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Endowments-regarding preferences ⋆

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Abstract

We consider a pure exchange economy model with endowments-regarding preferences, which means that demand functions and preferences depend not only on the own consumption of a consumer but also on other consumer's endowments. First, we study the particular case called wealth concern by Balasko (2015) when the consumers care about the wealth of the others. We generalise the result of Balasko by showing that most properties of the standard general equilibrium model without externalities are robust with respect to these kind of externalities if the external effect produced by only one agent is a wealth effect and not all. Next, we clarify under which sufficient conditions those properties hold true under the most general form of endowments externalities. Furthermore, we generalise the above sufficient condition to obtain generic regularity results in the economies with consumption and endowments externalities.

JEL classification: D50, D51, D62.

Key words: Endowment externalities, other-regarding preferences, regular economy, general equilibrium.

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1 Introduction

Debreu (1974) said that *"The observed state of an economy can be viewed as an equilibrium resulting from the interaction of a large number of agents with partially conflicting interests."* Thus, the theory of general economic equilibrium is a setting to describe and explain economic phenomena. However, most of the works assume exclusively that agents are selfish in the sense that they take their decisions ignoring the behaviours and opportunities of others. This assumption simplifies and clarifies arguments wonderfully. However, the presence of externalities on consumer behaviour is widely acknowledged in the economic literature and has been well documented from both empirical and experimental perspectives (for example Luttmer (2005) and Zizzo and Oswald (2001)). The question of whether properties of equilibrium with selfish preferences still hold true with the presence of externalities becomes an economically important question.

There is a large and growing number of papers on general equilibrium models with the presence of externalities. According to Laffont (1976) and Hammond (1998), the externalities appear not only in preferences but also in consumption sets. Studying the same kind of externalities, Bonnisseau and del Mercato (2010) showed that the genericity of regular equilibria holds true as long as the second-order external effects on the individual preferences are not too strong with respect to the direct effect of household consumption. Bonnisseau (2003) considered a wide class of non-ordered preferences with external effects. The author showed that nice properties of the equilibrium manifold and the regular economies (for example existence, the odd number of equilibria and genericity of regular economies) remain true under the geometric assumption on preferences. Besides, Balasko (2003) obtained that most properties of the standard equilibrium models can be extended to cases where preferences depend on prices, provided that the total resources are variable. Moreover, Fehr and Gächter (2000) and Sobel (2005) have emphasised the appropriateness of unselfish preferences introducing the popular name "other-regarding preferences" (ORP). A more recent paper by Dufwenberg et al. (2011) had shown that ORP will not affect market behaviour under a separability condition on preferences. This condition implies that the equilibria of economies with externalities coincide with the equilibria in a fictitious economy with selfish preferences.

In the present paper, building on the seminal contribution of Balasko (2015), we study a different kind of externalities called endowments externalities. Many behavioural economists and rational decision theorists agree that individuals behaviour depends on the reference point. In recent papers by Masatlioglu and Ok (2014) and Maltz (2020), the reference point belongs to the endowment's category, which contains the initial endowments. Moreover, there are many papers, for example, Heap et al. (2016), Hopkins and Kornienko (2010), and Hopkins (2018), which studied the effect of the endowment inequality or the distribution of endowment on the action of individuals such as contributing to public goods, making effort or doing gambling. It is, therefore, necessary to encompass the endowment dependence in the economic model.

Our contribution is at two levels. We first extend the paper of Balasko (2015) about wealth externalities where the consumers' behaviour is described by the demand functions. Balasko considered the case where the demand functions depend on the price and all other consumers' wealth. We consider the more general case where demand function depends on the other endowments and the wealth of only one agent. This asymmetric dependence allows us to obtain the same results as the one in Balasko (2015) by using the same method of analysis. On that account, the following properties of equilibrium can be directly extended to the model with wealth externalities under very general assumptions on individual demand functions: smooth equilibrium manifold, a ramified properness of the natural projection, genericity of regular economies. As usual, around the regular economies, a finite number of equilibrium and oddness of the number of equilibria, local continuity of the equilibrium selection mappings.

Even though the nice results of regularity is obtained without any additional assumption, we are still not enjoyable with this asymmetric dependence. Therefore, the second contribution aims at studying the general form of externalities in terms of endowments, which encompass wealth concern. In this part, the consumers are characterised by utility functions which depend on the endowments of all consumers. With this symmetric, Balasko's method cannot apply to this generalisation because of the complexity of the Jacobian matrix. Therefore, considering the seminal work by Smale (1974) and Villanacci et al. (2002), we use the so-called extended approach where equilibria are characterised by the first-order conditions and market clearing conditions. To the best of my knowledge and according to the example provided in Bonnisseau and del Mercato (2010), in this framework, an additional assumption is necessary to extend the above results. Therefore, we propose a new condition on utility functions, namely Assumption 12. Precisely, we assume that the first-order effects of the endowments on the marginal rate of substitution are small enough and even vanishing along the direction which keeps the wealth constant. This condition is clearly satisfied under the separability condition of Dufwenberg et al. (2011) and when the external effect of one agent on all others is a wealth effect. Under this additional assumption, we obtain that the equilibrium set is a smooth manifold and the set of regular economies is an open and full Lebesgue measure subset of the parameter space. While using the extended approach, we make a relationship with the aggregate excess demand approach. This creates an indirect way to study the genericity property by applying the aggregate excess demand approach in this general form of endowments externalities. We also generalise Assumption 12 to an exchange economy with both kinds of externalities: endowments and consumption to get the genericity of regular economies in this setting.

This paper is organised as follows. Section 2 is devoted to the definitions and notations regarding the economic environment of the exchange model. In section 3, we consider the model with demand functions depending on the other endowments and wealth of one consumer. We prove that the equilibrium set is a manifold and we deduce the genericity of regular economies. In section 4, we consider general endowment externalities, we state the key Assumption 12 and prove that it is sufficient to get the desired properties of equilibrium. Finally, in section 5, we conclude by generalising Assumption 12 to the exchange economy with two kinds of externalities: endowments and consumption. All technical proofs are gathered in Appendix A, except for Proposition 16, Lemma 17, Theorem 18, and Theorem 21.

2 The exchange model

In this section, we describe the framework of the exchange model: commodities, prices, consumers' characteristics, and initial endowments.

2.1 Commodities and prices

We consider a model with a finite number L of commodities. They are indicated by an index $l = 1, \dots, L$. So the commodity space is $\mathbb{R}^L$. A commodity bundle is a vector $x = (x^1, \dots, x^L) \in \mathbb{R}^L$. The component x^l represents the quantity of commodity l in the commodity bundle x. $x^{\setminus} = (x^1, \dots, x^{L-1}) \in \mathbb{R}^{L-1}$ is obtained from x by deleting the last coordinate x^L. For any vector $x = (x_1, \dots, x_i, \dots, x_m) \in \mathbb{R}^{Lm}$, we denoted $x_{-i} = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_m) \in \mathbb{R}^{L(m-1)}$ by deleting the i-th component x_i. Similarly, $x^{\setminus}_{-i}$ is obtained from $x^{\setminus} = (x^{\setminus}_1, \dots, x^{\setminus}_m)$ by deleting the i-th component $x^{\setminus}_i$.

A price for commodity l is a strictly positive number p^l. The price vector $p = (p^1, \dots, p^L) \in \mathbb{R}^L_{++}$ is normalized by choosing commodity L as *numéraire*, i.e., $p^L = 1$. The set of numéraire normalized prices is denoted by $\mathbb{S} = \mathbb{R}^{L-1}_{++} \times \{1\}$. The numéraire normalized price vector $p \in \mathbb{S}$ can be written as $p = (p^{\setminus}, 1)$, where $p^{\setminus} = (p^1, \dots, p^{L-1}) \in \mathbb{R}^{L-1}_{++}$ expresses the prices of the non-numéraire commodities.

2.2 Consumers and endowments

There is a finite number m of consumers, and each of them is indicated by an index $i = 1, \ldots, m$. Every consumer is economically characterized by her consumption set, her initial endowment of commodities and her demand function or her preferences.

The consumption set X_i of consumer i is commonly assumed to be convex and bounded from below. To easily use differential techniques, we assume that the consumption set of every consumer i is the strictly positive orthant, i.e., $X_i = \mathbb{R}^L_{++}$.

Every consumer i has an initial endowment $e_i = (e_i^1, \ldots, e_i^L) \in X_i$. The endowment space is the subset consisting of all commodity bundles, which, in principle, the consumers may own. The endowment space is $\Omega \subseteq \mathbb{R}^{Lm}$. In this paper, we consider the endowment space $\Omega = \prod_{i=1}^{m} X_i$. The m-tuple $e = (e_1, \ldots, e_m) \in \Omega$ represents the endowments of all consumers in the economy. Given the price vector $p \in \mathbb{S}$, consumer i's wealth w_i is the inner product $w_i = p \cdot e_i \in \mathbb{R}$. The vector e_{-i} and $e^{\setminus}_{-i}$ are defined in the same way as x_{-i} and $x^{\setminus}_{-i}$ above. The set Ω^{-i} denotes the space of endowment vectors of the consumers but the i-th one.

3 Endowment-wealth externalities

In this section, consumers are characterised by their demand functions, which depend on the other endowments and the wealth of only one consumer, namely consumer m. Even though this is asymmetric, there is an interesting economical example of this particular dependence. Many economists have pointed out that the poor and rich people behave very differently. And in some country, they define the richness or poorness depending on which called “median income”, for example, in the UK the poverty line is below 60% of the median income or in the US, they define the rich people, called upper class, by double of the median income. Therefore, we can imagine that the individual with the median wealth will play a role as the special consumer m. His initial wealth, which makes someone feel rich and others feel poor, then somehow affects the behavior of all other consumers. The other endowments can be considered as the reference category for consumers to make a choice.

Thanks to this particular consumer, who influences on the other demands only through her wealth, we can extend the results in Balasko (2015) by applying the same method without any additional assumption on the individual demand functions.

3.1 Individual demand functions and basic assumptions

The exchange model here is defined by the endowment space $\Omega = \mathbb{R}^{Lm}_{++}$ and the m-tuple of individual demand functions $(f_i)_{i=1,\ldots,m}$. The individual demand function depends not only on her endowments but also on the endowments of the others and the wealth of consumer m. Formally, the individual demand function of consumer i is the function from $\mathbb{S} \times \Omega^{-m} \times \mathbb{R}_{++}$ to $\mathbb{R}^L_{++}$:

$$f_i(p, e_1, \ldots, e_{m-1}, w_m)$$

where p is the price vector, e_j is the initial endowments of consumer j, and w_m is the wealth of consumer m.

