Ragnar Frisch 1933 model: And yet it rocks!

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Abstract

This article provides an analytical solution to Frisch's 1933 model. The Laplace transform and its inverse prove valuable to obtain an expression for the different components that make up Frisch's original solution. It also sheds a new light on the argument in Zambelli (2007), where the author argues that this model does not fluctuate. Unlike Zambelli, we show that we can obtain cyclical solutions. This work provides new insights on the vision contained in the model. It turns out that this vision was much larger than what is often remembered, in particular, we can show that Frisch tried to construct a model that would intertwine both cycles and growth. In addition, we are able to reconsider the link between Frisch's early work in statistics and the birth of macrodynamic models.

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1 Introduction: A reluctant horse

In 1969, Ragnar Frisch received the first "Nobel Prize" in economics (conjointly with Jan Tinbergen), "for having developed and applied dynamic models for the analysis of economic processes." One of his pioneering contribution in this respect was his 1933 article on "Propagation Problems and Impulse Problems in Dynamic Economics" (Frisch, 1933). Today, this article is still remembered and celebrated for his approach to macroeconomic phenomena. In particular, the way Frisch illustrated the separation between a propagation mechanism and an impulse mechanism has been one of the most enduring contribution of his article.

Frisch's idea was that the cycle in the data was not only the result of a periodic force acting from the exterior, as was often advanced during the 1920s (most famously by Moore, see for instance Moore (1921)). It is well known that to him, it was necessary to explain the fluctuations of the economy with a propagation mechanism, and the "impulses" ensured that the stationary equilibrium was never attained[1]. This embodied the idea that if you hit a rocking horse, or a pendulum, it will oscillate toward its resting position[2]. One does not need to push the horse continuously back and forth to obtain the oscillation: a new hit is only necessary from time to time to reinject some energy into the swaying of the toy. It was thus necessary to build a model that would produce fluctuations once it was hit, that is, removed from its equilibrium position. Frisch viewed the whole mechanism of impulses and propagation as a way to explain fluctuations without using a periodic source of energy, as did Moore and others with the rainfall cycle, the position of Venus or even the simple succession of seasons. This is still different from our modern understanding of the

[1] Boianovsky and Trautwein (2007) show that Frisch also introduced this distinction through his debate with Johan Åkerman.

[2] On the pendulum metaphor, and the way it was used in discussions between Frisch and Schumpeter, see Louçã (2001), who used an original correspondance between Frisch and Schumpeter to show their different opinions on the validity of the metaphor. On the definition of shocks by Frisch, and the role he played into establishing a modern understanding of them, see Duarte and Hoover (2012), who examine how the notion of shocks was transformed afterward, and "rediscovered" by the New Classicals, who insisted regularly on the necessity to find propagation mechanisms and to differentiate them from the impulses (see for instance Lucas and Sargent [1978] or Prescott [2006]). Legrand and Hagemann (2019) argue that the role of propagation mechanisms progressively lost importance afterward.

opposition between endogenous and exogenous, and can be linked to the idea of free and forced oscillations that was used at the same time by other econometricians[3]. It should also be noted that the equilibrium of Frisch's model, as in other early macrodynamic models, was a kind of systemic equilibrium, the market behind his economy being always in equilibrium[4].

Frisch set out to building his model following a tour of American universities in 1930-31[5]. Following his debate with John Maurice Clark on the role of the accelerator, he built a determinate model with enough relationships to solve for all endogenous variables, dynamic relations between them and strong stability properties, to ensure that his horse would return to its equilibrium.

His model did not have a lasting impact on macroeconomics, in large part because it completely missed the future development on the importance of employment and aggregate demand. However, it is an important step in the construction of macrodynamic models, if only because it was one of the first fully worked out model of the economy. Within this model, Frisch captured his vision of the economy, made of different "component cycles" and of a "secular trend", and saw the analytical output as a guide for what to search for in the economic data. This contribution of Frisch has not often been underlined, and sometimes misunderstood, mainly because of the intractability of his model.

Among those who have gone further than a restatement of Frisch's solutions of his model, two contributions stand out. Björn Thalberg (1990) presented a computer reconstruction of the model based on Frisch's own solution. He apparently saw that for Frisch's original parameters and initial conditions the model was not fluctuating, and found other

[3] See for instance Tinbergen ([1931], 1959), or in a different context Le Corbeiller (1933). Frisch had already written on this idea in 1931, in an article published in Norwegian and partly translated in Andvig (1981): "The bundle of phenomena we call business cycle is ... a complex we have to attack as composed of free oscillations if we as economists are ever to be able to understand it. The explanation of the cyclical character of the oscillations must be sought in the inner structure of the system." (Frisch, 1931 quoted in Andvig, 1981: 709).

[4] On this point and for a discussion of the different model building approaches in the 1930s, see our forthcoming book Assous and Carret (2021).

[5] On Frisch himself, particularly during the interwar, see the works of Andvig (1981) and Olav Bjerkholt (1995); Bjerkholt and Dupont-Kieffer (2010). Ariane Dupont-Kieffer (2003) showed in particular in her thesis the opposition of approaches between Frisch and Mitchell. On the origins of Frisch's model and for a detailed anaysis of the hypotheses of his model, see Dupont-Kieffer (2012). For a more general context of the work of Frisch in the econometric movement, see Louçã (2007).

initial conditions for which this cycle was clearly visible. He went on to change the parameters "to reduce the dampening effect and thus make the picture clearer" (Thalberg, 1990: 101). But Thalberg remained mainly interested in the "primary cycle" identified by Frisch, and did not explain why Frisch's initial parameters did not work, when they were aggregated with other components.

