Inheritance of Convexity for the P˜min-Restricted Game

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Inheritance of Convexity for the $\tilde{\mathcal{P}}_{\min}$-Restricted Game

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Abstract

We consider a restricted game on weighted graphs associated with minimum partitions. We replace in the classical definition of Myerson restricted game the connected components of any subgraph by the sub-components obtained with a specific partition $\tilde{\mathcal{P}}_{\min}$. This partition relies on the same principle as the partition $\mathcal{P}_{\min}$ introduced by Grabisch and Skoda (2012) but restricted to connected coalitions. More precisely, this new partition $\tilde{\mathcal{P}}_{\min}$ is induced by the deletion of the minimum weight edges in each connected component associated with a coalition. We provide a characterization of the graphs satisfying inheritance of convexity from the underlying game to the restricted game associated with $\tilde{\mathcal{P}}_{\min}$.

Keywords: cooperative game, convexity, graph-restricted game, graph partitions.

AMS Classification: 91A12, 91A43, 90C27, 05C75.

1 Introduction

We consider a finite set N of players with $|N| = n$. Let $\mathcal{P}$ be a correspondence on N associating to every subset $A \subseteq N$ a partition $\mathcal{P}(A)$ of A. For any game (N, v), Skoda (2017) defined the restricted game $(N, \overline{v})$ associated with $\mathcal{P}$ by:

$$\overline{v}(A) = \sum_{F \in \mathcal{P}(A)} v(F), \text{ for all } A \subseteq N. \tag{1}$$

We refer to this game as the $\mathcal{P}$-restricted game. v is the characteristic function of the underlying game, $v : 2^N \to \mathbb{R}$, $A \mapsto v(A)$ and satisfies $v(\emptyset) = 0$. Many correspondences have been considered in the literature to take into

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account communication or social restrictions. The first founding example is the Myerson's correspondence $\mathcal{P}_M$ associated with communication games (Myerson, 1977). Communication games are cooperative games (N, v) defined on the set of vertices N of an undirected graph $G = (N, E)$, where E is the set of edges. $\mathcal{P}_M$ associates to every coalition $A \subseteq N$ the partition of A into connected components. The $\mathcal{P}_M$-restricted game $(N, \overline{v})$, known as Myerson restricted game, takes into account the connectivity between players in G. Only connected coalitions are able to cooperate and get their initial values. Many other correspondences have been considered to define restricted games (see, e.g., Algaba et al. (2001); Bilbao (2000, 2003); Faigle (1989); Grabisch and Skoda (2012); Grabisch (2013)).

For a given correspondence, a classical problem is to study inheritance of a property from the initial game (N, v) to the restricted game $(N, \overline{v})$. Inheritance of properties has been thoroughly studied for the Myerson correspondence $\mathcal{P}_M$ (see Owen (1986); van den Nouweland and Borm (1991); Slikker (2000)). Inheritance of convexity is of special interest as it implies that lots of appealing properties are also inherited, for instance superadditivity, non-emptiness of the core, and that the Shapley value lies in the core. Skoda (2017) obtained an abstract characterization of inheritance of convexity for an arbitrary correspondence using a cyclic intersecting sequence condition on coalitions. This result implies some of the characterizations obtained for specific correspondences. In particular, it implies the characterization of inheritance of convexity for the Myerson correspondence by cycle-completeness of the underlying graph, established by van den Nouweland and Borm (1991). For correspondences associated with graphs, it is of course more interesting to find characterizations in terms of graph structures as for the Myerson correspondence.

Grabisch and Skoda (2012) introduced the correspondence $\mathcal{P}_{\min}$ for communication games on weighted graphs. A communication game on a weighted graph is a combination of a cooperative game (N, v) and a weighted graph $G = (N, E, w)$ which has the set of players as its vertices and where w is a weight function defined on the set E of edges of G. In this context, it is likely that players belonging to a given coalition are more or less prone to cooperate depending on the weights of their links. The correspondence $\mathcal{P}_{\min}$ takes into account connectedness of the players but a coalition gets its initial value under a stronger requirement. There must be some privileged relation between players to activate their cooperation. For a given coalition $A \subseteq N$, we denote by $E(A)$ the set of edges with both end-vertices in A, and by $\Sigma(A)$ the set of minimum weight edges in $E(A)$. It is assumed that two players have a privileged relation in a coalition $A \subseteq N$ if they are linked by an edge with weight strictly greater than the minimum edge-weight in the subgraph $G_A = (A, E(A))$. More precisely, the correspondence $\mathcal{P}_{\min}$ associates with any coalition $A \subseteq N$ the partition $\mathcal{P}_{\min}(A)$ of A into the connected components of the subgraph $(A, E(A) \setminus \Sigma(A))$. Then, the $\mathcal{P}_{\min}$-restricted game

$(N, \overline{v})$ is defined by:

$$\overline{v}(A) = \sum_{F \in \mathcal{P}_{\min}(A)} v(F), \text{ for all } A \subseteq N.$$

Grabisch and Skoda (2012) established three necessary conditions on the underlying graph G to have inheritance of convexity with the correspondence $\mathcal{P}_{\min}$. To establish these conditions, they only had to consider connected subsets. Hence, these conditions are valid assuming only $\mathcal{F}$-convexity which is a weaker condition than convexity introduced by Grabisch and Skoda (2012), obtained by restricting convexity to connected subsets. Skoda (2019) presented a characterization of inheritance of $\mathcal{F}$-convexity for $\mathcal{P}_{\min}$ by five necessary and sufficient conditions on the edge-weights of specific subgraphs. These subgraphs correspond to stars, paths, cycles, pans, and adjacent cycles of the underlying graph G. Finally, Skoda (2020) obtained a characterization of inheritance of convexity for $\mathcal{P}_{\min}$. As convexity implies $\mathcal{F}$-convexity, the conditions established by Skoda (2019) are necessary but they do not appear in the characterization of inheritance of convexity. Indeed, this last characterization relies on more straightforward conditions. This is in part due to the fact that inheritance of convexity restricts the edge-weights to at most three different values.

In this paper, we establish a characterization of inheritance of convexity for a new correspondence $\tilde{\mathcal{P}}_{\min}$. This correspondence is close to the correspondence $\mathcal{P}_{\min}$ as they coincide on connected coalitions of players. The correspondence $\tilde{\mathcal{P}}_{\min}$ follows the same pattern as the correspondence $\mathcal{P}_{\min}$ but it requires that players cooperate only if they are in a privileged relation relatively to the connected component they belong to. More precisely, let $A \subseteq N$ be a coalition of players and let $A_1, A_2, \ldots, A_p$ with $p \geq 1$ be the connected components of G_A. Then, $\tilde{\mathcal{P}}_{\min}(A) = \{\mathcal{P}_{\min}(A_1), \ldots, \mathcal{P}_{\min}(A_p)\}$ and the $\tilde{\mathcal{P}}_{\min}$-restricted game $(N, \hat{v})$ is defined by

$$\hat{v}(A) = \sum_{F \in \tilde{\mathcal{P}}_{\min}(A)} v(F), \text{ for all } A \subseteq N. \tag{2}$$

By definition of $\tilde{\mathcal{P}}_{\min}$, we also have the following relation:

$$\hat{v}(A) = \sum_{l=1}^{p} \overline{v}(A_l), \text{ for all } A \subseteq N. \tag{3}$$

Hence, the $\tilde{\mathcal{P}}_{\min}$-restricted game also corresponds to the Myerson restricted game associated with the $\mathcal{P}_{\min}$-restricted game. As a consequence, the $\tilde{\mathcal{P}}_{\min}$-restricted game $(N, \hat{v})$ and the $\mathcal{P}_{\min}$-restricted game $(N, \overline{v})$ assign the same values to connected coalitions. In particular, this implies that $\mathcal{F}$-convexity of the $\tilde{\mathcal{P}}_{\min}$-restricted game is equivalent to $\mathcal{F}$-convexity of the $\mathcal{P}_{\min}$-restricted game. As a result, the characterization of inheritance of $\mathcal{F}$-convexity obtained by Skoda (2019) for the correspondence $\mathcal{P}_{\min}$ is also valid for the correspondence $\tilde{\mathcal{P}}_{\min}$.

In the present paper, we establish a characterization of inheritance of convexity for the correspondence $\tilde{\mathcal{P}}_{\min}$. This characterization includes the five necessary and sufficient conditions on the edge-weights characterizing inheritance of $\mathcal{F}$-convexity. We have to add three new conditions to these five previous conditions to obtain a characterization of inheritance of classical convexity. These new conditions can be seen as reinforcements of the preceding conditions established on cycles, pans, and adjacent cycles in the characterization of $\mathcal{F}$-convexity. We prefer to state them separately as they are more specific than the previous ones. In particular, their necessity is obtained considering non-connected coalitions, whereas these last coalitions are not considered with $\mathcal{F}$-convexity. Moreover, our result implies that, in the case of cycle-free graphs, the two conditions on stars and paths are necessary and sufficient for inheritance of convexity with the correspondence $\tilde{\mathcal{P}}_{\min}$. A similar result holds for inheritance of $\mathcal{F}$-convexity with the correspondence $\mathcal{P}_{\min}$ as proved by Grabisch and Skoda (2012). Hence, there is equivalence between inheritance of $\mathcal{F}$-convexity with the correspondence $\mathcal{P}_{\min}$ and inheritance of convexity with $\tilde{\mathcal{P}}_{\min}$ in the case of cycle-free graphs. There is no such equivalence with inheritance of convexity with $\mathcal{P}_{\min}$. Indeed, Skoda (2020) proved that inheritance of convexity with $\mathcal{P}_{\min}$ restricts the number of different edge-weights to at most three even in the case of cycle-free graphs and there is no such limitation for inheritance of convexity with $\tilde{\mathcal{P}}_{\min}$. We also obtain that inheritance of convexity and inheritance of convexity restricted to the class of unanimity games are equivalent for the correspondence $\tilde{\mathcal{P}}_{\min}$. This last result also holds for $\mathcal{P}_{\min}$ (Skoda (2019)) and was already observed in a more general setting by Skoda (2017).

The article is organized as follows. In Section 2 we give preliminary definitions and results established by Grabisch and Skoda (2012). In particular, we recall the definition of convexity, $\mathcal{F}$-convexity and general conditions on a correspondence to have inheritance of superadditivity, convexity or $\mathcal{F}$-convexity. We recall in Section 3 the necessary and sufficient conditions on the graph and the weight vector w established by Skoda (2019) for inheritance of $\mathcal{F}$-convexity from the original game (N, v) to the $\mathcal{P}_{\min}$-restricted game $(N, \overline{v})$. Section 4 includes the main results ot the paper. We establish three supplementary necessary conditions on the edge-weights to have inheritance of convexity for the correspondence $\tilde{\mathcal{P}}_{\min}$. Then, we prove that these three conditions appended to the five previous conditions necessary for inheritance of $\mathcal{F}$-convexity are also sufficient to have inheritance of convexity for $\tilde{\mathcal{P}}_{\min}$. We conclude with some remarks and suggestions in Section 5.

2 Preliminary definitions and results

Let N be a given set with $|N| = n$. We denote by 2^N the set of all subsets of N. A game (N, v) is *zero-normalized* if $v(\{i\}) = 0$ for all $i \in N$. We recall that a game (N, v) is superadditive if, for all $A, B \in 2^N$ such that $A \cap B = \emptyset$, $v(A \cup B) \geq v(A) + v(B)$. For any given subset $\emptyset \neq S \subseteq N$, the unanimity game (N, u_S) is defined by:

$$u_S(A) = \begin{cases} 1 & \text{if } A \supseteq S, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

We note that u_S is superadditive for all $S \neq \emptyset$. Let X and Y be two given sets. A *correspondence* f with domain X and range Y is a map that associates to every element $x \in X$ a subset $f(x)$ of Y, i.e., a map from X to 2^Y. In this work we consider specific correspondences $\mathcal{P}$ with domain and range 2^N, such that for every subset $\emptyset \neq A \subseteq N$, the family $\mathcal{P}(A)$ of subsets of N corresponds to a partition of A. We set $\mathcal{P}(\emptyset) = \{\emptyset\}$.

For a given correspondence $\mathcal{P}$ on N and subsets $A \subseteq B \subseteq N$, we denote by $\mathcal{P}(B)_{|A}$ the restriction of the partition $\mathcal{P}(B)$ to A. For two given subsets A and B of N, $\mathcal{P}(A)$ is a refinement of $\mathcal{P}(B)$ if every block of $\mathcal{P}(A)$ is a subset of some block of $\mathcal{P}(B)$.

We recall the following results established by Grabisch and Skoda (2012). The first one gives general conditions on a correspondence $\mathcal{P}$ to have inheritance of superadditivity.

Theorem 1. *Let N be an arbitrary set and $\mathcal{P}$ a correspondence on 2^N. The following conditions are equivalent:*

1) The $\mathcal{P}$-restricted game $(N, \overline{u_S})$ is superadditive for all $\emptyset \neq S \subseteq N$.

2) $\mathcal{P}(A)$ is a refinement of $\mathcal{P}(B)_{|A}$ for all subsets $A \subseteq B \subseteq N$.

3) The $\mathcal{P}$-restricted game $(N, \overline{v})$ is superadditive for all superadditive game (N, v).

As $\mathcal{P}_{\min}(A)$ (resp. $\tilde{\mathcal{P}}_{\min}(A)$) is a refinement of $\mathcal{P}_{\min}(B)_{|A}$ (resp. $\tilde{\mathcal{P}}_{\min}(B)_{|A}$) for all subsets $A \subseteq B \subseteq N$, Theorem 1 implies the following result.

Corollary 2. *Let $G = (N, E, w)$ be an arbitrary weighted graph. The $\mathcal{P}_{\min}$-restricted game $(N, \overline{v})$ (resp. the $\tilde{\mathcal{P}}_{\min}$-restricted game $(N, \hat{v})$) is superadditive for every superadditive game (N, v).*

Let us consider a game (N, v). For arbitrary subsets A and B of N, we define the value:

$$\Delta v(A, B) := v(A \cup B) + v(A \cap B) - v(A) - v(B).$$

A game (N, v) is *convex* if its characteristic function v is supermodular, i.e., $\Delta v(A, B) \geq 0$ for all $A, B \in 2^N$. We note that u_S is supermodular for all $S \neq \emptyset$. Let $\mathcal{F}$ be a weakly union-closed family[1] of subsets of N such that $\emptyset \notin \mathcal{F}$. A game v on 2^N is said to be $\mathcal{F}$-convex if $\Delta v(A, B) \geq 0$, for all $A, B \in \mathcal{F}$ with $A \cap B \in \mathcal{F}$. Let us note that a game (N, v) is convex if and only if it is superadditive and $\mathcal{F}$-convex with $\mathcal{F} = 2^N \setminus \{\emptyset\}$. Of course, convexity implies $\mathcal{F}$-convexity. For any $i \in N$ and any subset $A \subseteq N \setminus \{i\}$, the derivative of v at A w.r.t i is defined by

$$\Delta_i v(A) := v(A \cup \{i\}) - v(A).$$

$\Delta_i v(A)$ is also known as the marginal contribution of player i w.r.t coalition A. If a game v on 2^N is $\mathcal{F}$-convex then, for all $i \in N$ and all $A \subseteq B \subseteq N \setminus \{i\}$ with A, B and $A \cup \{i\} \in \mathcal{F}$ we have:

$$\Delta_i v(B) \geq \Delta_i v(A). \tag{5}$$

For a given graph $G = (N, E)$, we say that a subset $A \subseteq N$ is connected if the induced graph $G_A = (A, E(A))$ is connected. The family of connected subsets of N is obviously weakly union-closed. For this last family, the following result holds.