We extend below classical assumptions on the individual demand functions to our framework:

Assumption 1
- *(S)* ***Smoothness*** *For all consumers* $i = 1, 2, .., m$ f_i *is smooth*[1] *.*
- *(W)* ***Walras law*** *For any* $b = (p, e_{-m}, w_m) \in \mathbb{S} \times \Omega^{-m} \times \mathbb{R}_{++}$, $p \cdot f_m(b) = w_m$ *and* $p \cdot f_i(b) = p \cdot e_i$ *for all* $i = 1, \ldots, m-1$.
- *(D)* ***Desirability*** *For at least one consumer* i, *for every sequence* $b^n = (p^n, e^n_1, \ldots, e^n_{m-1}, w^n_m) \subset \mathbb{S} \times \Omega^{-m} \times \mathbb{R}_{++}$ *such that the associated sequence* $(e^n_1, \ldots, e^n_{m-1}, w^n_m / \sum_{l=1}^{L} p^{n,l})$ *tends to the limit* $(e^0_1, \ldots, e^0_{m-1}, \tilde{w}^0_m) \in \Omega^{-m} \times \mathbb{R}_{++}$ *and* $(p^n / \sum_{l=1}^{L} p^{n,l})$ *tends to* $q^0 \in \mathbb{R}^L_+ \setminus \{0\}$ *and* $q^{0,l} = 0$ *for some* l, *then*
$$\limsup_{n \to +\infty} \|f_i(b^n)\| = +\infty.$$

Here we provide an example of the preferences which give the desired demand functions which satisfy Assumption 1. For all $i \neq m$, consumer i's preference is represented by a utility function $u_i(x_i, y) : \mathbb{R}^L_{++} \times \Omega^{-m} \times \mathbb{R}_{++} \to \mathbb{R}$ with $y = (e_{-m}, w_m) \in \Omega^{-m} \times \mathbb{R}_{++}$ and consumer m's preferences is represented by a utility function $u_m(x_m, e_{-m}) : \mathbb{R}^L_{++} \times \Omega^{-m} \to \mathbb{R}$. And for all i, u_i is smooth and differentiably strictly monotone, differentiably strictly quasi-concave with respect to her own consumption x_i, and satisfies the boundary condition, which is for every $x_i \in \mathbb{R}^L_{++}$ and for fixed other variables y, the closure of the upper set $\{x'_i \in X | u_i(x'_i, y) \leq u_i(x_i, y)\}$ is included in $\mathbb{R}^L_{++}$. Then, the maximization of $u_i(x_i, y)$ subject to the budget constraint $p \cdot x_i \leq p \cdot e_i$ has a unique solution which can be written as the form $f_i(p, e_{-m}, w_m)$. Moreover, this demand function satisfies (S), (W) and (D) as follows readily from classical consumer theory.

[1] That means f_i is in C^∞.

Following Debreu (1970), Balasko (1988), and Balasko (2015), we use the aggregate excess demand approach to study the equilibria and their properties.

The aggregate excess demand function is the mapping $z : \mathbb{S}\times\Omega \to \mathbb{R}^L$, $(p, e) \mapsto z(p, e)$, where $z(p, e)$ is equal to:

$$z(p,e) = \sum_{i=1}^{m} f_i\Big(p, e_1, \ldots, e_{m-1}, p \cdot e_m\Big) - \sum_{i=1}^{m} e_i. \qquad (*)$$

The pair $(p, e) \in \mathbb{S} \times \Omega$ is an equilibrium if $z(p, e) = 0$. The price vector $p \in \mathbb{S}$ is called the equilibrium price vector associated with the vector of the initial endowments (also called an economy) $e \in \Omega$. The set of equilibria E is the subset of $\mathbb{S} \times \Omega$ consisting of the equilibria (p, e). Note the equilibrium exists for all $e \in \Omega$ (see Theorem 5), which follows the set E is non-empty.

We define $z^{\backslash}(p, e) = \Big(z^1(p, e), \ldots, z^{L-1}(p, e)\Big)$. Then directly from the property (W) in Assumption 1 on the demand functions, the set E can be defined by the equation $z^{\backslash}(p, e) = 0$.

For a given endowments e, we define the partial mapping depending only on prices p as follows:

$$z_e^{\backslash} : \mathbb{R}_{++}^{L-1} \to \mathbb{R}^{L-1}, p^{\backslash} \mapsto z^{\backslash}(p, e)$$

Let $J(p, e)$ be the $(L-1) \times (L-1)$ Jacobian matrix of $z_e^{\backslash}$ at the equilibrium $(p, e) \in E$. The equilibrium $(p, e) \in E$ is critical (resp. regular) if $\det J(p, e) = 0$ (resp. $\neq 0$). We denote the set of critical equilibria by $\mathfrak{C}$, the set of regular equilibria by $\mathfrak{R}$. Straightforward from the definition, we have $\mathfrak{C} \cap \mathfrak{R} = \emptyset$ and $\mathfrak{C} \cup \mathfrak{R} = E$. The continuity of the mapping $(p, e) \to det J(p, e)$ implies that the critical equilibria set $\mathfrak{C}$ is a closed set.

In spirit of the works of Debreu (1970) and Mas-Collel (1985), singular and regular economies are defined as below.

Definition 2 *The economy $e \in \Omega$ is regular (resp. singular) if 0 is a regular (resp. singular) value for mapping $z_e^{\backslash}$. We denote $\mathcal{R}$ (resp. Σ) the set of regular (resp. singular) economies.*

3.3 The equilibrium manifold

In this subsection, the equilibrium set E is shown to be more than just a subset of the Cartesian product $\mathbb{S} \times \Omega$. In the classical exchange model without externalities, the equilibrium set E is a smooth submanifold of $\mathbb{S} \times \Omega$. This property is now extended our model.

Theorem 3 *Under Assumption 1, the equilibrium set E is a smooth submanifold of $\mathbb{S} \times \Omega$ of dimension Lm.*

Proof. See Appendix A. ■

Theorem 3 provides the characterisation of the small open neighborhoods of the points of E. Precisely, every point in E posses an open neighborhood diffeomorphic to all of the Euclidean space $\mathbb{R}^{Lm}$. The equilibrium set E then is locally connected, and even more locally path-connected.

3.4 Genericity of regular economies

Recall that the endowment space is $\Omega = \prod_{i=1}^{m} X_i = \mathbb{R}^{Lm}_{++}$. The natural projection is defined by the mapping $\pi : E \to \Omega,\ (p, e) \mapsto e$, that is the restriction of the projection mapping $pr : \mathbb{S} \times \Omega \to \Omega$ to the equilibrium set E. By definition, the economy $e \in \Omega$ is singular if it is the image of a critical equilibrium $(p, e) \in \mathfrak{C}$ by the natural projection π and the economy $e \in \Omega$ is regular if it is not singular. Therefore, we have $\Sigma = \pi(\mathfrak{C})$ and $\mathcal{R} = \Omega \setminus \Sigma$.

We know [2] that an economy $e \in \Omega$ is a regular economy if and only if e is a regular value for the natural projection π.

Thanks to this equivalent relation, we will study the set of regular equilibria in terms of regular values for the natural projection π. The equilibrium manifold E is the domain of the natural projection π. The fact that E is a smooth manifold allows to use the notion of smooth mappings and proper mappings on E, i.e., the preimage of any compact set is a compact set in E. [3]

Proposition 4 *Under Assumption 1, the natural projection $\pi : E \to \Omega$ is*

(i) smooth,
(ii) proper.

[2] See for example Balasko (2011) Chapter 6.
[3] See Chapter 4 in Villanacci et al. (2002) for other equivalent definitions.

Proof. See Appendix A. ■

Properness and smoothness of the mapping π allow us to study the size of singular economies Σ and extend the well-known properties of regular economies $\mathcal{R}$ to our model with endowments externalities. This is summarised in the following theorem.

Theorem 5 *Under Assumption 1, the following properties hold true.*

(i) The set of singular economies Σ is closed and of Lebesgue measure zero in Ω.
(ii) The set of regular economies $\mathcal{R} = \Omega \backslash \Sigma$ is open with full Lebesgue measure in Ω.
(iii) For every $e \in \Omega$, the set $\pi^{-1}(e)$ is non-empty and for every $e \in \mathcal{R}$, $\pi^{-1}(e)$ is finite and odd.
(iv) The restriction of the mapping π to the subset $\pi^{-1}(\mathcal{R})$ is a finite covering of $\mathcal{R}$.
(v) The number of elements of the preimage $\pi^{-1}(e)$ is locally constant over $\mathcal{R}$ and even over each path-connected component of $\mathcal{R}$.

Proof. See Appendix A. ■

These properties are the fundamental basis on which a comparative static analysis can be conducted for studying the dependence of the equilibrium on the parameters defining the economy, namely the vector of initial endowments $e \in \Omega$.

3.5 The economy with unbounded consumption

While our main setting is the strictly positive orthant, there is a large number of papers which consider the unbounded consumption sets, for example, Balasko (2011), Balasko (2003), and Sato (2010). The trivial example of an unbounded consumption set is the whole Euclidean space. We end this section by shortly talking about an unbounded economy which is the economy with $X_i = \mathbb{R}^L$ and the endowment space $\Omega = \mathbb{R}^{Lm}$.

The equilibrium set E is still a manifold; furthermore, we even obtain the global structure of the equilibrium manifold E as the following theorem.

Theorem 6 *In the economy with $\Omega = \mathbb{R}^{Lm}$ and $X_i = \mathbb{R}^L$ for all $i = 1, \dots, m$, the equilibrium manifold E is diffeomorphic to $\mathbb{R}^{Lm}$.*

Proof. See Appendix A. ■

Theorem 6 means that the equilibrium manifold has a similar structure as a Euclidean space in the sense of being diffeomorphic to $\mathbb{R}^{Lm}$. Thus, the equilibrium manifold enjoys interesting properties, namely, it is connected and contractible [4].

The idea to prove this theorem is very interesting. We follow Balasko (2011) by introducing the two useful mappings θ and ρ, which allow to parameterise the equilibrium set.

With $(p, e_1, \ldots, e_m) \in \mathbb{S} \times \Omega$ and $(w_1, \ldots, w_m) \in \mathbb{R}^m$, we define the mapping $\theta : \mathbb{S} \times \mathbb{R}^{Lm} \to \mathbb{S} \times \mathbb{R}^m \times \mathbb{R}^{(L-1)(m-1)}$ and the mapping $\rho : \mathbb{S} \times \mathbb{R}^m \times \mathbb{R}^{(L-1)(m-1)} \to \mathbb{S} \times \mathbb{R}^{Lm}$ as follow:

$$\theta(p, e) = (p, p \cdot e_1, \ldots, p \cdot e_m, e^{\setminus}_{-m}) \tag{1}$$

$$\rho(p, w_1, \ldots, w_m, e^{\setminus}_{-m}) = (p, e_1, \ldots, e_{m-1}, e_m) \tag{2}$$

where in equation (2) $e_i = (e^{\setminus}_i, w_i - p^{\setminus} \cdot e^{\setminus}_i)$ for $i = 1, 2, \ldots, m-1$ and

$$e_m = \sum_{i=1}^{m} f_i(p, e_{-m}, w_m) - \sum_{i=1}^{m-1} e_i \tag{3}$$

These two mappings are useful because the equilibrium manifold E is the image of ρ and the compositions of these two mappings the identity mappings: $\rho \circ \theta_{|E} = \mathrm{Id}_{|E}$ and $\theta \circ \rho = \mathrm{Id}$. These nice properties allow us to apply the following lemma [5] and obtain the result straightforwardly. For more detail, see Appendix A.

