 1, \ldots, I$ *and* $j = 1, \ldots, J$ *are positive and satisfy* $\sum_i \theta^{ij} = 1, \forall j$.

In this economy, each firm j maximizes its profit given the prices p of outputs and the prices q of inputs. Let

$$\Pi^j(p, q) = \max_{y \in \mathbb{R}^K_+} \{p \cdot F^j(y) - q \cdot y\}.$$

We observe that for any (p, q), $\Pi^j(p, q) \geq p \cdot F^j(0) - q \cdot 0 = 0$.

On the other hand, given the prices p of outputs and the prices q of inputs, each consumer i solves the problem

$$\max u^i(x^i) \text{ subject to } x^i \in \mathbb{R}^L_+ \text{ and } p \cdot x^i \leq p \cdot \omega^i + \sum_j \theta^{ij} \Pi^j(p, q) + q \cdot y_0^i.$$

We now introduce the definitions of equilibrium and feasible allocation for such an economy with production.

Definition 4. *An equilibrium is a list* $((x^{i*})_{i=1,\ldots,I}, (y^{j*})_{j=1,\ldots,J}, p^*, q^*)$ *that satisfies the properties: (i)* $p^* \gg 0, q^* \gg 0$*; (ii) given prices, households and firms maximize their utility and profit respectively; and (iii) all markets clear.*

[4] When $K = J = 0$, we recover the pure exchange economy.

Definition 5. *An allocation* $((x^i)_i, (y^j)_j)$ *is feasible if*

(i) $x^i \in \mathbb{R}^L_+$ *for any* $i = 1, \dots, I$, $y^j \in \mathbb{R}^K_+$, *for any* $j = 1, \dots, J$,

(ii) $\sum_{i=1}^I x^i \leq \sum_{i=1}^I \omega^i + \sum_{j=1}^J F^j(y^j)$,

(iii) $\sum_{j=1}^J y^j \leq \sum_{i=1}^I y_0^i$.

The set of feasible allocations is denoted by $\mathcal{F}$. It is convex and compact. We denote by X^i the set of allocations x^i such that there exist $(x^{-i}) \in (\mathbb{R}^L_+)^{I-1}$ and (y^j) which satisfy $((x^i, x^{-i}), (y^j)) \in \mathcal{F}$. We denote by Y^j the set of inputs (y^j) such that there exist allocations (x^i) which satisfy $((x^i), (y^j)) \in \mathcal{F}$. Note that all of these sets are convex, compact, and nonempty.

Let X be a closed ball of $\mathbb{R}^L_+$ that contains all the X^i (for $i = 1, \dots, I$) in its interior. Also, let Y be a closed ball of $\mathbb{R}^K_+$ that contains all the sets Y^j (for $j = 1, \dots, J$) in its interior.

We will consider an *intermediate economy* in which the consumption sets equal to X and the input sets equal to Y. In this economy, given prices p and q, the behavior of each firm j can be recast as: $\max_{y^j \in Y}\{p \cdot F^j(y^j) - q \cdot y^j\}$. Accordingly, the behavior of each consumer i can be recast as

$$\max u^i(x^i) \text{ subject to } x^i \in X \text{ and } p \cdot x^i \leq p \cdot \omega^i + \sum_j \theta^{ij}\Pi^j(p, q) + q \cdot y_0^i.$$

Definition 6. *An equilibrium of the intermediate economy is a list* $((x^{i*})_{i=1,\dots,I}, (y^{j*})_{j=1,\dots,J}, p^*, q^*)$ *that satisfies*

(i) $p^* \gg 0, q^* \gg 0$,

(ii) For any i, $x^{i*} \in X$ *and* $p^* \cdot x^{i*} = p^* \cdot \omega^i + \sum_j \theta^{ij}\Pi^j(p^*, q^*) + q^* \cdot y_0^i$,

(iii) For any i, $x^i \in X$, $p^* \cdot x^i \leq p^* \cdot \omega^i + \sum_j \theta^{ij}\Pi^j(p^*, q^*) + q^* \cdot y_0^i \Rightarrow u^i(x^i) \leq u^i(x^{i*})$,

(iv) For any j, $y^{j*} \in Y$ *and* $\Pi^j(p^*, q^*) = p^* \cdot F^j(y^{j*}) - q^* \cdot y^{j*}$,

(v) $\sum_{i=1}^I x^{i*} = \sum_{i=1}^I \omega^i + \sum_{j=1}^J F^j(y^{j*})$ *and* $\sum_{j=1}^J y^{j*} = \sum_{i=1}^I y_0^i$.

Since the utility functions and the production functions are strictly increasing, an equivalent definition can be reached by refining condition (v) in Definition 6. More precisely, an equilibrium in this intermediate economy is a list $((x^{i*})_{i=1,\dots,I}, (y^{j*})_{j=1,\dots,J}, p^*, q^*)$ that satisfies the conditions (i-iv) in Definition 6 together with

(vi') For any $l = 1, \dots, L$, $\sum_{i=1}^I x_l^{i*} - \left(\sum_{i=1}^I \omega_l^i + \sum_{j=1}^J F_l^j(y^{j*})\right) \leq 0$,

(vii') For any $k = 1, \dots, K, \sum_{j=1}^J y_k^{j*} - \sum_{i=1}^I y_{0,k}^i \leq 0$,

(viii') For any $l = 1, \dots, L$, $p_l^* \left(\sum_{i=1}^I x_l^{i*} - \left(\sum_{i=1}^I \omega_l^i + \sum_{j=1}^J F_l^j(y^{j*})\right)\right) = 0$, (viv') For any $k = 1, \dots, K$, $q^*_k \left(\sum_{j=1}^J y^{j*}_k - \sum_{i=1}^I y^i_{0,k} \right) = 0$.

The following remark is important for the analysis of the equilibrium existence.

Remark 2. *If (x^*, y^*) solves the problems of the consumers and the firms, then (x^*, y^*) satisfies **Weak Walras Law**:*

$$p \cdot \Big(\sum_i (x^{*i} - \omega^i) - \sum_j F^j(y^*) \Big) + q \cdot \Big(\sum_j y^{*j} - \sum_i y^i_0 \Big) \le 0. \quad (1)$$

*However, if $\sum_i (x^{*i} - \omega^i) - \sum_j F^j(y^*) \le 0$ and $\sum_j y^{*j} - \sum_i y^i_0 \le 0$, i.e., $(x^*, y^*) \in \mathcal{F}$, since the utility functions are strictly increasing and the feasible set $\mathcal{F}$ is in the interior of $X \times Y$, the allocation (x^*, y^*) satisfies **Walras Law**:*

$$p \cdot \Big(\sum_i (x^{*i} - \omega^i) - \sum_j F^j(y^*) \Big) + q \cdot \Big(\sum_j y^{*j} - \sum_i y^i_0 \Big) = 0. \quad (2)$$

We now use the Sperner lemma to prove the existence of an equilibrium for the intermediate economy. We will show that it is actually an equilibrium for the initial economy.

Proposition 1. *Under above assumptions, there exists an equilibrium in the intermediate economy.*

Proof. Let $\alpha > 0$.

Step 1. Consider the following transformed problem of the producer:

$$\Pi^{j,\alpha}(p, q) \quad = \max\{p \cdot F^j(y^j) - q \cdot y^j : y^j \in C^{j,\alpha}(p, q)\}$$

where $C^{j,\alpha}(p, q) = \{y \in Y : q \cdot y^j - p \cdot F^j(y^j) \le \alpha\}$. Let $\eta^{j,\alpha}(p, q) = \{y^j \in Y : p \cdot F^j(y^j) - q \cdot y^j = \Pi^{j,\alpha}(p, q)\}$. Since the production function is strictly concave, $\eta^{j,\alpha}$ is a single-valued mapping. We can directly prove, without using the Maximum Theorem (Berge, 1959), that $\eta^{j,\alpha}(p, q)$ is continuous in the set $\Delta \equiv \{(x_1, \dots, x_{L+K}) \ge 0 : \sum_{i=1}^{L+K} x_i = 1\}$. Indeed, let $(p, q) \in \Delta$ and denote $y^* = \eta^{j,\alpha}(p, q)$. We have that $p \cdot F^j(y^*) - q \cdot y^* \ge 0 > -\alpha$. Consider the sequence $(p^n, q^n) \in \Delta$ that converges to (p, q) when n tends to infinity. Let $y^n = \eta^{j,\alpha}(p^n, q^n)$. We have to prove that y^n converges to y^*. Since $C^{j,\alpha}(p, q)$ contains 0, we have $p \cdot F^j(y^*) - q \cdot y^* \ge 0$. Hence, for n large enough, we have $p^n \cdot F^j(y^*) - q^n \cdot y^* > -\alpha$.

Again, by definition, we have $\Pi^{j,\alpha}(p^n, q^n) = p^n \cdot F^j(y^n) - q^n \cdot y^n \ge 0 > -\alpha$ for any n.