Theorem 3. *Let $G = (N, E)$ be an arbitrary graph and let $\mathcal{F}$ be the family of connected subsets of N. The following conditions are equivalent:*

$$v \text{ is } \mathcal{F}\text{-convex}. \tag{6}$$

$$\begin{gathered} \Delta_i v(B) \geq \Delta_i v(A), \\ \forall i \in N, \forall A \subseteq B \subseteq N \setminus \{i\} \text{ with } A, B, \text{ and } A \cup \{i\} \in \mathcal{F}. \end{gathered} \tag{7}$$

The next theorem gives general conditions on a correspondence $\mathcal{P}$ to have inheritance of convexity for unanimity games.

Theorem 4. *Let N be an arbitrary set and $\mathcal{P}$ a correspondence on 2^N. Let $\mathcal{F}$ be a weakly union-closed family of subsets of N with $\emptyset \notin \mathcal{F}$. If the $\mathcal{P}$-restricted game $(N, \overline{u_S})$ is superadditive for all $\emptyset \neq S \subseteq N$, then the following conditions are equivalent.*

1) The $\mathcal{P}$-restricted game $(N, \overline{u_S})$ is $\mathcal{F}$-convex for all $\emptyset \neq S \subseteq N$.

2) For all $A, B \in \mathcal{F}$ with $A \cap B \in \mathcal{F}$, $\mathcal{P}(A \cap B) = \{A_j \cap B_k\,;\, A_j \in \mathcal{P}(A), B_k \in \mathcal{P}(B), A_j \cap B_k \neq \emptyset\}$.

[1] $\mathcal{F}$ is weakly union-closed if $A \cup B \in \mathcal{F}$ for all A, $B \in \mathcal{F}$ such that $A \cap B \neq \emptyset$ (Faigle et al., 2010). Weakly union-closed families were introduced and analysed by Algaba (1998) (see also Algaba et al. (2000)) and called union stable systems.

Moreover if $\mathcal{F} = 2^N \setminus \{\emptyset\}$ or if $\mathcal{F}$ corresponds to the set of connected subsets of a graph then 1) and 2) are equivalent to:

3) For all $i \in N$, for all $A \subseteq B \subseteq N \setminus \{i\}$ with A, B, and $A \cup \{i\} \in \mathcal{F}$, and for all $A' \in \mathcal{P}(A \cup \{i\})_{|A}$, $\mathcal{P}(A)_{|A'} = \mathcal{P}(B)_{|A'}$.

We finally recall the following lemma.

Lemma 5. *Let us consider $A, B \subseteq N$ and a partition $\{B_1, B_2, \ldots, B_p\}$ of B. Let $\mathcal{F}$ be a weakly union-closed family of subsets of N with $\emptyset \notin \mathcal{F}$. If A, B_i, and $A \cap B_i \in \mathcal{F}$ for all $i \in \{1, \ldots, p\}$, then for every $\mathcal{F}$-convex game (N, v) we have*

$$v(A \cup B) + \sum_{i=1}^{p} v(A \cap B_i) \geq v(A) + \sum_{i=1}^{p} v(B_i). \tag{8}$$

3 Inheritance of $\mathcal{F}$-convexity

Let $G = (N, E, w)$ be a weighted graph and let $\mathcal{F}$ be the family of connected subsets of N. We recall in this section necessary and sufficient conditions on the weight vector w established by Skoda (2019) for inheritance of $\mathcal{F}$-convexity from the original game (N, v) to the $\mathcal{P}_{\min}$-restricted game $(N, \overline{v})$. We denote by w_k or w_{ij} the weight of an edge $e_k = \{i, j\}$ in E.

A star S_k corresponds to a tree with one internal vertex and k leaves. We consider a star S_3 with vertices $1, 2, 3, 4$ and edges $e_1 = \{1, 2\}$, $e_2 = \{1, 3\}$ and $e_3 = \{1, 4\}$. Let us note that the edges $\{2, 3\}$, $\{3, 4\}$, or $\{2, 4\}$ may exist in G.

Star Condition. *For every star of type S_3 in G, the edge-weights satisfy*

$$w_1 \leq w_2 = w_3,$$

after renumbering the edges if necessary.

Path Condition. *For every path $\gamma = \{1, e_1, 2, e_2, 3, \ldots, m, e_m, m+1\}$ in G and for all i, j, k with $1 \leq i < j < k \leq m$, the edge-weights satisfy*

$$w_j \leq \max(w_i, w_k).$$

For a given cycle $C = \{1, e_1, 2, e_2, \ldots, m, e_m, 1\}$ with $m \geq 3$, we denote by $E(C)$ the set of edges $\{e_1, e_2, \ldots, e_m\}$ of C and by $\hat{E}(C)$ the set composed of $E(C)$ and of the chords of C in G.

Cycle Condition. *For every cycle $C = \{1, e_1, 2, e_2, \ldots, m, e_m, 1\}$ in G with $m \geq 3$, the edge-weights satisfy*

$$w_1 \leq w_2 \leq w_3 = \cdots = w_m = \hat{M},$$

after renumbering the edges if necessary, where $\hat{M} = \max_{e \in \hat{E}(C)} w(e)$. Moreover, $w(e) = w_2$ for all chord incident to 2, and $w(e) = \hat{M}$ for all $e \in \hat{E}(C)$ non-incident to 2.

For a given cycle C, an edge e in $\hat{E}(C)$ is a maximum weight edge of C if $w(e) = \max_{e \in \hat{E}(C)} w(e)$. Otherwise, e is a non-maximum weight edge of C. Moreover, we call maximum (resp. non-maximum) weight chord of C any maximum (resp. non-maximum) weight edge in $\hat{E}(C) \setminus E(C)$.

A *pan graph* is a connected graph corresponding to the union of a cycle and a path.

Pan Condition. *For every subgraph of G corresponding to the union of a cycle $C = \{1, e_1, 2, e_2, \ldots, e_m, 1\}$ with $m \geq 3$, and a path P such that there is an edge e in P with $w(e) \leq \min_{1 \leq k \leq m} w_k$ and $|V(C) \cap V(P)| = 1$, the edge-weights satisfy*

(a) either $w_1 = w_2 = w_3 = \cdots = w_m = \hat{M}$,

(b) or $w_1 = w_2 < w_3 = \cdots = w_m = \hat{M}$,

where $\hat{M} = \max_{e \in \hat{E}(C)} w(e)$. If Condition (b) is satisfied then $V(C) \cap V(P) = \{2\}$, and if moreover $w(e) < w_1$ then $\{1, 3\}$ is a maximum weight chord of C.

Two cycles are said adjacent if they share at least one common edge.

Adjacent Cycles Condition. *For all pairs $\{C, C'\}$ of adjacent cycles in G such that*

(a) $V(C) \setminus V(C') \neq \emptyset$ and $V(C') \setminus V(C) \neq \emptyset$,

(b) C has at most one non-maximum weight chord,

(c) C and C' have no maximum weight chord,

(d) C and C' have no common chord,

C and C' cannot have two common non-maximum weight edges. Moreover, C and C' have a unique common non-maximum weight edge e_1 if and only if there are non-maximum weight edges $e_2 \in E(C) \setminus E(C')$ and $e'_2 \in E(C') \setminus E(C)$ such that e_1, e_2, e'_2 are adjacent and

- $w_1 = w_2 = w'_2$ *if* $|E(C)| \geq 4$ *and* $|E(C')| \geq 4$.
- $w_1 = w_2 \geq w'_2$ *or* $w_1 = w'_2 \geq w_2$ *if* $|E(C)| = 3$ *or* $|E(C')| = 3$.

The following characterization of inheritance of $\mathcal{F}$-convexity was established by Skoda (2019) for the correspondence $\mathcal{P}_{\min}$. This result is also valid for the correspondence $\tilde{\mathcal{P}}_{\min}$ as the $\mathcal{P}_{\min}$-restricted game coincides with the $\tilde{\mathcal{P}}_{\min}$-restricted game on connected subsets.

Theorem 6. *Let $\mathcal{F}$ be the family of connected subsets of N. The $\mathcal{P}_{\min}$-restricted game $(N, \overline{v})$ (resp. the $\tilde{\mathcal{P}}_{\min}$-restricted game $(N, \hat{v})$) is $\mathcal{F}$-convex for every superadditive and $\mathcal{F}$-convex game (N, v) if and only if the Star, Path, Cycle, Pan, and Adjacent cycles conditions are satisfied.*

We finally recall two lemmas proved by Skoda (2019) valid when some of the previous necessary conditions are satisfied. The first one gives simple conditions ensuring that $\mathcal{P}_{\min}(A)$ is induced by $\mathcal{P}_{\min}(B)$ for any subsets $A \subseteq B \subseteq N$. The second one gives restrictions on the minimum edge-weights of subsets.

We say that an edge $e \in E$ is connected to a subset $A \subseteq N$, if there is a path in G joining e to A.

Lemma 7. *Let $\mathcal{F}$ be the family of connected subsets of N. Let us consider $A, B \in \mathcal{F}$ with $A \subseteq B \subseteq N$, $|A| \geq 2$, and $\sigma(A) = \sigma(B)$. Let us assume that*

1. *The Pan condition is satisfied.*
2. *$G_B = (B, E(B))$ is cycle-free or there exists an edge $e \in E$ connected to B with $w(e) < \sigma(B)$.*

Then

1. *$\mathcal{P}_{\min}(A) = \mathcal{P}_{\min}(B)_{|A}$.*
2. *For every $\mathcal{F}$-convex game (N, v), we have*

$$v(B) - \overline{v}(B) \geq v(A) - \overline{v}(A). \tag{9}$$

Lemma 8. *Let $\mathcal{F}$ be the family of connected subsets of N. Let us assume that the Path and Star conditions are satisfied. For all $i \in N$, for all $A \subseteq B \subseteq N \setminus \{i\}$ with A and B in $\mathcal{F}$, $|A| \geq 2$, and $E(A, i) \neq \emptyset$, we have*

1. *either $\sigma(A, i) \geq \sigma(A) \geq \sigma(B)$,*
2. *or $\sigma(A) = \sigma(B) > \sigma(A, i)$,*

where $\sigma(A, i) = \min_{e \in E(A,i)} w(e)$.

4 Inheritance of convexity for the $\tilde{\mathcal{P}}_{\min}$-restricted game

Throughout this section, $\mathcal{F}$ is the family of connected subsets.

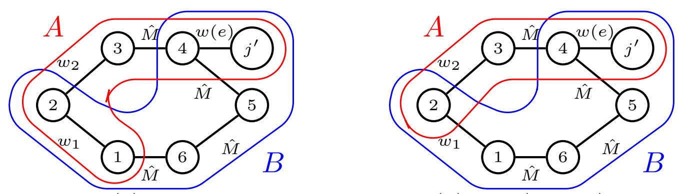
We get in this section a characterization of inheritance of convexity for the correspondence $\tilde{\mathcal{P}}_{\min}$. Of course, as the class of convex games is contained in the class of $\mathcal{F}$-convex games, the necessary conditions for inheritance of $\mathcal{F}$-convexity recalled in Section 3 are also necessary for inheritance of convexity. But these conditions are not sufficient and we establish three new necessary conditions.

Reinforced Cycle Condition. *For every cycle $C_m = \{1, e_1, 2, e_2, \ldots, m, e_m, 1\}$ with $m \geq 4$ and $w_3 = \ldots = w_m = \hat{M} = \max_{e \in \hat{E}(C_m)} w(e)$:*

1. *If $w_1 < w_3$ (resp. $w_2 < w_3$), then any edge incident to j with $4 \leq j \leq m-1$ (resp. $5 \leq j \leq m$) is linked to e_1 (resp. e_2) by an edge.*
2. *If $\max(w_1, w_2) < w_3$, then any edge incident to j with $4 \leq j \leq m$ is linked to 2 by an edge.*

Proposition 9. *If for all $\emptyset \neq S \subseteq N$ the $\tilde{\mathcal{P}}_{\min}$-restricted game $(N, \overline{u_S})$ is convex, then the Reinforced Cycle Condition is satisfied.*

Proof. Let us assume $w_1 < w_3$ (resp. $\max(w_1, w_2) < w_3$). Let us note that the proof with $w_2 < w_3$ is similar. Let us consider $e = \{j, j'\}$ with $4 \leq j \leq m$. If $e \in E(C_m)$, then $w(e) = \hat{M}$. Otherwise, the Star condition applied to $\{e, e_{j-1}, e_j\}$ implies $w(e) \leq \hat{M}$. The Path condition applied to $\{1, e_1, 2, e_2, \ldots e_{j-1}, j, e\}$ implies $\hat{M} = w_{j-1} \leq \max(w_1, w(e))$. As $w_1 < \hat{M}$, we get $w(e) = \hat{M}$. Let us assume $j = 4$ (the other cases are similar) and $\{1, 2, j, j'\}$ (resp. $\{2, j, j'\}$) non-connected. Let us consider $A = \{1, 2, 3, j, j'\}$ (resp. $A = \{2, 3, j, j'\}$) and $B = (V(C_m) \setminus \{3\}) \cup \{j'\}$ as represented in Figure 1a (resp. Figure 1b) with $m = 6$ and $j' \neq 5$. As $A \in \mathcal{F}$ and



(a) $w_1 < w_3$. (b) $\max(w_1, w_2) < w_3$.

Figure 1: e incident to 4.

$B \in \mathcal{F}$, we have $\tilde{\mathcal{P}}_{\min}(A) = \mathcal{P}_{\min}(A)$ and $\tilde{\mathcal{P}}_{\min}(B) = \mathcal{P}_{\min}(B)$. As $w_1 < \hat{M}$

(resp. $\max(w_1, w_2) < \hat{M}$), there are components A' in $\mathcal{P}_{\min}(A)$ and B' in $\mathcal{P}_{\min}(B)$ both containing $\{j, j'\}$. By Theorem 4 (applied with $\mathcal{F} = 2^N \setminus \{\emptyset\}$), $A' \cap B'$ is a component of $\tilde{\mathcal{P}}_{\min}(A \cap B)$ containing $\{j, j'\}$. But we have $\tilde{\mathcal{P}}_{\min}(A \cap B) = \{\{1\}, \{2\}, \{j\}, \{j'\}\}$ (resp. $\tilde{\mathcal{P}}_{\min}(A \cap B) = \{\{2\}, \{j\}, \{j'\}\}$), a contradiction. $\square$

Reinforced Pan Condition *For all connected subgraphs corresponding to the union of a cycle $C_m = \{1, e_1, 2, e_2, 3, \ldots, m, e_m, 1\}$ with $m \geq 4$, satisfying $w_1 \leq w_2 \leq w_3 = \ldots = w_m = \hat{M} = \max_{e \in \hat{E}(C)} w(e)$, and a path P containing an edge e with $w(e) < \hat{M}$ and $V(C_m) \cap V(P) = \{2\}$:*

(a) If $w(e) < w_1$, then any vertex j with $4 \leq j \leq m$ is linked to P by an edge in E.