Lemma 7 *Let $\varphi : X \to Y$ and $\psi : Y \to X$ be two smooth mappings between smooth manifolds such that the composition $\varphi \circ \psi : Y \to Y$ is the identity mapping id_Y. Then the set $Z = \psi(Y)$, the image of the mapping ψ, is a smooth submanifold of X that is diffeomorphic to Y.*

So the equilibrium set E is diffeomorphic to $\mathbb{S} \times \mathbb{R}^m \times \mathbb{R}^{(L-1)(m-1)}$, then diffeomorphic to $\mathbb{R}^{Lm}$. Moreover, thanks to the two mappings θ and ρ, we see explicitly a coordinate system of equilibrium (p, e), which is $\theta(p, e) = (p, p \cdot e_1, \ldots, p \cdot e_m, e^{\setminus}_{-m})$

Noticing that if the endowments space and the consumption sets are positive orthant as the main setting, i.e., $\Omega = \mathbb{R}^{Lm}_{++}$, then the mapping ρ is not well-defined. More precisely, the component $e^L_j = w_j - p^{\setminus} \cdot e^{\setminus}_j$ can be negative, then

[4] A space is contractible if it can be continuously shrunk to a point within this space.

[5] The proof of Lemma 7 can be found in Balasko (2011)

e_{-m} could be out of the set Ω^{-m}, which makes the demand function f_i undefined. Remark that in the classical model without externalities, the demand function of consumer i depends only on prices and his wealth so we have other ways [6] to prove the diffeomorphism property even for the economy where Ω is a product of open convex subsets of $\mathbb{R}^{Lm}$. However, those methods are difficult to apply here with the presence of externalities. Therefore, the diffeomorphism property of the equilibrium manifold E in the economy with endowment space $\Omega = \mathbb{R}^{Lm}_{++}$ and endowment externalities is still an open question for now.

Moreover, in this unbounded economy, all above properties in Proposition 4 and in Theorem 5 still hold true provided the following additional assumption.

- (B) ***Boundedness from below*** For any consumer $i = 1, \dots, m$, for any compact sets $K \subset \Omega$, the set $\{f_i(p, e_{-m}, p \cdot e_m) \mid (e_1, \dots, e_m) \in K \text{ and } p \in \mathbb{S}\}$ is bounded from below.

Remark that without the boundedness from below of the consumption set, equilibrium may not exist, and neither may the regular economies. Therefore, the above assumption is technically sufficient to ensure the existence result. Combining with Assumption 1, one can prove the properness of the natural projection. Then the other properties are proved by using the same argument in the proof of Theorem 5.

4 Full endowment externalities

In the previous section, the behaviour of consumers is affected by the endowments of the others and the wealth of one specific consumer (i.e, consumer m), which is a very particular function of the endowment. This particular dependence helps us to obtain the important properties of regular economies without any additional assumption; however, it is asymmetric. Therefore, in this section, we want to tackle endowment-regarding preferences in its full generality. Precisely, we study economies in which the behaviour of every consumer depends on the endowments of all consumers. So all consumers are treated symmetrically.

In the classical exchange model without externalities or the previous section, one can easily apply the aggregate excess demand approach to analyse the equilibrium properties. The reason is that the columns of the Jacobian matrix of the aggregate total demand with respect to the specific consumer (e.g., consumer m) are proportional to each other. Thus it is easy to extract a submatrix of this Jacobian matrix, which has a full row rank. However, with

[6] See for example Schecter (1979) and Balasko (1975).

this full endowments externalities, the demand functions are now in the very general form $f_i(p, p \cdot e_i, e)$. Thus, studying directly the rank of the Jacobian matrix $Dz^{\backslash}(p, e)$ is by no means an easy task.

Therefore, following works by Smale (1974), Villanacci et al. (2002), and Bonnisseau and del Mercato (2010), we use the extended approach, in which we use first-order conditions and market clearing conditions to characterise the set of equilibria. Using this extended approach, we can overcome the difficulty by directly introducing an additional assumption. Nevertheless, we also show that this approach is equivalent to the aggregate excess demand approach. Now we begin with some basic assumptions on the utility functions.

4.1 Utility functions and basic assumptions

Each consumer has preferences described by a utility function u_i. The economy now is $\mathcal{E} = (u_i, e_i)_{i=1,2,\dots,m}$ described by all initial endowments and m-tuple of utility functions. The endowment space and the consumption spaces are the strictly positive orthants: $\Omega = \mathbb{R}^{Lm}_{++}$ and $X_i = \mathbb{R}^L_{++}$ for all $i = 1, \dots, m$.

The utility function is a function from $\mathbb{R}^L_{++} \times \mathbb{S} \times \Omega$ to $\mathbb{R}$, that associates the consumer i's utility level $u_i(x_i, p, e)$ with the consumption x_i, the price vector p and the vector of all initial endowments $e = (e_1, \dots, e_m)$.

We extend below the classical assumptions on utility functions. We recall that S is the unit simplex of $\mathbb{R}^L$.

Assumption 8 *For all* $i = 1, 2, \dots, m$

(1) u_i *is* $\mathcal{C}^2$ *on* $\mathbb{R}^L_{++} \times \mathbb{S} \times \Omega$.
(2) For each $(p, e) \in \mathbb{S} \times \Omega$, *the function* $u_i(., p, e)$ *is differentiably strictly increasing, i.e., for every* $x_i \in \mathbb{R}^L_{++}$, $D_{x_i} u_i(x_i, p, e) \in \mathbb{R}^L_{++}$.
(3) For each $(p, e) \in \mathbb{S} \times \Omega$, *the function* $u_i(., p, e)$ *is differentiably strictly quasi-concave, i.e., for every* $x_i \in \mathbb{R}^L_{++}$, $D^2_{x_i} u_i(x_i, p, e)$ *is negative definite on* $\operatorname{Ker} D_{x_i} u_i(x_i, p, e)$.
(4) The mapping $\tilde{u}_i$ *defined by* $\tilde{u}_i(x_i, q, e) = u_i(x_i, (1/q^L)q, e)$ *and* $D\tilde{u}_i(x_i, q, e)$ *defined by* $D\tilde{u}_i(x_i, q, e) = Du_i(x_i, (1/q^L)q, e)$ *for all* $(x_i, q, e) \in \mathbb{R}^L_{++} \times (S \cap \mathbb{R}^L_{++}) \times \Omega$ *can be continuously extended to* $\mathbb{R}^L_{++} \times S \times \Omega$ *and, for each* $(q, e) \in S \times \Omega$,
   (i) for every $u \in \operatorname{Im} \tilde{u}_i(., q, e)$, $cl_{\mathbb{R}^L}\{x_i \in \mathbb{R}^L_{++} : \tilde{u}_i(x_i, q, e) \geq u\} \subseteq \mathbb{R}^L_{++}$;
   (ii) for every $x_i \in \mathbb{R}^L_{++}$, $D\tilde{u}_i(x_i, q, e) \in \mathbb{R}^L_{++}$.

We recall that the equilibrium is formally defined as follow.

Definition 9 *A vector* $(x, p^\backslash) = (x_1, \ldots, x_m, p^\backslash) \in \mathbb{R}^{Lm}_{++} \times \mathbb{R}^{L-1}_{++}$ *is an equilibrium for a vector of the initial endowments* $e = (e_1, \ldots, e_m) \in \Omega$ *if*

(1) for all $i = 1, 2, \ldots, m$*, consumer* i *solves the following problem.*

$$\max_{x_i \in \mathbb{R}^L_{++}} u_i(x_i, p, e) \quad s.t. \quad p \cdot x_i \le p \cdot e_i. \tag{P}$$

(2) $x \in \mathbb{R}^{Lm}_{++}$ *satisfies the market clearing condition*

$$\sum_{i=1}^{m} x_i = \sum_{i=1}^{m} e_i$$

Define the set of endogenous variables as $\Xi := \mathbb{R}^{Lm}_{++} \times \mathbb{R}^{L}_{++} \times \mathbb{R}^{L-1}_{++}$, with generic element $\xi := (x, \lambda, p^\backslash) := \big((x_i, \lambda_i)_{i=1,2,\ldots,m}, p^\backslash\big)$. By Assumption 8 and classical analysis of the optimization problem, we now describe the extended equilibria using the system of the Kuhn-Tucker conditions and market clearing condition as the following definition.

Definition 10 $\xi^* \in \Xi$ *is an extended equilibrium of the economy* $\mathcal{E}$ *if* $F_e(\xi^*) = 0$ *where the equilibrium function* $F_e : \Xi \to \mathbb{R}^{dim \Xi}$ *defined as:*

$$F_e(\xi) = \left(\left(F_e^{(i,1)}(\xi), F_e^{(i,2)}(\xi) \right)_{i=1,2,\ldots,m}, F_e^M(\xi) \right).$$

where for each $i = 1, 2, \ldots, m$

$F_e^{(i,1)}(\xi) := D_{x_i} u_i(x_i, p, e) - \lambda_i p$
$F_e^{(i,2)}(\xi) := -p \cdot (x_i - e_i)$

and $F_e^M(\xi) := \sum_{i=1}^{m} x_i^\backslash - \sum_{i=1}^{m} e_i^\backslash.$

Recall that in the aggregate excess demand approach, equilibria are considered as the solutions of the aggregate excess demand function $z : \mathbb{S} \times \Omega \to \mathbb{R}^L$, which is defined as

$$z(p, e) = \sum_{i=1}^{m} f_i(p, p \cdot e_i, e) - \sum_{i=1}^{m} e_i$$

where $f_i : \mathbb{S} \times \mathbb{R}_{++} \times \Omega \to \mathbb{R}^L_{++}$ is the demand function. Remark that under Assumption 8, it is easy to prove that the demand functions are $\mathcal{C}^1$ and satisfy the conditions (W) and (D) in Assumption 1. Therefore, applying the same proof as in point (iii) of Theorem 5, the equilibrium exists for all $e \in \Omega$.

In the excess demand approach, regular economy e is defined as the regular value of the mapping $z_e^\backslash : \mathbb{R}^{L-1}_{++} \to \mathbb{R}^{L-1}, p^\backslash \mapsto z^\backslash(p,e)$. In the extended approach, $e \in \Omega$ is a regular economy if 0 is a regular value of the mapping F_e where the function $F_e : \Xi \times \Omega \to \mathbb{R}^{dim\Xi}$ is defined as $F_e(\xi) := F(\xi, e)$. Precisely, $e \in \Omega$ is a regular economy if for each $\xi^* \in F_e^{-1}(0)$, the differential mapping $D_\xi F_e(\xi^*)$ is onto. We use the same word "regular economy" since the definitions in these two approaches are equivalent as a consequence of the following lemma.

Lemma 11 *Let (p^*, e^*) be an equilibrium and (ξ^*, e^*) be the associated extended equilibrium. Under Assumption 8, the Jacobian matrix $D_{p^\backslash} z^\backslash(p^*, e^*)$ has full rank if and only if the Jacobian matrix $D_{(x,\lambda,p^\backslash)} F(\xi^*, e^*)$ has full rank.*

Proof. See Appendix A. ■

Indeed, e is regular economy in the aggregate excess demand approach if and only if for any associated equilibrium prices p, $D_{p^\backslash} z^\backslash(p,e)$ has full rank if and only if $D_{(x,\lambda,p^\backslash)} F(\xi, e)$ has full rank if and only if e is regular economy in the extended approach. Therefore, we can use the same notation $\mathcal{R}$ for naming the set of regular economies without any confusion.

4.3 The additional assumption

To study the equilibrium properties and the genericity of regular economies, we introduce here the following additional assumption on endowment-regarding utility functions.