When $n \to +\infty$, we can assume $y^n \to \bar{y} \in Y$ and hence, $p \cdot F^j(\bar{y}) - q \cdot \bar{y} \ge -\alpha$. In other words, $\bar{y} \in C^{j,\alpha}(p, q)$. This implies

$$\Pi^{j,\alpha}(p, q) = p \cdot F^j(y^*) - q \cdot y^* \ge p \cdot F^j(\bar{y}) - q \cdot \bar{y}.$$

But, since $p^n \cdot F^j(y^*) - q^n \cdot y^* > -\alpha$, we have $y^* \in C^{j,\alpha}(p^n, q^n)$. Therefore,

$$\Pi^{j,\alpha}(p^n, q^n) = p^n \cdot F^j(y^n) - q^n \cdot y^n \ge p^n \cdot F^j(y^*) - q^n \cdot y^*.$$

Let $n \to +\infty$. We get

$$p \cdot F^j(\bar{y}) - q \cdot \bar{y} \ge p \cdot F^j(y^*) - q \cdot y^*.$$

Therefore, $\bar{y} = y^*$. We have proved that the mapping $\eta^{j,\alpha}$ is continuous. We then also get that the maximum profit $\Pi^{j,\alpha}$ is a continuous function.

Step 2. Consider also the transformed problem of the consumer:

$$\max u^i(x^i) \text{ subject to } x^i \in X,\ p \cdot x^i \leq p \cdot \omega^i + \sum_j \theta^{ij} \Pi^{j,\alpha}(p,q) + q \cdot y_0^i.$$

It is easy to see that the set $D^{i,\alpha}(p,q) = \{x^i : x^i \in X,\ p \cdot x^i \leq p \cdot \omega^i + \sum_j \theta^{ij} \Pi^{j,\alpha}(p,q) + q \cdot y_0^i\}$ is convex and compact. Moreover, it has a non-empty interior. Indeed, observe that $\Pi^{j,\alpha}(p,q) \geq 0$. If $p = 0$ then $q > 0$ and $q \cdot y_0^i > 0$. We have $0 < \sum_j \theta^{ij} \Pi^{j,\alpha}(p,q) + q \cdot y_0^i$. If $p \neq 0$, choose x^i close to ω^i and $x^i \ll \omega^i$. Then $p \cdot (x^i - \omega^i) < 0 \leq \sum_j \theta^{ij} \Pi^{j,\alpha}(p,q) + q \cdot y_0^i$.

For $(p,q) \in \Delta$ and $i = 1, \ldots, I$, we define

$$\xi^{\alpha,i}(p,q) = \{x^i \in X : u^i(x^i) \geq u^i(x'),\ \text{if } p \cdot x' \leq p \cdot \omega^i + \sum_j \theta^{ij} \Pi^{j,\alpha}(p,q) + q \cdot y_0^i\}. \tag{3}$$

The mapping $\xi^\alpha \equiv (\xi^{\alpha,i})_{i=1}^I$ is single-valued. We shall prove that ξ^α is continuous without using the Maximum Theorem (Berge, 1959).

Denote $x^{i*} = \xi^{\alpha,i}(p,q)$, we have $p \cdot x^{i*} \leq p \cdot \omega^i + \sum_j \theta^{ij} \Pi^{j,\alpha}(p,q) + q \cdot y_0^i$.

Let $(p^n, q^n) \in \Delta \to (p,q)$ when $n \to +\infty$. Denote $x^i(n) = \xi^i(p^n, q^n)$. We can assume $x^i(n) \to \bar{x}^{i,\alpha} \in X$. Since $p^n \cdot x^i(n) \leq p^n \cdot \omega^i + \sum_j \theta^{ij} \Pi^{j,\alpha}(p^n, q^n) + q^n \cdot y_0^i$, we have

$$p \cdot \bar{x}^i \leq p \cdot \omega^i + \sum_j \theta^{ij} \Pi^{j,\alpha}(p,q) + q \cdot y_0^i,$$

and hence $u^i(x^{i*}) \geq u^i(\bar{x}^i)$.

Let $z \in \text{int} D^{i,\alpha}(p,q)$, i.e. it satisfies $p \cdot z < p \cdot \omega^i + \sum_j \theta^{ij} \Pi^{j,\alpha}(p,q) + q \cdot y_0^i$. Then for n large enough, we have

$$p^n \cdot z < p^n \cdot \omega^i + \sum_j \theta^{ij} \Pi^{j,\alpha}(p^n, q^n) + q^n \cdot y_0^i.$$

This implies $u^i(x^i(n)) \geq u^i(z)$ for any n large enough. Hence $u^i(\bar{x}^i) \geq u^i(z)$. Actually this inequality holds for any z in the interior of $D^{i,\alpha}(p,q)$. Take $x_0 \in \text{int } D^{i,\alpha}(p,q)$. For any integer m define $z_m = \frac{1}{m} x_0 + (1 - \frac{1}{m}) x^{i*}$. Then z_m is in the interior of $D^{i,\alpha}(p,q)$. We have

$$\frac{1}{m} u^i(x_0) + (1 - \frac{1}{m}) u^i(x^{i*}) \leq u^i(z_m) \leq u^i(\bar{x}^i).$$

Let $m \to +\infty$. We get $u^i(x^{i*}) \leq u^i(\bar{x}^i)$. Hence $\bar{x}^i = x^{i*}$. We have proved that $\xi^{\alpha,i}$ is continuous.

Step 3. Denote $N = L + K$, $\pi = (p,q) \in \Delta$, and define the excess demand mappings:

$$\begin{aligned}
\xi^\alpha(\pi) &= \sum_{i=1}^I (\xi^{\alpha,i}(\pi) - \omega^i) - \sum_{j=1}^J F^j(\eta^{j,\alpha}(\pi)) \\
\eta^\alpha(\pi) &= \sum_{j=1}^J \eta^{j,\alpha}(\pi) - \sum_{i=1}^I y_0^i \\
\zeta(\pi) &= (\xi^\alpha(\pi), \eta^\alpha(\pi)).
\end{aligned}$$

According to Steps 1 and 2, the mapping ζ is continuous.

Step 4 (using the Sperner's lemma). We will use the Sperner lemma to prove that there exists $\pi^* \in \Delta$ such that $\zeta_j(\pi^*) \leq 0 \ \forall j$. Indeed, let $K > 0$ be an integer and consider a simplicial subdivision T^K of Δ such that $Mesh(T^K) < 1/K$ and define a labeling R as follows: For $\pi \in \Delta$, $R(\pi) = i$ where i satisfies $\zeta_i(\pi) \leq 0$. We can see that the labeling R is well-defined (because of Weak Walras Law) and satisfies Sperner condition.[5] Indeed, let $\pi \in \text{ ri}[[e^{i_1}, e^{i_2}, \dots, e^{i_m}]]$ where $m \leq N$, we have $\pi = \sum_{t=1}^m \lambda_t e^{i_t}$ where $\lambda_t > 0, \sum_{t=1}^m \lambda_t = 1$. By the Weak Walras Law, we have $\sum_{t=1}^m \lambda_t \zeta_{i_t}(\pi) \leq 0$. So, $R(\pi) \in \{i_1, i_2, \dots, i_m\}$ because, otherwise, we have $\zeta_{i_t}(\pi) > 0 \ \forall i_t \in \{i_1, i_2, \dots, i_m\}$ and hence $\sum_{t=1}^m \lambda_t \zeta_{i_t}(\pi) > 0$, which is a contradiction.

Applying the Sperner lemma, there exists a completely labeled subsimplex $[[\bar{\pi}^{K,1}, \bar{\pi}^{K,2}, \dots, \bar{\pi}^{K,n}]]$ such that $R(\bar{\pi}^{K,j}) = j$, i.e., $\zeta_j(\bar{\pi}^{K,j}) \leq 0, \forall j = 1, \dots, N$. Let K go to $+\infty$, the vertices $\{\bar{\pi}^{K,j}\}$ converge to the same point $\pi^* \in \Delta$. This point satisfies $\zeta_j(\pi^*) \leq 0 \ \forall j$.

Step 5. From Remark 2, Walras Law holds. Hence, $\sum_j \pi^*_j \zeta_j(\pi^*) = 0$ and we have actually $\pi^*_j \zeta_j(\pi^*) = 0, \forall j$.

Finally, we claim that $\Pi^{j,\alpha}(p^*, q^*) = \max\{p^* \cdot F^j(y^j) - q^* \cdot y^j : y^j \in Y\}$. Indeed, if there exists $y \in Y$ such that $p^* \cdot F^j(y) - q^* \cdot y > \Pi^{j,\alpha}(p^*, q^*) \geq 0$, then $q^* \cdot y - p^* F^j(y) < 0 < \alpha$ and that is a contradiction.

Condition $\Pi^{j,\alpha}(p^*, q^*) = \max\{p^* \cdot F^j(y^j) - q^* \cdot y^j : y^j \in Y\}$ and the definition of $\xi^{\alpha,i}(p, q)$ imply the optimality of consumers' allocation.

We have proved that there exists an equilibrium in the intermediate economy. □

The following proposition allows us to move from an equilibrium in the intermediate economy to an equilibrium in the initial economy.

