New Results for additive and multiplicative risk apportionment

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Abstract

We provide new characterizations of the preference for additive and multiplicative risk apportionment when risk ordering relies on stochastic dominance. We then point out a simple property of risk apportionment with additive risks: Quite generally, an observed preference for additive risk apportionment in a specific risk environment is preserved when the decision-maker is confronted to other risk situations, so long as the total order of stochastic dominance relationships among risk couples remains the same. The main objective of this paper is to check whether this simple property also holds for multiplicative risks environments. We explain why this is not the case in general, and then provide a set of conditions under which this property holds. We also show that it holds – and even more strongly – in the case of CRRA utility functions due to a particular feature of this family of utility functions.

Keywords: Additive vs. multiplicative risks, Constant relative risk aversion, Higher order risk aversion, Expected utility, Higher order risk apportionment, Stochastic dominance.

1. Introduction

The work of Eeckhoudt and Schlesinger (2006) provided a refreshing perspective for the study of decisions under risk. Until the publication of their seminal article, and since Friedman and Savage (1948) at least, it was common to start from an economic decision under risk (insurance, gambling, investment, consumption and saving, production, protection, among others) and to analyze it using utility theory. As a result, specific properties of the von Neumann-Morgenstern utility function were associated to specific traits of attitudes towards risk. A negative second derivative of the utility function reflected risk aversion (Friedman and Savage, 1948; Pratt, 1964). A positive third derivative reflected prudence and was associated to precautionary behavior (Kimball, 1990). A negative fourth derivative was defined as temperance and invoked to explain the demand for risky assets in the presence of background risks (Kimball, 1992).[1] In the same vein, a positive sign of the fifth derivative was more recently associated to edginess to explain the effects of background risk on precautionary saving (Lajeri-Chaherli, 2004). Similarly, an S-shaped utility function or a state-dependent utility function was used to rationalize simultaneous purchasing of insurance and lottery tickets, etc.

An alternative but less popular approach was initiated by Rothshild and Stiglitz (1970), who focused on properties of the probability distribution of outcomes and defined risk aversion as preference to avoid increases in risk, defined as a mean-preserving spread of the distribution. Their (second-order) approach was completed by Menezes *et al.* (1980) and by Menezes and Wang (2005) who introduced

[1] Notice in passing that Crainich *et al.* (2013) shows that even risk lover decision makers can be prudent while Menegatti (2014) points out that, under the assumptions of non-satiation and bounded marginal utility, temperance implies prudence and prudence implies risk aversion.

respectively downside risk aversion and aversion to outer risk by addressing third-order and fourth-order changes in the probability distribution of outcomes. All these authors provided also a link with utility theory, showing how their definitions could be associated to the signs of successive derivatives of the von Neumann-Morgenstern utility function.

Instead of focusing *ab initio* on an economic decision model or on the shape of the probability distribution of outcomes, Eeckhoudt and Schlesinger (2006) started from the behavior of individuals exposed to simple equal-probability lotteries. Their choices were then used to define risk aversion, prudence, temperance, and so on, and the term "risk apportionment" was coined to describe how these behavioral traits were reflected in individual choices. For example, temperance reflects the preference for not associating an additional zero-mean risk X_2 to a situation where the decision maker (DM) is already exposed to a prevailing zero-mean risk X_1. The $50-50$ lottery $[X_1, X_2]$ is preferred to the $50-50$ lottery $[0, X_1 + X_2]$. Preference for risk apportionment means preference for disaggregation of harms, given risk aversion. The beauty of this more primitive approach to attitudes towards risk is that it does not need familiarity with utility theory to be understood, although it is perfectly consistent with the traditional approach based on specific properties of the utility function. Preference for risk apportionment (or harms disaggregation) in successive increasingly complex lotteries translates into alternating signs of successive derivatives of the utility function (mixed risk aversion, as defined by Caballé and Pomansky, 1996).

As stressed by Eeckhoudt (2012), starting from these premises to analyze the direction of preferences under risk and to link them to successive derivatives of the utility function is appropriate. It is more robust than the traditional approach starting from an economic decision model (*e.g.* optimal saving), and deriving the sign of the n^{th}-order derivative of the utility function reconciling the model's results with expected DM's behavior. This latter approach may collapse as soon as changes are introduced in the structure of the economic decision model (see examples in Eeckhoudt, 2012).

The Eeckhoudt and Schlesinger (2006) approach, using zero-mean risks, received large support in experimental work (Deck and Schlesinger, 2010, 2014, 2018; Ebert and Wiesen, 2011; Trautmann and van de Kuilen, 2018; Attema *et al.*, 2019). It was also generalized by Eeckhoudt *et al.* (2009b) to any couple of risks linked together by properties of increases in risk or stochastic dominance at any order.[2] Indeed, considering four mutually independent risks X_1, Y_1, X_2 and Y_2 such that Y_i dominates X_i by s_i^{th}-order stochastic dominance for $i = 1, 2$, an expected utility maximizer with a mixed risk averse (MRA) utility function up to order $s = s_1 + s_2$ prefers the $50 - 50$ lottery $[X_1 + Y_2, Y_1 + X_2]$ to the $50 - 50$ lottery $[X_1 + X_2, Y_1 + Y_2]$.[3] In the former lottery risk apportionment holds. There is disaggregation of harms. In the latter lottery this is not the case. Instead of combining "good with bad" in the two

[2] Increases in risk introduced by Rothshild and Stiglitz (1970) – see above – were generalized to any order by Ekern (1980). Stochastic dominance was introduced by Hadar and Russel (1969) and Hanoch and Levy (1969) and extended to any order by Jean (1980). The generalization has shown that there is a correspondence between n^{th}-degree stochastic dominance and the preference for a non dominated risk by an expected utility maximizer with a mixed risk averse utility function up to order n. An equivalent but less strong relationship holds for increases in risk of order n (see Ekern, 1980).

[3] As a reminder, following the definition of Caballé and Pomansky (1996), a utility function u exhibits mixed risk aversion of order k if $(-1)^{k+1}\, u^{(k)} \geq 0$. Consequently, it exhibits mixed risk aversion up to order s if $(-1)^{k+1}\, u^{(k)} \geq 0$ for $k = 1, 2, \ldots, s$.

possible lottery outcomes, the lottery yields "bad with bad" in the first outcome and "good with good" in the second outcome.[4]

Eeckhoudt and Schlesinger (2006), as well as Eeckhoudt *et al.* (2009b), consider additive risks. In the above lotteries, the final outcomes are either $X_1 + Y_2$ and $Y_1 + X_2$ on the one hand, or $X_1 + X_2$ and $Y_1 + Y_2$ on the other hand. Subsequent research addressed risk apportionment for multiplicative risks. The analysis is thus restricted to non-negative random variables. Multiplicative risks are observed in various circumstances in economic and social life. For example, investing in an asset denominated in foreign currency exposes the domestic investor to two multiplicative risks, the risk of the asset itself and the risk of variations in the domestic currency value of the foreign currency. Similarly, taking a job with a variable income in a firm exposed to bankruptcy results for the wage earner in a range of final outcomes where the two risks interact multiplicatively. Wang and Li (2010) addressed risk apportionment with multiplicative risks specifically. Building on results obtained by Eeckhoudt *et al.* (2009a) in a related context – see also Eeckhoudt and Schlesinger (2008) – they reached the conclusion that there is a direct relation between multiplicative risk apportionment at order $n+1$ and the value of n^{th}-degree relative risk aversion.[5] A similar result was obtained by Chiu *et al.* (2012) using a model combining two additive risks with a multiplicative effect on the first risk, an n^{th}-degree shift of stochastic dominance on this risk, and a first-degree shift of stochastic dominance on the second risk. Again, the value of n^{th}-degree relative risk aversion is critical to conclude whether risk apportionment in the sense of Eeckhoudt and Schlesinger (2006) is obtained or not.[6]

This literature was completed by works addressing risk apportionment in a bivariate context, a context where the decision-maker's preferences are driven by the joint effects of two independent risks, for instance risks affecting wealth and health as in Eeckhoudt *et al.* (2007). In particular, Jokung (2011) and Denuit and Rey (2013) provided a new look at risk apportionment with additive or multiplicative risks by analyzing these cases as specific cases of risk apportionment in a bivariate context.

In this paper, we focus on additive and multiplicative risks and we start by providing homogeneous definitions for risk apportionment in these two risk contexts when the couples of risks are ranked by stochastic dominance. We then derive two theorems spelling out the exact conditions that a utility function must fulfill to yield additive or multiplicative risk apportionment. The first theorem, dealing with additive risks, provides an equivalence between risk apportionment and mixed risk aversion, instead of an implication as in Eeckhoudt *et al.* (2009b). The second theorem, dealing with multiplicative risks,

[4]Note that Eeckhoudt *et al.* (2009b) use expected utility theory and that the $50-50$ lotteries introduced to define risk apportionment may receive a different interpretation. The dominated risk X_i may result from an aggravation of the initial risk Y_i. If risk apportionment holds, the DM prefers to face one risk increase for sure instead of facing the prospect of possibly facing the two risk changes or none with probabilities one-half. In this sense, there is mutual aggravation of risk changes (Ebert *et al.*, 2017; Courbage *et al.*, 2018).

[5]Their definition of relative risk aversion at order $n+1$ corresponds to what is generally defined today as relative risk aversion at order n (see Section 4 below).

[6]Note that transforming the (positive) variables in the multiplicative case using logarithms to retrieve the additive case is not appropriate in this context. The reason is that if Y stochastically dominates X at order n, it is not guaranteed that $W = \ln Y$ dominates $Z = \ln X$ at the same order (or at all).

provides also an equivalence between risk apportionment and a property of the DM's utility function. Not surprisingly, this new condition is much more complex than the mere condition of mixed risk aversion obtained in the additive case.[7] It soon appears, however, that it is related to the value of the k^{th}-degree relative risk aversion for $k = 1, \ldots, s$, where s is a stochastic dominance order.