(b) If $w(e) < w_1 < \hat{M}$, then ($w_1 = w_2$ and) any vertex j with $4 \leq j \leq m$ is linked to 2 by an edge in E.

Proposition 10. *If for all $\emptyset \neq S \subseteq N$ the $\tilde{\mathcal{P}}_{\min}$-restricted game $(N, \overline{u_S})$ is convex, then the Reinforced Pan Condition is satisfied.*

Proof. We first prove that (a) is satisfied. Let us assume $j = 4$ (the other cases are similar). Let us assume $V(P) \cup \{4\}$ non-connected. Let us consider $A = V(P) \cup \{3, 4\}$ and $B = V(P) \cup (V(C_m) \setminus \{3\})$, as represented in Figure 2 with $m = 5$. As $A \in \mathcal{F}$ and $B \in \mathcal{F}$, we have $\tilde{\mathcal{P}}_{\min}(A) = \mathcal{P}_{\min}(A)$ and

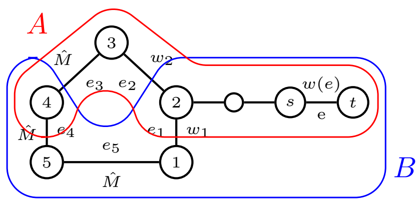


Figure 2: $w(e) < w_1 \leq w_2 \leq \hat{M}$.

$\tilde{\mathcal{P}}_{\min}(B) = \mathcal{P}_{\min}(B)$. As $w(e) < w_1$, there are components A' in $\mathcal{P}_{\min}(A)$ and B' in $\mathcal{P}_{\min}(B)$ containing $\{2, 4\}$. By Theorem 4 (applied with $\mathcal{F} = 2^N \setminus \{\emptyset\}$), $A' \cap B'$ is a component of $\tilde{\mathcal{P}}_{\min}(A \cap B)$ containing $\{2, 4\}$. But $\tilde{\mathcal{P}}_{\min}(A \cap B) = \{\{4\}, \mathcal{P}_{\min}(V(P))\}$, a contradiction.

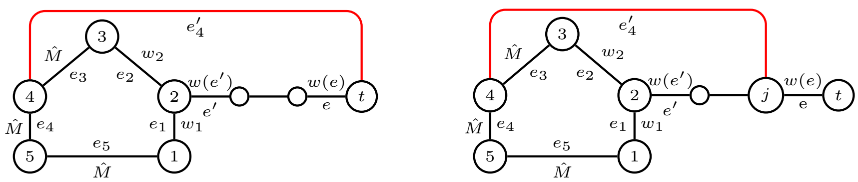
We now prove that (b) is satisfied. As $w(e) < w_1$, the Pan condition implies

$$w_1 = w_2 < w_3 = \cdots = w_m = \hat{M}. \tag{10}$$

Let us assume $j = 4$ (the other cases are similar) and $\{2, 4\} \notin E$. By (a), there exists at least one edge linking 4 to P. Let us consider an edge $e_4' = \{4, j\}$ with $j \in V(P) \setminus \{2\}$. Let t be the end-vertex of P different from 2 and let e' be the edge of P incident to 2 (e' may coincide with e). By the Star condition, we have $w(e') \leq w_1 = w_2$. Let C' be the cycle formed by $\{2, e_2, 3, e_3, 4, e'_4, j\} \cup P_{2,j}$. As $w(e') \leq w_2 < \hat{M} = w_3$, the Cycle condition implies

$$w(e) = \hat{M}, \ \forall e \in E(C') \setminus \{e', e_2\}. \tag{11}$$

Let us first assume $e' \neq e$. If $w(e') < w_1 = w_2$, then we can replace e by e'. Therefore, we can assume $w(e') = w_1 = w_2$. Let us assume $j = t$ as represented in Figure 3a. Then, e belongs to C' and has weight $w(e) < \hat{M}$, contradicting (11). Let us now assume $j \in V(P) \setminus \{2, t\}$, as represented in



(a) $j = t$

(b) $j \in V(P) \setminus \{2, t\}$

Figure 3: $w(e) < w(e') = w_1 = w_2 < \hat{M}$.

Figure 3b. As $w(e) < w(e') = w_2 < \hat{M}$, the Pan condition applied to the pan formed by C' and $P_{j,t}$ implies $j = 2$, a contradiction.

Let us now assume $e' = e$. We necessarily have $j = t$. Let us consider $A = \{2, 3, 4, t\}$ and $B = (V(C_m) \setminus \{3\}) \cup \{t\}$ as represented in Figure 4 with $m = 5$. As $A \in \mathcal{F}$ and $B \in \mathcal{F}$, we have $\tilde{\mathcal{P}}_{\min}(A) = \mathcal{P}_{\min}(A)$ and

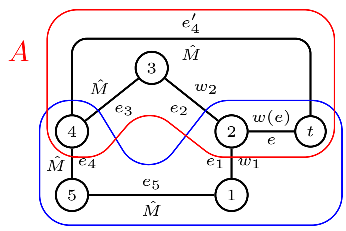


Figure 4: $w(e) < w_1 = w_2 < \hat{M}$.

$\tilde{\mathcal{P}}_{\min}(B) = \mathcal{P}_{\min}(B)$. As $w(e) < w_1 = w_2$, there are components A' in $\mathcal{P}_{\min}(A)$ and B' in $\mathcal{P}_{\min}(B)$ containing $\{2, 4\}$. By Theorem 4 (applied with $\mathcal{F} = 2^N \setminus \{\emptyset\}$), $A' \cap B'$ is a component of $\tilde{\mathcal{P}}_{\min}(A \cap B)$ containing $\{2, 4\}$. But $\tilde{\mathcal{P}}_{\min}(A \cap B) = \{\{2\}, \{4, t\}\}$, a contradiction. $\square$

Reinforced Adjacent Cycles Condition *For all pairs $\{C, C'\}$ of adjacent cycles in G such that one of the following conditions is satisfied*

1. $|E(C)| = |E(C')| = 4$ *and C and C' have two common non-*

maximum weight edges.

2. (a) $|E(C)| = |E(C')| = 5$, $|E(C) \cap E(C')| = 3$, *and* C *and* C' *have only one non-maximum weight-edge which is common to* C *and* C'.

   (b) *Setting* $C = \{1, e_1, 2, e_2, 3, e_3, 4, e_4, 5, e_5, 1\}$, *we have* $C' = \{1, e_1, 2, e_2, 3, e'_3, 4', e'_4, 5, e_5, 1\}$ *with* $4' \neq 4$, 4 *and* $4'$ *are not both linked to* 1 *or to* 2, *and* e_1 *is the unique non-maximum weight edge common to* C *and* C'.

There exists an edge linking $V(C) \setminus V(C')$ *to* $V(C') \setminus V(C)$.

Proposition 11. *If for all* $\emptyset \neq S \subseteq N$ *the* $\tilde{\mathcal{P}}_{\min}$*-restricted game* $(N, \overline{u_S})$ *is convex, then the Reinforced Adjacent Cycles Condition is satisfied.*

Proof. Let us first assume Condition 1 satisfied. Let us consider $C = \{1, e_1, 2, e_2, 3, e_3, 4, e_4, 1\}$ and $C' = \{1, e_1, 2, e_2, 3, e'_3, 4', e'_4, 1\}$ where e_1 and e_2 are non-maximum weight edges common to C and C'. By the Cycle condition, we have

$$\max(w_1, w_2) < w_3 = w_4 = w(e'_3) = w(e'_4) = \hat{M}, \tag{12}$$

where $\hat{M} = \max_{e \in \hat{E}(C)} w(e) = \max_{e \in \hat{E}(C')} w(e)$. Moreover, any chord of C or C' incident to 2 has a weight equal to $\max(w_1, w_2)$. Let us consider $A = \{1, 2, 4, 4'\}$ and $B = \{2, 3, 4, 4'\})$ as represented in Figure 6. Let us

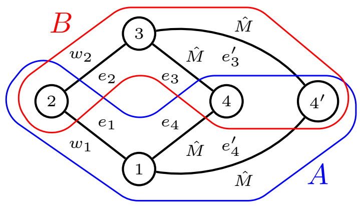


Figure 5: $\max(w_1, w_2) < \hat{M}$.

assume $\{4, 4'\} \notin E$. Let us note that $\{1, 3\}$, $\{2, 4\}$, and $\{2, 4'\}$ may exist. As $A \in \mathcal{F}$ and $B \in \mathcal{F}$, we have $\tilde{\mathcal{P}}_{\min}(A) = \mathcal{P}_{\min}(A)$ and $\tilde{\mathcal{P}}_{\min}(B) = \mathcal{P}_{\min}(B)$. As $\max(w_1, w_2) < \hat{M}$, there are components A' in $\mathcal{P}_{\min}(A)$ and B' in $\mathcal{P}_{\min}(B)$ both containing $\{4, 4'\}$. By Theorem 4 (applied with $\mathcal{F} = 2^N \setminus \{\emptyset\}$), $A' \cap B'$ is a component of $\tilde{\mathcal{P}}_{\min}(A \cap B)$ containing $\{4, 4'\}$. But we have $\tilde{\mathcal{P}}_{\min}(A \cap B) = \{\{2\}, \{4\}, \{4'\}\}$, a contradiction.

Let us now assume Condition 2 satisfied. Let us consider $C = \{1, e_1, 2, e_2, 3, e_3, 4, e_4, 5, e_5, 1\}$ and $C' = \{1, e_1, 2, e_2, 3, e'_3, 4', e'_4, 5, e_5, 1\}$ where e_1 is the unique non-maximum weight edge common to C and C'. By the Cycle condition, we have

$$w_1 < w_2 = w_3 = w_4 = w_5 = w(e'_3) = w(e'_4) = \hat{M}. \tag{13}$$

Moreover, any chord of C or C' has weight $\hat{M}$. Let us consider $A = \{1, 2, 4, 4', 5\}$ and $B = \{1, 2, 3, 4, 4'\}$ as represented in Figure 6. Let us

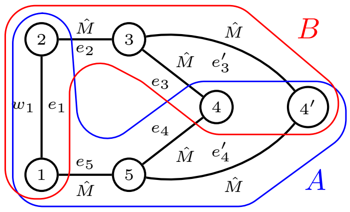


Figure 6: $w_1 < \hat{M}$.

assume $\{4, 4'\} \notin E$. As $A \in \mathcal{F}$ and $B \in \mathcal{F}$, we have $\tilde{\mathcal{P}}_{\min}(A) = \mathcal{P}_{\min}(A)$ and $\tilde{\mathcal{P}}_{\min}(B) = \mathcal{P}_{\min}(B)$. As $w_1 < \hat{M}$, there are components A' in $\mathcal{P}_{\min}(A)$ and B' in $\mathcal{P}_{\min}(B)$ both containing $\{4, 4'\}$. By Theorem 4 (applied with $\mathcal{F} = 2^N \setminus \{\emptyset\}$), $A' \cap B'$ is a component of $\tilde{\mathcal{P}}_{\min}(A \cap B)$ containing $\{4, 4'\}$. As 4 and $4'$ are neither both linked to 1 nor both linked to 2, they necessarily belong to distinct blocks of $\tilde{\mathcal{P}}_{\min}(A \cap B)$, a contradiction. $\square$

Theorem 12. *For every convex game (N, v), the $\tilde{\mathcal{P}}_{\min}$-restricted game $(N, \overline{v})$ is convex if and only if the Star, Path, Cycle, Pan, Adjacent Cycles, Reinforced Cycle, Reinforced Pan, and Reinforced Adjacent Cycles conditions are satisfied.*

We have already seen that these conditions are necessary. To prove their sufficiency, we will need the following propositions and lemmas.

Proposition 13. *Let us assume that the Path, Star, Cycle, Reinforced cycle and Reinforced pan conditions are satisfied. Let us consider $i \in N$ and $A \subseteq B \subseteq N \setminus \{i\}$ with $A \cup \{i\}$ and B in $\mathcal{F}$. Let us assume $A \notin \mathcal{F}$, and let A_1, A_2 be two connected components of A with $\sigma(A_1, i) < \sigma(A_2, i)$. If $\sigma(B) < \sigma(A_2, i)$, then*

1. *$|A_2| = 1$. Moreover, there exists a unique edge e_1 in $\Sigma(A_1, i)$ and setting $e_1 = \{i, j_1\}$ with $j_1 \in A_1$ and $A_2 = \{j_2\}$, there exist a vertex $k \in B$, and a cycle $C_4 = \{e_1, e_2, e_3, e_4\}$ with $e_2 = \{i, j_2\}$, $e_3 = \{j_2, k\}$, $e_4 = \{k, j_1\}$ and $w_2 = w_3 = \sigma(A_2, i)$, and $w_4 = \sigma(B)$.*

2. *$\{j_1\}$ and A_2 are in distinct blocks of $\mathcal{P}_{\min}(B)$. Moreover, any block of $\mathcal{P}_{\min}(A_1)$ belongs to a block of $\mathcal{P}_{\min}(B)$ different from the one containing A_2.*

3. *If A_3 is a third connected component of A and if the Reinforced adjacent cycles condition is satisfied, then $|A_3| = 1$, $\sigma(A_2, i) = \sigma(A_3, i)$, and A_2 and A_3 are in distinct blocks of $\mathcal{P}_{\min}(B)$.*

Proof. **1.** Let $e_1 = \{i, j_1\}$ (resp. $e_2 = \{i, j_2\}$) be an edge in $\Sigma(A_1, i)$ (resp. $\Sigma(A_2, i)$). As $\sigma(A_1, i) < \sigma(A_2, i)$, e_1 is necessarily unique by the Star condition. As $B \in \mathcal{F}$, there exists at least one path in G_B linking j_1 to j_2.