Proposition 2. *$((x^{i*})_{i=1,\dots,I}, (y^{j*})_{j=1,\dots,J}, p^*, q^*)$ is an equilibrium for the initial economy.*

Proof. First observe that if there exists $y \in \mathbb{R}^K_+$ such that

$$p^* \cdot F^j(y) - q^* \cdot y > p^* \cdot F^j(y^*) - q^* \cdot y^* = \Pi^{j,\alpha}(p^*, q^*) \geq 0,$$

then $q^* \cdot y - p^* F^j(y) < 0 < \alpha$ and that is a contradiction. By consequence, we get that

$$p^* \cdot F^j(y^*) - q^* \cdot y^* = \Pi^j(p^*, q^*) = \max\{p^* \cdot F^j(y^j) - q^* \cdot y^j : y^j \in \mathbb{R}^K_+\}.$$

Now fix some i and take $x \in \mathbb{R}^L_+$ satisfying $u^i(x) > u^i(x^{i*})$. We have to prove that $p^* \cdot x > p^* \cdot \omega^i + \sum_j \theta^{ij} \Pi^j(p^*, q^*) + q^* \cdot y^i_0$. Of course, this is the case if $x \in X$. We now consider the case where $x \notin X$. Since x^{i*} is in the interior of X, there exists $\lambda \in (0, 1)$ such that $\lambda x + (1-\lambda)x^{i*} \in X$. We have $u^i(\lambda x + (1-\lambda)x^{i*}) \geq \lambda u^i(x) + (1-\lambda)u^i(x^{i*}) > u^i(x^{i*})$. Hence, we have

$$p^* \cdot (\lambda x + (1-\lambda)x^{i*}) > p^* \cdot \omega^i + \sum_j \theta^{ij} \Pi^j(p^*, q^*) + q^* \cdot y^i_0 = p^* \cdot x^{i*}$$

$$\Leftrightarrow \lambda p^* \cdot x > \lambda p^* \cdot x^{i*} \Leftrightarrow p^* \cdot x > p^* \cdot x^{i*} = p^* \cdot \omega^i + \sum_j \theta^{ij} \Pi^j(p^*, q^*) + q^* \cdot y^i_0.$$

□

[5] This labeling is similar to that in Scarf (1982), page 1024.

Remark 3. *It is interesting to note that our proof of the existence of general equilibrium requires only the Sperner lemma and elementary mathematical results which were available before 1930. We also do not need to use the Maximum Theorem proven by Berge (1959).*

Remark 4. *When the utility and the production functions are only concave, the demand may not be single-valued and hence we cannot directly apply the Sperner lemma. In this case, the standard approach is to make use of the Kakutani fixed point theorem or the Gale-Nikaido-Debreu lemma. However, we can skip the use of the Kakutani fixed point theorem or the Gale-Nikaido-Debreu lemma by doing as follows. First, we approximate the utility and production functions by a family of strictly concave functions:*

$$\text{For each integer } N > 0, \text{ define } u^i_N(x) = u^i(x) + \frac{1}{N}v(x)\ , F^j_N(k) = F^j(k) + \frac{1}{N}G(k)$$

where v and G are strictly concave.

Second, applying Proposition 2, we have that: for any $N > 0$, there exists an equilibrium

$$\mathbf{e}_N \equiv \big((x^{i*}_N)_{i=1,\dots,I}, (y^{j*}_N)_{j=1,\dots,J}, p^*_N, q^*_N\big).$$

Third, let N go to infinity, there is an infinite subsequence $(N_t)_{t\geq 1}$ such that $\mathbf{e}_{N_t}$ converges to $\mathbf{e}$ when t goes to infinity. Last, we can prove that $\mathbf{e}$ is an equilibrium for the initial economy.

3.1.2 Equilibrium existence in an economy with financial assets

In this section, we use the Sperner lemma to prove the existence of an equilibrium in a two-period stochastic economy with incomplete financial markets. We consider both nominal and numéraire assets. We briefly present here some essential notions. For a full exposition, see Magill and Quinzii (1996) and Florenzano (1999).

Consider an economy with two periods ($t = 0$ and $t = 1$), L consumption goods, J financial assets, and I agents ($I \geq 2$). There is no uncertainty in period 0 while there are S possible states of nature in period 1. In period 0, each agent $i \leq I$ consumes and purchases assets. The consumption prices are denoted by $p_0 \in \mathbb{R}^L_+$ in the first period, $p_s \in \mathbb{R}^L_+$ in the state s of period 1.

Let $p \equiv (p_0, p_1, \dots, p_S)$. Each consumer has endowments of consumption good $\omega^i_0 \in \mathbb{R}^L_+$ in period 0 and $\omega^i_s \in \mathbb{R}^L_+$ in state s of period 1. Any agent i has a utility function $U^i(x^i_0, x^i_1, \dots x^i_S)$ where x^i_s is her consumption at state s. There is a matrix of returns depending on p of financial assets which is the same for any agent. Typically, if agent $i \leq I$ purchases z^i quantity of assets in period 0, then in period 1, at state s, she/he will obtain an income (positive or negative) $\sum_{j=1}^J R_{s,j}(p)z^j$. The returns $R(p)$ can be represented by a matrix

$$R = \begin{bmatrix} R_{1,1}(p) & R_{1,2}(p) & \dots & R_{1,J}(p) \\ R_{2,1}(p) & R_{2,2}(p) & \dots & R_{2,J}(p) \\ \vdots & \vdots & \ddots & \vdots \\ R_{S,1}(p) & R_{S,2}(p) & \dots & R_{S,J}(p) \end{bmatrix}.$$

We denote by $R_s(p) = (R_{s,1}(p), R_{s,2}(p), \dots, R_{s,J}(p))$ the s^{th} row of $R(p)$. Typically, the constraints faced by agent i are

$$p_0 \cdot (x^i_0 - \omega^i_0) + q \cdot z^i \leq 0,$$
$$p_s \cdot (x^i_s - \omega^i_s) \leq R_s(p) \cdot z^i \ \forall s = 1, \dots, S.$$

We make use of the following set of standard assumptions.

Assumption 2. *(i) For any $i = 1, \dots, I$, the consumption set is $X^i = \mathbb{R}_+^{L(S+1)}$, and the assets set is $Z^i = \mathbb{R}^J$.*
(ii) For any $i = 1, \dots, I$, $\omega_0^i \in \mathbb{R}_{++}^L$, $\omega_s^i \in \mathbb{R}_{++}^L$ for any state s in period 1.
(iii) The map $p \to R(p)$ is continuous and

- *either $R(p)$ is a positive constant matrix ($R_{s,j}(p) = R_{s,j} \geq 0$ $\forall s$, $\forall j$, $\forall p \in \Delta$), and $rank(R) = J$ (the nominal assets case).*
- *or, in the numéraire assets case, $R(p) = Q(p) \times G$ where G is a positive constant $S \times J$-matrix, $rank(G) = J$, and*

$$Q(p) = \begin{bmatrix} p_1 \cdot e & 0 & \dots & 0 \\ 0 & p_2 \cdot e & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & p_S \cdot e \end{bmatrix},$$

where $e \gg 0$ is a numéraire.[6]

(v) For any $i = 1, \dots, I$, U^i is strictly increasing, continuous, and strictly concave.

We now introduce the definitions of complete and incomplete asset markets, feasible allocations, and the notion of equilibrium in an economy with financial assets.

Definition 7. *The assets market is called complete if $S = J$ and incomplete if $S > J$.*

Definition 8. *Consider the economy*

$$\mathcal{E} = \big((U^i, X^i, Z^i, \omega^i), R\big)$$

An equilibrium of this economy is a list $\big((x^{i*}, z^{i*})_{i=1}^I, (p^*, q^*)\big)$ where $(x^{i*}, z^{i*})_{i=1}^I \in (X^i)^I \times (Z^i)^I$, $(p^*, q^*) \in \mathbb{R}_{++}^{L(S+1)} \times \mathbb{R}_{++}^J$ such that

(i) For any i, $(x^{i*}, z^{i*}) \in X^i \times Z^i$, $p_0^* \cdot (x_0^i - \omega_0^i) + q^* \cdot z^i = 0$, $p_s^* \cdot (x_s^i - \omega_s^i) = R_s(p^*) \cdot z^i$ $\forall s = 1, \dots, S$, and x^{i*} solves the problem

$$\max U^i(x_0^i, x_1^i, \dots, x_S^i) \text{ subject to: } x^i \in B^i(p^*, q^*) \tag{4a}$$

where we define

$$B^i(p, q) \equiv \{x^i \in X^i : \exists z^i \in Z^i, p_0 \cdot (x_0^i - \omega_0^i) + q \cdot z^i \leq 0$$
$$p_s \cdot (x_s^i - \omega_s^i) \leq R_s(p) \cdot z^i, s = 1, \dots, S\}$$

(ii) $\sum_{i=1}^I (x_s^{*i} - \omega_s^i) = 0$ for any $s = 0, 1, \dots, S$ and $\sum_{i=1}^I z^{*i} = 0$.

[6] Notice that if $p_s \cdot e > 0$ $\forall s \geq 1$, then $rank(R(p)) = rank(G)$.

Definition 9. *The allocations* $((x^i, z^i)_i) \in (X^i)^I \times (Z^i)^I$ *are feasible if (i)* $\sum_{i=1}^{I}(x^i - \omega^i) \leq 0$ *and (ii)* $\sum_{i=1}^{I} z^i = 0$.

Given $\alpha > 0$ *and define the sets* $F^\alpha = \{(x^i)_i \in (X^i)^I : \sum_{i=1}^{I}(x^i - \omega^i) \leq \alpha\}$. *Denote the projection of* F^α *on* X^i *by* $\widehat{X}^i$. *Let* B^c *be a ball of* $\mathbb{R}^L$, *centered at the origin, which contains any* $\hat{X}^i$ *in its interior.*

An intermediate economy is the economy

$$\widetilde{\mathcal{E}} = \big((U^i, \widetilde{X}^i, Z^i, \omega^i), R\big),$$

i.e., the consumption set is $\widetilde{X}^i = B^c$ for any i. An equilibrium in this intermediate economy is defined as in Definition 8.