Recognizing this problem, Stefano Zambelli claimed that Frisch's model was unable to fluctuate (Zambelli, 2007). Zambelli goes very far in his argument that Frisch's model was not oscillating, and draws some strong conclusions on this assumption. This is in spite of the fact that he does not offer a convincing explanation for the reasons behind the absence of oscillations: after explaining that Frisch's model was made up of the sum of an infinity of solutions of a sinusoidal type, Zambelli remarks that Frisch only presented the trajectories of the first four components of this series. If he had aggregated them with all other solutions, according to Zambelli, he would have seen a "straight line" instead of an oscillation (Zambelli, 2007: 153)[6]. Zambelli was then led to dub the cyclical solutions obtained by Frisch an "illusion", that would have completely undermined his approach, had it been seen by his contemporaries. However, we believe that the contentious conclusions drawn by Zambelli are not well-founded. While it may be possible that some components "cancel out" when they are aggregated, it would be quite amazing that an infinity of components would all cancel out for as wide a range of parameters as given by Zambelli in his appendix. There must be some other cause at work behind the absence of fluctuations, and we argue that we can only understand it by going back to Frisch's original approach.

Thus, in the following sections, we wish to underline the vision carried by Frisch's model by going back to his original approach to solve his equations. Although he was deprived of our modern tools of analysis, Frisch showed clearly that he understood well the behavior of the model. The algorithms he and his assistants[7] developed to obtain

[6]The only interpretation advanced by Zambelli to explain the disappearance of the cycles is that: "In summing harmonics it is well known that the result of the summation might not be cyclical at all, a trivial example being represented by two sinusoidal functions having the same amplitude and being out of phase for 180 degrees: the sum of the two harmonics is a constant function, a straight line." (Zambelli, 2007: 153).

[7]Frisch mentioned the help of two of his assistants in 1933, H. Holme and C. Thorbjörnsen, who

solutions were very clever; our approach only pays hommage to his by simplifying it. Doing so, we observe where exactly he strayed away from a complete understanding of his solutions.

2 New insights on Frisch's model

We propose an analytical solution to Frisch's model in the appendix. In this section, we explain the tools we use and where they came from, and give a general overview of the main points of this analysis to show how this solution throws light on Frisch's original intent. To apply our tools to Frisch's model, we start by reducing his system of three equations to one integro-differential equation with one unknown variable, the consumption x. Doing this, we obtain the following form, an integro-differential equation with a lag in the state variable and the integrand:

$$\dot{x}(t) + \lambda(r + \frac{s\mu}{\epsilon}) \cdot x(t) - \frac{\lambda s\mu}{\epsilon} \cdot x(t-\epsilon) + \frac{\lambda sm}{\epsilon} \int_{t-\epsilon}^{t} x(\tau) d\tau = c \tag{1}$$

In dealing with differential equations, a usual starting point is to pose a certain form of the solution, an *ansatz*. This often takes the form of an exponential function accepting a real or complex argument: real arguments will give a purely exponential explosion or return toward an equilibrium, while complex arguments will give rise to oscillations. In the early 1930s, the first econometricians built models that relied on mixed difference-differential equations[8] (see for instance Tinbergen ([1931] 1959) and Kalecki (1935)), that necessitated more developed tools of analysis. To solve his model, Frisch generalized the idea of an exponential solution, and posed that the general solution was made up of an infinite sum of exponential components.

We will follow him into this idea, because it is the only way to understand his thought process and the behavior of his model. But his process of solution although efficient, is

worked at the University Institute of Economics (Frisch, 1933: 16).

[8]These equations and their more general forms are now known as delay differential equations (DDE), and are still an ongoing area of research in mathematics and engineering.

particular to his equation and rather *ad hoc.* There are however some analytical tools that can give us the same infinite sum of components in a much more systematic way. It is notable that not long after the first macrodynamic models were built by Frisch, Kalecki and Tinbergen, those tools were introduced in *Econometrica* and the meetings of the *Econometric Society* by James and Belz (1938), who used a Fourier transform to obtain an infinite sum of solutions to a simple type of DDE[9]. The idea of Fourier decomposition and its numerous application in engineering and physics was well known in the interwar, and it may have to some extent inspired Frisch's own approach to his model. Their work however did not made an important impact, in part because of its difficulty. Shortly after, several mathematicians started working on the solution of those equations, most notably Richard Bellman, Kenneth Cooke, Edward Wright and N. D. Hayes.

The outcome of this research was a sort of textbook published by Bellman and Cooke at the Rand Corporation in 1963 (Bellman and Cooke, 1963), still a reference today. While Bellman is mostly remembered today among economists for inventing dynamic programming, he had been working on difference-differential equations since the 1940s. At the Rand, his early reports on difference-differential equations mentioned in particular the models of Kalecki (1935), Tinbergen ([1931] 1959) and the solutions worked out by Frisch and Holme (1935) and James and Belz (1938) as exemples of economic applications for those equations. The 1963 book also ended with a study of the stability of Kalecki's model, but its core was the analytical resolution of different classes of DDE via different approaches. One such approach developped by the authors, and the most important for our purpose, is the Laplace transform and its inverse.