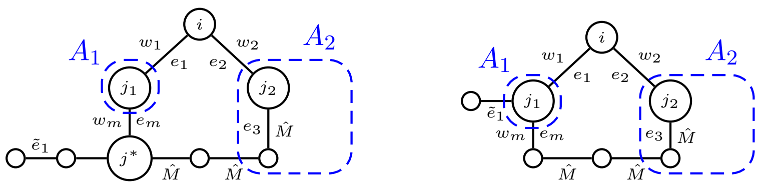
Let us first assume that there is no path in G_B linking j_1 to j_2 and containing at least one edge in $\Sigma(B)$. Let γ be a shortest path in G_B linking j_1 to j_2. $\{e_1\} \cup \gamma \cup \{e_2\}$ induces a cycle $C_m = \{e_1, e_2, \ldots, e_m\}$ with $m \geq 4$. Let $\tilde{e}_1$ be an edge in $\Sigma(B)$. As $E(\gamma) \cap \Sigma(B) = \emptyset$, $\tilde{e}_1$ cannot belong to $E(C_m)$. Moreover, $\tilde{e}_1$ cannot be a chord of C_m, otherwise there would be a path linking j_1 to j_2 and containing an edge in $\Sigma(B)$. Let P be a shortest path in G_B linking $\tilde{e}_1$ to a vertex j^* in γ (P may be reduced to j^*). We select $\tilde{e}_1$ such that P is as short as possible. As $w_1 = \sigma(A_1, i) < \sigma(A_2, i) = w_2$, the Cycle condition implies

$$w_1 < w_2 \leq w_3 = \cdots = w_{m-1} = \hat{M} = \max_{e \in \hat{E}(C_m)} w(e), \tag{14}$$

and

$$\begin{array}{ll} \text{either} & w_m < \hat{M} \text{ and } w_2 = \hat{M}, \\ \text{or} & w_m = \hat{M} \text{ and } w_2 \leq \hat{M}. \end{array} \tag{15}$$

If $j^* \neq j_1$ as represented in Figure 7a, then the Path condition implies $w_2 \leq \max(w_1, w(\tilde{e}_1))$. As $w_1 < w_2$ and as $w(\tilde{e}_1) = \sigma(B) < \sigma(A_2, i) = w_2$,



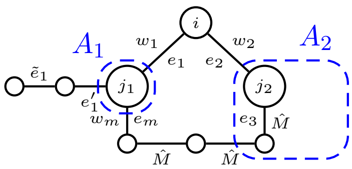
(a) $j^* \neq j_1$. (b) $j^* = j_1$.

Figure 7: $w_1 < w_2 \leq \hat{M}$ and $w_m \leq \hat{M}$.

we get a contradiction. We henceforth assume $j^* = j_1$.

Let us first assume $\tilde{e}_1$ incident to j_1 as represented in Figure 7b. Any edge in γ has weight strictly greater than $\sigma(B)$. As $w(\tilde{e}_1) = \sigma(B)$, the Star condition implies $w(\tilde{e}_1) < w_1 = w_m$. As $w_1 < \hat{M}$, we get $w_m < \hat{M}$, and (15) implies $w_2 = \hat{M}$. As $w(\tilde{e}_1) < w_1 = w_m < \hat{M}$, the Reinforced pan condition implies that j_2 is linked to j_1 by an edge, a contradiction.

Let us now assume $\tilde{e}_1$ non-incident to j_1. Let e'_1 be the edge of P incident to j_1 as represented in Figure 8. Let us assume $w(\tilde{e}_1) \geq w_1$. Then the Path condition implies $w(e'_1) \leq \max(w(\tilde{e}_1), w_1) = w(\tilde{e}_1)$. As $w(\tilde{e}_1) = \sigma(B)$, we necessarily have $w(e'_1) = \sigma(B)$. This contradicts the choice of $\tilde{e}_1$. Let us assume $w(\tilde{e}_1) < w_1$. As any edge in γ has weight strictly greater than $\sigma(B)$, we also have $w(\tilde{e}_1) < w_m$. If $w_m < \hat{M}$ and $w_2 = \hat{M}$, then by the



(a) $j^* = j_1$.

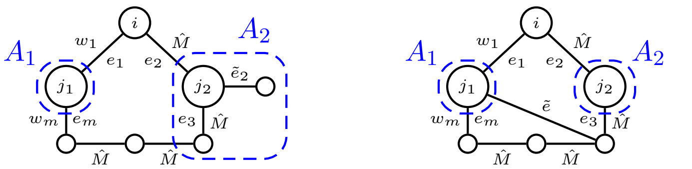
Figure 8: $w_1 < w_2 \leq \hat{M}$ and $w_m \leq \hat{M}$.

Reinforced pan condition j_2 is linked to j_1 by an edge, a contradiction. If $w_m = \hat{M}$ and $w_2 \leq \hat{M}$, then the Pan condition implies $w_1 = w_2$ (and $V(C_m) \cap V(P) = \{2\}$), a contradiction.

Let us now assume that there exists a path γ in G_B linking j_1 to j_2 and containing at least one edge in $\Sigma(B)$. We select γ as short as possible. $\{e_1\} \cup \gamma \cup \{e_2\}$ induces a cycle $C_m = \{e_1, e_2, \ldots, e_m\}$ with $m \geq 4$. Let $\tilde{e}_1$ be an edge of γ with $w(\tilde{e}_1) = \sigma(B)$. If $\tilde{e}_1$ is non-incident to j_1, then the Path condition implies $\sigma(A_2, i) = w_2 \leq \max(w_1, w(\tilde{e}_1)) = \max(\sigma(A_1, i), \sigma(B))$, a contradiction. If $\tilde{e}_1$ is incident to j_1, then $\tilde{e}_1 = e_m$. As $w_1 = \sigma(A_1, i) < \sigma(A_2, i) = w_2$ and as $w(\tilde{e}_1) = \sigma(B) < \sigma(A_2, i) = w_2$, the Cycle condition implies

$$(16) \qquad \max(w_1, w_m) < w_2 = \cdots = w_{m-1} = \hat{M} = \max_{e \in \hat{E}(C_m)} w(e).$$

Let us assume $|A_2| \geq 2$. Then, there is at least one edge $\tilde{e}_2$ in $E(A_2)$ incident to j_2 ($\tilde{e}_2$ may coincide with e_3), as represented in Figure 9a. By



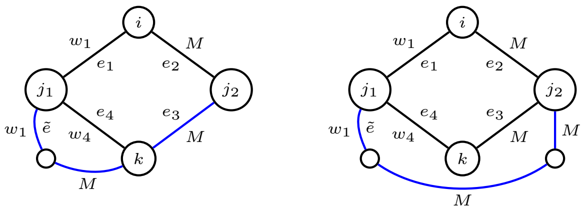
(a) $|A_2| \geq 2$. (b) $|A_2| = 1$.

Figure 9: $w_1 < \hat{M} = w_2$ and $w_m < \hat{M}$.

the Reinforced cycle condition, $\tilde{e}_2$ is linked to j_1 by an edge, and therefore A_2 is linked to A_1, a contradiction. Let us now assume $|A_2| = 1$. Then, by the Reinforced cycle condition, e_3 is linked to j_1 by an edge $\tilde{e}$. If $m \geq 5$ as represented in Figure 9b, then it contradicts the minimality of γ. Therefore, we necessarily have $m = 4$.

2. By Claim 1, we have $|A_2| = 1$. Moreover, there exists a unique edge $e_1 \in \Sigma(A_1, i)$ and setting $e_1 = \{i, j_1\}$ with $j_1 \in A_1$ and $A_2 = \{j_2\}$, there exists a vertex $k \in B$ and a cycle $C_4 = \{j_1, e_1, i, e_2, j_2, e_3, k, e_4, j_1\}$ with $w_2 = w_3 = \sigma(A_2, i)$, and $w_4 = \sigma(B)$. We set $\sigma(A_2, i) = M$.

Let us first assume that $\{j_1\}$ and A_2 belong to the same block B_j of $\mathcal{P}_{\min}(B)$. Then, there exists a path γ in G_{B_j} linking j_1 to j_2 as represented in Figure 10 and such that $w(e) > \sigma(B)$ for every edge e in γ. We select γ as short as possible. Let us assume $k \in \gamma$ (resp. $k \notin \gamma$) as represented in Figure 10a (resp. Figure 10b). Let $\tilde{e}$ be the edge of γ incident to j_1.



(a) $k \in \gamma$. (b) $k \notin \gamma$.

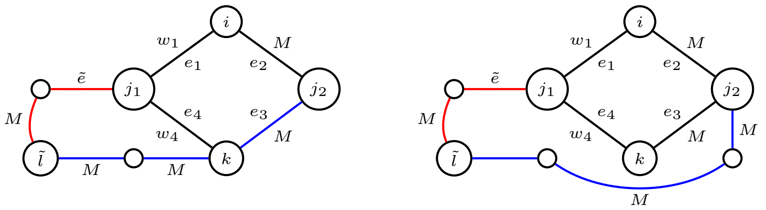
Figure 10: $w_4 = \sigma(B)$, $w_1 = \sigma(A_1, i) < M$ and $w(\tilde{e}) > \sigma(B)$.

As $w(\tilde{e}) > \sigma(B) = w_4$, the Star condition applied to $\{\tilde{e}, e_1, e_4\}$, implies $w(\tilde{e}) = w_1 > w_4$. We get

$$w_4 < w_1 = w(\tilde{e}) < w_2 = M. \tag{17}$$

Let $\tilde{C}$ be the cycle formed by $\{e_1\} \cup \{e_2\} \cup \gamma$. By (17), the Cycle condition implies $w(e) = M$ for every edge e in $\gamma \setminus \{\tilde{e}\}$ and $w(e) = w_1$ for any chord e of $\tilde{C}$ incident to j_1. In particular, e_4 cannot be a chord of $\tilde{C}$ and the situation represented in Figure 10a is not possible. Then, by (17) and the Reinforced pan condition, j_2 is linked to j_1 by an edge, a contradiction.

Let $A_{1,1}, A_{1,2}, \ldots, A_{1,p}$ be the blocks of $\mathcal{P}_{\min}(A_1)$. If $\sigma(B) < \sigma(A_1)$, then there exists a block B_j of $\mathcal{P}_{\min}(B)$ such that $A_{1,l} \subseteq B_j$ for all l, $1 \leq l \leq p$. As j_1 belongs to one block of $\mathcal{P}_{\min}(A_1)$, we get by the previous reasoning that j_2 belongs to a block of $\mathcal{P}_{\min}(B)$ distinct from B_j. We henceforth assume $\sigma(B) = \sigma(A_1)$. Let us assume the existence of a block $A_{1,l}$ of $\mathcal{P}_{\min}(A_1)$ such that $A_{1,l}$ and A_2 belong to the same block B_j of $\mathcal{P}_{\min}(B)$. By the previous reasoning, j_1 cannot belong to $A_{1,l}$. Let γ be a shortest path in G_{B_j} linking j_2 to a vertex $\tilde{l}$ in $A_{1,l}$ and such that $w(e) > \sigma(B)$ for every edge e in γ. Let γ' be a shortest path in G_{A_1} linking $\tilde{l}$ to j_1. We select $A_{1,l}$ and $\tilde{l} \in A_{1,l}$ such that γ and γ' are as short as possible. Let us note that γ cannot contain j_1, otherwise j_1 and j_2 belong to the same block of $\mathcal{P}_{\min}(B)$, a contradiction. Moreover, $\tilde{l}$ is the unique vertex common to γ and γ', otherwise it contradicts the choice of $\tilde{l}$ or $A_{1,l}$. Let us assume $k \in \gamma$ (resp. $k \notin \gamma$) as represented in Figure 11a (resp. Figure 11b). As $j_1 \notin A_{1,l}$, γ' contains at least one edge $\tilde{e}$ with weight $\sigma(A_1)$. As $w_1 < w_2 = M$ and as $w(e) > \sigma(B) = \sigma(A_1) = w(\tilde{e})$ for all edge e in γ, the Cycle condition applied to $\{e_1\} \cup \{e_2\} \cup \gamma \cup \gamma'$ implies that $\tilde{e}$ is incident to j_1 with $w(\tilde{e}) < M$ and $w(e) = M$ for any edge e in γ and in $\gamma' \setminus \{\tilde{e}\}$. As $\max(w_1, w(\tilde{e})) < M$, the

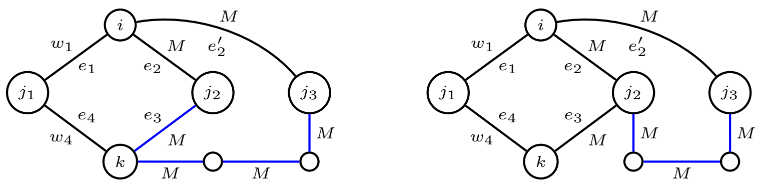


(a) $k \in \gamma$. (b) $k \notin \gamma$.

Figure 11: $w_1 = \sigma(A_1, i) < M$ and $w(\tilde{e}) = \sigma(A_1) = \sigma(B) = w_4$.

Reinforced Cycle condition implies that j_1 and j_2 are linked by an edge, a contradiction.

3. As $\sigma(A_1, i) < \sigma(A_2, i)$, the Star condition implies $\sigma(A_2, i) = \sigma(A_3, i)$ and there is a unique edge e_1 in $\Sigma(A_1, i)$. By Claim 1, we have $|A_2| = |A_3| = 1$. Let us set $e_1 = \{i, j_1\}$ with $j_1 \in A_1$, $A_2 = \{j_2\}$, and $A_3 = \{j_3\}$. By Claim 2, $\{j_1\}$ and A_2 (resp. A_3) are in distinct blocks of $\mathcal{P}_{\min}(B)$. Let us assume that A_2 and A_3 belong to the same block B_j of $\mathcal{P}_{\min}(B)$. By Claim 1, there exists a vertex $k \in B$ and a cycle $C_4 = \{j_1, e_1, i, e_2, j_2, e_3, k, e_4, j_1\}$ with $w_2 = w_3 = \sigma(A_2, i)$, and $w_4 = \sigma(B)$. Let us set $e'_2 = \{i, j_3\}$. As $w_1 = \sigma(A_1, i) < \sigma(A_2, i) = w_2$, the Star condition applied to $\{e_1, e_2, e'_2\}$ implies $w(e'_2) = \sigma(A_2, i)$. We set $\sigma(A_2, i) = M$. As $B_j \in \mathcal{P}_{\min}(B)$, there exists a path γ linking j_2 to j_3 in G_{B_j} with $w(e) > \sigma(B)$ for all edge e in γ. We select γ as short as possible. γ cannot contain j_1, otherwise j_1 and j_2 (resp. j_3) belong to the same block of $\mathcal{P}_{\min}(B)$, a contradiction. Let us assume $k \in \gamma$ (resp. $k \notin \gamma$) as represented in Figure 12a (resp. Figure 12b). As $w_1 < M$ and as $w_4 = \sigma(B) < M = w(e'_2)$, the Cycle condition applied



(a) $k \in \gamma$. (b) $k \notin \gamma$.