We then focus our attention on additive risks and we obtain a simple result for risk apportionment preferences of a DM in different situations. A DM displays a preference for risk apportionment when facing risk X_1 dominated by risk Y_1 at order s_1 and risk X_2 dominated by risk Y_2 at order s_2 if, and only if, he displays a preference for risk apportionment when facing two other couples of risks (X_1', Y_1') and (X_2', Y_2') related by stochastic dominance orderings at orders s_1' and s_2' respectively, provided that $s_1 + s_2 = s_1' + s_2'$. The contribution of this result to the literature is to emphasize a correspondence between risky choices by a DM in two different circumstances involving additive risk combinations, as long as the sum of stochastic dominance orders linking the risk combinations in each circumstance is the same.

Our main motivation is then to check whether this result still applies when the two risks combine multiplicatively. We show that the answer is negative in general and we explain why. We then proceed in two steps. Firstly, we show that preference for multiplicative risk apportionment when the two pairs of risks are characterized by stochastic dominance orders s_1 and s_2 is a sufficient condition for the preference for multiplicative risk apportionment when the stochastic dominance orders are $s_1' = s_1 - 1$ and $s_2' = s_2 + 1$, whatever $s_1 \leq s_2$. Secondly, we derive a simple condition under which the converse relation also holds when s_1 is either equal to one, *i.e.* in case of first-order stochastic dominance (FSD), or two, *i.e.* in case of second-order stochastic dominance (SSD). In both cases, relative risk aversion at order $k = 1, \ldots, s_2$ plays a decisive role. In the second case, in particular, the property under scrutiny holds whenever relative risk aversion at order k is decreasing, for $k = 1, \ldots, s_2$. Proceeding further, *i.e.* dealing with $s_1 \geq 2$, is technically challenging and does not lead to relevant economic interpretations.

As a consequence, we finally turn to the specific case where the DM's preferences are reflected in a Constant Relative Risk Aversion (CRRA) utility function – the function most commonly used in the literature. We obtain that our result derived for additive risks extends to multiplicative risks, independently of the stochastic dominance orders relating the two couples of risks X and Y, for all CRRA utility function whose relative risk aversion exceeds one.[8] This strong result is driven by a simple property of the CRRA utility function that is often neglected in the literature.

Our paper is organized as follows. Section 2 provides our definitions for additive and multiplicative risk apportionment and states our core characterization theorems (Theorems 1 and 2). Section 3 focuses on additive risks and derives our result of equivalence between risk apportionment at orders s_1 and s_2

[7]It is well known that generalizing results from the additive context to the multiplicative context is a highly non-trivial task as illustrated by an extensive literature (e.g. Franke *et al.*, 2006; Denuit and Rey, 2013; Gollier, 2019; Wang, 2019, among many others).

[8]The class of CRRA utility functions was already singled out by Malevergne and Rey (2010) as the unique functions that preserve preferences rankings under multiplicative background risks. Their results extends the work of Franke *et al.* (2006) on multiplicative background risks.

and risk apportionment at orders s_1' and s_2' when $s_1 + s_2 = s_1' + s_2'$ (Proposition 1). Section 4 turns towards risk apportionment in a multiplicative risks context and explains why our result from Section 3 does not generalize in the latter context. Section 5 first establishes that the preference for multiplicative risk apportionment at orders s_1 and $s_2 \geq s_1$ is sufficient for the preference to hold at orders $s_1' = s_1 - 1$ and $s_2' = s_2 + 1$ (Proposition 2). It then derives the converse result when $s_1 = 2$ (Proposition 3). Then, Section 6 turns to the specific case of CRRA utility functions and shows that our result from Section 5 holds generally in a multiplicative risks context, whatever s_1, s_2, s_1' and s_2' provided relative risk aversion is larger than one (Theorem 3 and Proposition 4). Section 7 concludes briefly. Most of the proofs are displayed in an Appendix at the end of the paper.

2. Additive and multiplicative risk apportionment

As in Eeckhoudt *et al.* (2009b), let $[A, B]$ denote a lottery that yields the risks A or B, each with probability one-half. Consider more particularly two pairs of risks, X_1 and Y_1 on the one hand, X_2 and Y_2 on the other hand. We assume that all risks have bounded supports contained within the interval $[a, b]$. Consider, for the time being, that risks are additive and the DM prefers Y_i to X_i for $i = 1, 2$. More formally, suppose that this preference reflects a stochastic dominance relationship: Y_i dominates X_i via s_i^{th}-order stochastic dominance for $i = 1, 2$.[9] In such a setting, using the terminology introduced by Eeckhoudt and Schlesinger (2006), additive risk apportionment may be defined as follows:

Definition 1 (Additive risk apportionment)**.** A DM exhibits preference for additive risk apportionment of order (s_1, s_2) if, given any set of mutually independent risks X_1, X_2, Y_1 and Y_2 such that Y_i dominates X_i via s_i^{th}-order stochastic dominance for $i = 1, 2$, he prefers the lottery $[X_1 + Y_2, Y_1 + X_2]$ to the lottery $[X_1 + X_2, Y_1 + Y_2]$.

In the former lottery, the risks are better “apportioned”. There is preference for harms disaggregation. The two “bad” risks X cannot occur in the same state of nature. The DM prefers to combine “bad with good” and “good with bad”, instead of “good with good” and “bad with bad”. Here, “bad” is defined as “stochastically dominated”.

Preference for harms disaggregation (risk apportionment) is defined without reference to utility theory. However, the comparison of risks uses the concept of stochastic dominance and it is well-known, since Hadar and Russel (1969), Whitmore (1970) and Jean (1980), that risk preferences when risks are compared using stochastic dominance reflect the shape of the DM's utility function. In the sequel, we denote by $u(x)$ the DM's utility function which will be assumed to be continuously differentiable at any desirable order. Its k^{th}-order derivative will be noted by $u^{(k)}$.

Relying on the theorem:

Theorem (Ingersoll, 1987, pp. 138-139)**.** *The following are equivalent:*

(i) Y dominates X via s^{th}-order stochastic dominance.

[9] The definition of s^{th}-order stochastic dominance is recalled in Appendix A.

(ii) $\mathrm{E}[u(Y)] \geq \mathrm{E}[u(X)]$ *for all utility function* u *such that* $(-1)^{k+1} u^{(k)} \geq 0$ *for* $k = 1, \ldots, s$.

we can rephrase Theorem 3 of Eeckhoudt *et al.* (2009b) as follows:

Theorem. *Suppose that* Y_i *dominates* X_i *via* s_i^{th}*-order stochastic dominance for* $i = 1, 2$. *Any MRA DM from order* 1 *to* $(s_1 + s_2)$ *prefers the lottery* $[X_1 + Y_2, Y_1 + X_2]$ *to the lottery* $[X_1 + X_2, Y_1 + Y_2]$.

This result provides a *sufficient* condition for additive risk apportionment but not an *equivalence*. However, based on lemma 1 in Chiu *et al.* (2012):[10]

Lemma (Chiu *et al.*, 2012, Lemma 1(ii))**.** *For all* X *and* Y *such that* X *dominates* Y *via* s^{th}*-order stochastic dominance,* $\mathrm{E}[u(X)] \geq \mathrm{E}[u(Y)]$ *if and only if* $(-1)^{k+1} u^{(k)} \geq 0$ *for* $k = 1, \ldots, s$,

we can actually state an equivalence between additive risk apportionment and mixed risk aversion:

Theorem 1. *For any set of four mutually independent risks* X_1, X_2, Y_1, Y_2 *such that* Y_i *dominates* X_i *via* s_i^{th}*-order stochastic dominance for* $i = 1, 2$, *a DM prefers the lottery* $[X_1 + Y_2, Y_1 + X_2]$ *to the lottery* $[X_1 + X_2, Y_1 + Y_2]$ if, and only if, *he is MRA from* 1 *to* $(s_1 + s_2)$.

Proof. See Appendix B. □

Thus, mixed risk aversion and additive risk apportionment are two faces of the same coin when the risks faced by the DM can be ordered using stochastic dominance.[11] Theorem 1 is stronger than Theorem 3 in Eeckhoudt *et al.* (2009b), featuring an implication (MRA implies additive risk apportionment).

Turning now to multiplicative risks and restricting our attention to risks defined over the interval $[a, b] \subset \mathbb{R}_+$, we first propose the following definition of multiplicative risk apportionment:

Definition 2 (Multiplicative risk apportionment)**.** A DM exhibits preference for multiplicative risk apportionment of order (s_1, s_2) if, given any set of mutually independent non negative risks X_1, X_2, Y_1 and Y_2 such that Y_i dominates X_i via s_i^{th}-order stochastic dominance for $i = 1, 2$, he prefers the lottery $[X_1 \cdot Y_2, Y_1 \cdot X_2]$ to the lottery $[X_1 \cdot X_2, Y_1 \cdot Y_2]$.

Characterization of multiplicative risk apportionment is more difficult and there is no general result in the literature, to our knowledge. Only specific cases were considered: Eeckhoudt *et al.* (2009a), Wang and Li (2010), Chiu *et al.* (2012), Denuit and Rey (2013), Wang (2019). Thus, we first need to propose a general characterization of multiplicative risk apportionment. This is the purpose of the following theorem.

Theorem 2. *For any set of four mutually independent non-negative risks* X_1, X_2, Y_1, Y_2 *such that* Y_i *dominates* X_i *via* s_i^{th}*-order stochastic dominance for* $i = 1, 2$, *a DM prefers the lottery* $[X_1 \cdot Y_2, Y_1 \cdot X_2]$ *to the lottery* $[X_1 \cdot X_2, Y_1 \cdot Y_2]$ if, and only if, *his utility function* u *satisfies*

$$(-1)^{k_1+k_2+1} \sum_{k=0}^{\min(k_1,k_2)} \frac{k_1! \cdot k_2!}{k!\,(k_1-k)!\,(k_2-k)!} \cdot x^{k_1+k_2-k} \cdot u^{(k_1+k_2-k)}(x) \geq 0\,, \qquad \forall x \geq 0\,,$$

[10] The lemma is not proved in Chiu *et al.* (2012). But a proof is available in Appendix A.