This analytical tool, a staple of modern engineering, allows one to solve very efficiently and simply a wide range of differential equations. Its true power is unleashed when applied more formally to intricate equations such as those involving differentials and delays. At its core, the transformation transports us into a new domain where differentials and integrals are transformed into operations of multiplication and division, making it much easier to find a solution. The difficulty then lies in going back to the original domain, the

[9]See Erreygers (2019) for an examination of their impact on the early work of the Econometric Society.

temporal domain where the solutions can describe their trajectories as a function of time. This transform is very close to the Fourier transform, but more adapted for the study of difference-differential equations. In particular, once applied to this type of equations, it gives a similar infinite sum of components, that can be expressed as a function of the parameters of the system, the roots of a characteristic equation, and the initial conditions determined by the model-builder.

The appendix gives the computation needed to obtain from equation (1), at the beginning of this section, via the Laplace transform and its inverse, the following solution for $t \geq \epsilon$ (where ϵ is the production lag):

$$x(t-\epsilon) = \frac{c}{\lambda(r+sm)} + k_1 e^{r_1 t} + \sum_{i=2}^{\infty} A_i e^{\alpha_i t} cos(\beta_i t + \phi_i) \tag{2}$$

There are three distinct terms in the right hand side of this equation, that make up the total solution of Frisch's system. On the left is an equilibrium level, determined as a function only of the parameters of the system. In the middle, we find an exponential function that will be stable or unstable, according to the parameters of the system, and that was dubbed by Frisch a "secular trend". On the right, we have an infinite sum of sinusoidal solutions, each with its own frequency and damping exponent. Both the trend and each of the cycles have their own amplitude determined by the initial conditions, and each cycle has its own phase, determined by the initial conditions as well. The appendix also presents our methodology to estimate the roots that give us the frequency and damping of each cycle, and it is with this equation that we are able to rebuild Frisch's solution and discuss the different trajectories of his model.

3 The Taming of the Horse: Cyclical solutions in Frisch's model

If we assume, as does Zambelli, that the model is initially in a state of equilibrium and that there is a shock on x, the consumption of 10%, above the equilibrium, after some time, we can trace the return to equilibrium with our equation. The difference with Zambelli is that we do not compute it step-by-step with an approximation of the integral, but we compute the sum of the first one thousand components of equation (2)[10]. Focusing on the first four components, we can give a decomposition that shows precisely what the problem was: the first component, the "trend" component that takes the form of a monotonic return to equilibrium, dominates largely the cycles, both in amplitude and in damping.

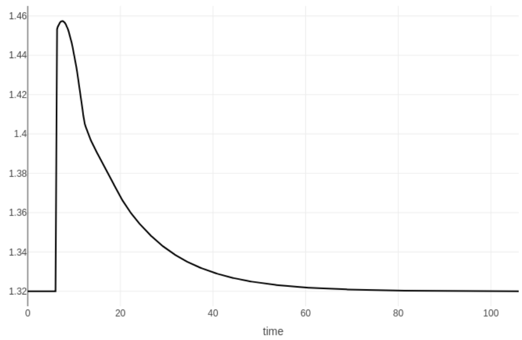

Figure 1: Solution of x(t) with the original parameters and 1000 components

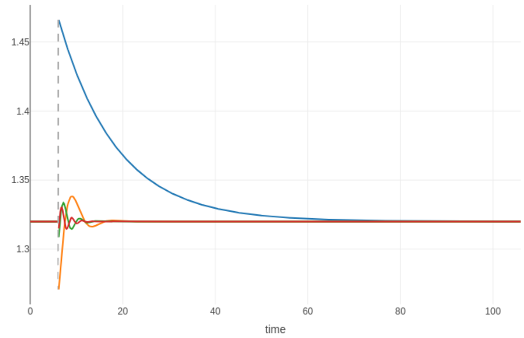

Figure 2: Decomposition with components 1 to 4 of equation (2)

Thus we are able to explain why Zambelli obtained a monotonic return to equilibrium: the huge damping exponents, as well as the small amplitude of the cycles, once they are aggregated, produces the result in figure 2. But why did it not bother Frisch? In the appendix, we compute a solution for x as a function of the initial conditions, the parameters of the system, and the roots of a characteristic polynomial. Doing so, we see clearly (see in particular equation (15)) that all the components will depend on the

[10] Using more components does not change significatively the solution; in fact using only one hundred components would have given us essentially the same picture, because their amplitude goes very quickly to zero.

same initial conditions. But Frisch poses two different initial conditions: one for the trend "x_0 shall be unity at origin" (Frisch, 1933: 18), and one for all the cycles "[w]e may, for instance, require that $x_1(0) = 0$ and $\dot{x}_1(0) = \frac{1}{2}$." (Frisch, 1933: 20)[11]. Hence it appears that Frisch did as if the first component (the trend) was independent of the other components (the cycles), and that he could impose different initial conditions on them, thereby artificially lowering the amplitude of the trend and making the cycles much more apparent. This was perceived by Thalberg (1990), who, as we noted, recognized that he had to change the initial conditions to show the cycles. If we use his initial condition of 1.25 (Thalberg, 1990: 115), we obtain the following figures showing the total solution and the decomposition of the "trend" and the first cycle.