Figure 12: $w_1 = \sigma(A_1, i) < M$ and $w_4 = \sigma(B) < M$.

to $\{e_4\} \cup \{e_1\} \cup \{e'_2\} \cup (\gamma \setminus \{e_3\})$ (resp. $\{e_4\} \cup \{e_1\} \cup \{e'_2\} \cup \gamma \cup \{e_3\}$) implies $w(e) = M$ for all e in γ. As $w_4 < M$, the Reinforced pan condition applied to the cycle formed by $\{e_2\} \cup \{e'_2\} \cup \gamma$ and the path formed by e_4 (resp. $\{e_4\} \cup \{e_3\}$) implies that j_3 is linked to a vertex in $\{j_1, k\}$ (resp. $\{j_1, k, j_2\}$) by an edge e. e is necessarily incident to k as represented in

Figure 13, otherwise it would link j_1 or A_2 to A_3. By the Star condition

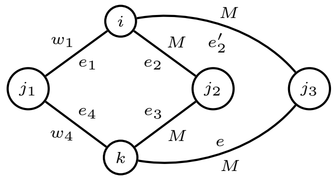


Figure 13: $w_1 < M$ and $w_4 < M$.

applied to $\{e, e_3, e_4\}$, e has weight M. Then, by the Reinforced adjacent cycles condition, there exists an edge linking j_2 to j_3, a contradiction. $\square$

Proposition 14. *Let us assume that the Path, Star, Cycle, Reinforced cycle and Reinforced pan conditions are satisfied. Let us consider $i \in N$ and $A \subseteq B \subseteq N \setminus \{i\}$ with $A \cup \{i\}$ and B in $\mathcal{F}$. Let us assume $A \notin \mathcal{F}$, and let A_1, A_2 be two connected components of A with $\sigma(A_1, i) = \sigma(A_2, i) = M$. If $|A_1| \geq 2$ and if $\sigma(A_1) < M$, then*

1. *$|A_2| = 1$ and $\sigma(B) = \sigma(A_1)$. Moreover, there exists a unique edge $\tilde{e}_1$ in $\Sigma(B)$ and setting $\tilde{e}_1 = \{j_1, k_1\}$ with j_1 and k_1 in A_1 and $A_2 = \{j_2\}$, there exist a vertex $k_2 \in B$, and a cycle $C_5 = \{k_1, \tilde{e}_1, j_1, e_2, i, e_3, j_2, e_4, k_2, e_5, k_1\}$ with $w(\tilde{e}_1) = \sigma(B)$ and $w_2 = w_3 = w_4 = w_5 = M$.*

2. *$\{j_1\}$ and A_2 are in distinct blocks of $\mathcal{P}_{\min}(B)$. Moreover, there is a unique block in $\mathcal{P}_{\min}(A_1)$ linked to i and this block contains j_1 and belongs to a block of $\mathcal{P}_{\min}(B)$ different from the one containing A_2.*

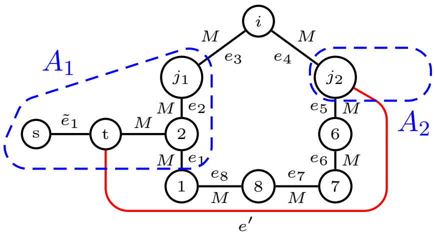
3. *If the Reinforced adjacent cycles condition is satisfied, then A_2 is unique.*

Proof. **1.** Let $\tilde{e}_1 = \{s, t\}$ be an edge in $E(A_1)$ with weight $w(\tilde{e}_1) < M$ (such an edge exists as $\sigma(A_1) < M$). Let P be a shortest path in G_{A_1} connecting $\tilde{e}_1$ to a vertex j_1 in A_1 such that $e'_1 = \{i, j_1\}$ belongs to $\Sigma(A_1, i)$. We select $\tilde{e}_1$ such that P is as short as possible (P may be reduced to j_1). We can assume $t \in P$. We have $w(e) \geq M$ for any edge e in P, otherwise we can change $\tilde{e}_1$. As $w(\tilde{e}_1) < M = w(e'_1)$, the Path condition implies $w(e) \leq \max(w(\tilde{e}_1), w(e'_1)) = M$ for any edge e in P. Therefore, we have

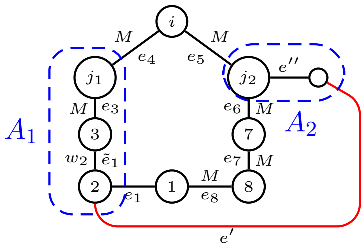
$$w(e) = M \text{ for any edge } e \text{ in } P. \tag{18}$$

Let $e'_2 = \{i, j_2\}$ be an edge in $\Sigma(A_2, i)$. Let γ be a shortest path in G_B linking j_2 to a vertex j^* in $V(P) \cup \{s\}$. Let us denote by P' the path formed by $P \cup \{\tilde{e}_1\}$ and by P'_{j_1,j^*} the subpath of P' linking j_1 to j^*. Then e'_1, e'_2, γ, and P'_{j_1,j^*} form a cycle $C_m = \{e_1, e_2, \dots, e_m\}$ with $m \geq 4$. We select j_1, P, and γ such that C_m is as short as possible.

Let us first assume $j^* \in V(P)$. Let us denote by P_{j_1,j^*} (resp. P_{t,j^*}) the subpath of P linking j_1 (resp. t) to j^*. The Path condition applied to $\{\tilde{e}_1\} \cup P_{t,j^*} \cup \gamma \cup \{e'_2\}$ implies $w(e) \leq \max(w(\tilde{e}_1), w(e'_2)) = M$ for any edge e in γ. Let us assume that there exists an edge e in γ with $w(e) < M$. If P_{t,j^*} contains at least one edge, then (18) and the Path condition imply $M \leq \max(w(\tilde{e}_1), w(e))$, a contradiction. Otherwise, P_{t,j^*} reduces to j^* and $\tilde{e}_1$ is incident to j^*. Then, by the Star condition, the edge in γ incident to j^* has weight M, and the Path condition also implies $M \leq \max(w(\tilde{e}_1), w(e))$, a contradiction. Hence, every edge in γ has weight M and C_m is a constant cycle. We can assume $j^* = 2$ and $2 \leq j_1 < i < j_2 \leq m$ as represented in Figure 14a, after renumbering if necessary. By the Reinforced pan condition,



(a) e' links $\{\tilde{e}_1\} \cup P_{t,2}$ to j_2.

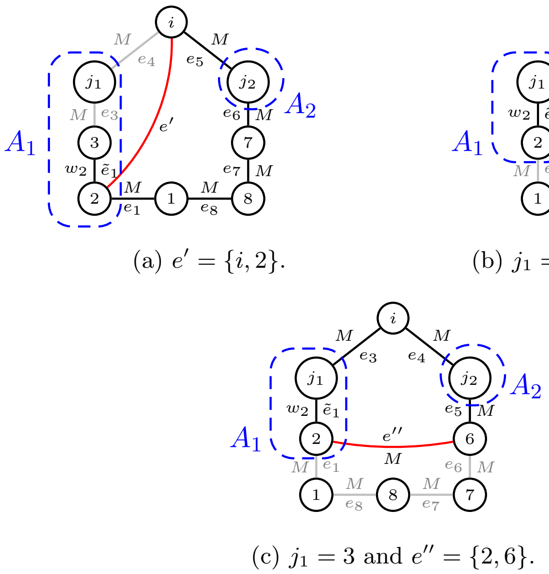


(b) $\tilde{e}_1 = e_2$ and $3 \leq j_1 < i < j_2 \leq m$.

Figure 14: $w(\tilde{e}_1) < M$.

there exists an edge e' in E linking $\{\tilde{e}_1\} \cup P_{t,2}$ to j_2, and therefore linking A_1 to A_2, a contradiction.

Let us now assume $j^* = s$. Then, we necessarily have $m \geq 5$. We can assume $\tilde{e}_1 = e_2$ with $s = 2$ and $t = 3$, and $3 \leq j_1 < i < j_2 \leq m$ as represented in Figure 14b. Moreover, by the Cycle condition, we have $w_2 < w_3 = \cdots = w_m = M$ and $w_1 \leq M$. If $|A_2| \geq 2$, then there exists an edge e'' in $E(A_2)$ incident to j_2. By the Reinforced cycle condition, there exists an edge linking e_2 to e'', and therefore linking A_1 to A_2, a contradiction. Thus, we necessarily have $|A_2| = 1$ with $A_2 = \{j_2\}$. If $w_1 < M$, then we have $\max(w_1, w_2) < M$ and by the Reinforced Cycle condition, the edge $\{i, j_2\}$ is linked to 2 by an edge e'. If $e' = \{j_2, 2\}$, then e' links A_1 to A_2, a contradiction. If $e' = \{i, 2\}$, then it contradicts the choice of j_1 and the minimality of C_m. Hence, we have $w_1 = M$. As $w_2 < M$, the Reinforced Cycle condition implies the existence of an edge e' (resp. e'') linking the edge $\{i, j_2\}$ (resp. $\{j_2, j_2+1\}$) to $\tilde{e}_1$. e' (resp. e'') is necessarily incident to i (resp. j_2+1), otherwise it would link A_2 to A_1. If e' is incident to 2 as represented in Figure 15a, then it contradicts the choice of j_1 (and the minimality of C_m). Hence, we have $e' = \{i, 3\}$ and this implies $j_1 = 3$ (otherwise it still contradicts the choice of j_1). If $e'' = \{3, j_2+1\}$ as represented in Figure 15b, then it contradicts the choice of γ. Thus, we necessarily have $e'' = \{2, j_2+1\}$ as represented in Figure 15c. If $m \geq 6$, then it contradicts the minimality of γ. Therefore, we necessarily have $m = 5$.



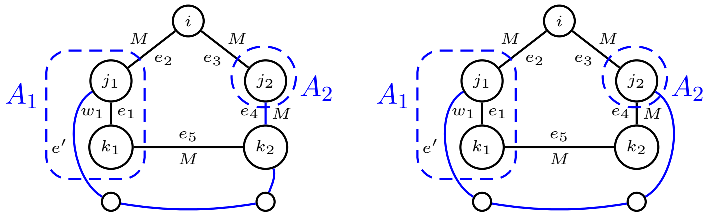
(a) $e' = \{i, 2\}$. (b) $j_1 = 3$ and $e'' = \{3, 6\}$. (c) $j_1 = 3$ and $e'' = \{2, 6\}$.

Figure 15: $w(\tilde{e}_1) = w_2 < M$ and $w_1 = M$.

Let us assume that there exist an edge $\tilde{e} \neq \tilde{e}_1$ in $E(B)$ with weight $w(\tilde{e}) \leq w(\tilde{e}_1)$. By the Cycle condition, any chord of C_5 has weight M. Thus, $\tilde{e}$ cannot belong to $\hat{E}(C_5)$. Let $\tilde{P}$ be a shortest path in G_B linking $\tilde{e}$ to C_5. As $w(\tilde{e}) \leq w(\tilde{e}_1) < M$, the Pan condition applied to the pan formed by C_5 and $\tilde{P} \cup \{\tilde{e}\}$ implies that one of the edges in $E(C_5)$ adjacent to $\tilde{e}_1$ should have weight $w(\tilde{e}_1)$, a contradiction. Therefore, we necessarily have $w(\tilde{e}_1) = \sigma(B) = \sigma(A_1)$ and $\Sigma(B) = \Sigma(A_1) = \{\tilde{e}_1\}$.

2. By Claim 1, we have $|A_2| = 1$ and $\sigma(B) = \sigma(A_1)$. Moreover, there exists a unique edge e_1 in $\Sigma(B) = \Sigma(A_1)$ and setting $e_1 = \{j_1, k_1\}$ with j_1 and k_1 in A_1 and $A_2 = \{j_2\}$, there exists a vertex k_2 in B and a cycle $C_5 = \{k_1, e_1, j_1, e_2, i, e_3, j_2, e_4, k_2, e_5, k_1\}$ with $w_1 = \sigma(A_1)$ and $w_2 = \cdots = w_5 = M$.

Let us first assume that $\{j_1\}$ and A_2 belong to the same block B_j of $\mathcal{P}_{\min}(B)$. Then, there exists a path γ in G_{B_j} linking j_1 to j_2 as represented in Figure 16 and such that $w(e) > \sigma(B)$ for every edge e in γ. We select γ as short as possible. Let $\tilde{C}$ be the cycle formed by $\{e_2\} \cup \{e_3\} \cup \gamma$. Any edge in $E(\tilde{C})$ has a weight strictly greater than $\sigma(A_1)$. By the Cycle condition, e_1 cannot be a chord of $\tilde{C}$. Thus, γ cannot contain k_1 but may contain k_2. Let us assume $k_2 \in \gamma$ (resp. $k_2 \notin \gamma$) as represented in Figure 16a (resp. Figure 16b). Let e' be the edge of γ incident to j_1. As $w_1 < M = w_2$, the Star condition applied to $\{e', e_1, e_2\}$ implies $w(e') = M$. Then, the Cycle condition applied to the cycle formed by $\{e_1\} \cup \{e_5\} \cup \gamma$ (resp. $\{e_1\} \cup \{e_5\} \cup \{e_4\} \cup \gamma$) implies $w(e) = M$ for any edge e in γ. Hence, $\tilde{C}$ is a constant cycle. As $w_1 = \sigma(A_1) < M$, the Reinforced pan condition implies

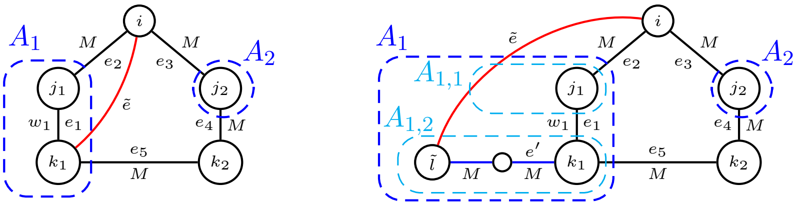


(a) $k_2 \in \gamma$. (b) $k_2 \notin \gamma$.

Figure 16: $w_1 = \sigma(A_1) < M = w_2 = \cdots = w_5$.

the existence of an edge linking j_2 to e_1, a contradiction.

As $\{j_1, k_1\}$ is the unique edge in $\Sigma(A_1)$, $\mathcal{P}_{\min}(A_1)$ contains at most two blocks. Let $A_{1,1}, \ldots, A_{1,p}$ with $p \leq 2$ be the blocks of $\mathcal{P}_{\min}(A_1)$ linked to i and let us assume $j_1 \in A_{1,1}$. Let us assume $p = 2$. Then, $A_{1,2}$ necessarily contains k_1. Let $\tilde{e} = \{i, \tilde{l}\}$ be an edge in $E(A_1, i)$ linking i to $A_{1,2}$. If $\tilde{l} = k_1$ as represented in Figure 17a, then as $w_1 < M = w_5$ the Star condition applied to $\{\tilde{e}, e_1, e_5\}$ implies $w(\tilde{e}) = M$. As $w_1 < M$, the Reinforced pan



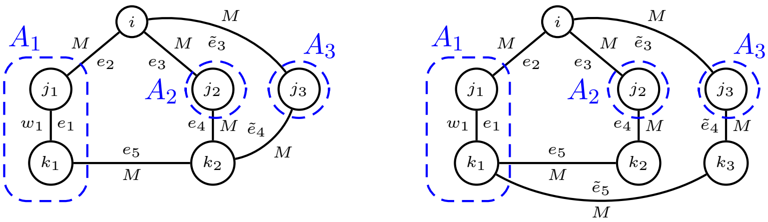
(a) $\tilde{l} = k_1$. (b) $\tilde{l} \neq k_1$.