Assumption 12 *Take any $(x,e,p) \in \mathbb{R}^{Lm}_{++} \times \Omega \times \mathbb{S}$. If for all $i = 1, \dots, m$ the gradient $D_{x_i} u_i(x_i, p, e)$ is proportional to the price vector p, then there exists at least one consumer k such that the following properties are satisfied.*[7]

For all $v_k \in \operatorname{Ker} D_{x_k} u_k(x_k, p, e)$:

$$v_k D^2_{x_k} u_k(x_k, p, e)(v_k) + v_k D^2_{e_k x_k} u_k(x_k, p, e)(v_k) < 0 \text{ whenever } v_k \neq 0, \tag{4}$$

$$v_k \left(D^2_{e_k x_i} u_i(x_i, p, e) \right)^t \text{ is proportional to } D_{x_i} u_i(x_i, p, e) \ \forall\ i \neq k. \tag{5}$$

This assumption is a global assumption on the preferences of all consumers with respect to the endowment externalities generated by only one consumer, namely consumer k. Note that we only consider the effect on the marginal rates of substitution and not on the welfare level.

[7] Without any loss of generality, the vector $v \in \mathbb{R}^n$ is treated as a row matrix. (v) denotes the transpose of v, which is a column matrix. Let A be a square matrix of size n. The formula $vA(v)$ is then treated as a normal product of three matrices.

Condition (4) holds true if $D^2_{x_k} u_k(x_k, p, e) + D^2_{e_k x_k} u_k(x_k, p, e)$ is negative definite on Ker $D_{x_k} u_k(x_k, p, e)$. From Point 3 of Assumption 8, we already know that the matrix $D^2_{x_k} u_k(x_k, p, e)$ is negative definite on Ker $D_{x_k} u_k(x_k, p, e)$. So we get the result if the norm of the matrix $D^2_{e_k x_k} u_k(x_k, p, e)$ is small enough with respect to the norm of $D^2_{x_k} u_k(x_k, p, e)$, which means that the endowment effect on the Marginal Rate of Substitution (MRS) is small with respect to the direct effect of the consumption.

For the second condition, $v_k \Big(D^2_{e_k x_i} u_i(x_i, p, e) \Big)^t$ is proportional to $D_{x_i} u_i(x_i, p, e)$ then it is proportional to the price vector p. This can be interpreted as the first-order of external effect on MRS of consumer $i \neq k$ is vanishing if we change the endowment e_k along the direction v_k which is orthogonal to the price p, that is a change along which the income the consumer k is constant.

Indeed, let us consider the marginal rate of substitution between commodities l and L, for any $l = 1, \ldots, L-1$: $MRS_i^{l,L} = \partial_{x_i^l} u_i / \partial_{x_i^L} u_i$. By a simple calculation, the directional derivative of $MRS_i^{l,L}$ with respect to e_k along the vector v_k is:

$$D_{e_k} MRS_i^{l,L}(v_k) = \frac{(D^2_{e_k x_k^l} u_i \cdot v_k) \partial_{x_i^L} u_i - (D^2_{e_k x_k^L} u_i \cdot v_k) \partial_{x_i^l} u_i}{(\partial_{x_i^L} u_i)^2}$$

Since $v_k \Big(D^2_{e_k x_i} u_i(x_i, p, e) \Big)^t$ is proportional to $D_{x_i} u_i(x_i, p, e)$, we have:

$$\frac{\sum_{l'=1}^{L} v_i^{l'} \partial^2_{e_k^{l'} x_k^l} u_i}{\partial_{x_i^l} u_i} = \frac{\sum_{l'=1}^{L} v_i^{l'} \partial^2_{e_k^{l'} x_k^L} u_i}{\partial_{x_i^L} u_i}$$

i.e.,

$$\frac{D^2_{e_k x_k^l} u_i \cdot v_k}{\partial_{x_i^l} u_i} = \frac{D^2_{e_k x_k^L} u_i \cdot v_k}{\partial_{x_i^L} u_i}$$

Thus, $(D^2_{e_k x_k^l} u_i \cdot v_k) \partial_{x_i^L} u_i = (D^2_{e_k x_k^L} u_i \cdot v_k) \partial_{x_i^l} u_i$, which implies $D_{e_k} MRS_i^{l,L}(v_k) = 0$.

Furthermore, Condition (5) allows us to evaluate the effect of endowments e_k on the demand of the others consumers as presented in the following proposition.

Proposition 13 *Under Assumption 12, the first-order effect of the endowments of consumer k on the demand of the others is vanishing if we change the endowment e_k along a direction orthogonal to the price vector p.*

Proof. See Appendix A. ∎

Therefore, Assumption 12 put some restrictions on the effect of a change in the endowment e_k of consumer k on the behaviour of all consumers. Precisely, the first-order effect on marginal rates of substitution and also the demand of all other consumers is vanishing. In some sense, only a change in the wealth of consumer k affects the behaviour of the other consumers. Assumption 12 means that the effect of the endowments of consumer k is first-order equivalent to a wealth effect. That is the reason why Assumption 12 is satisfied in the exchange economy with endowment-wealth externalities and, a fortiori, without any externalities or with wealth concern as in Balasko (2015).

In the case of Smale (1974) with no externalities at all, Assumption 12 is clearly satisfied since the derivative $D^2_{e_k x_i} u_i(x_i, p, e)$ is equal to 0 for any $k, i = 1, \ldots, m$.

In Balasko (2015) with wealth-concern, the utility functions have the following form:

$$u_i(x_i, p, e) = V_i(x_i, w_{-i}), \quad \text{where } w_j = p \cdot e_j \ \forall j$$

Considering any consumer k, we have $\partial^2_{e_k^{l'} x_i^l} u_i(x_i, p, e) = p^{l'} \partial^2_{w_k x_i^l} V_i(x_i, w_{-i})$ for any $i \neq k$ and for any $1 \leq l', l \leq L$. Then for all $l = 1, \ldots, L$ and for all $i \neq k$ we get:

$$D^2_{e_k x_i^l} u_i(x_i, p, e)(v_k) = \partial^2_{w_k x_i^l} V_i(x_i, w_{-i})(p \cdot v_k)$$

Since v_k is taken in the orthogonal complement of the price vector p, $p \cdot v_k = 0$. Thus, for any $i \neq k$, $D^2_{e_k x_i^l} u_i(x_i, p, e)(v_k) = 0$, and then $v_k \Big(D^2_{e_k x_i} u_i(x_i, p, e) \Big)^t = 0$, which is obviously proportional to $D_{x_i} u_i(x_i, p, e)$. So Condition (5) holds true. Condition (4) holds also true since

$$D^2_{e_k x_k} u_k(x_k, p, e) = D^2_{e_k x_k} V_k(x_k, w_{-k}) = 0.$$

So clearly we get $v_k D^2_{e_k x_k} u_k(x_k, p, e)(v_k) = 0$. Hence, the strict quasi-concavity of u_k implies the desired inequality. Therefore, Assumption 12 is valid with wealth-concern and even the two conditions (4) and (5) hold true for all consumers.

In section 3, we extend the Balasko's paper by considering the individual demand function depending on the endowment of $m - 1$ consumers and on wealth of consumer m. Then, the utility functions are written in the following form

$$\begin{aligned} \forall i = 1, \ldots, m-1 \quad & u_i(x_i, p, e) = V_i(x_i, e_1, \ldots, e_{m-1}, w_m), \ \text{where } w_m = p \cdot e_m \\ \text{and} \quad & u_m(x_m, p, e) = V_m(x_m, e_{-m}). \end{aligned}$$

Following the same argument as above, the two conditions (4) and (5) holds true with $k = m$. Therefore, Assumption 12 is also valid.

Before ending this subsection, we compare Assumption 12 with the corresponding assumption in Bonnisseau and del Mercato (2010). The authors studied the case of the consumption externalities, i.e. the preference of consumer i is represented by the function $u_i(x_i, x_{-i})$ from $\mathbb{R}^L_{++} \times \mathbb{R}^{L(m-1)}_{++}$ to $\mathbb{R}$. The externalities are different but the interpretations of external effects are similar. The additional assumption introduced in Bonnisseau and del Mercato (2010) is the following one.

Assumption 14 *Let* $(x, v) \in \mathbb{R}^{Lm}_{++} \times \mathbb{R}^{Lm}$. *If* $\sum_{i=1}^{m} v_i = 0$, *for all consumer* i, $v_i \in \operatorname{Ker} D_{x_i} u_i(x_i, x_{-i})$ *and the gradients* $\left(D_{x_i} u_i(x_i, x_{-i})\right)_{i=1,\ldots,m}$ *are collinear, then the following inequality is satisfied for all consumer* $k = 1, .., m$.

$$\sum_{i=1}^{m} v_i D^2_{x_i x_k} u_k(x_k, x_{-k})(v_k) < 0 \quad \textit{whenever } v_k \neq 0. \tag{6}$$

The forms of the two assumptions are quite different but by deepening the analysis of Inequality (6), we show that the two assumptions are actually similar.

Indeed, Inequality (6) can be rewritten as

$$A(v) = v_k D^2_{x_k} u_k(x_k, x_{-k})(v_k) + \sum_{i \neq k}^{m} v_k \left(D^2_{x_i x_k} u_k(x_k, x_{-k})\right)^t (v_i) < 0.$$

For a given v_k, $A(v_{-k}, v_k)$ is an affine function with respect to (v_{-k}). So Inequality (6) means that the affine function $A(\cdot, v_k)$ is bounded from above by 0 on the affine space

$$F = \{(v_{-k}) \in \mathbb{R}^{L(m-1)} | \sum_{i \neq k} v_i = -v_k, v_i \in \operatorname{Ker} D_{x_k} u_k(x_k, x_{-k}) \text{ for all } i \neq k\}.$$

Therefore, we must have $\left(v_k \left(D^2_{x_i x_k} u_k(x_k, x_{-k})\right)^t\right)_{i \neq k} \in F^{\perp}$, which implies that there exists $a_i \in \mathbb{R}$ and $b \in \operatorname{Ker} D_{x_k} u_k(x_k, x_{-k})$ such that

$$v_k \left(D^2_{x_i x_k} u_k(x_k, x_{-k})\right)^t = a_i D_{x_k} u_k(x_k, x_{-k}) + b \text{ for all } i \neq k.$$

This means there exist $m-1$ linear mappings (α_i) from $\operatorname{Ker} D_{x_k} u_k(x_k, x_{-k})$ to $\mathbb{R}$ and a linear mapping β from $\operatorname{Ker} D_{x_k} u_k(x_k, x_{-k})$ to itself such that for all $v_k \in \operatorname{Ker} D_{x_k} u_k(x_k, x_{-k})$,

$$v_k \left(D^2_{x_i x_k} u_k(x_k, x_{-k})\right)^t = \alpha_i(v_k) D_{x_k} u_k(x_k, x_{-k}) + \beta(v_k) \text{ for all } i \neq k.$$

If we plug this formula in the above inequality, since $v_i \in \operatorname{Ker} D_{x_k} u_k(x_k, x_{-k})$ for all i and $\sum_{i \neq k}^{m} v_i = -v_k$, we get:

$$v_k D^2_{x_k} u_k(x_k, x_{-k})(v_k) + \beta(v_k)(\sum_{i \neq k}^{m} v_i) = v_k D^2_{x_k} u_k(x_k, x_{-k})(v_k) - \beta(v_k)(v_k) < 0$$

which can be interpreted similarly as Condition (4).

4.4 Equilbrium manifold and genericity of regular economies

Assumption 12 plays an important role as a sufficient condition to obtain the following key lemma of this subsection.

Lemma 15 *Under Assumption 8 and Assumption 12, 0 is a regular value for $F(\xi, e)$.*

Proof. See Appendix A. ∎

Directly from Lemma 15 and the Regular Value Theorem, we have the following proposition.