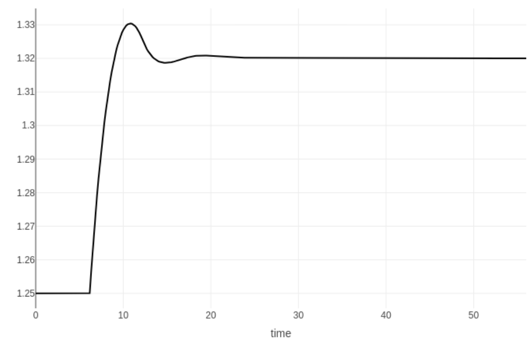

Figure 3: Solution of x(t) with Thalberg (1990) initial conditions

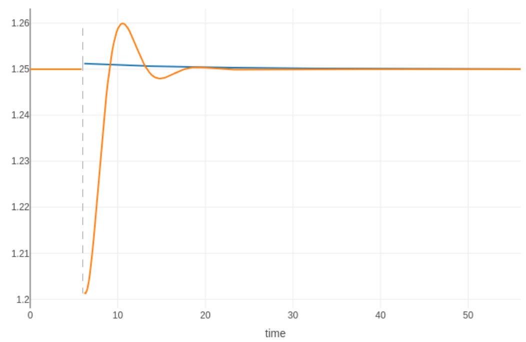

Figure 4: Decomposition with components 1 and 2 of equation (2)

We see clearly that the "trend" has all but disappeared, while the cycles become clearly visible, and their initial offset (phase) provides the "lift" toward the equilibrium level. This explains that Thalberg did not have the same conclusions as Zambelli. However, the latter argues that his result, the absence of fluctuations, will hold for a wide range of parameters. Going back to his methodology of displacing our variable from its equilibrium, and looking at its return towards this equilibrium position, does this result really hold? We find that this is not the case. Indeed we can find parameters so that i) the first component will have a small enough amplitude, such that the cycles underneath will appear, and ii) the damping exponent of the first component, that is, the quickness of the return to equilibrium, will be lower for the first component than for the first few cycles.

[11] He then repeats the same condition for x_2, thus implying that he used it for all cycles.

One such combination of parameters, that remains within the somewhat arbitrary bounds given by Zambelli in his appendix, can be obtained just by increasing λ, for instance to 0.3.

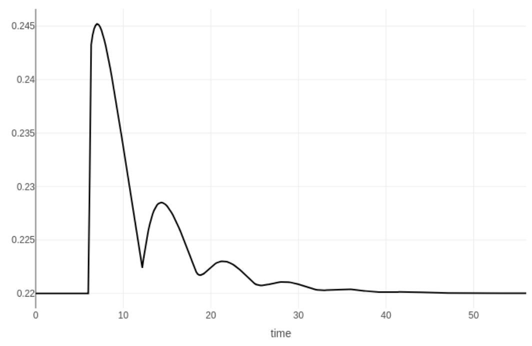

Figure 5: Solution of x(t) with $\lambda = 0.3$, 1000 components

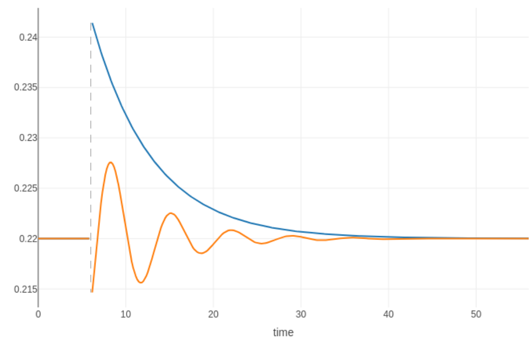

Figure 6: Components 1 and 2 (trend and major cycle)

In figure 5 and 6, we see that the amplitude of the first cycle has grown relatively to the "trend" component, such that the return to equilibrium is clearly of a cyclical form, with v-shaped recoveries ever so smaller. Still within the bounds given by Zambelli, we can change more parameters to obtain an even clearer cycle, where the damping out of the "trend" is quicker than the damping of the first cycle; for instance, with $m = 1$, $\lambda = 0.3$, $r = 1$, $s = 2$ and $\mu = 15$, we obtain the following figures.

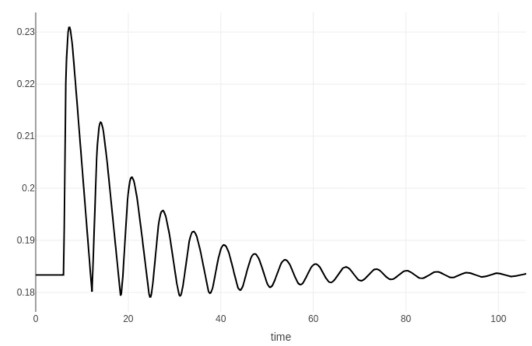

Figure 7: Solution of x(t), see parameters in the text, 1000 components

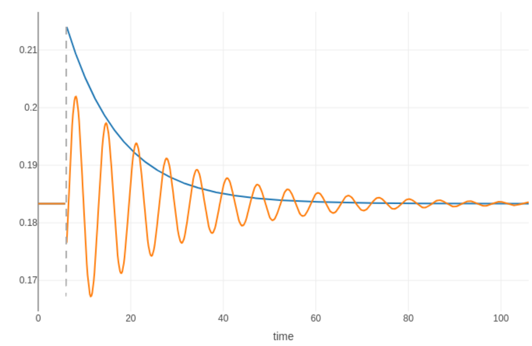

Figure 8: Components 1 and 2 (trend and major cycle)

This result is very important, because it means that Frisch's model can output a trend component which will leave room for some cycles. This is only possible because of the complexity of equation (1), and this result is hardly possible in a simpler difference-differential model such as that of Kalecki (1935) or Tinbergen (1931)[12]. As a final example, to underline Frisch's original vision of trend and cycles, we can give a solution showing the type of "transitory growth" that was present in his original model. To do this, we need to abandon the framework adopted by Zambelli of supposing that the economy starts at the equilibrium, and that a shock displaces it away from this equilibrium, that was used to observe the shape of the return to equilibrium.