Figure 17: $w_1 = \sigma(A_1) < M$.

condition applied to the pan formed by $\{e_3, e_4, e_5, \tilde{e}\}$ and e_1 implies that j_2 is linked to j_1 or k_1, a contradiction. We henceforth assume $\tilde{l} \neq k_1$. Let γ' be a shortest path in $G_{A_{1,2}}$ linking $\tilde{l}$ to k_1 as represented in Figure 17b. Let e' be the edge of γ' incident to k_1. As $w_1 < M = w_5$, the Star condition applied to $\{e_1, e_5, e'\}$ implies $w(e') = M$. Then, by the Cycle condition applied to the cycle formed by $\{\tilde{e}, e_2, e_1\} \cup \gamma'$, we get $w(\tilde{e}) = M$ and $w(e) = M$ for every edge in γ'. Finally, by the Reinforced pan condition applied to the pan formed by $\{\tilde{e}, e_3, e_4, e_5\} \cup \gamma'$ and e_1, there is an edge linking j_2 to j_1 or k_1, a contradiction. Hence, we necessarily have $p = 1$ and $A_{1,1}$ is the unique block of $\mathcal{P}_{\min}(A_1)$ linked to i. As $j_1 \in A_{1,1}$, $A_{1,1}$ and A_2 are in distinct blocks of $\mathcal{P}_{\min}(B)$ by the previous reasoning.

3. By contradiction, let A_3 be a third connected component of A satisfying $\sigma(A_3, i) = M$. By Claim 1, we have $|A_3| = 1$. Moreover, there exists a unique edge e_1 in $\Sigma(B) = \Sigma(A_1)$ and setting $e_1 = \{j_1, k_1\}$ with j_1 and k_1 in A_1, and $A_2 = \{j_2\}$ (resp. $A_3 = \{j_3\}$), there exists a vertex

k_2 (resp. k_3) in B and a cycle $C_5 = \{k_1, e_1, j_1, e_2, i, e_3, j_2, e_4, k_2, e_5, k_1\}$ (resp. $\tilde{C}_5 = \{k_1, e_1, j_1, e_2, i, \tilde{e}_3, j_3, \tilde{e}_4, k_3, \tilde{e}_5, k_1\}$) with $w_1 = \sigma(A_1)$ and $w_2 = w_3 = \cdots = w_5 = M$ (resp. $w_2 = w(\tilde{e}_3) = \cdots = w(\tilde{e}_5) = M$). We may have $k_2 = k_3$ and $e_5 = \tilde{e}_5$. Let us assume $k_2 \neq k_3$ as represented in Figure 18b. As $w_1 < M$, the Reinforced pan condition applied to



(a) $k_2 = k_3$.

(b) $k_2 \neq k_3$.

Figure 18: $w_1 = \sigma(A_1) < M$.

$\{e_5, e_4, e_3, \tilde{e}_3, \tilde{e}_4, \tilde{e}_5\}$ and e_1 implies that j_2 is linked to j_1 or k_1, a contradiction. Let us now assume $k_2 = k_3$ as represented in Figure 18a. $\{j_1, j_2\}$ and $\{k_1, j_2\}$ (resp. $\{j_1, j_3\}$ and $\{k_1, j_3\}$) cannot exist, otherwise A_1 and A_2 (resp. A_3) are not distinct components. Then, by the Reinforced adjacent cycles condition, there exists an edge linking j_2 to j_3, a contradiction. $\square$

Lemma 15. *Let us assume that the Path, Star, Cycle, Pan, Adjacent cycles, Reinforced cycle, Reinforced pan, and Reinforced adjacent cycles conditions are satisfied. Let us consider $i \in N$ and $A \subseteq B \subseteq N \setminus \{i\}$ with $A \cup \{i\}$ and B in $\mathcal{F}$. Let us assume $A \notin \mathcal{F}$, and let $A_1, A_2, \ldots, A_p$ with $p \geq 2$ be the connected components of A. We set $K_A = \{2, \ldots, p\}$. Let us assume that one of the following conditions is satisfied:*

1. $|A_1| = 1$ *and* $\sigma(B) \leq \sigma(A_1, i) < \sigma(A_2, i) \leq \cdots \leq \sigma(A_p, i)$.
2. $|A_1| \geq 2$ *and either*

$$\sigma(B) \leq \sigma(A_1) \leq \sigma(A_1, i) < \sigma(A_2, i) \leq \cdots \leq \sigma(A_p, i), \tag{19}$$

or

$$\sigma(B) \leq \sigma(A_1) < \sigma(A_1, i) = \sigma(A_2, i) = \cdots = \sigma(A_p, i). \tag{20}$$

Let $A_{1,1}, A_{1,2}, \ldots, A_{1,k}$ (resp. $B_1, B_2, \ldots, B_q$) be the blocks of $\mathcal{P}_{\min}(A_1)$ (resp. $\mathcal{P}_{\min}(B)$). Let J_{A_1} (resp. J_B) be the set of indices $j \in \{1, \ldots, k\}$ (resp. $j \in \{1, \ldots, q\}$) such that $A_{1,j}$ (resp. B_j) is linked to i by an edge e in $E(A_1, i)$ (resp. $E(B, i)$) with weight $w(e) > \sigma(A_1)$ (resp. $w(e) > \sigma(B)$). We set $J_{A_1} = \emptyset$ if $|A_1| = 1$. Then, we have

$$A_{1,j} \subseteq B_j, \forall j \in J_{A_1}, \tag{21}$$

(22) $$|A_j| = 1 \text{ and } A_j \subseteq B_j,\ \forall j \in K_A,$$

with $J_{A_1} \cup K_A \subseteq J_B$ and $J_{A_1} \cap K_A = \emptyset$, after renumbering if necessary. Moreover, if (20) is satisfied, then $|J_{A_1}| = 1$ and $|K_A| = 1$.

Proof. Let us first assume $|A_1| = 1$. Then, $J_{A_1} = \emptyset$ and (21) is satisfied. Claims 1 and 3 of Proposition 13 applied to $A = A_1 \cup \bigcup_{k\in K_A} A_k$ and B imply (22). Let us now assume $|A_1| \geq 2$. As $\sigma(A_1, i) \geq \sigma(A_1)$, we have $\sigma(A_1 \cup \{i\}) = \sigma(A_1)$. Then $A' = \bigcup_{j\in J_{A_1}} A_{1,j}$ is a block of $\mathcal{P}_{\min}(A_1 \cup \{i\})_{|A_1}$ and $\mathcal{P}_{\min}(A_1)_{|A'} = \bigcup_{j\in J_{A_1}} \{A_{1,j}\}$. By Theorem 6, there is inheritance of $\mathcal{F}$-convexity for superadditive games with the correspondence $\mathcal{P}_{\min}$. In particular, there is inheritance of $\mathcal{F}$-convexity for unanimity games. By Corollary 2, there is inheritance of superadditivity with the correspondence $\mathcal{P}_{\min}$. Then, Theorem 4 implies $\mathcal{P}_{\min}(A_1)_{|A'} = \mathcal{P}_{\min}(B)_{|A'}$ and therefore (21) is satisfied. If (19) is satisfied, then Claims 1, 2, and 3 of Proposition 13 imply (22). If (20) is satisfied, then Claims 1, 2, and 3 of Proposition 14 imply (22) and $|J_{A_1}| = |K_A| = 1$. □

Lemma 16. *Let us assume that the Path, Star, and Cycle conditions are satisfied. Let us consider $i \in N$ and $A \subseteq B \subseteq N \setminus \{i\}$ with $A \cup \{i\}$ and B in $\mathcal{F}$. Let $A_1, A_2, \ldots, A_p$ with $p \geq 1$ be the connected components of A. Let us assume $\sigma(A_k, i) \leq \sigma(B)$ for all k, $1 \leq k \leq p$, and $\sigma(A_k) = \sigma(B)$ for all k with $|A_k| \geq 2$. Then, each block of $\mathcal{P}_{\min}(B)$ meets at most one subset A_k of A.*

Proof. If $p = 1$, then the result is trivially satisfied. Let us assume $p \geq 2$.

Let us consider a connected component A_k with $1 \leq k \leq p$. Let $e_1 = \{i, j\}$ be an edge in $\Sigma(A_k, i)$. Let e_m be an edge in $\Sigma(B)$ and let γ be a shortest path in G_B linking e_m to j as represented in Figure 19. We select

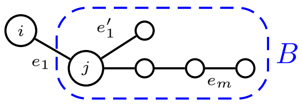


Figure 19: $w_1 = \sigma(A_k, i) \leq \sigma(B) = w_m$.

e_m such that γ is as short as possible. Then, γ necessarily reduces to j, otherwise the Path condition applied to $\{e_1\} \cup \gamma \cup \{e_m\}$ implies $w(e) \leq \max(w_1, w_m) = \max(\sigma(A_k, i), \sigma(B)) = \sigma(B)$ and therefore $w(e) = \sigma(B)$ for any edge e in γ, contradicting the choice of e_m. If there exists an edge $e'_1 \neq e_m$ in $E(B)$ incident to j, then the Star condition applied to $\{e_1, e'_1, e_m\}$ implies $w(e'_1) \leq w_m = \sigma(B)$. We get that any edge in $E(B)$ incident to j has weight $\sigma(B)$. In particular, $\{j\}$ corresponds to a block of $\mathcal{P}_{\min}(B)$. Thus, if $|A_k| = 1$, then $A_k = \{j\}$ is a block of $\mathcal{P}_{\min}(B)$.

Let us now consider two connected components A_j and A_k with $1 \leq j < k \leq p$, and let us assume that there is a block $F \in \mathcal{P}_{\min}(B)$ such that

$F \cap A_j \neq \emptyset$ and $F \cap A_k \neq \emptyset$. Let $e_1 = \{i, j_1\}$ (resp. $e_2 = \{i, j_2\}$) be an edge in $\Sigma(A_j, i)$ (resp. $\Sigma(A_k, i)$). By the previous reasoning, we necessarily have $|A_j| \geq 2$ and $|A_k| \geq 2$. Moreover, any edge in $E(B)$ incident to j_1 (resp. j_2) has weight $\sigma(B)$ and $\{j_1\}$ (resp. $\{j_2\}$) is a block of $\mathcal{P}_{\min}(B)$ distinct from F. Let k_1 (resp. k_2) be a vertex of $A_j \cap F$ (resp. $A_k \cap F$) and let γ_1 (resp. γ_2) be a shortest path linking j_1 to k_1 in G_{A_j} (resp. j_2 to k_2 in G_{A_k}). We select k_1 and k_2 such that γ_1 and γ_2 are as short as possible. As $F \in \mathcal{P}_{\min}(B)$ and as $\{j_1\}$ and $\{j_2\}$ are also blocks of $\mathcal{P}_{\min}(B)$, γ_1 (resp. γ_2) contains at least one edge. Let $\tilde{e}_1$ (resp. $\tilde{e}_2$) be the edge of γ_1 (resp. γ_2) incident to j_1 (resp. j_2). As $F \in \mathcal{P}_{\min}(B)$, there exists a path $\tilde{\gamma}$ in G_F from k_1 to k_2 with $w(e) > \sigma(B)$ for each edge e in $\tilde{\gamma}$. We select $\tilde{\gamma}$ as short as possible. We obtain a simple cycle $C_m = \{e_1\} \cup \gamma_1 \cup \tilde{\gamma} \cup \gamma_2 \cup \{e_2\}$ as represented in Figure 20. Then, e_1, e_2, $\tilde{e}_1$, and $\tilde{e}_2$ are four edges with weights strictly

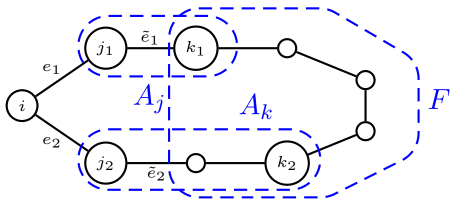


Figure 20: $C_m = \{e_1\} \cup \gamma_j \cup \tilde{\gamma} \cup \gamma_k \cup \{e_2\}$.

smaller than the edges of $\tilde{\gamma}$, contradicting the Cycle condition. Therefore, each block $F \in \mathcal{P}_{\min}(B)$ meets at most one subset A_k of A. $\square$

Lemma 17. *Let us assume that the Path, Star, Cycle, Pan, Reinforced pan, and Reinforced cycle conditions are satisfied. Let us consider $i \in N$ and $A \subseteq B \subseteq N \setminus \{i\}$ with $A \cup \{i\}$ and B in $\mathcal{F}$. Let $A_1, A_2, \ldots, A_p$ with $p \geq 2$ be the connected components of A. Let us assume that one of the following conditions is satisfied*

$$\sigma(A_k, i) = M < \sigma(B),\ \forall k,\ 1 \leq k \leq p, \tag{23}$$

or

$$\sigma(A_1, i) < \sigma(B) \text{ and } \sigma(A_1, i) < \sigma(A_k, i) = M,\ \forall k,\ 2 \leq k \leq p. \tag{24}$$

Then, for every convex game (N, v), we have

$$\sum_{F \in \mathcal{P}_{\min}(B)_{|A}} v(F) = \sum_{k=1}^{p} \overline{v}(A_k). \tag{25}$$

Proof. Let us first assume (23) satisfied. Then, Lemma 8 implies

$$\sigma(A_k, i) < \sigma(A_k) = \sigma(B),\ \forall k,\ 1 \leq k \leq p, \ \text{ s.t. } |A_k| \geq 2. \tag{26}$$

By (23), (26), and Lemma 16, we get

$$\mathcal{P}_{\min}(B)_{|A} = \{\mathcal{P}_{\min}(B)_{|A_1}, \mathcal{P}_{\min}(B)_{|A_2}, \dots, \mathcal{P}_{\min}(B)_{|A_p}\}. \tag{27}$$

Let us consider a subset A_k with $1 \leq k \leq p$. As $\sigma(A_1, i) < \sigma(B)$, there exists an edge e_0 in $E(A_1, i)$ connected to B with weight $w(e_0) < \sigma(B)$. If $|A_k| \geq 2$, then Lemma 7 applied to $A_k \subseteq B$ implies $\mathcal{P}_{\min}(B)_{|A_k} = \mathcal{P}_{\min}(A_k)$. If $|A_k| = 1$, then we obviously have $\mathcal{P}_{\min}(B)_{|A_k} = \mathcal{P}_{\min}(A_k)$. Therefore, (27) implies (25).