We aim to provide a new proof (by using the Sperner lemma) of the following result:

Proposition 3. *Consider the economy* $\mathcal{E}$. *Under above assumptions, for any list* $(\lambda_0, \lambda_1, \ldots, \lambda_S)$ *with* $\lambda_0 = 1, \lambda_S > 0, s = 1, \ldots, S$, *there exists an equilibrium* $\big((x^{i*}, z^{i*})_{i=1}^{I}, (p^*, q^*)\big)$ *with* $p^* \in \Delta$, *and, more importantly,* $q^* = \sum_{s=1}^{S} \lambda_s R_s(p^*)$, *i.e.,* $q_j^* = \sum_{s=1}^{S} \lambda_s R_{s,j}(p^*)$, $\forall j = 1, \ldots, J$.

Comments. This result is similar to Theorem 1 in Cass (2006) or Theorem 7.1 in Florenzano (1999) for the case of nominal assets. Our contribution is that we do not require that the returns are nominal as Cass (2006) and Florenzano (1999) did. Before presenting our proof, we point out some important consequences of Proposition 3:

- *Continuum of equilibria.* In the case of nominal assets where the return matrix is constant, we have $q^* = \sum_{s=1}^{S} \lambda_s R_s$, and, hence, there is a continuum of equilibrium asset prices.

  While the property $q^* = \sum_{s=1}^{S} \lambda_s R_s$ is well-known in the case of nominal assets, our paper is the first to show a similar property ($q^* = \sum_{s=1}^{S} \lambda_s R_s(p^*)$) in the case of numéraire assets. Since prices q^*, p^* depend on $\lambda \equiv (\lambda_s)_{s=1}^{S}$, we can rewrite that $q^*(\lambda) = \sum_{s=1}^{S} \lambda_s R_s\big(p^*(\lambda)\big)$. From this, we can prove that there is a continuum of equilibrium prices (p^*, q^*).[7]

- *Equilibrium price versus no-arbitrage price.* For the nominal assets, an equilibrium always exists, and an asset price is an asset equilibrium price if and only if it is a no arbitrage price. Indeed, take a no-arbitrage price. Using the Cass trick we obtain an equilibrium. Conversely, for any financial equilibrium, under the assumption that the utility functions are strictly increasing, the first order conditions show that an equilibrium asset price is a no-arbitrage price.

  However, we do not have this equivalence in the numéraire case. Indeed, in this case, the set of no-arbitrage prices is $\{q : q = \sum_{s=1}^{S} \lambda_s R_s(p), \lambda_s > 0, \forall s \geq 1, p \in \Delta\}$. If q is an equilibrium price, then by the first order conditions, it is a no-arbitrage price. The converse is not always true. Indeed, if $q = \sum_{s=1}^{S} \lambda_s R_s(p)$ with $\lambda_s > 0, \forall s \geq 1, p \in \Delta$, it is not sure that this q is an equilibrium price (because the return matrix depends on price p).

[7] Indeed, let us consider two lists of weights $(\lambda_s)_{s=1}^{S}$ and $(\tilde{\lambda}_s)_{s=1}^{S}$ such that $\lambda_1 \neq \tilde{\lambda}_1$ and $\lambda_s = \tilde{\lambda}_s, \forall s > 1$. We claim that $(p^*, q^*) \neq (\tilde{p}^*, \tilde{q}^*)$. Suppose the contrary that $(p^*, q^*) = (\tilde{p}^*, \tilde{q}^*)$, we then have $\sum_s(\lambda_s - \tilde{\lambda}_s)R_s(p^*) = 0$ which implies that $R_1^j(p^*) = 0, \forall j = 1, \ldots, J$. We get a contradiction since $p^* \gg 0$.

Proof of Proposition 3. Observe that, by using the same argument in the proof of Proposition 2 in Section 3.1.1, we can prove that an equilibrium of the intermediate economy is indeed an equilibrium for the initial economy. As such, it remains to prove the existence of equilibrium in the intermediate economy $\widetilde{\mathcal{E}}$. To do so, we proceed in two steps. First, we use the Sperner lemma to prove that there exists actually a *Cass equilibrium*. Second, from this Cass equilibrium, we construct an equilibrium for the intermediate economy.

We now define and prove the existence of a Cass equilibrium.

Definition 10. *A Cass equilibrium associated with* $(\lambda_0, \lambda_1, \ldots, \lambda_S), \lambda_0 = 1, \lambda_s > 0, \forall s \geq 1$ *is a list* $((\bar{x}^i)_{i=1}^I, (\bar{z}^i)_{i=2}^I, (\bar{p}, \bar{q}))$ *such that* $\big((\bar{x}^i)_{i=1}^I, (\bar{z}^i)_{i=2}^I\big) \in (B^c)^I \times (\mathbb{R}^J)^{I-1}$, $(\bar{p}, \bar{q}) \in \mathbb{R}^{L(1+S)}_{++} \times \mathbb{R}^J_{++}$ *where*

(i) $\bar{x}^1$ *solves the consumer* 1 *problem under the constraints:* $x^1 \in B^c$, $\bar{p}' \cdot (x^1 - \omega^1) \leq 0$, *where* $\bar{p}' = (\bar{p}_0, \lambda_1 \bar{p}_1, \ldots, \lambda_s \bar{p}_s)$.

(ii) For $i = 2, \ldots, I$, *we have* $\bar{p}_0 \cdot (\bar{x}^i_0 - \omega^i_0) + \bar{q} \cdot \bar{z}^i = 0$, $\bar{p}_s \cdot (\bar{x}^i_s - \omega^i_s) = R_s(\bar{p}) \cdot \bar{z}^i$, $\forall s \geq 1$, *and* $\bar{x}^i$ *solves the consumer i's problem*

$$\max U^i(x^i_0, x^i_1, \ldots, x^i_S) \text{ subject to: } x^i \in B^i_{B^c}(\bar{p}, \bar{q})$$

where $B^i_{B^c}(\bar{p}, \bar{q}) \equiv \{x^i \in B^c : \exists z^i \in \mathbb{R}^J : \bar{p}_0 \cdot (x^i_0 - \omega^i_0) + \bar{q} \cdot z^i \leq 0, \bar{p}_s \cdot (x^i_s - \omega^i_s) \leq R_s(\bar{p}) \cdot z^i, \forall s \geq 1\}$.

(iii) $\bar{q} = \sum_s \lambda_s R_s(\bar{p})$.

(iv) $\sum_{i=1}^I (\bar{x}^i - \omega^i) = 0$.

Lemma 2. *There exists a Cass equilibrium associated with* $(\lambda_0, \lambda_1, \ldots, \lambda_S), \lambda_0 = 1, \lambda_s > 0, \forall s \geq 1$.

Proof. Let $p = (p_0, p_1, \ldots, p_S) \in \Delta$ where Δ denotes the unit-simplex of $\mathbb{R}^{L(S+1)}$. Define $p' = (p_0, \lambda_1 p_1, \ldots, \lambda_s p_s)$. Let $\tilde{\lambda} = \min_s \lambda_s$. Let ϵ be such that $0 < \epsilon < \frac{\alpha \tilde{\lambda}}{(I-1)}$. Define the following ϵ-returns matrix $R'(p, \epsilon)$: $R'(p, \epsilon) = R(p) + H(\epsilon) \times G$ where

$$H(\epsilon) = \begin{bmatrix} \epsilon & 0 & \ldots & 0 \\ 0 & \epsilon & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \epsilon \end{bmatrix}.$$

Obviously, $R'(p, 0) = R(p)$ and $R'(p, \epsilon)$ is of rank J for any $\epsilon > 0$.

Consider the problem of agent 1:

$$\max U^1(x^1) \text{ subject to } x^1 \in B^1_{B^c}(p) \equiv \{x^1 \in B^c : p' \cdot (x^1 - \omega^1) \leq 0\}.$$

Any agent i ($i \geq 2$) solves the following problem:

$$\max U^i(x^i) \text{ subject to: } x^i \in B^{i,\epsilon}_{B^c}(p),$$

$$\text{where } B^{i,\epsilon}_{B^c}(p) \equiv \Big\{x^i \in B^c : \exists z^i \in \mathbb{R}^J : p_0 \cdot (x^i_0 - \omega^i_0) + \big(\sum_s \lambda_s R'_s(p, \epsilon)\big) \cdot z^i \leq \epsilon, \quad p_s \cdot (x^i_s - \omega^i_s) \leq R'_s(p, \epsilon) \cdot z^i \ \forall s \geq 1\Big\}.$$

These optimization problems have continuous objective functions. Since $B^1_{B^c}(p)$ and $B^{i,\epsilon}_{B^c}(p)$ are compact, these problems have a solution. The budget set of agent 1 has a nonempty interior since $p \in \Delta$. To prove the budget sets of the agents $i \geq 2$ have nonempty interiors, we observe that $x^i_s = \omega^i_s$, $s = 0, 1, \ldots, S$ and $z^i > 0$ such that $\sum_s \lambda_s R'_s(p, \epsilon) z^i < \epsilon$ are in the interior of these budget sets. By combining with the fact that the utility functions are strictly concave, the optimal values $x^{*1}, x^{*2}_\epsilon \ldots, x^{*I}_\epsilon$ are continuous functions with respect to p.[8] For any p, we have

$$p' \cdot \sum_{i=1}^{I} (x^{*i}_\epsilon(p) - \omega^i) \leq (I-1)\epsilon,$$

where, by convention, we denote $x^{*1}_\epsilon \equiv x^{*1}$.