[11] If X_i represents an s_i^{th}-order increase in risk with respect to Y_i, results in Ekern (1980) and Eeckhoudt *et al.* (2009b, Corollary 1) show that additive risk apportionment holds if the DM is $(s_1 + s_2)^{\text{th}}$ risk averse: $(-1)^{1+s1+s2} u^{(s_1+s_2)} \geq 0$.

for $k_1 = 1, 2, \ldots, s_1$ *and* $k_2 = 1, 2, \ldots, s_2$.

Proof. See Appendix C. $\square$

The condition in Theorem 2 is not very appealing, at first sight, compared with the simpler relation obtained in the case of additive risk apportionment. But setting $s_1 = 1$ (and $s_2 \geq 1$), that is restricting our attention to FSD for the relationship between X_1 and Y_1 as in Denuit and Rey (2013, Proposition 5.4(ii)), the general condition provided by Theorem 2 simplifies to the well-know condition

$$\begin{aligned}
&(-1)^{1+k_2+1} \sum_{k=0}^{1} \frac{k_2!}{k!\,(k_2-k)!} \cdot x^{1+k_2-k} u^{(1+k_2-k)}(x) \geq 0 \\
\Longleftrightarrow\ &(-1)^{k_2} \cdot \left[x^{k_2+1} u^{(k_2+1)}(x) + k_2 \cdot x^{k_2} u^{(k_2)}(x) \right] \geq 0\,, \\
\Longleftrightarrow\ &(-1)^{k_2+1} \cdot u^{(k_2)}(x) \left[r_u^{(k_2)}(x) - k_2 \right] \geq 0\,, \qquad (1) \\
\Longleftrightarrow\ &r_u^{(k_2)}(x) \geq k_2\,,
\end{aligned}$$

for all $x \geq 0$ and $k_2 = 1, 2, \ldots, s_2$ provided that $(-1)^{k_2+1} u^{(k_2)} > 0$, *i.e.*, the DM is MRA of order k_2. In the derivation above, we have introduced the index of relative risk aversion of order k_2

$$r_u^{(k_2)}(x) := -x \cdot \frac{u^{(k_2+1)}(x)}{u^{(k_2)}(x)}\,.$$

Remark 1. Assuming Equation (1) holds:

$$\begin{aligned}
(-1)^{k_2+1} u^{(k_2)} > 0 \implies\ &r_u^{(k_2)} \geq k_2\,, \\
\implies\ &r_u^{(k_2)} > 0\,, \\
\implies\ &(-1)^{k_2+2} u^{(k_2+1)} > 0\,.
\end{aligned}$$

Hence, given $u^{(1)} > 0$, a straightforward induction shows that preference for multiplicative risk apportionment of order $(1, s_2)$ implies mixed risk aversion from order 1 to $s_2 + 1$ and, then, preference for *additive* risk apportionment at the same order $(1, s_2)$ as a consequence of Theorem 1.

3. Preserved preferences for additive risks

Consider an MRA DM from 1 to $s_1 + s_2$. Theorem 1 tells us that such a DM exhibits preference for risk apportionment in the presence of additive risks. Faced with the two pairs of risks defined in the preceding section, his expected utility satisfies:

$$\frac{1}{2}\mathrm{E}\left[u\left(X_1 + Y_2\right)\right] + \frac{1}{2}\mathrm{E}\left[u\left(Y_1 + X_2\right)\right] \geq \frac{1}{2}\mathrm{E}\left[u\left(X_1 + X_2\right)\right] + \frac{1}{2}\mathrm{E}\left[u\left(Y_1 + Y_2\right)\right]\,. \qquad (2)$$

Equation (2) means that the DM prefers to disaggregate bad outcomes rather than aggregate them. In what follows, we denote this disaggregation preference relation by $\mathrm{PD}_A\left\{\left[(X_1, Y_1)\,;(X_2, Y_2)\right]_{(s_1,s_2)}\right\}$ or, more compactly, by $\mathrm{PD}_A(s_1, s_2)$ when an explicit reference to the pairs of risks is not necessary.

Let us consider a DM with an MRA utility function from 1 to 4. We consider a first set of two pairs of mutually independent risks, X_1 and Y_1, X_2 and Y_2, such that Y_i dominates X_i by second-order stochastic dominance for $i = 1, 2$: $X_1 \preceq_{2-SD} Y_1$ and $X_2 \preceq_{2-SD} Y_2$. The total dominance order is equal to

4 ($s_1+s_2=2+2=4$). Theorem 1 states that $\text{PD}_A(2,2)$ holds for this DM. Let us now consider a second set of two pairs of mutually independent risks, X_1' and Y_1', X_2' and Y_2', such that Y_1' dominates X_1' by first-order stochastic dominance and Y_2' dominates X_2' by third-order stochastic dominance: $X_1' \preceq_{1-SD} Y_1'$ and $X_2' \preceq_{3-SD} Y_2'$. The total dominance order is also equal to 4 ($s_1'+s_2'=1+3=4$). Relying on Theorem 1, we obtain that $\text{PD}_A(1,3)$ also holds for this DM since $s_1'+s_2'=4$, and thus $u^{(s_1+s_2)}=u^{\left(s_1'+s_2'\right)}=u^{(4)}$. Preference ranking is preserved when the first set of risks is replaced by the second set, and vice-versa.

We understand that this result can easily be extended to the more general case $s_1+s_2=s_1'+s_2'$ with $s_1\neq s_1'$ and $s_2\neq s_2'$. This is our first proposition:

Proposition 1. *Given an expected utility maximizing DM, the following equivalence holds*

$$\text{PD}_A(s_1,s_2) \Longleftrightarrow \text{PD}_A(s_1',s_2')\ ,$$

whenever $s_1+s_2=s_1'+s_2'$.

Example. Consider a winegrower in the South of Europe. Her activity in the current period ($t=0$) is risky and generates a revenue Y_1. Due to trade conflicts between the EU and the USA, the risk of this activity will increase and become X_1 at date $t=1$, with $X_1 \preceq_{3-SD} Y_1$, a third-degree stochastic dominance shift. To diversify her activity, the winegrower has the opportunity to start an oil production, using grape stones, at a cost of X_2 in the current period and sell it to an agricultural cooperative for a certain positive revenue Y_2 at date $t=1$. Thus, $X_2 \preceq_{1-SD} Y_2$. Assume that she could, alternatively, borrow the amount Y_2 in the current period and reimburse the amount X_2 at date $t=1$. Following Eeckhoudt *et al.* (2009b, p. 999), *i.e.*, reinterpreting the $50-50$ lottery $[A,B]$ as sequential revenues of A at $t=0$ and B at $t=1$, and assuming moreover a zero interest rate and no time-discounting, Theorem 1 implies that the winegrower will prefer to start an oil production *if, and only if,* she is MRA from 1 to 4. Indeed,

$$[(Y_1+X_2),(X_1+Y_2)] \succeq_{4-SD} [(Y_1+Y_2),(X_1+X_2)]\ .$$

Investing in oil production at a cost X_2 is preferred to borrowing Y_2 in the current period and reimbursing X_2 at $t=1$, even if $-X_2<Y_2$. (Note, that an MRA winegrower from 1 to 4 is prudent and temperant).

A few years later, the winegrower has retired and she enjoys the revenue Y_1' of a diversified asset portfolio at period $t=n$. Due to coming elections, the risk of this portfolio is bound to increase by an SSD shift and will be X_1' at period $t=n+1$ ($X_1' \preceq_{2-SD} Y_1'$). Unfortunately, the winegrower has just lost her mother and a remote uncle. She is the single heir of these two relatives and the legacies should be organized with her lawyer before the end of period $t=n+1$. The bequest to be received from the uncle (X_2') is more uncertain than the bequest from the mother (Y_2'). Indeed, the winegrower estimate is $X_2' \preceq_{2-SD} Y_2'$. Proposition 1 predicts that the winegrower will instruct the lawyer to start with the uncle's bequest and delay the mother's bequest until period $t=n+1$:

$$[(Y_1'+X_2'),(X_1'+Y_2')] \succeq_{4-SD} [(Y_1'+Y_2'),(X_1'+X_2')]\ .$$

Remark 2. Definition 1 introduced the notion of additive risk apportionment *of order* (s_1,s_2), where s_1 and s_2 refer to the stochastic dominance orders of the two pairs of risk involved in the definition. As a consequence of Proposition 1, we see that it makes sense to talk about additive risk apportionment *of order* s, with $s = s_1 + s_2$, since only the total order s matters and not s_1 and s_2 separately.

Proposition 1 deals with additive risks and allows to make behavioral predictions in various circumstances, as the above example shows. Our question is then the following: does the result still hold when risks are multiplicative? Let us denote by $\mathrm{PD}_M \left\{ [(X_1, Y_1) ; (X_2, Y_2)]_{(s_1,s_2)} \right\}$ – and $\mathrm{PD}_M (s_1, s_2)$ for short – the disaggregation preference relationship when risks interact multiplicatively:

$$\frac{1}{2}\mathrm{E}\left[u\left(X_1 \cdot Y_2\right)\right] + \frac{1}{2}\mathrm{E}\left[u\left(Y_1 \cdot X_2\right)\right] \geq \frac{1}{2}\mathrm{E}\left[u\left(X_1 \cdot X_2\right)\right] + \frac{1}{2}\mathrm{E}\left[u\left(Y_1 \cdot Y_2\right)\right] , \tag{3}$$

i.e., the preference to disaggregate bad outcomes instead of aggregating them when outcomes interact in a multiplicative form. Assume that $s_1' + s_2' = s_1 + s_2 = s$, with $s_1 \neq s_1'$ and $s_2 \neq s_2'$. Is preference ranking preserved when we replace $[(X_1, Y_1); (X_2, Y_2)]$ by $[(X_1', Y_1') ; (X_2', Y_2')]$ and conversely? More formally is $\mathrm{PD}_M (s_1, s_2)$ equivalent to $\mathrm{PD}_M (s_1', s_2')$?