Proposition 16 *Under Assumption 8 and Assumption 12, $F^{-1}(0)$ is a $\mathcal{C}^1$ submanifold of $\Xi \times \Omega$ of dimension Lm.*

Once again, we recall that the pair $(p, e) \in \mathbb{S} \times \Omega$ is an equilibrium if $z(p, e) = 0$. The set of equilibria is denoted by E. Remark that under Assumption 8, the demand function f_i obtained from the maximization problem (P) satisfies Walras law and Desirability. Thus the pair $(p, e) \in \mathbb{S} \times \Omega$ is an equilibrium if and only if $z^{\setminus}(p, e) = 0$.

Same argument as Lemma 11, we establish a relationship between rank of two Jacobian matrices $D_e z^{\setminus}(p, e)$ and $D_{(x,\lambda,e)} F(\xi, e)$ as the next lemma, which helps us to study the equilibrium set E.

Lemma 17 *Let (p^*, e^*) be an equilibrium and (ξ^*, e^*) be the associated extended equilibrium. Under Assumption 8, for all $j = 1, \ldots, m$, the Jacobian matrix $D_{e_j} z^{\setminus}(p^*, e^*)$ has full rank if and only if the Jacobian matrix $D_{(x,\lambda,e_j)} F(\xi^*, e^*)$ has full rank.*

From Lemma 15, $\operatorname{rank} D_{x,\lambda,e_k} F(\xi^*, e^*) = (L+1)m + L - 1$, which follows $\operatorname{rank} D_{e_k} z^{\setminus}(p^*, e^*) = L - 1$, i.e., full row rank. As in the beginning of this section, we recall that in the classical model without extenalities and Balasko (2015), it is easy to extract a submatrix of Jacobian matrix of aggregate excess demand which has full row rank. And now, thanks to Lemma 17, we can extract such a submatrix and this submatrix is related to the endowments e_k

of consumer k. The special consumer k in Assumption 12 plays the important role as consumer m in Section 3.

The matrix $Dz^{\setminus}(p^*, e^*)$ now clearly has full row rank, which implies 0 is a regular value for mapping $z^{\setminus}$. Therefore, besides the classical assumption, Assumption 12 helps us to recapture the local structure of the equilibrium set E as the following theorem.

Theorem 18 *Under Assumption 8 and Assumption 12, the equilibrium set E is a Lm-dimensional submanifold of $\mathbb{S} \times \Omega$.*

Recall that the demand functions $f_i(p, p \cdot e_i, e)$ is $\mathcal{C}^1$ and satisfies Walras law (W) and Desirability (D). And thanks to Theorem 18, the equilibrium set E is still a Lm-dimension manifold. Therefore, it suffices to reproduce the argument in the previous section for the genericity property of the regular economies. Here we provide another argument using in the extended approach to obtain those results.

Let us introduce the restriction Π to $F^{-1}(0)$ of the projection of $\Xi \times \Omega$ onto Ω,

$$\Pi : (\xi, e) \in F^{-1}(0) \to \Pi(\xi, e) := e \in \Omega$$

The important property of Π is given in the following lemma.

Lemma 19 *Under Assumption 8, the projection mapping $\Pi : F^{-1}(0) \to \Omega,\ (\xi, e) \mapsto e$ is proper.*

Proof. See Appendix A. ■

We now can state the important theorem of generic regularity.

Theorem 20 *Under Assumption 8 and Assumption 12, the set of regular economies $\mathcal{R}$ is an open and full Lebesgue measure subset of Ω.*

Proof. See Appendix A. ■

In addition, by directly applying the Stack of Records Theorem (see Guillemin and Pollack (1974)), we can obtain the important properties of the regular economies, which are similar to Theorem 5.

Theorem 21 *Under Assumption 8 and Assumption 12, for given $e \in \mathcal{R}$ we have*

1. *there exists $r \in \mathbb{N} \setminus \{0\}$ such that $F_e^{-1}(0) = \{\xi^1, \dots, \xi^r\}$;*
2. *there exist an open neighborhood I of e in Ω, and for each $i = 1, 2, \dots, r$, an open neighborhood U_i of ξ^i in Ξ and a continuous function $g_i : I \to U_i$ such that*

(a) $U_i \cap U_j = \emptyset$ *if* $i \neq j$,
(b) g_i *is* C^1 *and* $g_i(e) = \xi^i$,
(c) *for all* $e' \in I$, $F_{e'}^{-1}(0) = \{g_i(e') : \ i = 1, \ldots, r\}$,
(d) *the economies* $e' \in I$ *are regular.*

5 Conclusion and remark

In this paper, we show that under very general assumptions on the demand functions, almost all critical important properties of the equilibria and the regular economies are unaltered in the model with endowment-wealth concern. We also propose the additional assumption to ensure that these properties hold true in the exchange model with a general form of endowments externalities. We remark that in the classical exchange model without externalities and also in the exchange economy with only endowment externalities as in Section 3 and 4, our key additional assumption concerns only one "particular" consumer whereas with standard other-regarding preferences, all consumers are involved as it appears in Assumption 14 above.

This paper can be extended in several ways. One interesting question is to find the weakest possible sufficient condition for the generic regularity of equilibria with endowments externalities. Our assumption still implies some restrictions on externalities as the demand remains unchanged along with the directions which have no effects on the wealth of consumers. Thus, the next step might be to have a deeper analysis of the external wealth effect.

Another research domain is to study the equilibrium with more general forms of externalities. Remark that along with this paper, the endowments externalities allow us to charaterise the competitive equilibrium by only one equation $z(p, e) = 0$. This has many advantages since studying the equilibrium: endowments, prices, and consumption now reduces to look at prices and endowments. The latter's dimension is much less than the former's. With the appearance of other externalities, we cannot use this aggregate excess demand approach. For example, the utility function of consumer i depends on the consumption of the others, the endowments of all consumers and the prices vector: $u_i(x_i, x_{-i}, p, e)$. We can find the demand function; however, this demand function depends on the consumption of the other. The aggregate demand function is not the function of prices and endowment anymore. The consumption enters it in a complicated way. The extended approach then is a good approach for this general form of externalities. However, this line of research still need further research.

Besides, in the above economy with the utility function $u_i(x_i, x_{-i}, p, e)$, we can obtain the generic of regular economies and the associated nice properties under some classical assumptions on the utility functions and the following additional assumption.

Assumption 22 *Let $(x, e, v, p) \in \mathbb{R}^{Lm}_{++} \times \Omega \times \mathbb{R}^{Lm} \times \mathbb{S}$. If for all consumer i, $v_i \in \operatorname{Ker} D_{x_i} u_i(x_i, x_{-i}, p, e)$ and the gradient $D_{x_i} u_i(x_i, x_{-i}, p, e)$ is proportional to the prices vector p, then the following property is satisfied for all consumers $k = 1, .., m$.*

$$v_k D^2_{e_k x_k} u_k(x_k, x_{-k}, p, e)(v_k) + v_k D^2_{x_k} u_k(x_k, x_{-k}, p, e)(v_k) < 0 \text{ whenever } v_k \neq 0,$$
$$\Big(v_i D^2_{x_k x_i} u_i(x_i, x_{-i}, p, e) + v_i D^2_{e_k x_i} u_i(x_i, x_{-i}, p, e)\Big)(v_k) = 0 \text{ for all } i \neq k.$$

Assumption 22 has the same spirit as Assumption 12 and Assumption 14 in the sense that the effect of externalities on the marginal rates of substitution of a consumer is small enough with respect to the effect of a change of her own consumption. These assumptions together bring the idea of how to unify the sufficient condition for the genericity of regular economies. This is also an interesting research field beside the existence result.

Appendix A

Proof of Theorem 3. By definition, the equilibrium set E is the preimage of $0 \in \mathbb{R}^{L-1}$ by the mapping $z^\setminus : \mathbb{S} \times \Omega \to \mathbb{R}$. We apply the Regular Value Theorem to the mapping $z^\setminus$.

It suffices to prove that the element $0 \in \mathbb{R}^{L-1}$ is a regular value of the smooth mapping $z^\setminus$. This is equivalent to prove that the derivative of the mapping $(p, e) \to z^\setminus(p, e)$ has full rank. Therefore, we manage to prove that the Jacobian matrix of $z^\setminus$ at $(p, e) \in \mathbb{S} \times \Omega$ has rank $L - 1$ since that matrix has $L - 1$ rows and $Lm + L - 1$ columns.

To prove the rank property, it suffices to extract a submatrix that has rank $L - 1$ from the Jacobian matrix. Let us look at the $(L - 1) \times L$ block defined by derivatives of $z^\setminus$ with respect to the coordinates $e^1_m, \dots, e^L_m$ of e_m, the endowment of consumer m. In the computation, we apply the chain rule and the fact that consumer i's demand depends on the m's wealth. This gives us the matrix.

$$\begin{pmatrix} p_1 \sum_i \dfrac{\partial f_i^1}{\partial w_m} - 1 & \dots & p_{L-1} \sum_i \dfrac{\partial f_i^1}{\partial w_m} & \sum_i \dfrac{\partial f_i^1}{\partial w_m} \\ p_1 \sum_i \dfrac{\partial f_i^2}{\partial w_m} & \cdots & p_{L-1} \sum_i \dfrac{\partial f_i^2}{\partial w_m} & \sum_i \dfrac{\partial f_i^2}{\partial w_m} \\ \vdots & \ddots & \vdots & \vdots \\ p_1 \sum_i \dfrac{\partial f_i^{L-1}}{\partial w_m} & \cdots & p_{L-1} \sum_i \dfrac{\partial f_i^{L-1}}{\partial w_m} - 1 & \sum_i \dfrac{\partial f_i^{L-1}}{\partial w_m} \end{pmatrix}$$

We multiply the last column by p_l and subtract it from the l-th column for all $l = 1, \dots, L-1$. This yields the $(L-1) \times L$ matrix

$$\begin{pmatrix} -1 & 0 & \cdots & 0 & \sum_i \dfrac{\partial f_i^1}{\partial w_m} \\ 0 & -1 & \cdots & 0 & \sum_i \dfrac{\partial f_i^2}{\partial w_m} \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & \sum_i \dfrac{\partial f_i^{L-1}}{\partial w_m} \end{pmatrix}$$

which has the same rank. The rank of this new matrix is clearly equal to $L-1$ since the block made of its first $L-1$ columns obviously has rank $L-1$. Therefore, from the Regular Value Theorem, the equilibrium set E is a smooth submanifold of $\mathbb{S} \times \Omega$ of dimension Lm. ■

Proof of Theorem 6.

We first prove the two following useful properties of the mappings ρ and θ defined in Subsection 3.3.

i) If $(p^*, e^*) \in E$ then $\rho \circ \theta(p^*, e^*) = (p^*, e^*)$. Indeed, take any $(p^*, e^*) \in E$. From (1) and (2) we have:

$$\theta(p^*, e^*) = (p^*, p^* \cdot e_1^*, \dots, p^* \cdot e_m^*, e_{-m}^{*\backslash}) = (p, w_1^*, \dots, w_m^*, e_{-m}^{*\backslash})$$
$$\rho\Big(\theta(p^*, e^*)\Big) = \rho(p^*, w_1^*, \dots, w_m^*, e_{-m}^{*\backslash}) = (p^*, e_1, \dots, e_{m-1}, e_m)$$

where for all i: $w_i^* = p^* \cdot e_i^*$, for $i = 1, 2, \dots, m-1$, $e_i = (e_i^{*\backslash}, w_i^* - p^{*\backslash} \cdot e_i^{*\backslash})$ and $e_m = \sum_{i=1}^{m} f_i(p^*, e_{-m}, w_m^*) - \sum_{i=1}^{m-1} e_i$.
Since $p^* = (p^{*\backslash}, 1)$ and $w_i^* = p^* \cdot e_i^*$, we have $w_i^* - p^{*\backslash} \cdot e_i^{*\backslash} = e_i^{*L}$, which implies $e_i = e_i^*$ for $i = 1, 2, .., m-1$. Also from the very definition of the excess demand function z, it is clear that $z(p^*, e^*) = 0$ since $(p^*, e^*) \in E$.