Define the excess demand mapping ξ by

$$\xi(p) = \sum_{i=1}^{I} (x^{*i}_\epsilon(p) - \omega^i).$$

It is obvious that $\forall p \in \Delta, p' \cdot \xi(p) \leq (I-1)\epsilon$.

(Using the Sperner lemma) Denote $N = (S+1)L$. Let $K > 0$ be an integer and consider a simplicial subdivision T^K of the unit-simplex Δ of $\mathbb{R}^N$ such that $Mesh(T^K) < 1/K$. We define the following labeling r. For any $p \in \Delta, r(p) = t$ if $\xi_t(p) \leq \frac{(I-1)\epsilon}{\tilde{\lambda}}$. Such a labeling is well defined. Moreover, it satisfies Sperner condition. Indeed, we see that:

- For $t \in \{1, \ldots, N\}$. If $p = e^t$ (recall that e^t is a unit-vector of $\mathbb{R}^N$), then $(I-1)\epsilon \geq \lambda_t e^t \cdot \xi(e^t) = \lambda_t \xi_t(e^t)$. This implies $\xi(e^t) \leq \frac{(I-1)\epsilon}{\lambda_t} \leq \frac{(I-1)\epsilon}{\tilde{\lambda}}$. We label $r(e^t) = t$.
- If $p \in [[e^{i_1}, \ldots, e^{i_m}]]$ with $m < N$, then $(I-1)\epsilon \geq p' \cdot \xi(p) = \sum_{q \in \{i_1, \ldots, i_m\}} \lambda_q p_q \xi_q(p)$. There must exist $v \in \{i_1, \ldots, i_m\}$ with $\xi_v(\pi) \leq \frac{(I-1)\epsilon}{\tilde{\lambda}}$. We label $r(p) = v$ with some $v \in \{i_1, \ldots, i_m\}$.

So, the labeling r satisfies Sperner condition. According to the Sperner lemma, there exists a completely labeled subsimplex $[[\bar{p}^1(K), \ldots, \bar{p}^N(K)]]$, i.e., $\xi_t(\bar{p}^t(K)) \leq \frac{(I-1)\epsilon}{\tilde{\lambda}}$, $\forall t = 1, \ldots, N$.

Observe that

$$\forall t = 1, \ldots, N, \ \sum_{i=1}^{I} \left(x^{*i}_\epsilon(\bar{p}^t(K)) - \omega^i \right) \leq \frac{(I-1)\epsilon}{\tilde{\lambda}} < \alpha. \tag{5}$$

Let $K \to +\infty$. Then, for any $t \in \{1, \ldots, N\}$, $\bar{p}^t(K) \to p^*(\epsilon) \in \Delta$. We have $\xi_v(p^*(\epsilon)) \leq \frac{(I-1)\epsilon}{\tilde{\lambda}} < \alpha$, for all v. It follows from (5) that

$$\sum_{i=1}^{I} \left(x^{*i}_\epsilon(p^*(\epsilon)) - \omega^i \right) \leq \frac{(I-1)\epsilon}{\tilde{\lambda}} < \alpha. \tag{6}$$

This implies that for any i, $x^{*i}_\epsilon(p^*(\epsilon))$ is uniformly bounded from above when ϵ is small.

[8] We can prove this continuity by applying the Maximum Theorem (Berge, 1959) or adapting our argument in Step 2 of the proof of Proposition 1.

Write $p^*(\epsilon) = (p^*_0(\epsilon), p^*_1(\epsilon), \dots, p^*_S(\epsilon)), p'^*(\epsilon) = (p^*_0(\epsilon), \lambda_1 p^*_1(\epsilon), \dots, \lambda_S p^*_S(\epsilon))$. Because of (6) and the fact that utility functions are strictly increasing, we obtain

$$p'^*(\epsilon) \cdot (x^{*1}(p^*(\epsilon)) - \omega^1) = 0, \tag{7}$$

which implies $p'^*(\epsilon) \gg 0 \Leftrightarrow p^*(\epsilon) \gg 0$.

For any agent $i \geq 2$, there exists z^{*i}_ϵ such that (notice that budget constraints are binding because of (6))

$$p^*_0(\epsilon) \cdot (x^{*i}_{\epsilon,0}(p^*(\epsilon)) - \omega^i_s) + (\sum_s \lambda_s R'_s(p^*(\epsilon), \epsilon) z^{*i}_\epsilon = \epsilon,$$
$$p^*_s(\epsilon) \cdot (x^{*i}_{\epsilon,s}(p^*(\epsilon)) - \omega^i_s) = R'_s(p^*(\epsilon), \epsilon) \cdot z^{*i}_\epsilon, s = 1, \dots, S.$$

Since $R'(p^*(\epsilon), \epsilon)$ is of rank J for ϵ small enough, z^{*i}_ϵ is unique.

Let $\epsilon \to 0$, without loss of generality, we can assume that $p^*(\epsilon) \to \bar{p} \in \Delta$. Since x^{*1} is continuous, we have $x^{*1}(p^*(\epsilon)) \to \bar{x}^1 \equiv x^{*1}(\bar{p})$. From this, we have $\bar{p} \gg 0$ (because $x^{*i}_\epsilon(p^*(\epsilon))$ is uniformly bounded, see (6)).

We next claim that there is $\beta > 0$ such that $\|z^{*i}(p^*(\epsilon))\| \leq \beta$ when ϵ is small enough. Indeed, suppose that there exists a sequence $(z^{*i}_{k_n})$, with $(k_n)_{n=1,2,\dots}$ being a decreasing sequence converging to zero, and $\|z^{*i}_{k_n}\| \to +\infty$ when $n \to +\infty$. We have, for any n, $\forall s = 1, \dots, S$, $p^*_s(k_n) \cdot (x^{*i}(p^*(k_n)) - \omega^i_s) = R'_s(p^*(k_n), k_n) \cdot z^{*i}_{k_n}$. Then,

$$\frac{p^*_s(k_n) \cdot (x^{*i}_{k_n}(p^*(k_n)) - \omega^i_s)}{\|z^{*i}_{k_n}\|} = R'_s(p^*(k_n), k_n) \cdot \frac{z^{*i}_{k_n}}{\|z^{*i}_{k_n}\|} \quad \forall s = 1, \dots, S, \forall n \tag{8}$$

Let $n \to +\infty$, we can suppose $\frac{z^{*i}_{k_n}}{\|z^{*i}_{k_n}\|} \to \zeta \neq 0$. Since $\lim_{n\to\infty} p^*(k_n) = \bar{p}$, we have $\lim_{n\to\infty} R'_s(p^*(k_n), k_n) = R(\bar{p})$. Therefore, we get that $0 = R_s(\bar{p}) \cdot \zeta = 0 \ \forall s = 1, \dots, S$. Since $\bar{p} \gg 0$, we have $\bar{p}_s \cdot e > 0 \ \forall s \geq 1$, and hence the matrix $R(\bar{p})$ is of rank J. This implies that $\zeta = 0$, which is a contradiction. Therefore, there is $\beta > 0$ such that $\|z^{*i}_\epsilon\| \leq \beta$ when ϵ is small enough. Hence, we can assume that z^{*i}_ϵ converges when ϵ goes to zero.

To sum up, when $\epsilon \to 0$, we can assume that $p^*(\epsilon) \to \bar{p} \in \Delta$, $x^{*1}(p^*(\epsilon)) \to \bar{x}^1 \equiv x^{*1}(\bar{p})$, $\bar{p} \gg 0$. For $i \geq 2$, $x^{*i}_\epsilon(p^*(\epsilon)) \to \bar{x}^i, z^{*i}(p^*(\epsilon)) \to \bar{z}^i$.

Let $\bar{p}' = (\bar{p}_0, \lambda_1 \bar{p}_1, \dots, \lambda_S \bar{p}_s)$. Note that from (6) that $\sum_{i=1}^I (\bar{x}^i - \omega^i) \leq 0$ and from (7) that $\bar{p}' \cdot (\sum_{i=1}^I (\bar{x}^i - \omega^i) = 0 \Rightarrow \bar{p}_p \sum_i (\bar{x}^i_p - \omega^i_p) = 0, p = 1, \dots, N$. Since $\bar{p} \gg 0$, we deduce that $\sum_{i=1}^I (\bar{x}^i_p - \omega^i_p) = 0, \forall p = 1, \dots, N$, or equivalently $\sum_{i=1}^I (\bar{x}^i - \omega^i) = 0$.

The last step: prove the optimality of $\bar{x}^i$ for each $i \geq 2$. To do so, assume that there is $x^i \in B^i_{B^c}(\bar{p}, \bar{q})$ such that $U^i(x^i) > U^i(\bar{x}^i)$. Without loss of generality, we can assume that $x^i_s \gg 0 \ \forall s$.[9]

Since $x^i \in B^i_{B^c}(\bar{p}, \bar{q})$, we take any $z^i \in \mathbb{R}^J$ such that $\bar{p}_0 \cdot (x^i_0 - \omega^i_0) + \bar{q} \cdot z^i \leq 0, \bar{p}_s \cdot (x^i_s - \omega^i_s) \leq R_s(\bar{p}) \cdot z^i \ \forall s \geq 1$. Notice that $x^i_s \gg 0, \forall s$. So, without loss of generality, we can assume that[10]

$$\bar{p}_0 \cdot (x^i_0 - \omega^i_0) + \bar{q} \cdot z^i < 0, \quad \bar{p}_s \cdot (x^i_s - \omega^i_s) < R_s(\bar{p}) \cdot z^i, \forall s \geq 1.$$

[9]Indeed, we can introduce $x^i(\lambda)$ by $x^i_s(\lambda) = (1 - \lambda) x^i_s + \lambda \omega_s$. Then, $x^i_s(\lambda) \gg 0$ because $\omega_s \gg 0, \forall s$. Moreover, we can choose $\lambda > 0$ small enough so that $U^i(x^i(\lambda)) > U^i(\bar{x}^i)$.