Example (*Continued*). Assume that the winegrower faces an exchange risk, in addition to the increased commercial risk (from Y_1 to X_1). The currency risk may be imperfectly hedged using forward exchange contracts (these must be renewed after some months at a random forward rate). However, if the forward rate Y_2 represents an SSD improvement, compared to the spot rate X_2 ($X_2 \preceq_{2-SD} Y_2$), and if the winegrower's utility satisfies the condition of Theorem 2 for $s_1 + s_2 = 3 + 2 = 5$, we get:

$$\mathrm{E}\left[u\left(Y_1 \cdot X_2\right)\right] + \mathrm{E}\left[u\left(X_1 \cdot Y_2\right)\right] \geq \mathrm{E}\left[u\left(Y_1 \cdot Y_2\right)\right] + \mathrm{E}\left[u\left(X_1 \cdot X_2\right)\right] .$$

The winegrower will choose to hedge the future more risky revenue with forward contracts, instead of taking the currency risk (given her preference for the current less risky commercial income). Does it mean that she will also decide to use index futures when her diversified asset portfolio faces an increase in risk due to coming elections, once retired, if $s_1' + s_2' = 2 + 3 = 5$? This is the question we address in the rest of this article.

In Section 4, we show that the answer is negative in the general case and we explain why. In section 5, we are able to define two specific cases where the answer is positive, but it appears difficult to proceed further. For this reason, in Section 6, we restrict our attention to CRRA utility functions. In this case, we show that the result holds and we explain why. We also show that with CRRA utility functions an even stronger result holds.

4. Preserved preferences for multiplicative risks – Preliminary remarks

Consider equations (2) and (3) above, defining risk apportionment in the expected utility framework, respectively for additive risks and for multiplicative risks. Jokung (2011) and Denuit and Rey (2013) consider these two cases as special cases of a bivariate context where the utility function $V(x_1, x_2)$ takes the forms $u(x_1 + x_2)$ and $u(x_1 \cdot x_2)$ respectively. Using results in Denuit *et al.* (1999), it turns out that risk apportionment holds when the bivariate utility function satisfies

$$(-1)^{k_1+k_2+1} V^{(k_1,k_2)} \geq 0, \quad \forall k_1 = 1, \ldots, s_1 \text{ and } k_2 = 1, \ldots, s_2 \,, \tag{4}$$

where $V^{(k_1,k_2)}$ denotes the cross-derivative of order (k_1, k_2) of the bivariate utility function V.

Remark 3. In the additive risks context, the condition (4) simplifies because $V^{(k_1,k_2)}(x_1,x_2) = u^{(k_1+k_2)}(x_1+x_2)$, and thus $V^{(s_1,s_2)} = V^{\left(s_1',s_2'\right)}$ when $s_1+s_2 = s_1'+s_2'$. That is why we obtain Proposition 1, *i.e.*, the result of preserved preference ranking when replacing $\{(X_1,Y_1);(X_2,Y_2)\}_{(s_1,s_2)}$ by $\{(X_1',Y_1');(X_2',Y_2')\}_{\left(s_1',s_2'\right)}$ and conversely.

Remark 4. Whether the utility function reads $u(x_1+x_2)$ or $u(x_1 \cdot x_2)$, $\operatorname{sgn} V^{(k_1,k_2)} = \operatorname{sgn} V^{(k_2,k_1)}$ for all k_1 and k_2 since the utility function is symmetric in its arguments x_1 and x_2. For instance, in the additive case, we have $V^{(3,2)}(x_1,x_2) = u^{(5)}(x_1+x_2) = V^{(2,3)}(x_1,x_2)$. In the multiplicative case, the equality of the derivatives does not hold, but their signs remain the same, given that x_1 and x_2 are both assumed positive. For instance, $V^{(3,2)}(x_1,x_2) = x_2 \cdot \Omega(x_1 \cdot x_2)$ and $V^{(2,3)}(x_1,x_2) = x_1\Omega(x_1 \cdot x_2)$, where $\Omega(x) \coloneqq x^2 u^{(5)}(x) + 6xu^{(4)}(x) + 6u^{(3)}(x)$. Hence the two relations $\mathrm{PD}_A(s_1,s_2) \iff \mathrm{PD}_A(s_2,s_1)$ and $\mathrm{PD}_M(s_1,s_2) \iff \mathrm{PD}_M(s_2,s_1)$ hold trivially.

Remark 5. However, in the multiplicative case, the signs of the cross-derivatives are not necessarily equal for a given total order $k = k_1+k_2 = k_1'+k_2' > 3$, with $k_1 \neq k_1' \neq k_2$ and $k_2 \neq k_2' \neq k_1$. Consider the above example again. With $V^{(3,2)}$ and $V^{(2,3)}$, $k_1+k_2 = 5$. But a total derivative order of 5 can also be obtained by $1+4$ and $4+1$ in the case of the cross-derivatives $V^{(1,4)}(x_1,x_2)$ and $V^{(4,1)}(x_1,x_2)$. These two derivatives are respectively equal to $x_1^3\Phi(x_1 \cdot x_2)$ and to $x_2^3\Phi(x_1 \cdot x_2)$, where $\Phi(x) \coloneqq xu^{(5)}(x) + 4u^{(4)}(x)$. They have same signs as already noticed in Remark 4. But since $\Phi \neq \Omega$, $\operatorname{sgn} V^{(2,3)}$ and $\operatorname{sgn} V^{(1,4)}$ may differ so that the result stated in Proposition 1 for additive risks does not generalize to multiplicative risks.

Remark 6. Several authors remarked that relative risk aversion, and its higher order generalization, play a central role when risks are multiplicative (see Eeckhoudt and Schlesinger, 2008; Eeckhoudt *et al.*, 2009a; Wang and Li, 2010; Chiu *et al.*, 2012; Denuit and Rey, 2013; Wang, 2019). More specifically, starting from Eeckhoudt and Schlesinger (2006), and using lotteries with multiplicative risks defined by Eeckhoudt *et al.* (2009a), Wang and Li (2010) show that multiplicative risk apportionment of order $(1,k)$ occurs if and only if k^{th}-degree relative risk aversion $r_u^{(k)}$ exceeds k:[12]

$$r_u^{(k)} \geq k\,. \tag{5}$$

We observe then the important following point. An MRA DM from 1 to s, who prefers to disaggregate risks X_1, X_2, Y_1, Y_2 with $X_1 \preceq_{s_1-SD} Y_1$ and $X_2 \preceq_{s_2-SD} Y_2$, and $s_1+s_2 = s$, when risks are additive does not necessarily disaggregate them when they are multiplicative. Indeed, an MRA utility function does not necessarily satisfy $r_u^{(k)} \geq k$.[13] As a consequence the preservation of preference for *multiplicative* risk apportionment cannot be inherited from the preservation of preference for *additive* risk apportionment (we will show later that the converse actually holds, as already noted in Remark 1).

[12]Wang and Li (2010) talk about multiplicative *risk apportionment of order* $k+1$ which is, in fact, a special case of multiplicative risk apportionment of order $(1,k)$ as defined in the present paper.

[13]Consider, for instance, the CARA utility function (negative exponential). It is an MRA utility function (up to infinity). A DM with such a function will prefer to combine good with bad in lotteries with additive risks. But, for this function, $r_u^{(k)}(x) = cx$, for all k, with c the CARA coefficient. Thus, $r_u^{(k)} \geq k$ if, and only if, $x \geq \frac{k}{c}$. The same DM will not necessarily combine good with bad in lotteries with multiplicative risks.

5. Preserved preference for multiplicative risks – Some additional results

Denuit and Rey (2013, Proposition 5.4(ii)) show that they can adapt Condition (4) to the multiplicative case if they set $s_1 = 1$ (and $s_2 \geq 1$). They show that the condition $(-1)^{(1+k+1)} V^{(1,k)} \geq 0$ for all k reads $r_u^{(k)} \geq k$ for all k when risks are multiplicative. Specifically, they consider the degenerate lotteries $X_1 = a$ and $Y_1 = b$ where a and b are two positive constants such that $a < b$ (then $X_1 \preceq_{1-SD} Y_1$ holds obviously). They obtain the following result (see also Eeckhoudt *et al.*, 2009a; Chiu *et al.*, 2012): The inequality

$$\frac{1}{2}\mathrm{E}\left[u\left(aY_2\right)\right] + \frac{1}{2}\mathrm{E}\left[u\left(bX_2\right)\right] \geq \frac{1}{2}\mathrm{E}\left[u\left(aX_2\right)\right] + \frac{1}{2}\mathrm{E}\left[u\left(bY_2\right)\right] \tag{6}$$

holds for all utility functions u that satisfy $r_u^{(k)} \geq k$ for all $k = 1, \ldots, s_2$. They explain that they cannot extend their result to higher values of s_1, because for a given total order $s = s_1 + s_2$, the signs of the higher cross-derivatives of V are not necessarily the same, i.e., $(-1)^{(1+s_2+1)} V^{(1,s_2)} \geq 0$ is not the equivalent to $(-1)^{(2+s_2-1+1)} V^{(2,s_2-1)} \geq 0$ for $s_2 \geq 3$ (see remark 5).

Before stating the main result of this section, let us first stress the following relation between preference for multiplicative risk apportionment of orders (s_1, s_2) and $(s_1 - 1, s_2 + 1)$. Based on Theorem 2, we get:

Proposition 2. *For an expected utility maximizing DM, the following implication holds:*

$$\mathrm{PD}_M\left(s_1, s_2\right) \implies \mathrm{PD}_M\left(s_1 - 1, s_2 + 1\right) ,$$

whenever $s_2 \geq s_1$.