The last component e_m then equals to e_m^* by the equation:

$$e_m = \sum_{i=1}^{m} f_i(p^*, e_1^*, \ldots, e_{m-1}^*, p^* \cdot e_m^*) - \sum_{i=1}^{m-1} e_i^* = \sum_{i=1}^{m} e_i^* - \sum_{i=1}^{m-1} e_i^* = e_m^*.$$

So we have $\rho\Big(\theta(p^*, e^*)\Big) = (p^*, e^*)$

ii) $\theta \circ \rho = \mathrm{Id}_{\mathbb{S}\times\mathbb{R}^m\times\mathbb{R}^{(L-1)(m-1)}}$.
To get this equality, we are using the same computations as above and the Walras law (W) in Assumption 1, which is $p \cdot f_i(b) = p \cdot e_i$ for all $i = 1, \ldots, m-1$ and $p \cdot f_m(b) = w_m$.

Now, our strategy is to show that the equilibrium manifold E is the image of the mapping ρ, and then to apply Lemma 7.

We easily see that both mappings θ and ρ are smooth. We already have the equality:

$$\theta \circ \rho = \mathrm{Id}_{\mathbb{S}\times\mathbb{R}^m\times\mathbb{R}^{(L-1)(m-1)}}$$

Take any $(p, w_1, \ldots, w_m, e_1, \ldots, e_{m-1}) \in \mathbb{S} \times \mathbb{R}^m \times \mathbb{R}^{(L-1)(m-1)}$. We prove that $(p, e_1, \ldots, e_m) \in E$. The formula defining ρ implies that $w_i = p \cdot e_i$ for all $i = 1, \ldots, m$. The inner product of equality (3) with the price vector $p \in \mathbb{S}$ combined with the Walras law (W) implies $w_m = p \cdot e_m$. Then, equality (3) can be reformulated as:

$$\sum_{i=1}^{m} f_i(p, e_{-m}, p \cdot e_m) = \sum_{i=1}^{m} e_i$$

which is the equilibrium equation. This proves the inclusion $\mathrm{Im}(\rho) \subset E$.

Now let $(p, e) \in E$. From (i) above, we have the equality $\rho \circ \theta(p, e) = (p, e)$, which implies the other inclusion $E \subset \mathrm{Im}(\rho)$. Therefore, by Lemma 7 , E is a smooth submanifold of $\mathbb{S} \times \Omega$ diffeomorphic to the set $\mathbb{S} \times \mathbb{R}^m \times \mathbb{R}^{(L-1)(m-1)}$. And since $\mathbb{S} = \mathbb{R}^{L-1}_{++} \times \{1\}$, the set $\mathbb{S} \times \mathbb{R}^m \times \mathbb{R}^{(L-1)(m-1)}$ is diffeomorphic to $\mathbb{R}^{Lm}$, this implies the result. ■

Proof of Proposition 4.

i) Smoothness. The natural projection $\pi : E \to \Omega$ is the composition of two mappings: the canonical embedding $E \to \mathbb{S} \times \Omega$ (i.e., the mapping $(p, e) \to (p, e)$), which is smooth because E is a smooth submanifold of $\mathbb{S} \times \Omega$ (Theorem 3), and the projection mapping $\mathbb{S} \times \Omega \to \Omega$ (i.e., the projection $(p, e) \to e$), which is itself a smooth mapping. The natural projection is therefore smooth as the composition of two smooth mappings.

ii) Properness. We prove that the preimage of every compact set is compact. Let K be a compact subset of the parameter space Ω. In order to prove that the set $\pi^{-1}(K)$ is compact, it is sufficient to show that every sequence (p^n, e^n) in $\pi^{-1}(K)$ has a convergent subsequence.

For this, we consider the auxiliary sequence $(q^n = p^n / \sum_{l=1}^{L} p^{n,l}, e^n)$, which remains in $S \times K$, which is a compact set. So, we can assume without loss of generality that there exists a subsequence still denoted by (q^n, e^n) converging to a limit $(q^*, e^*) \in S \times K$.

Since the sequence $(e^n)_{n \geq 0}$ is included in K, a compact set, it is bounded. Therefore, there exists $B \in \mathbb{R}^L_{++}$ such that the inequality $e_1^n + \ldots + e_m^n \leq B$ is satisfied for all n.

By the market clearing condition, we have the inequality:

$$0 \leq f_i(p^n, e_{-m}^n, p^n \cdot e_m^n) \leq B \text{ for any } i$$

Assume now the limit q^* does not belong to $\mathbb{R}^L_{++}$. Pick consumer i whose demand function f_i satisfied Desirability (D). Let $w_m^n = p^n \cdot e_m^n$. We remark that the sequence $(w_m^n / \sum_{l=1}^{L} p^{n,l} = q^n \cdot e_m^n)$ converges to $\tilde{w}_m^* = q^* \cdot e_m^* > 0$ since $e_m^* \in \mathbb{R}^L_{++}$ and $q^* \in \mathbb{R}^L_{+} \setminus \{0\}$.

So the sequence $b^n = (p^n, e_{-m}^n, w_m^n)$ satisfies the condition of the Desirability Assumption (D). Then it leads to $\limsup_{n \to +\infty} \|f_i(b^n)\| = +\infty$, which contradicts the above inequality. Therefore, the limit q^* belongs to $\mathbb{R}^L_{++}$ and the sequence (p^n) converges to $p^* = (1/q^{*,L})q^*$.

We conclude that the pair (p^*, e^*) is an equilibrium by the continuity of the aggregate excess demand function $(p, e) \to z(p, e)$. The equilibrium (p^*, e^*) therefore belongs to the preimage $\pi^{-1}(K)$. ■

Proof of Theorem 5. Almost all properties can be found in Chapter 7 in Balasko (2011). Nevertheless, for sake of completeness, we give the proofs here.

i) We already know by straightforward application of Sard's Theorem that the singular economies Σ is a set of Lebesgue measure zero in Ω. Moreover, the set of critical equilibria $\mathfrak{C}$ is closed, from which follows the set $\Sigma = \pi(\mathfrak{C})$ is closed by properness of mapping π.

ii) The set of regular economies $\mathcal{R} = \Omega \setminus \Sigma$ is the complement of the null closed subset Σ of space Ω. It is therefore open and full Lebesgue measure in Ω.

iii) The mapping $\pi : E \to \Omega$ is a smooth mapping between two smooth manifolds with the same dimension $dimE = dim\Omega = Lm$ and $e \in \mathcal{R}$ is a regular value of π. Therefore, by the Regular Value Theorem the preimage $\pi^{-1}(e)$ is a smooth submanifold of E with dimension 0, i.e, just a discrete set.

By properness of π, the preimage $\pi^{-1}(e)$ is compact. Therefore, the preimage $\pi^{-1}(e)$ is a finite set as a discrete and compact topological space. The proof of the existence and oddness is quite long so we present it at the end.

iv) By applying the Inverse Function Theorem at the regular point (p, e) for the natural projection $\pi : E \to \Omega$, there exist an open neighborhood $U \subset E$ of the regular equilibrium $(p, e) \in \mathfrak{R}$ such that $V = \pi(U)$ is open in E and the restriction $\pi|U : U \to V$ is a diffeomorphism.

We study the finite covering property of π in the sense that there exist an open neighborhood $U \subset \mathcal{R}$ of regular economy $e \in \mathcal{R}$ such that the preimage $\pi^{-1}(U)$ is the union of a finite number of pairwise disjoint open sets V_n such that the restriction $\pi_n : V_n \to U$ of the mapping π is a diffeomorphism for all n. This is direct implication of Stack of Records Theorem.

v) Let choose U and $(V_k)_{k=1,\dots,n}$ as in previous property. We compose the mapping $\pi_k^{-1} : U \to V_k$ with the embedding mapping $\mathbb{S} \times \Omega \to \mathbb{S}$ to define the smooth mapping $s_k : U \to \mathbb{S}$. Then we have

$$\pi^{-1}(e') = \bigcup_{1 \le k \le n} \{\pi_k(e')\} = \bigcup_{1 \le k \le n} \{(s_k(e'), e')\}$$

for all $e' \in U$.

This result immediately implies that the number of elements of the preimage $\pi^{-1}(e)$ is locally constant over $\mathcal{R}$.

For the property over the connected component, we recall that the connected component of a point in a topological space is the largest connected set containing that point. Let $N(e) = \#\pi^{-1}(e)$ denote the number of equilibria of regular economy $e \in \mathcal{R}$.This defines a function $N : \mathcal{R} \to \mathbb{N}$. Let us equip this set with the discrete topology, the topology where each subset is open (and also closed). The above result tells us for $e \in \mathcal{R}$, there exists an open neighborhood U of e contained in $\mathcal{R}$ where the number of equilibria $N(e)$ is constant. The function N is therefore said to be locally constant on $\mathcal{R}$. Our result then comes from the property that a locally constant function is necessarily constant on every connected component of its domain.

Proof of existence and oddness:

We consider an auxiliary classical exchange economy [8] where all demand functions $(f_i'(p, w_i))_{i=1}^m$ are identical and defined by $f_i'(p, w_i) = \left(w_i / L p^l\right)_{l=1}^L$. It is known that there is only one equilibrium for this economy whatever are the initial endowment, all economies are regular and the modulo 2 degree of the

[8] These demands derived from a standard Cobb-Douglas utility functions.

excess demand function ζ is equal to 1. Following contributions by Villanacci et al. (2002), Villanacci and Zenginobuz (2005) and del Mercato (2006), we now prove that the degree of z is equal to the degree of ζ by using Homotopy Theorem.

Theorem 23 (Homotopy Theorem) *Let M, N be $\mathcal{C}^2$ (boundaryless) manifolds with the same dimension. Let f, g be two $\mathcal{C}^2$ mappings from M to N with the homotopy $F : M \times [0,1] \to N$. Assume that:*

1) y is a regular value for f and $\#f^{-1}(y)$ is odd.
2) $F^{-1}(y)$ is compact

Then $g^{-1}(y)$ is compact and non-empty. Moreover, if y is also regular value for g then $\deg_2(f, y) = \deg_2(g, y)$.

Take any $e \in \Omega = \mathbb{R}^{Lm}_{++}$. We prove that there exists an homotopy mapping between $z^{\setminus}_e$ and $\zeta^{\setminus}_e$.

Consider the following homotopy $G(t, p) : [0, 1] \times \mathbb{S} \to \mathbb{R}^{L-1}$ as:

$$G(t, p) = tz^{\setminus}_e(p) + (1 - t)\zeta^{\setminus}_e(p)$$

Clearly, G is continuous and we have $G(0, .) = \zeta^{\setminus}_e$ and $G(1, .) = z^{\setminus}_e(p)$. We now prove that $G^{-1}(0)$ is compact.