[10]Indeed, we can define $x^{i'}$ by $x^{i'}_{s,l} = x^i_{s,l} - \tau \ \forall s = 0, \dots, S, \forall l = 1, \dots, L$ where $\tau > 0$ small enough so that $U^i(x^{i'}) > U^i(\bar{x}^i)$.

Thus, we can choose $\epsilon > 0$ small enough such that

$$p_0^*(\epsilon) \cdot (x_0^i - \omega_0^i) + \bar{R}_s'(p^*(\epsilon), \epsilon) \cdot z^i < \epsilon, \quad p_s^*(\epsilon) \cdot (x_s^i - \omega_s^i) < R_s'(p^*(\epsilon), \epsilon) \cdot z^i, \forall s \geq 1.$$

By the optimality of $x_\epsilon^{*i}(p^*(\epsilon))$, we have $U^i(x^i) \leq U^i\Big(x_\epsilon^{*i}(p^*(\epsilon))\Big)$. Let $\epsilon \to 0$, we get that $U^i(x^i) \leq U^i(\bar{x}^i)$, which is a contradiction. We have proved the existence of a *Cass equilibrium.* □

We now move from Cass equilibrium to an equilibrium in the intermediate economy.

Lemma 3. *There exists an equilibrium in the intermediate economy with* $\bar{q} = \sum_s \lambda_s R_s(\bar{p})$.

Proof. Since $\sum_{i=1}^I (\bar{x}_s^i - \omega_s^i) = 0$, $\forall s \geq 1$, we get that

$$\forall s \geq 1\ , 0 = \lambda_s \bar{p}_s \cdot \sum_{i=1}^I (\bar{x}_s^i - \omega_s^i) = \lambda_s \bar{p}_s \cdot (\bar{x}_s^1 - \omega_s^1) + \lambda_s \bar{p}_s \cdot \sum_{i=2}^I (\bar{x}_s^i - \omega_s^i).$$

Denote $\bar{z}^1 = -\sum_{i \geq 2} \bar{z}^i$. We have $\bar{p}_s \cdot \sum_{i=2}^I (\bar{x}_s^i - \omega_s^i) = R_s(\bar{p}) \cdot \bar{z}^1$ which implies that

$$\sum_{s \geq 1} \lambda_s \bar{p}_s \cdot (\bar{x}_s^1 - \omega_s^1) = \Big(\sum_s \lambda_s R_s(\bar{p})\Big) \cdot \bar{z}^1 = \bar{q} \cdot \bar{z}^1.$$

By combining this with the fact that $\bar{p}_0 \cdot (\bar{x}_0^1 - \omega_0^1) + \sum_{s \geq 1} \lambda_s \bar{p}_s \cdot (\bar{x}_s^1 - \omega_s^1) = 0$, we get that $\bar{p}_0 \cdot (\bar{x}_0^1 - \omega_0^1) + \bar{q} \cdot \bar{z}^1 = 0$.

It is easy to prove the optimality of $\bar{x}^1$.

□

□

3.2 Using Sperner's lemma to prove the Gale-Nikaido-Debreu lemma

In Section 3.1, the consumers have utility functions and the firms have production functions. When we consider preference orders for the consumers and production sets for the firms, the demands of the consumers or of the firms are not necessarily single-valued. In this case, the customary proofs of the equilibrium existence make use of either the Gale-Nikaido-Debreu lemma (Debreu, 1956, 1959; Gale, 1955; Nikaido, 1956) or the Gale and Mas-Colell lemma (Gale and Mas-Colell, 1975, 1979) whose proofs, in turn, require the Kakutani fixed point theorem or the Knaster-Kuratowski-Mazurkiewicz lemma. In what follows, we use the Sperner lemma and well-known mathematical results to prove several versions of the Gale-Nikaido-Debreu lemma.

Let us start with the following version (Theorem 1 in Debreu (1959), page 82).

Lemma 4 (Gale-Nikaido-Debreu lemma)**.** *Let* Δ *be the unit-simplex of* $\mathbb{R}^N$. *Let* ζ *be an upper semi-continuous correspondence with non-empty, compact, convex values from* Δ *into* $\mathbb{R}^N$. *Suppose* ζ *satisfies the following condition:*

$$\forall p \in \Delta,\ \forall z \in \zeta(p), p \cdot z \leq 0. \tag{9}$$

Then there exists $\bar{p} \in \Delta$ *such that* $\zeta(\bar{p}) \cap \mathbb{R}_-^N \neq \emptyset$.

Proof of Lemma 4. Let $A = \max\{\|z\|_1 : z \in \zeta(\Delta)\}$.

Step 0. Let $\epsilon \in (0,1)$. Since Δ is compact, there exists a finite covering of Δ with a finite family of open balls $\big(B\left(x^i(\epsilon), \epsilon\right)\big)_{i=1,\ldots,I(\epsilon)}$. Take a partition of unity subordinate to the family $\big(B\left(x^i(\epsilon), \epsilon\right)\big)_{i=1,\ldots,I(\epsilon)}$, i.e. a family of continuous non-negative real functions $(\alpha_i)_{i=1,\ldots,I(\epsilon)}$ from Δ in $\mathbb{R}_+$ such that Supp $\alpha_i \subset B\left(x^i(\epsilon), \epsilon\right), \forall i$ and $\sum_{i=1}^{I(\epsilon)} \alpha_i(x) = 1, \forall x \in \Delta$.[11] Take $y^i(\epsilon) \in \zeta(x^i(\epsilon))\ \forall i$.

Step 1. We define the function $f^\epsilon : \Delta \to \Delta$ by $f^\epsilon(x) = \sum_{i=1}^{I(\epsilon)} \alpha_i(x) y^i(\epsilon)$. This function is continuous.

Step 2. We claim that: $x \cdot f^\epsilon(x) \leq \epsilon A\ \forall x \in \Delta$. Let $x \in \Delta$, there exists a set $J(x) \subset \{1, \ldots, I(\epsilon)\}$ such that $x \in \cap_{i \in J(x)} B\left(x^i(\epsilon), \epsilon\right)$. We have $f^\epsilon(x) = \sum_{i \in J(x)} \alpha_i(x) y^i(\epsilon)$ with $\sum_{i \in J(x)} \alpha_i(x) = 1$. We have

$$\forall i \in J(x), x^i(\epsilon) = x + \epsilon u^i(x), \text{ with some } u^i(x) \in B(0,1),$$

which implies that: $\forall i \in J(x), y^i(\epsilon) \in \zeta(x^i(\epsilon)) = \zeta(x + \epsilon u^i(x)) \subset \zeta\big(B(x,\epsilon)\big)$. By consequence, $f^\epsilon(x) \in co\Big(\zeta\big(B(x,\epsilon)\big)\Big)$. According to Carathéodory's convexity theorem,[12] we have a decomposition

$$f^\epsilon(x) = \sum_{i=1}^{N+1} \beta_i(x,\epsilon) \tilde{y}^i(x,\epsilon),$$

with $\tilde{y}^i(x,\epsilon) \in \zeta(x + \epsilon u^i)$ where $u^i \in B(0,1)$, $\beta_i(x,\epsilon) \geq 0$,, and $\sum_{i=1}^{N+1} \beta_i(x,\epsilon) = 1$. From this, we have

$$\begin{aligned} x \cdot f^\epsilon(x) &= \sum_{i=1}^{N+1} \beta_i(x,\epsilon)(x + \epsilon u^i) \cdot \tilde{y}^i(x,\epsilon) - \epsilon \sum_{i=1}^{N+1} \beta_i(x,\epsilon) u^i \cdot \tilde{y}^i \\ &\leq \epsilon \sum_{i=1}^{N+1} \beta_i(x,\epsilon) \|u^i\| \cdot \|\tilde{y}^i\| \leq \epsilon A \sum_{i=1}^{N+1} \beta_i(x,\epsilon) = \epsilon A \end{aligned}$$

since $(x + \epsilon u^i) \cdot \tilde{y}^i(x,\epsilon) \leq 0$ (see condition (9)), $\|u^i\| \leq 1$ and $\|\tilde{y}^i\| \leq A$.

Step 3. We prove that:

$$\forall x \in \Delta, \exists i,\ f_i^\epsilon(x) \leq \epsilon A. \tag{10}$$

Indeed, if $\forall i, f_i^\epsilon(x) > \epsilon A$, then $\epsilon A < \sum_i x_i f_i^\epsilon(x) = x \cdot f^\epsilon(x) \leq \epsilon A$, which is a contradiction.

Step 4 (using the Sperner lemma). Let $K > 0$ be an integer and consider a simplicial subdivision T^K of the unit-simplex Δ of $\mathbb{R}^N$ such that $Mesh(T^K) < 1/K$ and define the labeling R as follows:

$$\forall x \in \Delta,\ R(x) = i, \text{ if } f_i^\epsilon(x) \leq \epsilon A.$$

[11] For the notion of partition of unity, see, for instance, Aliprantis and Border (2006)'s Section 2.19.