Let's suppose that the economy was artificially maintained at a lower level than its equilibrium level for some time (at least a time equal to θ). In the following two figures, we kept the same parameters as in the previous plots:

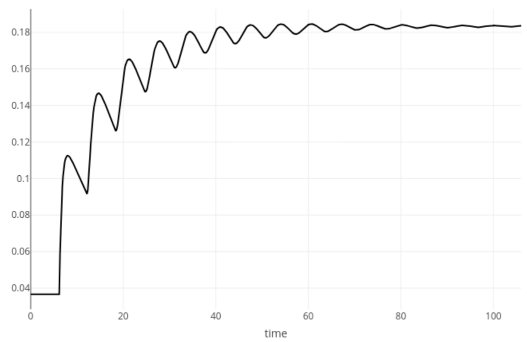

Figure 9: Solution of x(t), see parameters in the text, 1000 components

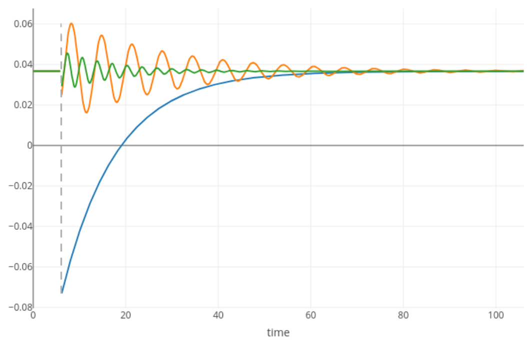

Figure 10: Components 1, 2 and 3 (trend and two cycles)

Figures 9 and 10 are the closest one to the vision originally exposed by Frisch in his article: there is a trend component that brings us toward the equilibrium level of the economy from the level where the economy was stuck, a primary cycle with a period of about 6.5 years, and a secondary cycle with a period of about 3.2 years. The primary

[12]The shared visions as well as differences of those models is detailed in the forthcoming book (Assous and Carret, 2021), where we apply the type of analysis made here to the models built in the 1930s and 1940s by Hamburger, Tinbergen and others in Europe, as well as Samuelson and Lange in the United States (see Assous and Carret (2020) on the latter).

cycle is clearly when it is superposed on the trend component, because it has a relatively high magnitude (compared to the trend and other cyclical components), and because it is less damped than the trend component.

4 Final remarks: Frisch's achievements in their context

Frisch's approach is somewhat peculiar for a modern reader, and can only be understood in the context of what was being done before the econometricians began building macrodynamic models. During the 1920s, the study of the business cycle was essentially empirical, and was driven by several methodologies dealing with the explosion in availability of economic data. One of the big ideas of the 1920s was that the business cycle could be disaggregated into several components and that it was possible to analyze the structure of lags, relative importance and frequencies of those cycles to make predictions on the future phase of the cycle. This idea stemmed largely from the discovery during the second part of the 19th century of cycles of different length, and it was built upon and applied in particular by Warren Persons, who established the famous tool of the "economic barometer" and the ABC curves[13].

While Persons' method relied on the elimination of seasonal variation and trend and the computations of correlations between time series, another related method relied on the frequency domain decomposition of economic time series into components of varying phase, amplitude and period[14]. The techniques employed date back from the groundbreaking work of Joseph Fourier in the early 19th century, who explained how almost any curve could be synthesized as the sum of a number of simple periodic curves (sinusoidal curves), giving birth to Fourier analysis (and a host of related tools), a central branch of modern mathematics and physics. The periodogram analysis as it was then called was first employed in economics by Moore and Beveridge in the 1910s and 1920s, and one

[13]See in particular the first chapters of Morgan (1995) on the developments of the empirical approach to the business cycle.

[14]On the development of the periodogram analysis see in particular Davis (1941, chapter 1), and (Cargill, 1974)

of its main advocate well into the 1940s was Harold T. Davis, one of the founder of the Cowles Commission.

Frisch himself worked on this idea in the late 1920s, and tried to devise a new method of component estimation where the amplitude and phase of each frequency could be varying (Frisch, 1928). His ideas on this subject, while acknowledged by Davis, did not have an important impact on the development of spectral analysis, in part because other advances were made at the same time, and also because interest was already shifting under the critiques of Schumpeter and Mitchell. Morgan also argues convincingly that it was abandoned because it could not represent in any way the *relationships* between the variables, something that Frisch was very much conscious of (Morgan, 1990: 38). Nevertheless, the periodogram analysis influenced undubitably the way that the first macrodynamic models were built, something that Morgan hints at in her book where the description of Frisch's famous model follows the explanation of his statistical decomposition method[15].

Indeed, Frisch took advantage of the fact that models mixing difference and differential equations gave rise to an infinity of components to decompose the solution of his models into a trend components and cyclical components featuring different periods, dampings, phases and amplitudes. This analytical approach mirrored the statistical approach of the 1920s: instead of estimating the cycles' features directly from the data, a theoretical model was built and (loosely) calibrated, and it was solved such that the different components of the general solution would be apparent, and could be compared to the cycles periodicty already well studied.