Indeed, take any sequence $(t^n, p^n)_{n\geq 0} \subseteq G^{-1}(0)$. We prove that there exists a subsequence converging to a point in $[0, 1] \times \mathbb{S}$. We have:

$$t^n z^{\setminus}_e(p^n) + (1 - t^n)\zeta^{\setminus}_e(p^n) = 0 \tag{7}$$

which is equivalent to :

$$t^n \sum_i f^{\setminus}_i(p^n, e_{-m}, p^n \cdot e_m) + (1 - t^n) \sum_i f'^{\setminus}_i(p^n, p^n \cdot e_i) = t^n \sum_i e^{\setminus}_i + (1 - t^n) \sum_i e^{\setminus}_i = \sum_i e^{\setminus}_i$$

Note that $p^{n,L} = 1$ and all these demand functions $(f_i)_{i=1,\ldots,m}$ and $(f'_i)_{i=1,\ldots,m}$ satisfy the Walras law (W) so above equation is equivalent to the following equation:

$$t^n \sum_i f_i(p^n, e_{-m}, p^n \cdot e_m) + (1 - t^n) \sum_i f'_i(p^n, p^n \cdot e_i) = \sum_i e_i \tag{8}$$

Since $(t^n, q^n = p^n / \sum_{l=1}^L p^{n,l})_{n\geq 0} \subseteq [0, 1] \times S$, which is compact, there exists a subsequence converging to a point $(t^*, q^*) \in [0, 1] \times S$. We prove that $q^* \in \mathbb{R}^L_{++}$. Note that this implies that the sequence (p^n) converges to $p^* = (1/q^{*,L})q^*$. If it is not true, as in the proof of Proposition 4, using the Desirability (D) assumption satisfied by the demand functions f_i and f'_i of Consumer i, we get a contradiction with (8) since $\limsup \|t^n f_i(p^n, e_{-m}, p^n \cdot e_m) + (1-t^n) f_i'(p^n, p^n \cdot e_i)\| = +\infty$.

We know that 0 is a regular value for the mapping $\zeta_e^{\setminus}$ and $\#(\zeta_e^{\setminus})^{-1}(0) = 1$, which is an odd number. Therefore, from the Homotopy Theorem, $(z_e^{\setminus})^{-1}(0)$ is compact and non-empty so the equilibrium exists for any $e \in \Omega$. In other words, the set $\pi^{-1}(e)$ is non-empty for all $e \in \Omega$.

Moreover, if e is regular economy in our model. Then 0 is a regular value for both two mappings $z_e^{\setminus}$ and $\zeta_e^{\setminus}$. Therefore, the modulo 2 degree of these two mappings $z_e^{\setminus}$ and $\zeta_e^{\setminus}$ are equal. Since, $\deg_2(\zeta_e^{\setminus}) = 1$, thus $\deg_2(z_e^{\setminus}) = 1$, i.e., $\#(z_e^{\setminus})^{-1}(0) = 1 \ (mod\ 2)$. That is, the number of equilibrium price vectors for regular economy e is odd. In other words, the set $\pi^{-1}(e)$ has an odd number of elements. ■

Proof of Lemma 11. We fix e^* and define the function $F_i : \mathbb{R}^{L+1}_{++} \times \mathbb{R}^{L-1}_{++} \to \mathbb{R}^{L+1}$ as

$$F_i(x_i, \lambda_i, p^{\setminus}) = (D_{x_i} u_i(x_i, p, e^*) - \lambda_i p, -p \cdot (x_i - e_i^*)).$$

Under Assumption 8, at an equilibrium (p^*, e^*), the matrix $D_{(x_i,\lambda_i)} F_i$ is invertible thanks to the fact that u_i is differentiably strictly quasi-concave with respect to x_i. By the Implicit Function Theorem, there exist an open set $U \subset \mathbb{R}^{L-1}_{++}$ containing $p^{\setminus *}$ and a unique C^1 function $(\widehat{x}_i, \widehat{\lambda}_i) : U \to \mathbb{R}^{L+1}$ such that $(\widehat{x}_i(p^{*\setminus}), \widehat{\lambda}_i(p^{*\setminus})) = (x_i^*, \lambda^*)$ and for each $p^{\setminus} \in U$, $F_i(\widehat{x}_i(p^{\setminus}), \widehat{\lambda}_i(p^{\setminus}), p^{\setminus}) = 0$. Note that under Assumption 8, for any price p, there exist a unique pair $(x_i, \lambda_i) \in \mathbb{R}^{L+1}_{++}$ such that $F_i(x_i, \lambda_i, p^{\setminus}) = 0$, which implies that x_i is the demand of consumer i. Therefore, $\widehat{x}_i(p^{\setminus})$ coincides with the demand of consumer i, i.e., $\widehat{x}_i(p^{\setminus}) = f_i(p, p \cdot e_i^*, e^*)$. Again by the Implicit Function Theorem, we have:

$$D_{p^{\setminus}}(f_i, \lambda_i)(p^{*\setminus}) = D_{p^{\setminus}}(\widehat{x}_i, \widehat{\lambda}_i)(p^{*\setminus}) = -[D_{x_i,\lambda_i} F_i(x_i^*, \lambda_i^*, p^{*\setminus})]^{-1} D_{p^{\setminus}} F_i(x_i^*, \lambda_i^*, p^{*\setminus})$$

The Jacobian matrix $D_{x,\lambda,p^{\setminus}} F(\xi^*, e^*)$ can be written as follows:

$$\begin{pmatrix}
D_{x_1,\lambda_1}F_1 & & & & & D_{p^{\setminus}}F_1 \\
& \ddots & & & & \\
& & D_{x_i,\lambda_i}F_i & & & D_{p^{\setminus}}F_i \\
& & & \ddots & & \\
& & & & D_{x_m,\lambda_m}F_m & D_{p^{\setminus}}F_m \\
\tilde{I}|0 & \cdots & \tilde{I}|0 & \cdots & \tilde{I}|0 & 0_{L-1}
\end{pmatrix}$$

where $\tilde{I} = [I_{L-1}|0]$ [9] is the matrix representation of projection mapping Pr :

[9] I_{L-1} is the identity square matrix of size $L-1$ and $\tilde{I} = [I_{L-1}|0]$ is the $(L-1) \times L$

$\mathbb{R}^L \to \mathbb{R}^{L-1} : x \mapsto x^\backslash$. For all i, the matrix $D_{(x_i,\lambda_i)}F_i$ is invertible so the following square matrix has full rank $(L+1)m + L - 1$.

$$\begin{pmatrix} -[D_{x_1,\lambda_1}F_1]^{-1} & & & & & \\ & \ddots & & & & \\ & & -[D_{x_i,\lambda_i}F_i]^{-1} & & & \\ & & & \ddots & & \\ & & & & -[D_{x_m,\lambda_m}F_m]^{-1} & \\ & & & & & I_{L-1} \end{pmatrix}$$

Multiplying this matrix with $D_{x,\lambda,p^\backslash}F(\xi^*, e^*)$ give us the following matrix, which has the same rank as the matrix $D_{x,\lambda,p^\backslash}F(\xi^*, e^*)$.

$$\begin{pmatrix} -I_{L+1} & & & & & -[D_{x_1,\lambda_1}F_1]^{-1}D_{p^\backslash}F_1 \\ & \ddots & & & & \\ & & -I_{L+1} & & & -[D_{x_i,\lambda_i}F_i]^{-1}D_{p^\backslash}F_i \\ & & & \ddots & & \\ & & & & -I_{L+1} & -[D_{x_m,\lambda_m}F_m]^{-1}D_{p^\backslash}F_m \\ \tilde{I}|0 & \cdots & \tilde{I}|0 & \cdots & \tilde{I}|0 & 0_{L-1} \end{pmatrix}$$

For all $j = 1, \ldots, L-1$, by adding rows $j + k(L+1)$ for $k = 0, \ldots, m-1$ into the row $j + m(L+1)$, we obtain the following matrix, which has the same rank as $D_{x,\lambda,p^\backslash}F(\xi^*, e^*)$:

$$\begin{pmatrix} -I_{L+1} & & & & & -[D_{x_1,\lambda_1}F_1]^{-1}D_{p^\backslash}F_1 \\ & \ddots & & & & \\ & & -I_{L+1} & & & -[D_{x_i,\lambda_i}F_i]^{-1}D_{p^\backslash}F_i \\ & & & \ddots & & \\ & & & & -I_{L+1} & -[D_{x_m,\lambda_m}F_m]^{-1}D_{p^\backslash}F_m \\ 0 & \cdots & 0 & \cdots & 0 & -D_{p^\backslash}z^\backslash \end{pmatrix}$$

Clearly from the structure of the above matrix, we deduce that:

$$\text{rank}D_{p^\backslash}z^\backslash(p^*, e^*) + (L+1)m = \text{rank}D_{x,\lambda,p^\backslash}F(x^*, \lambda^*, e^*)$$

---

matrix built by adding a column of 0 to the matrix I_{L-1}.

So the Jacobian matrix $D_{p\backslash}z^{\backslash}(p^*, e^*)$ has full rank, i.e., rank$D_{p\backslash}z^{\backslash}(p^*, e^*) = L-1$ if and only if rank$D_{x,\lambda,p\backslash}F(x^*, \lambda^*, e^*) = (L+1)m + L - 1$, i.e., the Jacobian matrix $D_{x,\lambda,p\backslash}F(x^*, \lambda^*, e^*)$ has full rank. ■

Proof of Proposition 13. We recall that $f_i(p, p\cdot e_i, e)$ denotes the demand of consumer i by solving problem (P). Remark that $D_{e_k}f_i(p, p\cdot e_i, e)(v_k)$ is the directional derivative with respect to e_k along vector v_k. Once again, using the Implicit Function Theorem, we have

$$D_{e_k}f_i(p, p\cdot e_i, e) = -[I_L|0][D_{x_i,\lambda_i}F_i(x_i, \lambda_i, e)]^{-1}D_{e_k}F_i(x_i, \lambda_i, e)$$

where $F_i(x_i, \lambda_i, e) = (D_{x_i}u_i(x_i, p, e) - \lambda_i p, -p\cdot(x_i - e_i))$. By a simple computation, we get

$$D_{e_k}f_i(p, p\cdot e_i, e)(v_k) = \begin{pmatrix} D^2_{e_k x_i}u_i(x_i, p, e) \\ 0 \end{pmatrix}$$

Therefore,

$$D_{e_k}F_i(x_i, \lambda_i, e)(v_k) = -[I_L|0][D_{x_i,\lambda_i}F_i(x_i, \lambda_i, e)]^{-1}\begin{pmatrix} D^2_{e_k x_i}u_i(x_i, p, e)(v_k) \\ 0 \end{pmatrix}$$

Since $v_k\Big(D^2_{e_k x_i}u_i(x_i, p, e)\Big)^t$ is proportional to $D_{x_i}u_i(x_i, p, e)$ so proportional to the prices vector p, there exists $\gamma \in \mathbb{R}$ such that $v_k\Big(D^2_{e_k x_i}u_i(x_i, p, e)\Big)^t = \gamma p$ or equivalently $D^2_{e_k x_i}u_i(x_i, p, e)(v_k) = \gamma p^t$. The directional derivative now becomes

$$D_{e_k}F_i(x_i, \lambda_i, e)(v_k) = -[I_L|0][D_{x_i,\lambda_i}F_i(x_i, \lambda_i, e)]^{-1}\begin{pmatrix} \gamma p^t \\ 0 \end{pmatrix}$$