[12] Carathéodory (1907)'s convexity Theorem states that: In an n-dimensional vector space, every vector in the convex hull of a nonempty set can be written as a convex combination using no more than $n+1$ vectors from the set. For a simple proof, see Florenzano and Le Van (2001)'s Proposition 1.1.2 or Aliprantis and Border (2006)'s Theorem 5.32.

According to (10), this labeling is well-defined. It also satisfies the Sperner condition

$$x \in [[e^{i_1}, \ldots, e^{i_m}]] \Rightarrow R(x) = i \in \{i_1, \ldots, i_m\}$$

Indeed, if $f_i^\epsilon(x) > \epsilon A, \forall i \in \{i_1, \ldots, i_m\}$, then $\epsilon A \geq x \cdot f^\epsilon(x) = \sum_{i \in \{i_1,\ldots,i_m\}} x_i f_i^\epsilon(x) > \epsilon \sum_{i \in \{i_1,\ldots,i_m\}} x_i = A\epsilon$, which is a contradiction.

The Sperner lemma implies that there exists a completely labeled subsimplex $[[x^{K,1}, \ldots, x^{K,N}]]$ with $R(x^{K,l}) = l, \forall l = 1, \ldots, N$, i.e., $f_l^\epsilon(x^{K,l}) \leq \epsilon A, \forall l = 1, \ldots, N$.

Let $K \to +\infty$, there is a subsequence (K_t) such that

$$\forall l, x^{K_t,l} \to x^\epsilon \in \Delta, \quad f^\epsilon(x^{K_t,l}) \to f^\epsilon(x^\epsilon)$$

and, therefore, $f_l^\epsilon(x^\epsilon) \leq \epsilon A, \forall l = 1, \ldots, N$.

Step 5. Since $\big(B\left(x^i(\epsilon), \epsilon\right)\big)_{i=1,\ldots,I(\epsilon)}$ is a covering of Δ, there exists a set $J(x^\epsilon) \subset \{1, \ldots, I(\epsilon)\}$ such that $x \in \cap_{i \in J(x^\epsilon)} B\left(x^i(\epsilon), \epsilon\right)$. We have $f^\epsilon(x^\epsilon) = \sum_{i \in J(x^\epsilon)} \alpha_i(x^\epsilon) y^i(x^\epsilon)$ with $\sum_{i \in J(x^\epsilon)} \alpha_i(x^\epsilon) = 1$. By using Carathéodory's convexity theorem as we have done in Step 2, we get a decomposition

$$f^\epsilon(x^\epsilon) = \sum_{i=1}^{N+1} \beta_i(x^\epsilon) \tilde{y}^i(x^\epsilon)$$

with $\tilde{y}^i(x^\epsilon) \in \zeta\big(B(x^\epsilon, \epsilon)\big)$, $\beta_i(x^\epsilon) \geq 0$, $\sum_{i=1}^{N+1} \beta_i(x^\epsilon) = 1$.

Step 6. Let $\epsilon \to 0$, without loss of generality, we can assume that

$$x^\epsilon \to \bar{x} \in \Delta, \quad \beta_i(x^\epsilon) \to \bar{\beta}_i \geq 0, \sum_{i=1}^{N+1} \bar{\beta}_i = 1,$$
$$\tilde{y}^i(x^\epsilon) \to \bar{y}^i \in \zeta(\bar{x}), \forall i = 1, \ldots, N+1.$$

Therefore, we have

$$f^\epsilon(x^\epsilon) \overset{\epsilon \to 0}{\longrightarrow} \bar{z} = \sum_{i=1}^{N+1} \bar{\beta}_i \bar{y}^i \in \zeta(\bar{x}) \text{ (because } \zeta(\bar{x}) \text{ is convex).}$$

Moreover, the condition $f_l^\epsilon(x^\epsilon) \leq \epsilon A$, $\forall l = 1, \ldots, N$ implies that $\bar{z}_l \leq 0, \forall l = 1, \ldots, N$. Define $\bar{p} \equiv \bar{x}$, we have $\zeta(\bar{p}) \cap \mathbb{R}^N_- \neq \emptyset$ because $\bar{z} \in \zeta(\bar{p}) \cap \mathbb{R}^N_-$. The proof is over.

□

From Lemma 4, we can additionally derive two stronger versions of the Gale-Nikaido-Debreu lemma. Each of them is stated and proved below.

Lemma 5. *Let Δ be the unit-simplex of $\mathbb{R}^N$. Let ζ be an upper semicontinuous correspondence with nonempty, compact, convex values from Δ into $\mathbb{R}^N$. Suppose ζ satisfies the condition*

$$\forall p \in \Delta, \; \exists z \in \zeta(p) \text{ which satisfies } p \cdot z \leq 0.$$

Then there exists $\bar{p} \in \Delta$ such that $\zeta(\bar{p}) \cap \mathbb{R}^N_- \neq \emptyset$.

Proof. For $p \in \Delta$, let $\tilde{\zeta}(p) = \{z \in \zeta(p) : z \cdot p \leq 0\}$. The correspondence $\tilde{\zeta}$ is upper semicontinuous, convex, and compact valued from Δ into $\mathbb{R}^N$. It satisfies the assumptions of Lemma 4. Hence there exist $\bar{p}$ and $\bar{z} \in \tilde{\zeta}(\bar{p}) \subset \zeta(\bar{p})$, such that $\bar{z} \leq 0$. □

Lemma 6. *Let Δ be the unit-simplex of $\mathbb{R}^N$. Let ζ be an upper semicontinuous correspondence with nonempty, compact, convex values from Δ into $\mathbb{R}^N$. Suppose ζ satisfies the condition*

$$\forall p \in \Delta,\ \forall z \in \zeta(p),\ \text{we have } p \cdot z = 0.$$

Then there exist $\bar{p}$, $\bar{z} \in \zeta(\bar{p})$ such that (1) $\bar{z} \leq 0$, and (2) $\forall i = 1, \ldots, N, \bar{p}_i \neq 0 \Rightarrow \bar{z}_i = 0$.

Proof. Since "$\forall p \in \Delta,\ \forall z \in \zeta(p), p \cdot z = 0$" $\Rightarrow$ "$\forall p \in \Delta,\ \forall z \in \zeta(p), p \cdot z \leq 0$", from Lemma 4, there exist $\bar{p}$ and $\bar{z} \in \zeta(\bar{p})$ such that $\bar{z} \leq 0$. Since $\bar{p} \cdot \bar{z} = 0$, the conclusion is immediate. □

Remark 5. *Florenzano (2003) (Lemma 2.1.1) provides another version of the Gale-Nikaido-Debreu lemma (her proof of this result makes use of the separation and the Brouwer fixed point theorems). However, the point $\bar{p}$ in her Lemma 2.1.1 is not proved to be different from zero. In Lemma 4 and Lemma 5, the price $\bar{p}$ is in the unit-simplex and hence not equal to zero (see Florenzano (1982), Florenzano and Le Van (1986) for more detailed discussions).*

Remark 6 (The Kakutani fixed point theorem and the Gale-Nikaido-Debreu lemma)**.** *We emphasize that the Kakutani fixed point theorem can be obtained as a corollary of the Gale-Nikaido-Debreu lemma. We prove this by adapting the argument of Uzawa (1962) for continuous mapping.*

Let ζ be an upper semicontinuous correspondence, with non-empty convex compact values from Δ into itself. Define, for $p \in \Delta$,

$$\psi(p) = \Big\{y : y = z - \frac{p \cdot z}{\sum_{i=1}^N p_i^2} p,\ \text{with } z \in \zeta(p)\Big\}$$

One can check that ψ is upper semicontinuous and convex valued. Moreover, for any $p \in \Delta$, any $y \in \psi(p)$, we have $p \cdot y = 0$. Hence, from Lemma 6, there exist $\bar{p} \in \Delta$ and $\bar{y} \in \psi(\bar{p})$ which satisfy $\bar{y} \leq 0$, and $\forall i = 1, \ldots, N, \bar{p}_i \neq 0 \Rightarrow \bar{y}_i = 0$. In other words, there exist $\bar{p} \in \Delta$ and $\bar{z} \in \zeta(\bar{p})$ satisfying two conditions:

1. $\forall i = 1, \ldots, N,\ \bar{z}_i \leq \frac{\bar{p} \cdot \bar{z}}{\sum_{i=1}^N \bar{p}_i^2} \bar{p}_i$.

2. $\forall i = 1, \ldots, N,\ \bar{p}_i \neq 0 \Rightarrow \bar{z}_i = \frac{\bar{p} \cdot \bar{z}}{\sum_{i=1}^N \bar{p}_i^2} \bar{p}_i$.

Hence, if $\bar{p}_i = 0$, we have $0 \leq \bar{z}_i \leq 0$ which in turn implies that $\bar{z}_i = 0$. Let $\mu = \frac{\bar{p} \cdot \bar{z}}{\sum_{i=1}^N \bar{p}_i^2}$. We obtain that $\bar{z}_i = \mu \bar{p}_i$ for any $i = 1, \ldots, N$. Since $\bar{z} \in \Delta, \bar{p} \in \Delta$, we have $\mu = 1$. Hence, $\bar{p} = \bar{z} \in \zeta(\bar{p})$.

Notice that Florenzano (1982) (see her Proposition 2) also proves the Kakutani fixed point theorem from the Gale-Nikaido-Debreu lemma but she considers for the unit ball instead of the simplex Δ and she makes use of the separation theorem.