Let us now assume (24) satisfied. As $\sigma(A_1, i) < \sigma(B)$, the previous reasoning applied to $A_1 \subseteq B$ gives

$$\sum_{F \in \mathcal{P}_{\min}(B)_{|A_1}} v(F) = \overline{v}(A_1). \tag{28}$$

Let us assume $\sigma(B) < M$. By (24), we get

$$\sigma(A_1, i) < \sigma(B) < M = \sigma(A_k, i), \, \forall k, \, 2 \leq k \leq p. \tag{29}$$

Let K_A be the set of indices $k \in \{2, \dots, p\}$. By (29), Claims 1, 2, and 3 of Proposition 13 applied to $A_1 \cup \bigcup_{k \in K_A} A_k$ and B imply

$$|A_k| = 1 \text{ for all } k \in K_A, \tag{30}$$

and each subset A_k is included in a distinct block of $\mathcal{P}_{\min}(B)$. Moreover, the blocks of $\mathcal{P}_{\min}(A_1)$ belong to blocks of $\mathcal{P}_{\min}(B)$ different from the ones containing the subsets A_k with $k \in K_A$. In particular, this implies that A_1 has always an empty intersection with the block of $\mathcal{P}_{\min}(B)$ containing a subset A_k with $k \in K_A$. We get

$$\sum_{F \in \mathcal{P}_{\min}(B)_{|A}} v(F) = \sum_{F \in \mathcal{P}_{\min}(B)_{|A_1}} v(F) + \sum_{k \in K_A} v(A_k). \tag{31}$$

By (30), we have $v(A_k) = \overline{v}(A_k)$ for all $k \in K_A$. Then, (28) and (31) give (25).

Let us now assume $M \leq \sigma(B)$. By (24), we get

$$\sigma(A_1, i) < \sigma(A_k, i) \leq M \leq \sigma(B), \, \forall k, \, 2 \leq k \leq p. \tag{32}$$

(32) and Lemma 8 imply

$$\sigma(A_k, i) \leq \sigma(A_k) = \sigma(B), \, \forall k, \, 1 \leq k \leq p, \text{ s.t. } |A_k| \geq 2. \tag{33}$$

Then, (32), (33), and Lemma 16 applied to $\bigcup_{k=1}^{p} A_k$ and B imply (27) and we can conclude as in the first case. $\square$

For any $i \in N$ and for any subset $A \subseteq N \setminus \{i\}$, we define $M(A, i) = \max_{e \in E(A,i)} w(e)$.

Lemma 18. *Let us assume that the Star and Path conditions are satisfied. Let us consider $i \in N$ and $A \subseteq N \setminus \{i\}$ with $A \notin \mathcal{F}$ but $A \cup \{i\}$ in $\mathcal{F}$. Let $A_1, A_2, \ldots, A_p$ be the connected components of A. We assume*

$$\sigma(A,i) = \sigma(A_1,i) \leq \sigma(A_2,i) = \ldots = \sigma(A_p,i) = M(A,i). \tag{34}$$

1. *If $\sigma(A,i) < M(A,i)$, then we have*

$$M(A,i) \leq \sigma(A_k), \forall k,\ 2 \leq k \leq p,\ \text{ s.t. } |A_k| \geq 2. \tag{35}$$

2. *If $\sigma(A,i) = M(A,i)$ and if $|A_k| \geq 2$ for at least one index k with $1 \leq k \leq p$, then we have*

$$\max(\sigma(A_1), M(A,i)) \leq \sigma(A_k), \forall k,\ 2 \leq k \leq p,\ \text{ s.t. } |A_k| \geq 2, \tag{36}$$

after renumbering if necessary.

Proof. Let us first assume $\sigma(A,i) < M(A,i)$. By the Star condition, there is a unique edge e_0 in $E(A,i)$ with $w(e_0) = \sigma(A,i)$ and $w(e) = M(A,i)$ for all $e \in E(A,i) \setminus \{e_0\}$. By (34), we have $e_0 \in E(A_1,i)$. Let us consider a subset A_k with $2 \leq k \leq p$ and $|A_k| \geq 2$. Let $\tilde{e} = \{i, j_k\}$ be an edge in $E(A_k, i)$, and let e_k be an edge in $\Sigma(A_k)$. Let γ_k be a path in G_{A_k} linking e_k to j_k as represented in Figure 21 (γ_k may be reduced to j_k). The Path condition

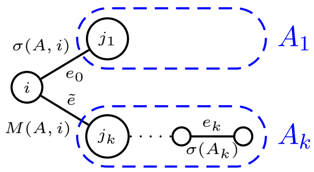


Figure 21: Path induced by $e_0 \cup \tilde{e} \cup \gamma_k \cup e_k$.

applied to $e_0 \cup \tilde{e} \cup \gamma_k \cup e_k$ implies $w(\tilde{e}) \leq \max(w(e_0), w(e_k))$. As $w(e_0) = \sigma(A,i) < M(A,i) = w(\tilde{e})$ and as $w(e_k) = \sigma(A_k)$, we get $M(A,i) \leq \sigma(A_k)$. This implies (35).

Let us now assume $\sigma(A,i) = M(A,i)$. By (34), we get $\sigma(A_k,i) = M(A,i)$ for all k, $1 \leq k \leq p$. We can assume $\sigma(A_1) \leq \sigma(A_k)$ for all k with $|A_k| \geq 2$, after renumbering if necessary. Let us consider a subset A_k with $2 \leq k \leq p$ and $|A_k| \geq 2$. Let e_1 be an edge in $\Sigma(A_1)$ and let e_k be an edge in $\Sigma(A_k)$. There is a path linking e_1 to e_k and passing through i. As the edges incident to i have weight $M(A,i)$, the Path condition implies $M(A,i) \leq \max(\sigma(A_1), \sigma(A_k))$ and (36) is satisfied. ∎

Proof of Theorem 12. Let us consider a convex game (N,v), $i \in N$ and subsets $A \subseteq B \subseteq N \setminus \{i\}$. We have to prove that the following inequality is satisfied:

$$\Delta_i \hat{v}(B) \geq \Delta_i \hat{v}(A). \tag{37}$$

We first show that we can w.l.o.g. make several restrictions on the sets A and B. By Corollary 2 and Theorem 6, $(N, \overline{v})$ and $(N, \hat{v})$ are superadditive and $\mathcal{F}$-convex. Let $A_1, A_2, \ldots, A_p$ (resp. $B_1, B_2, \ldots, B_q$) with $p \geq 1$ (resp. $q \geq 1$) be the connected components of A (resp. B). Let J_A (resp. $\overline{J_A}$) be the set of indices $j \in \{1, \ldots, p\}$ such that A_j is linked (resp. non-linked) to i by an edge. If $J_A = \emptyset$, then $\Delta_i \hat{v}(A) = \hat{v}(\{i\})$ and (37) is satisfied by the superadditivity of $(N, \hat{v})$. If $J_A \neq \emptyset$, then setting $A' = \bigcup_{j \in J_A} A_j$, we get

$$\Delta_i \hat{v}(A) = \overline{v}(A' \cup \{i\}) - \sum_{j \in J_A} \overline{v}(A_j) = \hat{v}(A' \cup \{i\}) - \hat{v}(A') = \Delta_i \hat{v}(A'). \tag{38}$$

Therefore we can substitute A for A' and assume $A \cup \{i\}$ connected. Similarly, we can assume $B \cup \{i\}$ connected. Moreover, we can assume $p \geq 2$, otherwise the superadditivity and $\mathcal{F}$-convexity of $(N, \hat{v})$ implies (37). We can also w.l.o.g. assume

$$A \cap B_j \neq \emptyset, \; \forall j, \; 1 \leq j \leq q. \tag{39}$$

Indeed, let us assume $A \cap B_j \neq \emptyset$ for all j, $1 \leq j \leq m$ and $A \cap B_j = \emptyset$ for all j, $m+1 \leq j \leq q$ for some index m with $1 \leq m \leq q$. Setting $B' = \bigcup_{j=1}^{m} B_j$ and $B'' = \bigcup_{j=m+1}^{q} B_j$, we have

$$\Delta_i \hat{v}(B) = \overline{v}(B' \cup B'' \cup \{i\}) - \sum_{j=1}^{q} \overline{v}(B_j). \tag{40}$$

By the superadditivity of $(N, \overline{v})$, we have $\overline{v}(B' \cup B'' \cup \{i\}) \geq \overline{v}(B' \cup \{i\}) + \sum_{j=m+1}^{q} \overline{v}(B_j)$. Then (40) implies

$$\Delta_i \hat{v}(B) \geq \overline{v}(B' \cup \{i\}) - \sum_{j=1}^{m} \overline{v}(B_j) = \hat{v}(B' \cup \{i\}) - \hat{v}(B') = \Delta_i \hat{v}(B'). \tag{41}$$

By (41) and as $A \subseteq B'$, (37) is satisfied if the following inequality is satisfied

$$\Delta_i \hat{v}(B') \geq \Delta_i \hat{v}(A). \tag{42}$$

Therefore we can replace B by B'.

Let us first assume $|B \setminus A| = 1$. Then, we have $B = A \cup \{j\}$ with $j \in N \setminus (A \cup \{i\})$ and (37) is equivalent to

$$\hat{v}(A \cup \{i, j\}) - \hat{v}(A \cup \{j\}) \geq \hat{v}(A \cup \{i\}) - \hat{v}(A). \tag{43}$$

We can assume j linked to A by at least one edge, otherwise (43) is satisfied by superadditivity of $(N, \hat{v})$. Let $A_1, A_2, \ldots, A_{p'}$ with $1 \leq p' \leq p$ be the components of A linked to j by an edge. Then, the connected components of B are $B_1, B_2, \ldots, B_{p-p'+1}$ with $B_1 = \bigcup_{k=1}^{p'} A_k \cup \{j\}$, and $B_l = A_{p'+l-1}$ for all l, $2 \leq l \leq p - p' + 1$. (43) is equivalent to

$$\overline{v}(B \cup \{i\}) - \overline{v}(B_1) \geq \overline{v}(A \cup \{i\}) - \sum_{k=1}^{p'} \overline{v}(A_k). \tag{44}$$

As $\bigcup_{k=1}^{p'} A_k \cup \{i\}$, $A \cup \{i\}$, $B_1 \cup \{i\}$, and $B \cup \{i\}$ are connected, the $\mathcal{F}$-convexity of $(N, \overline{v})$ implies

$$\overline{v}(B \cup \{i\}) + \overline{v}\left(\bigcup_{k=1}^{p'} A_k \cup \{i\}\right) \geq \overline{v}(A \cup \{i\}) + \overline{v}(B_1 \cup \{i\}). \tag{45}$$

By (45), (44) is satisfied if the following inequality is satisfied:

$$\overline{v}(B_1 \cup \{i\}) - \overline{v}(B_1) \geq \overline{v}\left(\bigcup_{k=1}^{p'} A_k \cup \{i\}\right) - \sum_{k=1}^{p'} \overline{v}(A_k). \tag{46}$$

(46) is equivalent to

$$\hat{v}(B_1 \cup \{i\}) - \hat{v}(B_1) \geq \hat{v}\left(\bigcup_{k=1}^{p'} A_k \cup \{i\}\right) - \hat{v}\left(\bigcup_{k=1}^{p'} A_k\right). \tag{47}$$

Thus, it is sufficient to prove that (47) is satisfied and we can assume each component of A linked to j and replace p' by p and B_1 by B.

If $\sigma(B, i) < M(B, i)$, then the Star condition implies the existence of a unique edge $e_0 \in E(B, i)$ with $w(e_0) = \sigma(B, i)$ and $w(e) = M(B, i)$ for all $e \in E(B, i) \setminus \{e_0\}$. Otherwise, we have $w(e) = \sigma(B, i) = M(B, i)$ for all $e \in E(B, i)$. As $p \geq 2$, $E(A, i)$ contains at least two edges. Hence, we have $M(A, i) = M(B, i)$ in both cases. Setting $M = M(A, i) = M(B, i)$, we can always assume

$$\sigma(B, i) \leq \sigma(A, i) = \sigma(A_1, i) \leq \sigma(A_2, i) = \ldots = \sigma(A_p, i) = M, \tag{48}$$

after renumbering if necessary. If $\sigma(A, i) < M$, then we have $\sigma(A, i) = \sigma(B, i)$ and $e_0 \in E(A_1, i)$.

We consider several cases.

Case 1 We assume $\sigma(A, i) < M$. Then, we also have $\sigma(B, i) < M$ and (48) implies

$$\sigma(A, i) = \sigma(A_1, i) < \sigma(A_k, i) = M, \ \forall k, \ 2 \leq k \leq p. \tag{49}$$

Moreover, e_0 links i to A_1 and we necessarily have

$$\sigma(B, i) = \sigma(A_1, i) = \sigma(A, i). \tag{50}$$

As $\sigma(A,i) < M$, Claim 1 of Lemma 18 implies

$$M \leq \sigma(A_k),\ \forall k,\ 2 \leq k \leq p,\ \text{ s.t. } |A_k| \geq 2. \tag{51}$$

Subcase 1.1 We assume $\sigma(B) \leq \sigma(B,i)$. Then, we have

$$\Sigma(B \cup \{i\}) = \left\{ \begin{array}{cl} \Sigma(B) & \text{if } \sigma(B) < \sigma(B,i), \\ \Sigma(B) \cup \{e_0\} & \text{if } \sigma(B) = \sigma(B,i). \end{array} \right. \tag{52}$$

By (49) and (50), we obtain

$$\sigma(B) \leq \sigma(A_1,i) < \sigma(A_k,i) = M,\ \forall k,\ 2 \leq k \leq p. \tag{53}$$

As $E(A_1,i) \neq \emptyset$, (53) and Lemma 8 applied to $A_1 \subseteq B$ imply

$$\sigma(B) \leq \sigma(A_1) \leq \sigma(A_1,i) < \sigma(A_k,i) = M,\ \forall k,\ 2 \leq k \leq p,\ \text{if}\, |A_1| \geq 2. \tag{54}$$

Then, (51), (53), and (54) imply

$$\Sigma(A \cup \{i\}) = \left\{ \begin{array}{cl} \Sigma(A_1) & \text{if } |A_1| \geq 2 \text{ and } \sigma(A_1) < \sigma(A_1,i), \\ \Sigma(A_1) \cup \{e_0\} & \text{if } |A_1| \geq 2 \text{ and } \sigma(A_1) = \sigma(A_1,i), \\ \{e_0\} & \text{if } |A_1| = 1. \end{array} \right. \tag{55}$$

Let us assume $|A_1| \geq 2$. Let $A_{1,1}, A_{1,2}, \ldots, A_{1,k_1}$ (resp. $B_1, B_2, \ldots, B_{l_1}$) be the blocks of $\mathcal{P}_{\min}(A_1)$ (resp. $\mathcal{P}_{\min}(B)$) connected to i by an edge in $E(A_1,i)$ (resp. $E(B,i)$). Let J_{A_1} (resp. J_B) be the set of indices $j \in \{1,\ldots,k_1\}$ (resp. $j \in \{1,\ldots,l_1\}$) such that $E(A_{1,j},i)$ (resp. $E(B_j,i)$) contains at least one edge with weight strictly greater than $\sigma(A_1)$ (resp. $\sigma(B)$). J_{A_1} (resp. J_B) may be empty. Let K_A be the set of indices $k \in \{2,\ldots,p\}$. Setting $\tilde{A} = \bigcup_{j \in J_{A_1}} A_{1,j} \cup \bigcup_{k \in K_A} A_k$ and $\tilde{B} = \bigcup_{j \in J_B} B_j$, we have

$$\Delta_i \hat{v}(A) = v(\tilde{A} \cup \{i\}) - \sum_{j \in J_{A_1}} v(A_{1,j}) - \sum_{k \in K_A} \overline{v}(A_k). \tag{56}$$

and

$$\Delta_i \hat{v}(B) = v(\tilde{B} \cup \{i\}) - \sum_{j \in J_B} v(B_j). \tag{57}$$