In this context, Frisch's approach to his model becomes a lot more understandable. He viewed the output of his model as symmetric to what was done on economic series via the periodogram analysis, and he pushed this idea a bit too far when he came to consider that the components he obtained could depend on different initial conditions. After Frisch, the idea of using the components of the model never really gained any traction. Tinbergen and Kalecki, using simpler versions of difference-differential equations, kept only

[15]This continuity between Frisch's statistical decomposition techniques and the components of his model has also been noted by Boumans (1995), who underlined that Frisch sought to explain how the different components of the business cycle were aggregated together.

the first component and looked for parameters that would make it oscillate. They were (mostly) justified in doing so, because of the simpler form of their equations. After them, Allen (1959: 302) called these higher terms "spurious", "arising because of the rigid and unrealistic assumption of a fixed time-delay", and, reflecting on this period, Samuelson himself acknowledged that he was stumped by the meaning of this components: "As a young student, what I found mystifying was the meaning of the infinite number of sinusoidal components of Frisch's more transcendental mixed difference-differential equation" (Samuelson, 1974: 9). James and Belz (1938), proposed to interpret them as the overtone or harmonics of a fundamental note, drawing a justified analogy with physical processes. But while we can interpret the different solutions arising, for instance, from a system of springs and masses (via the concept of normal modes), such an interpretation has never been given in economics, and the idea was largely abandoned.

It remains that, in 1933, Frisch saw in mixed difference and differential equation a way to represent both growth and fluctuations. Only by going back to this process of thought are we able to understand the author's intention, and where it could fail. Thus the analytical approach adopted here to find Frisch's original solution helped us avoid the pitfalls of numerical computations of the type used by Zambelli, and brought us back to the original spirit of Frisch's 1933 contribution. This allowed us to show the richness and complexity of his model, and whip his reluctant horse into rocking again.

5 Appendix - Solving the model with a shifted Laplace transform

The equations of Frisch's model are the following[16]:

$$\dot{x}_t = c - \lambda(r \cdot x_t + s \cdot z_t) \tag{3}$$

$$y_t = m \cdot x_t + \mu \cdot \dot{x}_t \tag{4}$$

$$z_t = \frac{1}{\epsilon}\int_{t-\epsilon}^{t} y(\tau)d\tau \tag{5}$$

As a preliminary, we can compute the equilibrium of the system, which will help us verify later on that our computations are right. After at least ϵ time at the equilibrium, we know that the rate of change will be zero, and the past values will be the same as the equilibrium values. We have thus three equations to determine the three unknown equilibrium values:

$$c = \lambda(r \cdot \bar{x} + s\bar{z}) \tag{6}$$

$$\bar{y} = m \cdot \bar{x} \tag{7}$$

$$\bar{z} = \bar{y} \tag{8}$$

Solving for the equilibrium values, we get that

$$\bar{x} = \frac{c}{\lambda(r + sm)} \tag{9}$$

$$\bar{y} = \bar{z} = \frac{mc}{\lambda(r + sm)} \tag{10}$$

To find solutions satisfying this system, we reduce it to one equation by successive replacements: first, we replace y in z_t and we obtain $z_t = \frac{1}{\epsilon}\int_{t-\epsilon}^{t} m \cdot x(\tau) + \mu \cdot \dot{x}(\tau)d\tau$. This can be simplified as $z_t = \frac{m}{\epsilon}\int_{t-\epsilon}^{t} x(\tau)d\tau + \frac{\mu}{\epsilon} \cdot (x(t) - x(t-\epsilon))$.

[16] Frisch's notation of z_t is rather confusing. This form makes clear that we want the integral of y over the interval $t-\epsilon$ to t, and computations yield the same result as Frisch.

Putting this into (3), we obtain $\dot{x}_t = c - \lambda \left[r \cdot x_t + s \cdot \left(\frac{m}{\epsilon} \int_{t-\epsilon}^{t} x(\tau) d\tau + \frac{\mu}{\epsilon} \cdot (x(t) - x(t-\epsilon)) \right) \right]$

Combining and rearranging terms, we obtain equation (1), an integro-differential equation with a delay in the state variable and the integral:

$$\dot{x}_t + \lambda(r + \frac{s\mu}{\epsilon}) \cdot x_t - \frac{\lambda s \mu}{\epsilon} \cdot x(t-\epsilon) + \frac{\lambda s m}{\epsilon} \int_{t-\epsilon}^{t} x(\tau) d\tau = c$$

We can easily check that the equilibrium value of this equation is the same as x in the system above. If we manage to find a solution for this equation, it will then just be a matter of differentiating it and replacing into the system (3)-(5) to obtain the other variables.

To simplify computations, we pose that $a = \lambda(r + \frac{s\mu}{\epsilon})$, $b = -\frac{\lambda s \mu}{\epsilon}$ and $d = \frac{\lambda s m}{\epsilon}$.

This equation needs two type of initial conditions: an initial condition on the state variable, and an initial condition for its evolution during the time $0 \leq t < \epsilon$. The importance of the two initial conditions is clear in the numerator of equation (12). In fact it was from in the treatment of those initial conditions that Frisch committed his error in solving the model, as we show in the text.