Proof. See Appendix D. □

As a consequence of Proposition 2, by an obvious recursion, we also have

Corollary 1. *For an expected utility maximizing DM, the following implication holds:*

$$\mathrm{PD}_M\left(s_1, s_2\right) \implies \mathrm{PD}_M\left(s_1 - k, s_2 + k\right) ,$$

for all $k < s_1$*, whenever* $s_2 \geq s_1$.

With $k = s_1 - 1$, Corollary 1 shows that $\mathrm{PD}_M\left(s_1, s_2\right)$ implies $\mathrm{PD}_M\left(1, s_1 + s_2 - 1\right)$ while Remark 1 showed that $\mathrm{PD}_M\left(1, s_1 + s_2 - 1\right)$ implies mixed risk aversion of order 1 to $s_1 + s_2$. Hence, $\mathrm{PD}_M\left(s_1, s_2\right)$ also implies mixed risk aversion of order 1 to $s_1 + s_2$ so that, by use of Theorem 1, we can state

Corollary 1'. *For an expected utility maximizing DM, the following implication holds:*

$$\mathrm{PD}_M\left(s_1, s_2\right) \implies \mathrm{PD}_A\left(s_1, s_2\right) .$$

As stressed by Remark 6 the converse is not true. It shows that preference for *multiplicative* risk apportionment is much more demanding than preference for *additive* risk apportionment.

Since mixed risk aversion appears as a necessary requirement for multiplicative risk apportionment, we assume that this condition holds from now on. Hence, under this assumption, we can express Theorem 2 in a slightly more convenient form:

Corollary 2. *For any set of four mutually independent non-negative risks* X_1, X_2, Y_1, Y_2 *such that* Y_i *dominates* X_i *via* s_i^{th}*-order stochastic dominance for* $i = 1, 2$*, an MRA DM from* 1 *to* $s_2 = \max(s_1, s_2)$ *prefers the lottery* $[X_1 \cdot Y_2, Y_1 \cdot X_2]$ *to the lottery* $[X_1 \cdot X_2, Y_1 \cdot Y_2]$ if, and only if, *his utility function* u *satisfies*

$$(-1)^{k_1} \left[\sum_{k=0}^{k_1} \frac{(-k_1)_k}{k!\,(k_2 - k_1 + 1)_k} \cdot \left(\prod_{i=0}^{k-1} r_u^{(k_2+i)} \right) \right] \geq 0\,,$$

for $k_1 = 1, 2, \ldots, s_1$*, and* $k_2 = k_1, k_1 + 1, \ldots, s_2$*,*

where $(a)_n \coloneqq a \cdot (a+1) \cdots (a+n-1)$ denotes Pochhammer's symbol (NIST Digital Library of Mathematical Functions, 2020, Eq. 5.2.4).

Proof. See Appendix E. □

The necessary and sufficient condition of Corollary 2 is more appealing than the condition provided by Theorem 2 because it involves the higher-order coefficients of relative risk aversion, which play a decisive role in multiplicative risk apportionment.

As an illustration, consider the case $s_1 = 2$ (and $s_2 \geq 2$). This case goes beyond the restrictions set by Denuit and Rey (2013) in the opening of this section ($s_1 = 1$ and $s_2 \geq 1$). Corollary 2 leads to the two following conditions, provided that u is MRA from order 1 to s_2:

1. for $k_1 = 1$, $r_u^{(k_2)} \geq k_2$ for $k_2 = 1, 2, \ldots, s_2$,
2. for $k_1 = 2$ and $k_2 = 2, \ldots, s_2$, the condition in Corollary 2 reads

$$\begin{aligned} & \sum_{k=0}^{2} \frac{(-2)_k}{k!\,(k_2 - 1)_k} \cdot \left(\prod_{i=0}^{k-1} r_u^{(k_2+i)} \right) \geq 0\,, \\ \iff & 1 - \frac{2}{k_2 - 1} \cdot r_u^{(k_2)} + \frac{1}{k_2\,(k_2 - 1)} \cdot r_u^{(k_2)} \cdot r_u^{(k_2+1)} \geq 0\,, \\ \iff & r_u^{(k_2)} \cdot \left(r_u^{(k_2+1)} - 2k_2 \right) + k_2\,(k_2 - 1) \geq 0\,, \\ \iff & r_u^{(k_2)} \cdot \left[r_u^{(k_2+1)} - (k_2 + 1) \right] \geq (k_2 - 1) \cdot \left(r_u^{(k_2)} - k_2 \right)\,, \end{aligned} \tag{7}$$

The case $k_1 = 2$ and $k_2 = 1$ has also to be considered but it leads to a condition already accounted for at step 1, namely $r_u^{(2)} \geq 2$.

Case 1 above retrieves the condition of Proposition 5.4(ii) in Denuit and Rey (2013), *i.e.*, the case where $s_1 = 1$ and $s_2 \geq 1$. However, the condition obtained in case 2 is new.

We can remark that the set of conditions in Case 1 are direct consequences of the set of conditions in Case 2, given $r_u^{(1)} \geq 1$. Indeed, given $r_u^{(1)} \geq 1$, the condition $r_u^{(2)} \geq 2$ (in Case 1) is equivalent to the condition in Case 2 for $k_2 = 1$ (setting $k_2 = 1$ in Equation (7) yields $r_u^{(1)} \cdot \left[r_u^{(2)} - 2 \right] \geq 0$). Then, by a straightforward recursion, the set of Conditions (7) shows that $r_u^{(k_2)} \geq k_2 \implies r_u^{(k_2+1)} \geq k_2 + 1$ for all $k_2 \leq s_2$ so that the set of conditions in Case 1 holds as soon as ($\imath$) $r_u^{(1)} \geq 1$ and ($\imath\imath$) the Condition (7) holds for $k_2 = 1, \ldots, s_2$ (and not just $k_2 = 2, \ldots, s_2$). We can then state the following Corollary:

Corollary 2'. *Let* u *be an MRA utility function from order* 1 *to* s*. It exhibits multiplicative risk apportionment of orders* $(2, s)$ if, and only if,

1. $r_u^{(1)} \geq 1$,

2. $r_u^{(k)}\left[r_u^{(k+1)}-(k+1)\right]\geq (k-1)\left[r_u^{(k)}-k\right]$ *for* $k=1,2,\ldots,s$.

Further, as a consequence of Corollary 2' above and Proposition 5.4(ii) in Denuit and Rey (2013), considering an MRA DM from 1 to $s+1$, we know that if the equivalence between multiplicative risk apportionment of order $(1,s+1)$ and multiplicative risk apportionment of order $(2,s)$ holds, *i.e.*, $\mathrm{PD}_M(1,s+1) \iff \mathrm{PD}_M(2,s)$, the equivalence relation

$$r_u^{(k)}\geq k,\,\forall k\leq s+1 \iff \begin{cases} r_u^{(1)}\geq 1\,, \\ r_u^{(k)}\left[r_u^{(k+1)}-(k+1)\right]\geq (k-1)\left[r_u^{(k)}-k\right],\,\forall k\leq s \end{cases},$$

also holds and vice-versa.

First, as stated by Proposition 2 and illustrated before Corollary 2', the relation $\mathrm{PD}_M(1,s+1) \Longleftarrow \mathrm{PD}_M(2,s)$ holds since, under the second condition of this Corollary,

$$r_u^{(k)}\geq k \implies r_u^{(k+1)}\geq k+1\,,\forall k\leq s\,,$$

and $r_u^{(1)}\geq 1$.

Let us now consider the converse relation, *i.e.*, $\mathrm{PD}_M(1,s+1) \implies \mathrm{PD}_M(2,s)$. Notice that

$$r_u^{(k)}\geq k,\,\forall k\leq s+1 \iff \begin{cases} r_u^{(1)}\geq 1\,, \\ r_u^{(k)}\left[r_u^{(k+1)}-(k+1)\right]\geq k\left[r_u^{(k+1)}-(k+1)\right],\,\forall k\leq s. \end{cases}$$

Thus, in order to conclude that $\mathrm{PD}_M(1,s+1) \implies \mathrm{PD}_M(2,s)$, we need the condition

$$k\left[r_u^{(k+1)}-(k+1)\right]\geq (k-1)\left[r_u^{(k)}-k\right],\,\forall k\leq s,$$

to hold, that is,

$$k\left(r_u^{(k+1)}-r_u^{(k)}\right)\geq k-\left(r_u^{(k)}-k\right),\,\forall k\leq s,$$

whenever $r_u^{(k)}\geq k$, $\forall k\leq s+1$.

This later condition is not easy to interpret but given $r_u^{(k)}\geq k$, $\forall k\leq s+1$, it is enough to have

$$k\left(r_u^{(k+1)}-r_u^{(k)}\right)\geq k,\,\forall k\leq s,$$

or, equivalently,

$$r_u^{(k+1)}-r_u^{(k)}\geq 1,\,\forall k\leq s.$$

The above inequality is interesting because, as recalled by Wang (2019), given $r_u^{(k)}\geq 0$, it means that the k^{th}-order relative risk aversion coefficient is decreasing in its argument:

$$r_u^{(k+1)}-r_u^{(k)}\geq 1 \overset{r_u^{(k)}\geq 0}{\iff} \frac{dr_u^{(k)}(x)}{dx}\leq 0\,.$$

The above results can then be summed up in the following proposition:

Proposition 3. *For an expected utility maximizing DM with decreasing relative risk aversion coefficients of order* 1 *to* s *the following holds*

$$\mathrm{PD}_M(1,s+1) \iff \mathrm{PD}_M(2,s)\,.$$

This proposition provides new conditions for the equivalence between multiplicative risk apportionment at total order $s+2$, when $s_1 = 1$ and $s_2 = s+1$ for the first set of risks, and when $s'_1 = 2$ and $s'_2 = s$ for the second set of risks. It shows that

$$\text{PD}_M(1,3) \iff \text{PD}_M(2,2) ,$$
$$\text{PD}_M(1,4) \iff \text{PD}_M(2,3) ,$$
$$\text{PD}_M(1,5) \iff \text{PD}_M(2,4) .$$

Thus Proposition 3 allows us to extend Proposition 1, which dealt with additive risks to the case of multiplicative risks, although only in a limited context, with the first stochastic dominance relationship either equal to one (FSD) or two (SSD).[14] Proceeding further, with $s_2 \geq s_1 \geq 3$ and, more generally, deriving the converse of the implication displayed in Proposition 2 would require more and more stringent conditions which quickly become very difficult to express. For this reason, we turn now to a specific case, assuming a CRRA utility function.