Let us now assume $|A_1| = 1$. Then, we have $v(A_1) = \overline{v}(A_1) = 0$ and (56) is also satisfied setting $J_{A_1} = \emptyset$. By (56) and (57), (37) is equivalent to

$$v(\tilde{B} \cup \{i\}) - \sum_{j \in J_B} v(B_j) \geq v(\tilde{A} \cup \{i\}) - \sum_{j \in J_{A_1}} v(A_{1,j}) - \sum_{k \in K_A} \overline{v}(A_k). \tag{58}$$

By (53) and (54), Lemma 15 applied to $A_1 \cup \bigcup_{k \in K_A} A_k$ and B implies

$$A_{1,j} \subseteq B_j,\ \forall j \in J_{A_1}, \tag{59}$$

and

$$|A_j| = 1 \text{ and } A_j \subseteq B_j,\ \forall j \in K_A, \tag{60}$$

with $J_{A_1} \cup K_A \subseteq J_B$ and $J_{A_1} \cap K_A = \emptyset$, after renumbering if necessary. (59) and (60) imply $\tilde{A} \subseteq \tilde{B}$ and

$$\tilde{A} \cap B_j = \begin{cases} A_{1,j} & \forall j \in J_{A_1}, \\ A_j & \forall j \in K_A, \\ \emptyset & \forall j \in J_B \setminus (J_{A_1} \cup K_A). \end{cases} \tag{61}$$

As $\tilde{A} \subseteq \tilde{B}$, the convexity of (N, v) implies

$$v(\tilde{B} \cup \{i\}) - v(\tilde{B}) \geq v(\tilde{A} \cup \{i\}) - v(\tilde{A}). \tag{62}$$

Applying Lemma 5 with $\mathcal{F} = 2^N \setminus \{\emptyset\}$ to the subsets $\tilde{A}$, $\tilde{B}$, and to the partition $\{\{B_j \mid j \in J_B\}\}$ of $\tilde{B}$, we obtain

$$v(\tilde{B}) - \sum_{j \in J_B} v(B_j) \geq v(\tilde{A}) - \sum_{j \in J_B} v(\tilde{A} \cap B_j). \tag{63}$$

By (60), we have $|A_j| = 1$ and therefore $v(A_j) = \overline{v}(A_j)$ for all j in K_A. Then, (61) implies

$$\sum_{j \in J_B} v(\tilde{A} \cap B_j) = \sum_{j \in J_{A_1}} v(A_{1,j}) + \sum_{k \in K_A} \overline{v}(A_k). \tag{64}$$

(62), (63), and (64) imply (58).

Subcase 1.2 We now assume $\sigma(B, i) < \sigma(B)$. (50) implies

$$\sigma(B, i) = \sigma(A_1, i) = \sigma(A, i) < \sigma(B). \tag{65}$$

We get $\Sigma(B \cup \{i\}) = \Sigma(A \cup \{i\}) = \{e_0\}$. As $p \geq 2$, $E(B, i) \setminus \{e_0\}$ contains at least one edge. We obtain

$$\Delta_i \hat{v}(B) = v(B \cup \{i\}) - \overline{v}(B). \tag{66}$$

Setting $\tilde{A} = \bigcup_{j=2}^{p} A_j$, we have

$$\Delta_i \hat{v}(A) = \begin{cases} v(A \cup \{i\}) - \sum\limits_{k=1}^{p} \overline{v}(A_k) & \text{if } E(A_1, i) \setminus \{e_0\} \neq \emptyset, \\ v(A_1) + v(\tilde{A} \cup \{i\}) - \sum\limits_{k=1}^{p} \overline{v}(A_k) & \text{otherwise.} \end{cases} \tag{67}$$

The superadditivity of (N, v) implies

$$v(A \cup \{i\}) \geq v(A_1) + v(\tilde{A} \cup \{i\}). \tag{68}$$

By (66), (67), and (68), (37) is satisfied if the following inequality is satisfied:

(69) $$v(B \cup \{i\}) - \overline{v}(B) \geq v(A \cup \{i\}) - \sum_{k=1}^{p} \overline{v}(A_k).$$

By (49) and (65), Lemma 17 implies

(70) $$\sum_{F \in \mathcal{P}_{\min}(B)_{|A}} v(F) = \sum_{k=1}^{p} \overline{v}(A_k).$$

The convexity of (N, v) implies

(71) $$v(B \cup \{i\}) - v(B) \geq v(A \cup \{i\}) - v(A).$$

Applying Lemma 5 with $\mathcal{F} = 2^N \setminus \{\emptyset\}$ to A, B, and to the partition $\{\mathcal{P}_{\min}(B)\}$ of B, we obtain

(72) $$v(B) - \overline{v}(B) \geq v(A) - \sum_{F \in \mathcal{P}_{\min}(B)_{|A}} v(F).$$

By (70), (72) is equivalent to

(73) $$v(B) - \overline{v}(B) \geq v(A) - \sum_{k=1}^{p} \overline{v}(A_k).$$

(71) and (73) imply (69).

Case 2 We assume $\sigma(A, i) = M$. If $|A_k| = 1$ for all k with $1 \leq k \leq p$, then we have $\Sigma(A \cup \{i\}) = E(A, i)$. We get $\Delta_i \hat{v}(A) = \hat{v}(\{i\})$ and (37) is satisfied by the superadditivity of $(N, \hat{v})$. We henceforth assume that there is at least one index k such that $|A_k| \geq 2$.

Subcase 2.1 We assume $\sigma(A, i) = M \leq \sigma(A)$. By (48), we get

(74) $$\sigma(A_k, i) = M, \ \forall k, \ 1 \leq k \leq p,$$

and

(75) $$\sigma(A_k, i) = M \leq \sigma(A_k), \ \forall k, \ 1 \leq k \leq p, \ \text{ s.t. } |A_k| \geq 2.$$

(74) and (75) imply $\Sigma(A \cup \{i\}) = E(A, i) \cup \bigcup_{k \,|\, \sigma(A_k) = M} \Sigma(A_k)$. This implies

(76) $$\Delta_i \hat{v}(A) = \sum_{\{k \,:\, \sigma(A_k) > M\}} v(A_k) - \overline{v}(A_k).$$

We can assume that there is at least one index k such that $\sigma(A_k) > M$, otherwise we have $\Delta_i \hat{v}(A) = 0$ and (37) is trivially satisfied by superadditivity of $(N, \hat{v})$. By Lemma 8 and (75), we get

$$\sigma(A_k, i) = M < \sigma(A_k) = \sigma(B),\ \forall k,\ 1 \leq k \leq p,\ \text{ s.t. } \sigma(A_k) > M. \tag{77}$$

By (77), we can henceforth assume $\sigma(B) > M$. If $\sigma(B, i) < M$ (resp. $\sigma(B, i) = M$), then we have $\Sigma(B \cup \{i\}) = \{e_0\}$ (resp. $\Sigma(B \cup \{i\}) = E(B, i)$). As $\sigma(A, i) = M$, we necessarily have $e_0 = \{i, j\}$. We get

$$\Delta_i \hat{v}(B) = \begin{cases} v(B \cup \{i\}) - \overline{v}(B) & \text{if } \sigma(B, i) < M, \\ v(B) - \overline{v}(B) & \text{if } \sigma(B, i) = M. \end{cases} \tag{78}$$

The superadditivity of (N, v) implies

$$v(B \cup \{i\}) - \overline{v}(B) \geq v(B) - \overline{v}(B). \tag{79}$$

By (76), (78), and (79), (37) is satisfied if the following inequality is satisfied:

$$v(B) - \overline{v}(B) \geq \sum_{\{k \,:\, \sigma(A_k) > M\}} v(A_k) - \overline{v}(A_k). \tag{80}$$

Applying Lemma 5 with $\mathcal{F} = 2^N \setminus \{\emptyset\}$ to the subsets $\tilde{A} = \bigcup_{\{k \,:\, \sigma(A_k) > M\}} A_k$ and B, and to the partition $\mathcal{P}_{\min}(B)$ of B, we obtain

$$v(B) - \overline{v}(B) \geq v(\tilde{A}) - \sum_{F \in \mathcal{P}_{\min}(B)_{|\tilde{A}}} v(F). \tag{81}$$

By superadditivity of (N, v), (81) implies

$$v(B) - \overline{v}(B) \geq \sum_{\{k \,:\, \sigma(A_k) > M\}} v(A_k) - \sum_{F \in \mathcal{P}_{\min}(B)_{|\tilde{A}}} v(F). \tag{82}$$

As $\sigma(B) > M$, there exists an edge $e \in E$ connected to B with weight $w(e) < \sigma(B)$. Then, as (77) is satisfied, Lemma 17 applied to $\tilde{A}$ and B implies

$$\sum_{F \in \mathcal{P}_{\min}(B)_{|\tilde{A}}} v(F) = \sum_{\{k \,:\, \sigma(A_k) > M\}} \overline{v}(A_k). \tag{83}$$

(82) and (83) imply (80) and therefore (37) is satisfied.

Subcase 2.2 We assume $\sigma(A) < \sigma(A, i) = M$.
Let us assume $\sigma(B, i) < M$. Then, $e_0 = \{i, j\}$ is the only edge in $E(B, i)$ with $w(e_0) = \sigma(B, i)$ and $w(e) = M$ for all $e \in E(B, i) \setminus \{e_0\}$. We can assume w.l.o.g. $\sigma(A_1) = \sigma(A)$. Let us consider an edge $e_1 = \{i, j_1\}$ with $j_1 \in A_1$ and an edge $\tilde{e}_1$ in $\Sigma(A_1)$. Let γ be a shortest path in G_{A_1} linking j_1

to an end-vertex of $\tilde{e}_1$ (γ may be reduced to j_1). Then the Path condition applied to $\{e_0\} \cup \{e_1\} \cup \gamma \cup \{\tilde{e}_1\}$ implies $M = w_1 \leq \max(w(e_0), w(\tilde{e}_1))$, a contradiction. Thus, we necessarily have $\sigma(B,i) = \sigma(A,i) = M$ and (48) implies

$$\sigma(B,i) = \sigma(A,i) = \sigma(A_1,i) = \cdots = \sigma(A_p,i) = M. \tag{84}$$

Moreover, as $\sigma(B) \leq \sigma(A) < M$, Claim 2 of Lemma 18 implies

$$\sigma(A) = \sigma(A_1) < M \leq \sigma(A_k),\ \forall k,\ 2 \leq k \leq p, \text{ s.t. } |A_k| \geq 2, \tag{85}$$

with $|A_1| \geq 2$, after renumbering if necessary. As $\sigma(A_1) < M$, (84) implies

$$\sigma(B) \leq \sigma(A_1) < M = \sigma(A_k,i),\ \forall k,\ 1 \leq k \leq p. \tag{86}$$

(85) and (86) imply $\Sigma(A \cup \{i\}) = \Sigma(A_1)$ and $\Sigma(B \cup \{i\}) = \Sigma(B)$. Let $A_{1,1}, A_{1,2}, \ldots, A_{1,k_1}$ (resp. $B_1, B_2, \ldots, B_{l_1}$) be the blocks of $\mathcal{P}_{\min}(A_1)$ (resp. $\mathcal{P}_{\min}(B)$) linked to i by an edge in $E(A_1,i)$ (resp. $E(B,i)$). Let us set $J_{A_1} = \{1,\ldots,k_1\}$ and $J_B = \{1,\ldots,l_1\}$. Let K_A be the set of indices $k \in \{2,\ldots,p\}$. Setting $\tilde{A} = \bigcup_{j \in J_{A_1}} A_{1,j} \cup \bigcup_{k \in K_A} A_k$ and $\tilde{B} = \bigcup_{j \in J_B} B_j$, we get

$$\Delta_i \hat{v}(A) = v(\tilde{A} \cup \{i\}) - \sum_{j \in J_{A_1}} v(A_{1,j}) - \sum_{k \in K_A} \overline{v}(A_k), \tag{87}$$

and

$$\Delta_i \hat{v}(B) = v(\tilde{B} \cup \{i\}) - \sum_{j \in J_B} v(B_j). \tag{88}$$

By (86), Lemma 15 applied to $A_1 \cup \bigcup_{k \in K_A} A_k$ and B implies $|J_{A_1}| = |K_A| = 1$, and

$$A_{1,1} \subseteq B_1, \tag{89}$$

and

$$|A_2| = 1 \text{ and } A_2 \subseteq B_2, \tag{90}$$

after renumbering if necessary. (89) and (90) correspond exactly to (59) and (60) and we can conclude as in Subcase 1.1.

We can finally prove that (37) is satisfied by induction on $|B \setminus A|$. By the previous reasoning, it is true if $|B \setminus A| = 1$. If $|B \setminus A| \geq 2$, then we can select a vertex $j \in B \setminus A$ and set $B' = B \setminus \{j\}$. Applying (37) to $B' \subseteq B \subseteq N \setminus \{i\}$, we get

$$\Delta_i \hat{v}(B) \geq \Delta_i \hat{v}(B'). \tag{91}$$

By induction, we also have

$$\Delta_i \hat{v}(B') \geq \Delta_i \hat{v}(A). \tag{92}$$

(91) and (92) imply (37). □

5 Conclusion

Our main result gives a characterization of the weighted graphs satisfying inheritance of convexity for the correspondence $\tilde{\mathcal{P}}_{\min}$. This characterization implies strong restrictions on the structure of the underlying graph and the relative positions of edge-weights. But it also highlights that the class of graphs satisfying inheritance of convexity with $\tilde{\mathcal{P}}_{\min}$ is much larger than the one satisfying inheritance of convexity with $\mathcal{P}_{\min}$. Indeed, the characterization of inheritance of convexity for the correspondence $\mathcal{P}_{\min}$ obtained by Skoda (2020) implies a restriction of the edge-weights to at most three different values and cycle-completeness of large subgraphs is required. In contrast, inheritance of convexity for the correspondence $\tilde{\mathcal{P}}_{\min}$ allows an arbitrary number of edge-weights. Moreover, the reinforced conditions on cycles and pans imply the existence of some specific chords in some cycles but cycle-completeness does not come into play. Skoda (2020) established that inheritance of convexity for $\mathcal{P}_{\min}$ can be verified in polynomial time. It would be interesting to investigate the complexity of the problem of deciding whether inheritance of convexity is satisfied for the correspondence $\tilde{\mathcal{P}}_{\min}$. It is already known that inheritance of $\mathcal{F}$-convexity can be checked in polynomial time (Skoda (2016)). Therefore, the first five conditions of our characterization can also be checked in polynomial time. It can be easily seen that the Reinforced Adjacent Cycles condition can be checked in polynomial time as we only have to consider cycles of size 4 or 5. It remains to study the complexity of checking the Reinforced Cycle and the Reinforced Pan conditions. It seems more difficult as there is no obvious limitation on the number or size of paths and cycles involved in these last conditions.

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