The definition of the Laplace transform is $\mathscr{L}\{f(t)\} = \int_0^\infty f(t) e^{-st} dt = F(s)$, and in order to have a solution to our equation in terms of the initial conditions from 0 to ϵ, we shift this definition by ϵ to obtain $\int_\epsilon^\infty f(t) e^{-st} dt$ (this follows Bellman and Cooke (1963) approach and avoids expressing our solution as a function of a negative time from $-\epsilon$ to 0). The integral in equation (1) has to be slightly modified to simplify the application of our transform. We separate it in two parts from 0 to t and from $t - \epsilon$ to 0, invert the second part and change variables to obtain:

$$\dot{x}(t) + a \cdot x(t) + b \cdot x(t-\epsilon) + d \int_0^t x(\tau) d\tau - d \int_\epsilon^t x(\tau - \epsilon) d\tau = c \tag{11}$$

Applying our transform to this equation term by term is a rather straightforward, if unwieldy, computation. $\dot{x}(t)$ will give us an initial condition on $x(\epsilon)$ as can be expected.

For $b \cdot x(t-\epsilon)$ we use a change of variable to make apparent the initial condition:

$$b\int_{\epsilon}^{\infty} x(t-\epsilon)e^{-st}dt = be^{-s\epsilon}\int_{0}^{\infty} x(t)e^{-st}dt = be^{-s\epsilon}\left[\int_{0}^{\epsilon} x(t)e^{-st}dt + \int_{\epsilon}^{\infty} x(t)e^{-st}dt\right]$$

The first integral in the brackets is our initial condition on the development of $x(t)$ during the period 0 to ϵ. The second is the definition of the transform, which we will call $X(s)$.

We integrate our two integrals using integration by parts. For the first one, we have:

$$\begin{aligned} d\int_{\epsilon}^{\infty}\int_{0}^{t} x(\tau)d\tau\; e^{-st}dt &= d\left[-\frac{1}{s}e^{-st}\int_{0}^{t} x(\tau)d\tau\Big|_{\epsilon}^{\infty} + \frac{1}{s}\int_{\epsilon}^{\infty} x(t)e^{-st}dt\right] \\ &= \frac{d}{s}\left[e^{-s\epsilon}\int_{0}^{\epsilon} x(\tau)d\tau + \int_{\epsilon}^{\infty} x(t)e^{-st}dt\right] \end{aligned}$$

Where the evaluation on the first line of the left term in the brackets is 0 when $t=\infty$. For the second one, we obtain a rather similar expression, with the lag:

$$\begin{aligned} d\int_{\epsilon}^{\infty}\int_{\epsilon}^{t} x(\tau-\epsilon)d\tau\; e^{-st}dt &= -d\left[-\frac{1}{s}e^{-st}\int_{\epsilon}^{t} x(\tau-\epsilon)d\tau\Big|_{\epsilon}^{\infty} + \frac{1}{s}\int_{\epsilon}^{\infty} x(t-\epsilon)e^{-st}dt\right] \\ &= -\frac{d}{s}e^{-s\epsilon}\left[\int_{0}^{\epsilon} x(t)e^{-st}dt + \int_{\epsilon}^{\infty} x(t)e^{-st}dt\right] \end{aligned}$$

This time the left term inside the bracket will vanish once evaluated, and we are left with the right term which is similar to the third term of equation (11), which we computed above, giving us the solution on the second line.

With $X(s) = \int_{\epsilon}^{\infty} x(t)e^{-st}dt$ and $p_0(s) = \int_{0}^{\epsilon} x(t)e^{-st}dt$, we can replace our computations in equation (11) to obtain:

$$sX - x(\epsilon)e^{-s\epsilon} + aX + be^{-s\epsilon}(p_0 + X) + \frac{d}{s}(e^{-s\epsilon}p_0 + X) - \frac{d}{s}e^{-s\epsilon}(p_0 + X) = \frac{c}{s}e^{-s\epsilon}$$

Grouping terms we have that:

$$X(s + a + be^{-e\epsilon} + \frac{d}{s} - \frac{d}{s}e^{-s\epsilon}) + p_0e^{-s\epsilon}(b + \frac{d}{s} - \frac{d}{s}) - x(\epsilon)e^{-s\epsilon} = \frac{c}{s}e^{-s\epsilon}$$

Which gives us our final expression:

$$X(s) = \frac{e^{-\epsilon s}\left(c - b \cdot s \cdot p_0 + s \cdot x(\epsilon)\right)}{s^2 + as + bse^{-\epsilon s} + d - de^{-\epsilon s}} \tag{12}$$

where we have multiplied both $x(\epsilon)$ and p_0 by $\frac{s}{s}$.

The formula for the inverse Laplace transform is $f(t) = \frac{1}{2\pi i} lim_{T\to\infty} \int_{\gamma-Ti}^{\gamma+Ti} F(s)e^{st}ds$. Following Bellman and Cooke[17], we compute this contour by shifting it to the left and taking into account the singularities we meet. We start by defining our contour integral:

$$x(t) = \frac{1}{2\pi i}\int_{(b)} \frac{e^{-\epsilon s}\left(c - b \cdot s \cdot \int_0^{\epsilon} x(t)e^{-s\cdot t}dt + s \cdot x(\epsilon)\right)}{s^2 + as + bse^{-\epsilon s} + d - de^{-\epsilon s}} e^{st}ds \tag{13}$$

where (b) is our vertical contour.