6. Multiplicative risks and CRRA utility functions

The utility functions u of the CRRA family satisfy

$$\frac{d}{dx}\Big(\frac{-x \cdot u''(x)}{u'(x)}\Big) = 0 ,$$

and they read

$$u(x) = \begin{cases} \frac{x^{1-\gamma}}{1-\gamma}, & \gamma > 0,\ \gamma \neq 1 , \\ \ln x, & \gamma = 1 . \end{cases} \tag{8}$$

In this case, the relative risk aversion index, $r_u^{(1)}$, is equal to the constant γ:

$$\frac{-x \cdot u''(x)}{u'(x)} = \gamma ,$$

and

Property 1. *For a CRRA utility function u, with $\gamma > 0$,*

$$r_u^{(k+1)} = r_u^{(k)} + 1 ,$$

for all $k \geq 1$.

Proof. See Appendix F. □

From Property 1, one easily derives the following

Property 2. *For a CRRA utility function u, with $\gamma > 0$,*

$$r_u^{(1)} > 1 \iff r_u^{(k)} > k ,$$

for all $k \geq 2$.

[14] Note that, by Remark 4, the equivalence displayed in Proposition 3 is also valid if it is the second stochastic dominance relationship that is equal to one or two.

The equivalence $r_u^{(1)} > 1 \iff r_u^{(2)} > 2$ already appears in the literature (see, for instance, Eeckhoudt *et al.*, 2005), but the generalization to order $k > 1$ is a direct consequence of Property 1 which has been neglected so far in the literature (to our knowledge).

It is well-known that CRRA utility functions are MRA (at all order), hence, they satisfy *additive* risk apportionment at any order (see Theorem 1). Additionally, based on the above Properties and on Corollary 2, we can show that preference for harms disaggregation extends to the case of multiplicative risks whenever $\gamma \geq 1$.

Theorem 3. *For any set of four mutually independent non-negative risks X_1, X_2, Y_1 and Y_2 such that Y_i dominates X_i via s_i^{th} order stochastic dominance for $i = 1, 2$, a DM with a CRRA utility function prefers the lottery $[X_1 \cdot Y_2; Y_1 \cdot X_2]$ to the lottery $[X_1 \cdot X_2; Y_1 \cdot Y_2]$* if, and only if, $r_u^{(1)} = \gamma \geq 1$.

Proof. See Appendix G. □

Remark 7. Of course, the converse result holds, *i.e.* a DM with a CRRA utility function prefers the lottery $[X_1 \cdot X_2; Y_1 \cdot Y_2]$ to the lottery $[X_1 \cdot Y_2; Y_1 \cdot X_2]$ *if, and only if,* $r_u^{(1)} = \gamma \leq 1$. It means that a CRRA DM with $\gamma \leq 1$ exhibits preference for multiplicative risks aggregation. Additionally, in the specific case $\gamma = 1$, *i.e.* for a logarithmic utility function, the DM is indifferent to both lotteries. The latter result is obvious and insofar as

$$\begin{aligned}
\frac{1}{2}\mathrm{E}\left[\ln\left(X_1 \cdot Y_2\right)\right] + \frac{1}{2}\mathrm{E}\left[\ln\left(X_2 \cdot Y_1\right)\right] &= \frac{1}{2}\mathrm{E}\left[\ln\left(X_1\right)\right] + \frac{1}{2}\mathrm{E}\left[\ln\left(Y_2\right)\right] + \frac{1}{2}\mathrm{E}\left[\ln\left(X_2\right)\right] + \frac{1}{2}\mathrm{E}\left[\ln\left(Y_1\right)\right] \ , \\
&= \frac{1}{2}\mathrm{E}\left[\ln\left(X_1\right)\right] + \frac{1}{2}\mathrm{E}\left[\ln\left(X_2\right)\right] + \frac{1}{2}\mathrm{E}\left[\ln\left(Y_1\right)\right] + \frac{1}{2}\mathrm{E}\left[\ln\left(Y_2\right)\right] \ , \\
&= \frac{1}{2}\mathrm{E}\left[\ln\left(X_1 \cdot X_2\right)\right] + \frac{1}{2}\mathrm{E}\left[\ln\left(Y_1 \cdot Y_2\right)\right] \ .
\end{aligned}$$

Notice that the restriction spelt out in Proposition 3 about the value of s_1 (or s_2) does not apply any more while the requirements expressed by Equation (5) about the value of the relative risk aversion at orders greater than 1 are automatically satisfied as soon as $r_u^{(1)} \geq 1$. For instance, if $s_2 = 10$, and $s_1 = 5$, multiplicative risk apportionment holds assuming a CRRA utility function with $X_1 \preceq_{5-SD} Y_1$ and $X_2 \preceq_{10-SD} Y_2$ provided $r_u^{(1)} \geq 1$.

We can therefore extend the result in Proposition 3 and state our Proposition 4.

Proposition 4. *Given a DM with a CRRA utility function u, the following equivalence holds*

$$\mathrm{PD}_M\left(s_1, s_2\right) \Longleftrightarrow \mathrm{PD}_M\left(s_1', s_2'\right) \ ,$$

whatever s_1, s_2, s_1' and s_2'.

Proof. Theorem 3 holds for all CRRA utility function u verifying $r_u^{(1)} \geq 1$ whatever the stochastic dominance orders (s_1, s_2). Consequently, if preference for multiplicative risk apportionment holds for orders (s_1, s_2), *i.e.* $\mathrm{PD}_M(s_1, s_2)$, it also holds for multiplicative risk apportionment of orders (s_1', s_2'), *i.e.* $\mathrm{PD}_M(s_1', s_2')$, since both are equivalent to $r_u^{(1)} \geq 1$. □

Remark 8. Proposition 4 shows that the disaggregation preference relation is preserved when the first set of risks $\left[(X_1, Y_1); (X_2, Y_2)\right]_{(s_1, s_2)}$ is replaced by the second set of risks $\left[(X_1', Y_1'); (X_2', Y_2')\right]_{(s_1', s_2')}$ and vice-versa: $\mathrm{PD}_M(s_1, s_2) \iff \mathrm{PD}_M(s_1', s_2')$ for all total order levels, *i.e.*, such that $s_1' + s_2' = s_1 + s_2$ or $s_1' + s_2' \neq s_1 + s_2$. This remark underlines the *power* of CRRA utility functions.

7. Conclusion

In this paper, we start by providing homogeneous definitions of additive and multiplicative risk apportionment and we derive two new theorems spelling out the exact conditions for observing these two behavioral features. We then extend the Eeckhoudt *et al.* (2009b) results on risk apportionment with two additive risks by showing that, for an MRA DM, risk apportionment at total order $s = s_1 + s_2$ implies risk apportionment at total order $s' = s_1' + s_2'$ provided that $s = s'$. Our main motivation is then to check whether this simple property holds in a multiplicative risk context. We show that this is not the case, in general. However, elaborating on the basis of our theorem for multiplicative risk apportionment, we are able to characterize specific cases where the property holds. We also show that the property is recovered when we restrict the analysis to CRRA utility functions. In this case, the property is even stronger, as it holds more generally for all $s' \neq s$, as well for multiplicative risks as for additive risks, provided relative risk aversion is larger than 1. We are able to relate this last result to a property of constant relative risk aversion functions neglected so far: relative risk aversion at order $n+1$ is equal to relative risk aversion at order n, plus one.

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Appendix

Appendix A. A reminder on stochastic dominance and the proof of Lemma 1(ii) in Chiu *et al.* (2012)

Let F_Z denote the cumulative distribution function of the random variable Z defined over the interval $[a, b] \subset \mathbb{R}$, meaning that

$$\Pr(Z \leq a) = 0 \qquad \text{and} \qquad \Pr(Z \leq b) = 1\,.$$

We introduce the notation

$$F_Z^{(1)}(t) \coloneqq F_Z(t) \quad \text{and} \quad F_Z^{(k+1)}(t) \coloneqq \int_a^t F_Z^{(k)}(t')\,dt'\,.$$

Then (Jean, 1980; Ingersoll, 1987)

Definition (s^{th}-order stochastic dominance)**.** The random variable X weakly dominates the random variable Y in the sense of s^{th}-order stochastic dominance, *i.e.* $X \succcurlyeq_{s-SD} Y$, if

$$F_X^{(s)}(t) \le F_Y^{(s)}(t)\,, \quad \forall t \in [a,b]\,, \tag{A.1}$$

and

$$F_X^{(k)}(b) \le F_Y^{(k)}(b)\,, \quad \forall k \in \{1,\dots,s-1\}\,. \tag{A.2}$$

Chiu *et al.* (2012) state the following equivalence:

Lemma (Chiu *et al.*, 2012, lemma 1(ii))**.** *For all X and Y such that X dominates Y via s^{th}-order stochastic dominance,* $\mathrm{E}[u(X)] \ge \mathrm{E}[u(Y)]$ if and only *if* $(-1)^{k+1} u^{(k)} \ge 0$ *for* $k = 1,\dots,s$.

Formally, the lemma reads

$$(-1)^{k+1} u^{(k)} \ge 0, \quad \forall k \in \{1,\dots s\} \iff \mathrm{E}[u(X)] \ge \mathrm{E}[u(Y)]\,, \quad \forall X, Y \text{ such that } X \succcurlyeq_{s-SD} Y\,.$$

Since this result is not proved by Chiu *et al.* (2012) nor by Ingersoll (1987), as referred to by Chiu *et al.* (2012), we provide a short proof below.