Using the explicit form of $D_{x_i,\lambda_i}F_i(x_i, \lambda_i, e)$, we have the following equality.

$$D_{x_i,\lambda_i}F_i(x_i, \lambda_i, e)\begin{pmatrix} 0 \\ -\gamma \end{pmatrix} = \begin{pmatrix} D^2_{x_i}u_i(x_i, p, e) & -p^t \\ p & 0 \end{pmatrix}\begin{pmatrix} 0 \\ -\gamma \end{pmatrix} = \begin{pmatrix} \gamma p^t \\ 0 \end{pmatrix}$$

Multiplying both sides of the above equality by $-[I_L|0][D_{x_i,\lambda_i}F_i(x_i, \lambda_i, e)]^{-1}$ we obtain $D_{e_k}f_i(x_i, \lambda_i, e)(v_k) = 0$, which concludes the proof. ■

Proof of Lemma 15. We have to show that for each $(\xi^*, e^*) \in F^{-1}(0)$, the Jacobian matrix $D_{\xi,e}F(\xi^*, e^*)$ is onto, i.e., has full row rank. We remark that (x^*, e^*, p^*) fulfills the condition in Assumption 12, namely the gradient $D_{x_i}u_i(x_i^*, p^*, e^*)$ is proportional to price vector p^* for all i. Without loss of generality, we assume that $k = m$ in Assumption 12, so the Condition (4) now becomes

$$v_m D^2_{x_m} u_m(x_m, p, e)(v_m) + v_m D^2_{e_m x_m} u_m(x_m, p, e)(v_m) < 0 \tag{9}$$

if $v_m \in \operatorname{Ker} D_{x_m} u_m(x_m, p, e) \setminus \{0\}$ and the Condition (5) now becomes

$$\text{and } v_i D^2_{e_m x_i} u_i(x_i, p, e)(v_m) = 0 \ \forall\ i < m. \tag{10}$$

where $v_i \in \operatorname{Ker} D_{x_i} u_i(x_i, e)$ for all i.

Let $\Delta := (\Delta x_i, \Delta\lambda_i)_{i=1,2,\ldots,m}, \Delta p^{\setminus}) \in \mathbb{R}^{(L+1)m} \times \mathbb{R}^{L-1}$. Our target is to show that $\Delta.D_{\xi,e} F(\xi^*, e^*) = 0$ implies $\Delta = 0$. Actually, we will prove the stronger statement that $\Delta.D_{((x_i,\lambda_i)_{i=1,2,\ldots,m}, e_m)} F(\xi^*, e^*) = 0$ implies $\Delta = 0$. We now write in detail this system.

$$\begin{cases} \Delta x_i D^2_{x_i} u_i(x_i^*, p^*, e^*) - \Delta\lambda_i p^* + \Delta p^{\setminus}[I_{L-1}|0] = 0, \ \forall i & (11) \\ -\Delta x_i \cdot p^* = 0, \ \forall i & (12) \\ \sum_{i=1}^{m} \Delta x_i D^2_{e_m x_i} u_i(x_i^*, p^*, e^*) + \Delta\lambda_m p^* - \Delta p^{\setminus}[I_{L-1}|0] = 0 & (13) \end{cases}$$

Adding Equations (11) and (13) for consumer m then multiplying with Δx_m, we get

$$\Delta x_m D^2_{x_m} u_m(x_m^*, p^*, e^*)(\Delta x_m) + \sum_{i=1}^{m} \Delta x_i D^2_{e_m x_i} u_i(x_i^*, p^*, e^*)(\Delta x_m) = 0 \tag{14}$$

From Equation (12), Δx_i is orthogonal to p^*, so $\Delta x_i \in \operatorname{Ker} D_{x_i} u_i(x_i^*, p^*, e^*)$ for all $i = 1, 2, \ldots, m$. Thus, from Condition 5 in Assumption 12,

$$\Delta x_i D^2_{e_m x_i} u_i(x_i^*, p^*, e^*)(\Delta x_m) = 0 \text{ for all } i \neq m.$$

So Equation 14 becomes

$$\Delta x_m D^2_{x_m} u_m(x_m^*, p^*, e^*)(\Delta x_m) + \Delta x_m D^2_{e_m x_m} u_m(x_m^*, p^*, e^*)(\Delta x_m) = 0 \tag{15}$$

Then, from Condition 4 in Assumption 12, we get $\Delta x_m = 0$.

From $\Delta x_m = 0$, Equation (11) for $i = m$ implies $\Delta\lambda_m = 0$ and also $\Delta p^{\setminus} = 0$. So Equations (11) and (12) become

$$\begin{cases} \Delta x_i D^2_{x_i} u_i(x_i^*, p^*, e^*) - \Delta\lambda_i p^* = 0, \ \forall i & (16) \\ -\Delta x_i \cdot p^* = 0, \ \forall i & (17) \end{cases}$$

from which, one deduces that $\Delta x_i D^2_{x_i} u_i(x_i^*, p^*, e^*)(\Delta x_i) = 0$ for all i. Therefore, since u_i is differentiably strictly quasi-concave, $\Delta x_i = 0$. Then $\Delta\lambda_i = 0\, \forall i$. So $\Delta = 0$. ■

Proof of Lemma 19. We will prove that any sequence $(\xi^n, e^n)_{n\in\mathbb{N}} \subset F^{-1}(0)$, up to a subsequence, converges to an element in $F^{-1}(0)$, knowing that the sequence $(e^n)_{n\in\mathbb{N}}$ converges to $e^* \in \Omega$. We recall that $\xi^n = (x^n, \lambda^n, p^{n\backslash})$ where $x^n = (x_1^n, \ldots, x_m^n) \in \mathbb{R}^{Lm}_{++}$, $\lambda^n = (\lambda_1^n, \ldots, \lambda_m^n) \in \mathbb{R}^m_{++}$ and $p^{n\backslash} \in \mathbb{R}^{L-1}_{++}$. Let $q^n = (1/\sum_{l=1}^L p^{ln})p^n)$.

- $(x^n, q^n)_{n\in\mathbb{N}}$ admits a subsequence converging to $(x^*, q^*) \in \mathbb{R}^{Lm}_{++} \times S$.

From $F^M(\xi^n, e^n) = 0$ and $F^{(i,2)}(\xi^n, e^n) = 0$, $x_j^n = \sum_i e_i^n - \sum_{i\neq j} x_i^n \leq \sum_i e_i^n$ for each $j = 1, \ldots, m$. Then $(x^n)_{n\in\mathbb{N}}$ is bounded from above since $(e^n)_{n\in\mathbb{N}}$ converges and it is bounded from below by zero. (q^n) belongs to S, which is compact, so $(x^n, q^n)_{n\in\mathbb{N}}$ has a converging subsequence, again denoted (x^n, q^n), which converges to $(x^*, q^*) \in \mathbb{R}^{Lm}_{+} \times S$. Moreover, from the definition of x^n, we have $u_i(x_i^n, p^n, e^n) \geq u_i(e_i^n, p^n, e^n)$ for any n and any i. Define $\mathbf{1} := (1, 1, \ldots, 1) \in \mathbb{R}^L_{++}$. From Point 2 of Assumption 8, we have for each $\epsilon > 0$, $u_i(x_i^n + \epsilon\mathbf{1}, p^n, e^n) \geq u_i(e_i^n, p^n, e^n)$. So, from the definition of $\tilde{u}_i$ in Point 4 of Assumption 8, $\tilde{u}_i(x_i^n + \epsilon\mathbf{1}, q^n, e^n) \geq \tilde{u}_i(e_i^n, q^n, e^n)$. Taking the limit on n, since (x_i^n, q^n, e^n) converges to $(x_i^*, q^*, e^*) \in \mathbb{R}^{Lm}_{+} \times S \times \Omega$, and $\tilde{u}_i$ is continuous, we get $\tilde{u}_i(x_i^* + \epsilon\mathbf{1}, q^*, e^*) \geq \tilde{u}_i(e_i^*, q^*, e^*) := \underline{u}_i$ for any $\epsilon > 0$. By point 4 of Assumption 8, $x_i^* \in \mathbb{R}^L_{++}$ since x_i^* belongs to the set $cl_{\mathbb{R}^L}\{x_i \in \mathbb{R}^L_{++} : \tilde{u}_i(x_i, q^*, e^*) \geq \underline{u}_i\}$.

- $(\lambda^n)_{n\in\mathbb{N}}$ converges to $\lambda^* \in \mathbb{R}^m_{++}$.

Since for all $n \in \mathbb{N}$ $p^{n,L} = 1$ and $F^{(i,1)}(\xi^n, e^n) = 0$, we have $D^L_{x_i} u_i(x_i^n, p^n, e^n) = D^L_{x_i} \tilde{u}_i(x_i^n, q^n, e^n) = \lambda_i^n$. By taking the limit on both sides, $D^L_{x_i} \tilde{u}_i(x_i^*, q^*, e^*) = \lambda_i^* > 0$. The strict inequality comes from Point 4 of Assumption 8.

- $(p^{n\backslash})_{n\in\mathbb{N}}$ converges to $p^{*\backslash} \in \mathbb{R}^{L-1}_{++}$.

Again from $F^{(i,1)}(\xi^n, e^n) = 0$, $D^{\backslash}_{x_i} u_i(x_i^n, p^n, e^n) = D^{\backslash}_{x_i} \tilde{u}_i(x_i^n, q^n, e^n) = \lambda_i^n p^{n\backslash}$. Therefore, taking the limit, we get

$$\lim_{n\to+\infty} p^{n\backslash} = \lim_{n\to+\infty} \frac{D^{\backslash}_{x_i} \tilde{u}_i(x_i^n, q^n, e^n)}{\lambda_i^n} = \frac{D^{\backslash}_{x_i} \tilde{u}_i(x_i^*, q^*, e^*)}{\lambda_i^*} = p^{*\backslash}.$$

and $p^{*\backslash} \in \mathbb{R}^{L-1}_{++}$ from Point 4 of Assumption 8.

Finally, since $(\xi^n, e^n)_{n\in\mathbb{N}} \subset F^{-1}(0)$ and F is continuous, we get $F(\xi^*, e^*) = 0$, i.e., $(\xi^*, e^*) \in F^{-1}(0)$, which ends our proof. ■

Proof of Theorem 20. From Lemma 15, 0 is a regular value of the mapping F. Then the Tranversality Theorem implies that there exists Ω^*, a full measure subset of Ω, such that $\forall e \in \Omega^*$, 0 is a regular value of F_e. Therefore, the set of regular economies $\mathcal{R}$ is full measure since it contains Ω^*.

Define $G = \{(\xi^*, e^*) \in \Xi \times \Omega,\ (\xi^*, e^*) \in F^{-1}(0) \mid \text{rank} D_\xi F(\xi^*, e^*) < \dim\Xi\}$. Then we can write $\mathcal{R} = \Omega \setminus \Pi(G)$ where Π is defined in Lemma 19. An element (ξ, e) of G is characterised by the fact that the determinant of all the square submatrices of $D_\xi F(\xi, e)$ of dimension $\dim\Xi$ is equal to 0. Thus, the set G is closed since the determinant is a continuous function. Therefore, $\Pi(G)$ is closed since Π is proper, which implies the set $\mathcal{R}$ is open. ■

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