Our contour will be equal to the sum of residue times $2\pi i$, which will cancel the $\frac{1}{2\pi i}$ of the inverse formula, so that we are just left with the sum of residues. Because the last expression can be written in the form $x(t) = \frac{1}{2\pi i}\int_{(b)} \frac{g(s)}{h(s)}ds$, where

$$f(s) = \frac{g(s)}{h(s)} = \frac{\left(c - b \cdot s \cdot \int_0^{\epsilon} x(t)e^{-s\cdot t}dt + s \cdot x(\epsilon)\right) e^{s(t-\epsilon)}}{s^2 + as + bse^{-\epsilon s} + d - de^{-\epsilon s}} \tag{14}$$

and because all poles arising from the denominator will be simple, we have that at a simple pole c, $Res(f, c) = lim_{z\to c}(z - c)f(z) = \frac{g(c)}{h'(c)}$. Thus our sum of residues will rise from this expression, once we are able to compute the zeros of our denominator[18].

This allows us to give a final expression for $x(t)$ as a sum of components, each one with its own amplitude (and phase for cyclical components) arising from the initial conditions:

$$x(t) = \sum_0^{\infty} \frac{c - br_i p_0(t) + r_i x(\epsilon)}{2r_i + a + be^{-\epsilon r_i} - b\epsilon r_i e^{-\epsilon r_i} + d\epsilon e^{-\epsilon r_i}} e^{r_i t}u(t-\epsilon) = \sum_0^{\infty} k_i e^{r_i t}u(t-\epsilon) \tag{15}$$

[17]See Chapter 1 in Bellman and Cooke (1963) for a discussion of the inversion algorithm, and Chapter 3, section 7 for a detailed application.

[18]Bellman and Cooke show that to the left of any vertical line in the left-half plane, the residues will always give rise to ever smaller components (in amplitude and damping), at least for retarded and neutral DDE (our equation is a transformation of a neutral DDE). This result does not hold for forward DDE, that is, when the lag is negative.

Where $u(t)$ is the Heaviside step function and the r_i are zeros of the characteristic polynomial $h(s)$. We take the sum from 0 to ∞, but the conjugate of each complex root is also a solution, and its coefficient is $\bar{k_i}$.

All that remains to do now is to find a procedure to obtain the zeros of our characteristic polynomial[19]. First of, we can remark that our polynomial always has a trivial root at $r_0 = 0$. This root will give us an equilibrium level for this system, and the reader will not be surprised to see that this level is equal to $\frac{c}{a+b+d\epsilon} = \frac{c}{\lambda(r+sm)}$, the same equilibrium we previously computed.

To find other roots of this polynomial, we can remark that $lim_{s\to\infty} h(s) = s^2 + as(1 + \theta_1(s)) + bse^{-\epsilon s}(1 - \theta_2(s))$, where both θ_1 and θ_2 will tend to zero as s grows to infinity. This means that for large s, the simpler expression $s + a + be^{-\epsilon s} = 0$ will be a good approximation of our roots. Because this is a transcendental equation, we will have an infinity of solutions to this equation, but we can give a closed form solution with the Lambert W function[20]. We first change variables and pose $s = \frac{w}{\epsilon} - a$. Replacing, we have that $\frac{w}{\epsilon} = -be^{-w+a\epsilon}$ and rearranging to have the Lambert form $we^w = -\epsilon be^{a\epsilon}$, which means that $w = W_k(-\epsilon be^{a\epsilon})$ and finally $s = \frac{W_k(-\epsilon be^{a\epsilon})}{\epsilon} - a$, where $k = 0, 1, 2...\infty$ is the branch of the Lambert function giving us an approximation of the value of w for this branch. Because ϵ is positive and b is negative, the expression inside W_k will always be positive. We know that in this case there will always be one nontrivial real root (this is the "trend" identified by Frisch in 1933), and an infinity of complex roots that will give us an infinity of cyclical solution. The general solution of x will be the sum (superposition) of all these solutions. This gives us an initial guess that is improved with Newton's algorithm, giving us the same results as Frisch (and his assistants) for the roots of the four first components[21].

Inserting those solutions in equation (15), we obtain the complete solution given in

[19]The reader will note that h is indeed the characteristic polynomial, that we could have obtained for instance by inserting a solution of the form $Ae^{\lambda t}$ in equation (1).

[20]See Corless et al. for the definition of this function and an algorithm to obtain its solutions.

[21]Note that our approximation is working well for cyclical components, but can fail for the lowest component, a real root. In this case we broaden our search sequentially.

the text as equation (2) (expressed here with the Heaviside step function):

$$x(t) = \frac{c}{\lambda(r + sm)}u(t - \epsilon) + k_1 e^{r_1 t}u(t - \epsilon) + \sum_{i=2}^{\infty} A_i e^{\alpha_i t} cos(\beta_i t + \phi_i)u(t - \epsilon)$$

Where r_1 is a real root, and the terms in the sum are all sinusoidal functions, with damping and period given by $r_i = \alpha_i + \beta_i j$. The roots and the initial conditions determine together all the k_i, giving the amplitude of the sinusoidal $A_i = 2 \cdot |k_i|$ and its phase $\phi_i = arg(2 \cdot k_i)$ (we get a factor of two because of the complex conjugate). In the case of the pure exponential, the amplitude is simply k_1.

The code used to make the figures is available at `https://gist.github.com/placardo/2c0832e815dcf1f7918fbfd140d57ba5`

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