Proof of the lemma. **Sufficiency.** To prove the lemma, we start with the sufficient part, that is, we assume $(-1)^{k+1} u^{(k)} \ge 0$ for all $k \in \{1,\dots s\}$. Following Whitmore (1970), we set $H \coloneqq F_X - F_Y$. For simplicity, we assume that u is s times continuously differentiable. Then, s integrations by parts yield

$$\begin{aligned}
\mathrm{E}[u(X)] - \mathrm{E}[u(Y)] &= \int_a^b u(t)\, dF_X^{(1)}(t) - \int_a^b u(t)\, dF_Y^{(1)}(t)\ , \\
&= \int_a^b u(t)\, dH^{(1)}(t)\ , \\
&= \Big[u(t)\, H^{(1)}(t)\Big]_a^b - \int_a^b u^{(1)}(t)\, dH^{(2)}(t)\ , \\
&= \Big[u(t)\, H^{(1)}(t) - u^{(1)}(t)\, H^{(2)}(t)\Big]_a^b + \int_a^b u^{(2)}(t)\, dH^{(3)}(t)\ , \\
&= \cdots \\
&= -\sum_{k=0}^{s-1} \Big[(-1)^{k+1} u^{(k)}(t)\, H^{(k+1)}(t)\Big]_a^b - (-1)^{s+1} \int_a^b u^{(s)}(t)\, dH^{(s+1)}(t)\ , \\
&= -\sum_{k=1}^{s-1} (-1)^{k+1} u^{(k)}(b)\, H^{(k+1)}(b) - (-1)^{s+1} \int_a^b u^{(s)}(t)\, dH^{(s+1)}(t)\ .
\end{aligned}$$

The last equality results from the fact that $H^{(k)}(a) = 0$ for all k and $H^{(1)}(b) = 0$.

For all X and Y such that $X \succcurlyeq_{s-SD} Y$, Equations (A.1-A.2) imply $H^{(k)}(b) \le 0$ for all $k \in \{1,\dots,s\}$. Hence, given $(-1)^{k+1} u^{(k)} \ge 0$ for all $k \in \{1,\dots s\}$,

$$-\sum_{k=1}^{s-1} (-1)^{k+1} u^{(k)}(b)\, H^{(k+1)}(b) \ge 0\,,$$

and

$$\mathrm{E}[u(X)] - \mathrm{E}[u(Y)] \ge -(-1)^{s+1} \int_a^b u^{(s)}(t)\, dH^{(s+1)}(t)\ .$$

Then, Equation (A.1) shows that $H^{(s+1)}$ is a decreasing function – its derivative, $H^{(s)}$, is negative – so that

$$\int_a^b (-1)^{s+1} u^{(s)}(t)\, dH^{(s+1)}(t) \leq 0\,,$$

since $(-1)^{s+1} u^{(s)} \geq 0$, and

$$\mathrm{E}[u(X)] - \mathrm{E}[u(Y)] \geq 0\,,$$

which concludes the proof of the sufficient part of the lemma.

Necessity. Let us now prove the converse *ad absurdum*. To this aim, let us assume that there exists an integer $k^* \in \{1, \ldots, s\}$ such that $(-1)^{k^*+1} u^{(k^*)} \ngeq 0$. We show that it implies the existence of a couple of random variables (X, Y) such that $X \succcurlyeq_{s-SD} Y$ and $\mathrm{E}[u(X)] < \mathrm{E}[u(Y)]$.

Given the existence of $k^* \in \{1, \ldots, s\}$ such that $(-1)^{k^*+1} u^{(k^*)} \ngeq 0$ and $u^{(k^*)}$ is a continuous function (by assumption), there is an interval $[a^*, b^*] \subset [a, b]$, with $a^* < b^*$ such that

$$(-1)^{k^*+1} u^{(k^*)}(t) < 0, \quad \forall t \in [a^*, b^*]\,,$$

Let us consider a couple of random variables (X^*, Y^*), such that $X^* \succcurlyeq_{s-SD} Y^*$, satisfying

$$\Pr(X^* \leq a^*) = \Pr(Y^* \leq a^*) = 0 \qquad \text{and} \qquad \Pr(X^* \leq b^*) = \Pr(Y^* \leq b^*) = 1\,,$$

that is, whose domain is restricted to $[a^*, b^*]$, and

$$F_{X^*}^{(k)}(b^*) = F_{Y^*}^{(k)}(b^*)\,, \quad \forall k \in \{1, \ldots k^*\}\,,$$

namely which share the same moments up to the order k^*.

Defining

$$H^{(1)} \coloneqq F_{X^*} - F_{Y^*}\,,$$

k^* integrations by parts yield

$$\begin{aligned}
\mathrm{E}[u(X^*)] - \mathrm{E}[u(Y^*)] &= \int_{a^*}^{b^*} u(t)\, dH^{(1)}(t)\,, \\
&= -\sum_{k=0}^{k^*-1} \left[(-1)^{k+1} u^{(k)}(t)\, H^{(k+1)}(t)\right]_{a^*}^{b^*} - (-1)^{k^*+1} \int_{a^*}^{b^*} u^{(k^*)}(t)\, dH^{(k^*+1)}(t)\,, \\
&= -(-1)^{k^*+1} \int_{a^*}^{b^*} u^{(k^*)}(t)\, dH^{(k^*+1)}(t)\,.
\end{aligned}$$

The last equality holds because $H^{(k)}(a^*) = H^{(k)}(b^*) = 0$ for all $k \leq k^*$. Since $(-1)^{k^*+1} u^{(k^*)}(t) < 0$ for all $t \in [a^*, b^*]$ and $H^{(k^*+1)}$ is a decreasing function

$$(-1)^{k^*+1} \int_{a^*}^{b^*} u^{(k^*)}(t)\, dH^{(k^*+1)}(t) > 0\,,$$

and

$$\mathrm{E}[u(X^*)] - \mathrm{E}[u(Y^*)] < 0\,.$$

Hence, the existence of an integer $k^* \in \{1, \ldots, s\}$ such that $(-1)^{k^*+1} u^{(k^*)} \ngeq 0$ does imply the existence of a couple of random variables (X^*, Y^*) such that $X^* \succcurlyeq_{s-SD} Y^*$ and $\mathrm{E}[u(X^*)] < \mathrm{E}[u(Y^*)]$, which concludes the proof. □

Appendix B. Proof of Theorem 1

Let X_1, X_2, Y_1, Y_2 be any four mutually independent risks such that $X_1 \preceq_{s_1-SD} Y_1$ and $X_2 \preceq_{s_2-SD} Y_2$. A DM exhibits preference for *additive risk apportionment* if (and only if) he prefers the lottery $[X_1 + Y_2, Y_1 + X_2]$ to the lottery $[X_1 + X_2, Y_1 + Y_2]$, namely

$$\frac{1}{2} \cdot \mathrm{E}\left[u\left(X_1 + Y_2\right)\right] + \frac{1}{2} \cdot \mathrm{E}\left[u\left(Y_1 + X_2\right)\right] \geq \frac{1}{2} \cdot \mathrm{E}\left[u\left(X_1 + X_2\right)\right] + \frac{1}{2} \cdot \mathrm{E}\left[u\left(Y_1 + Y_2\right)\right] .$$

This relation is equivalent to

$$\mathrm{E}\left[u\left(Y_1 + X_2\right)\right] - \mathrm{E}\left[u\left(Y_1 + Y_2\right)\right] \geq \mathrm{E}\left[u\left(X_1 + X_2\right)\right] - \mathrm{E}\left[u\left(X_1 + Y_2\right)\right] . \tag{B.1}$$

Hence, introducing

$$w\left(t\right) \coloneqq \mathrm{E}\left[u\left(t + X_2\right)\right] - \mathrm{E}\left[u\left(t + Y_2\right)\right] , \tag{B.2}$$

equation (B.1) reads

$$\mathrm{E}\left[w\left(Y_1\right)\right] \geq \mathrm{E}\left[w\left(X_1\right)\right] .$$

According to Lemma 1 in Chiu *et al.* (2012) (see also Ingersoll, 1987), this last inequality holds for all $X_1 \preceq_{s_1-SD} Y_1$ *if, and only if,*

$$\left(-1\right)^{k_1+1} w^{(k_1)} \geq 0, \quad \text{for } k_1 = 1, 2, \ldots, s_1 .$$

Replacing w by its expression given by equation (B.2), we get

$$\mathrm{E}\left[\left(-1\right)^{k_1+1} u^{(k_1)}\left(t + X_2\right)\right] \geq \mathrm{E}\left[\left(-1\right)^{k_1+1} u^{(k_1)}\left(t + Y_2\right)\right], \quad \text{for } k_1 = 1, 2, \ldots, s_1 .$$

Again, relying on Lemma 1 in Chiu *et al.* (2012), this relation holds for all $X_2 \preceq_{s_2-SD} Y_2$ *if, and only if,*

$$\left(-1\right)^{k_1+k_2+1} u^{(k_1+k_2)} \geq 0, \quad \text{for } k_1 = 1, 2, \ldots, s_1 \text{ and } k_2 = 1, 2, \ldots, s_2 .$$

Of course, given k_1 and k_2 only enter in the relation above in terms of their sum, we equivalently have

$$\left(-1\right)^{k+1} u^{(k)} \geq 0, \quad \text{for } k = 1, 2, \ldots, s_1 + s_2 .$$