3.3 Using Sperner's lemma to prove fixed point theorems

The Brouwer fixed point theorem is considered as one of the most fundamental results in topology. The Kakutani fixed point theorem is a generalization of the Brouwer fixed point theorem for the case of set-valued functions. These two theorems have a wide application across different fields of mathematics and economics. We now formally state the Kakutani fixed point theorem and use the Sperner lemma to prove it.

Theorem 1. *(Kakutani) Let ζ be an upper semi continuous correspondence, with non empty convex compact values from a non-empty convex, compact set $V \subset \mathbb{R}^N$ into itself. Then there exists a fixed point x, i.e. $x \in \zeta(x)$.*

Proof. Without loss of generality, we prove this theorem for the case where the set V is the unit-simplex Δ of $\mathbb{R}^N$.

Let $\epsilon > 0$ be given. Since Δ is compact, there exists a finite covering of Δ with a finite family of open balls $\big(B\left(x^i(\epsilon), \epsilon\right)\big)_{i=1,\ldots,I(\epsilon)}$. Take a partition of unity subordinate to the family $\big(B\left(x^i(\epsilon), \epsilon\right)\big)_{i=1,\ldots,I(\epsilon)}$, i.e. a family of continuous non-negative real functions $(\alpha_i)_{i=1,\ldots,I(\epsilon)}$ from Δ in $\mathbb{R}_+$ such that $Supp(\alpha_i) \subset B\left(x^i(\epsilon), \epsilon\right), \forall i$ and $\sum_{i=1}^{I(\epsilon)} \alpha_i(x) = 1, \forall x \in \Delta$.

Take $y^i(\epsilon) \in \zeta(x^i(\epsilon)), \forall i$ and define the function $f^\epsilon : \Delta \to \Delta$ by $f^\epsilon(x) = \sum_{i=1}^{I(\epsilon)} \alpha_i(x) y^i(\epsilon)$. This function is continuous.

Let $K > 0$ be an integer and consider a simplicial subdivision T^K such that $Mesh(T^K) < 1/K$ (see Remark 1). We define a labeling R as follows:[13]

$$\text{for } x \in \Delta, R(x) = l, \text{ if } x_l \geq f_l^\epsilon(x). \tag{11}$$

This labeling is well defined because $\sum_l x_l = \sum_l f_l^\epsilon(x) = 1$. Moreover, this labeling satisfies the Sperner condition. Indeed, take $x \in \mathrm{ri}[[e^{i_1}, \ldots, e^{i_r}]]$ (recall that $(e^i)_i$ are the unit-vectors of $\mathbb{R}^N$.) We claim that $R(x) \in \{i_1, \ldots, i_r\}$. If not, $x_l < f_l^\epsilon(x), \forall l \in \{i_1, \ldots, i_r\}$ and we get a contradiction:

$$1 = \sum_{l \in \{i_1,\ldots,i_r\}} x_l < \sum_{l \in \{i_1,\ldots,i_r\}} f_l^\epsilon(x) \leq 1.$$

According to the Sperner lemma, there exists a completely labeled subsimplex $S^K = [[x^{K,1}, \ldots, x^{K,N}]]$, with $x_l^{K,l} \geq f_l^\epsilon(x^{K,l})\ \forall l = 1, \ldots, N$.

Let $K \to +\infty$, there exists a subsequence $(K_t)_{t \geq 1}$ such that $x^{K_t,l}$ converges to x^l for any $l = 1, \ldots, N$. Since $Mesh(T^K)$ tends to zero, we must have $x^1 = x^2 = \cdots = x^N$. Let $x^*(\epsilon)$ be this point. By continuity, we have $f^\epsilon(x^{K_t,l}) \to f^\epsilon(x^*(\epsilon))\ \forall l$. Since $x_l^*(\epsilon) \geq f_l^\epsilon(x^*(\epsilon))\ \forall l$, we get $x^*(\epsilon) = f^\epsilon(x^*(\epsilon))$.

Since $\big(B\left(x^i(\epsilon), \epsilon\right)\big)_{i=1,\ldots,I(\epsilon)}$ is a covering of Δ, we have $x^*(\epsilon) \in \cap_{i \in J(\epsilon)} B\left(x^i(\epsilon), \epsilon\right)$, where $J(\epsilon) \subset \{1, \ldots, I(\epsilon)\}$. Hence

$$x^*(\epsilon) = f^\epsilon(x^*(\epsilon)) = \sum_{i \in J(\epsilon)} \alpha_i(x^*(\epsilon)) y^i(\epsilon) \tag{12a}$$

$$\text{with} \sum_{i \in J(\epsilon)} \alpha_i(x^*(\epsilon)) = 1, y^i(\epsilon) \in \zeta(x^i(\epsilon)), \forall i \in J(\epsilon). \tag{12b}$$

[13] This labeling is similar to that in Scarf (1967) and Border (1985).

Observe that $\forall i \in J(\epsilon), x^i(\epsilon) \in B(x^*(\epsilon), \epsilon) \subset \mathbb{R}^N$. Therefore, $y^i(\epsilon) \in \zeta\big(B(x^*(\epsilon), \epsilon)\big)$ and $f^\epsilon(x^*(\epsilon)) \in co\Big(\zeta\big(B(x^*(\epsilon), \epsilon)\big)\Big)$. From Carathéodory's convexity theorem, we have a decomposition

$$f^\epsilon(x^*(\epsilon)) = \sum_{i=1}^{N+1} \beta_i(x^*(\epsilon))\tilde{y}^i(x^*(\epsilon)) \tag{13}$$

with $\tilde{y}^i(x^*(\epsilon)) \in \zeta\big(B(x^*(\epsilon), \epsilon)\big)$, $\beta_i(x^*(\epsilon)) \geq 0$, $\sum_{i=1}^{N+1} \beta_i(x^*(\epsilon)) = 1$.

Let $\epsilon \to 0$. Without loss of generality, we can assume $x^*(\epsilon) \to \bar{x} \in \Delta$, $\beta_i(x^*(\epsilon)) \to \bar{\beta}_i \geq 0$, $\sum_{i=1}^{N+1} \bar{\beta}_i = 1$, and $\tilde{y}^i(x^*(\epsilon)) \to \bar{y}^i \in \zeta(\bar{x}), \forall i = 1, \dots, N+1$. This implies $\bar{x} = \sum_{i=1}^{N+1} \bar{\beta}_i \bar{y}^i$. Since $\zeta(\bar{x})$ is convex, we get $\bar{x} \in \zeta(\bar{x})$. The proof of the Kakutani fixed point theorem is, therefore, over. □

The Brouwer fixed point theorem, stated below, is a corollary of the Kakutani fixed point theorem when ζ is a single valued mapping.

Corollary 1. *(Brouwer) Let ϕ be a continuous mapping from a non-empty convex compact set into itself. Then there exists a fixed point x, i.e. $x = \phi(x)$.*

Remark 7. *In the literature, the Brouwer fixed point theorem has been used to prove the Kakutani fixed point theorem. Indeed, the original proof of the Kakutani fixed point theorem in Kakutani (1941) relies on the application of the Brouwer fixed point theorem to single-valued mappings approximating the given set-valued mapping. For a pedagogical purpose, we summarize here the proof of Kakutani. Let S^n be the n-th barycentric simplicial subdivision of Δ. For each vertex x^n of S^n, take an arbitrary point $y^n \in \zeta(x^n)$. This mapping can be extended linearly to a continuous point-to-point mapping $x \to \phi_n(x)$ of Δ to itself. Applying the Brouwer fixed point theorem, there exists $x_n \in \Delta$ such that $x_n = \phi_n(x_n)$. Let n tend to infinity, there is a subsequence of (x_n) converging to a point x^* which is actually a fixed-point of ζ.*

Florenzano (1981), in Proposition 2, also makes use the Brouwer fixed point theorem to prove the Kakutani fixed point theorem. More precisely, for any $\epsilon > 0$, Florenzano considers a covering of Δ by a finite family of open balls and defines the function f^ϵ as in our above proof. By applying the Brouwer fixed point theorem, f^ϵ has a fixed point x^ϵ. Let $\epsilon \to 0$, then $x^\epsilon \to \bar{x}$. To prove that $\bar{x} \in \zeta(\bar{x})$, assume that this is not a case, then apply the Separation Theorem to the sets $\{\bar{x}\}$ and $\zeta(\bar{x})$ to get a contradiction.

We proceed as in Florenzano (1981) but use the Sperner lemma to get a fixed point x^ϵ of the function f^ϵ. Let $\epsilon \to 0$, then $x^\epsilon \to \bar{x}$. To prove that $\bar{x} \in \zeta(\bar{x})$, we proceed differently. More precisely, we apply Carathéodory's convexity theorem to get a decomposition (13) of $f^\epsilon(x^*(\epsilon))$. When $\epsilon \to 0$, x can be expressed as a convex combination of elements which belong $\zeta(\bar{x})$. So, $\bar{x} \in \zeta(\bar{x})$.

4 Conclusion

We have used the Sperner lemma and elementary mathematical results to prove the existence of general equilibrium for an economy with production and for another economy with incomplete financial markets. We have also made use of the Sperner lemma to provide a new proof of the Gale-Nikaido-Debreu lemma and the Kakutani fixed point theorem.

It is interesting to notice that, by using the Sperner lemma and algorithms of a combinatorial nature, we can approximate the equilibrium price (see Scarf and Hansen (1973), Scarf (1982) for more details). By consequence, we hope that our paper provides a fresh alternative way in studying the equilibrium existence, and, potentially, in computing economic equilibria.

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