Appendix C. Proof of Theorem 2

Let X_1, X_2, Y_1, Y_2 be any four mutually independent non-negative risks such that $X_1 \preceq_{s_1-SD} Y_1$ and $X_2 \preceq_{s_2-SD} Y_2$. A DM exhibits preference for *multiplicative risk apportionment* if (and only if) he prefers the lottery $[X_1 \cdot Y_2, Y_1 \cdot X_2]$ to the lottery $[X_1 \cdot X_2, Y_1 \cdot Y_2]$, namely

$$\frac{1}{2} \cdot \mathrm{E}\left[u\left(X_1 \cdot Y_2\right)\right] + \frac{1}{2} \cdot \mathrm{E}\left[u\left(Y_1 \cdot X_2\right)\right] \geq \frac{1}{2} \cdot \mathrm{E}\left[u\left(X_1 \cdot X_2\right)\right] + \frac{1}{2} \cdot \mathrm{E}\left[u\left(Y_1 \cdot Y_2\right)\right] .$$

This relation is equivalent to

$$\mathrm{E}\left[u\left(Y_1 \cdot X_2\right)\right] - \mathrm{E}\left[u\left(Y_1 \cdot Y_2\right)\right] \geq \mathrm{E}\left[u\left(X_1 \cdot X_2\right)\right] - \mathrm{E}\left[u\left(X_1 \cdot Y_2\right)\right] . \tag{C.1}$$

Hence, introducing

$$w\left(t\right) \coloneqq \mathrm{E}\left[u\left(t \cdot X_2\right)\right] - \mathrm{E}\left[u\left(t \cdot Y_2\right)\right] , \tag{C.2}$$

with $t \geq 0$, Equation (C.1) reads

$$\mathrm{E}\left[w\left(Y_1\right)\right] \geq \mathrm{E}\left[w\left(X_1\right)\right] \,.$$

According to Lemma 1 in Chiu *et al.* (2012), this last inequality holds for all $X_1 \preceq_{s_1-SD} Y_1$ *if, and only if,*

$$(-1)^{k_1+1}\, w^{(k_1)} \geq 0, \quad \text{for } k_1 = 1, 2, \ldots, s_1 \,.$$

Replacing w by its expression given by Equation (C.2), we get

$$\forall t > 0, \quad \mathrm{E}\left[(-1)^{k_1+1}\, X_2^{k_1} \cdot u^{(k_1)}\left(t \cdot X_2\right)\right] \geq \mathrm{E}\left[(-1)^{k_1+1}\, Y_2^{k_1} \cdot u^{(k_1)}\left(t \cdot Y_2\right)\right],$$

for $k_1 = 1, 2, \ldots, s_1$. Again, relying on Lemma 1 in Chiu *et al.* (2012), this relation holds for all $X_2 \preceq_{s_2-SD} Y_2$ *if, and only if,*

$$\forall t > 0, z > 0, \quad (-1)^{k_1+k_2+1} \frac{\partial^{k_2}}{\partial z^{k_2}} \left[z^{k_1} \cdot u^{(k_1)}\left(t \cdot z\right)\right] \geq 0,$$

for $k_1 = 1, 2, \ldots, s_1$ and $k_2 = 1, 2, \ldots, s_2$.

Now, relying on Leibniz formula, we can express the k_2^{th}-order derivative as follows

$$\begin{aligned}
\frac{\partial^{k_2}}{\partial z^{k_2}} \left[z^{k_1} \cdot u^{(k_1)}\left(t \cdot z\right)\right] &= \sum_{k=0}^{k_2} \binom{k_2}{k} \left(\frac{d^k}{dz^k} z^{k_1}\right) \cdot \left(\frac{d^{k_2-k}}{dz^{k_2-k}} u^{(k_1)}\left(t \cdot z\right)\right), \\
&= \sum_{k=0}^{\min(k_1,k_2)} \frac{k_1! \cdot k_2!}{k!\,(k_1-k)!\,(k_2-k)!} \cdot z^{k_1-k} \cdot t^{k_2-k} u^{(k_1+k_2-k)}\left(t \cdot z\right),
\end{aligned}$$

so that preference for multiplicative risk apportionment holds if, and only if,

$$\forall t, z > 0, \quad (-1)^{k_1+k_2+1} \sum_{k=0}^{\min(k_1,k_2)} \frac{k_1! \cdot k_2!}{k!\,(k_1-k)!\,(k_2-k)!} \cdot z^{k_1-k} \cdot t^{k_2-k} u^{(k_1+k_2-k)}\left(t \cdot z\right) \geq 0\,,$$

for $k_1 = 1, 2, \ldots, s_1$ and $k_2 = 1, 2, \ldots, s_2$. Of course, since t and z are non-negative, the above condition is equivalent to

$$\forall t, z > 0, \quad (-1)^{k_1+k_2+1} \sum_{k=0}^{\min(k_1,k_2)} \frac{k_1! \cdot k_2!}{k!\,(k_1-k)!\,(k_2-k)!} \cdot \left(t \cdot z\right)^{k_1+k_2-k} u^{(k_1+k_2-k)}\left(t \cdot z\right) \geq 0\,,$$

or

$$\forall x > 0, \quad (-1)^{k_1+k_2+1} \sum_{k=0}^{\min(k_1,k_2)} \frac{k_1! \cdot k_2!}{k!\,(k_1-k)!\,(k_2-k)!} \cdot x^{k_1+k_2-k} u^{(k_1+k_2-k)}\left(x\right) \geq 0\,, \tag{C.3}$$

hence the result.

Appendix D. Proof of Proposition 2

Proposition 2 results from the auxiliary Lemma that follows (see Appendix H)

Lemma 1. *Given a $(k_1 + k_2 + 1)$-differentiable utility function u,*

$$Q_u^{(k_1,k_2+1)} = Q_u^{(k_1+1,k_2)} + \left(k_2 - k_1\right) \cdot Q_u^{(k_1,k_2)}\,,$$

whatever k_1 and k_2,

where

$$Q_u^{(k_1,k_2)}(x) := (-1)^{k_1+k_2+1} \sum_{k=0}^{\min(k_1,k_2)} \frac{k_1! \cdot k_2!}{k!\,(k_1-k)!\,(k_2-k)!} \cdot x^{k_1+k_2-k} u^{(k_1+k_2-k)}(x) \; .$$

Proof of Proposition 2. Given Theorem 2, with $s_1 \le s_2$,

$$\text{PD}_M(s_1, s_2) \iff \left\{ Q_u^{(k_1,k_2)} \ge 0,\ \forall k_1 \le s_1, \forall k_2 = k_1, \ldots, s_2 \right\} ,$$

and

$$\text{PD}_M(s_1-1, s_2+1) \iff \left\{ Q_u^{(k_1,k_2)} \ge 0,\ \forall k_1 \le s_1-1, \forall k_2 = k_1, \ldots, s_2+1 \right\} .$$

We then have to prove that the first set of relations implies the second one. The implication is obvious for all the common values of k_1 and k_2 in the two sets of relations. Hence, we just have to prove that the first set of relations, that is, $\left\{ Q_u^{(k_1,k_2)} \ge 0,\ \forall k_1 \le s_1, \forall k_2 = k_1, \ldots, s_2 \right\}$, implies $\left\{ Q_u^{(k_1,s_2+1)} \ge 0,\ \forall k_1 = 1, 2, \ldots, s_1-1 \right\}$.

As a consequence of Lemma 1,

$$Q_u^{(k_1,s_2+1)} = Q_u^{(k_1+1,s_2)} + (s_2 - k_1) \cdot Q_u^{(k_1,s_2)} ,$$

meaning that $Q_u^{(k_1+1,s_2)} \ge 0$ and $Q_u^{(k_1,s_2)} \ge 0$ for $k_1 \le s_1 - 1$ is enough to ensure $Q_u^{(k_1,s_2+1)} \ge 0$ for $k_1 = 1, 2, \ldots, s_1 - 1$, which concludes the proof. □

Appendix E. Proof of Corollary 2

Accounting for the relation

$$u^{(k)}(x) = -x^{-1} \cdot r_u^{(k-1)}(x) \cdot u^{(k-1)}(x) ,$$

we obtain

$$u^{(k_1+k_2-k)}(x) = (-1)^{k_1-k} x^{k-k_1} \cdot \left(\prod_{i=0}^{k_1-k-1} r_u^{(k_2+i)}(x) \right) \cdot u^{(k_2)}(x) \; .$$

We can assume, without loss of generality, that $k_1 = \min(k_1, k_2)$. Hence, by substitution of the above relation in Theorem 2 (or Eq. C.3), we obtain the following alternative expression of the necessary and sufficient condition for multiplicative risk apportionment:

$$(-1)^{k_2+1} u^{(k_2)}(x) \left[\sum_{k=0}^{k_1} \frac{(-1)^k k_1! \cdot k_2!}{k!\,(k_1-k)!\,(k_2-k)!} \cdot \left(\prod_{i=0}^{k_1-k-1} r_u^{(k_2+i)}(x) \right) \right] \ge 0 \, , \quad \forall x > 0 \, , \tag{E.1}$$

for all $k_1 = 1, \ldots, s_1$ and $k_2 = k_1, \ldots, s_2$, with the convention $\prod_{i=0}^{-1} u_i = 1$ whatever u_i. Assuming $(-1)^{k_2+1} u^{(k_2)} \ge 0$, by the change of index $k \leftrightarrow k_1 - k$, we get

$$(-1)^{k_1} \sum_{k=0}^{k_1} \frac{(-1)^k k_1! \cdot k_2!}{k!\,(k_1-k)!\,(k_2-k_1+k)!} \cdot \left(\prod_{i=0}^{k-1} r_u^{(k_2+i)}(x) \right) \ge 0 \, , \qquad \forall x > 0 \, .$$

We can now rewrite this condition by use of Pochhammer's symbol $(a)_n = a \cdot (a+1) \cdots (a+n-1)$ and get

$$(-1)^{k_1} \cdot \frac{k_2!}{(k_2-k_1)!} \sum_{k=0}^{k_1} \frac{(-k_1)_k}{k!\,(k_2-k_1+1)_k} \cdot \left(\prod_{i=0}^{k-1} r_u^{(k_2+i)}(x) \right) \ge 0 \, , \qquad \forall x > 0 \, ,$$

which concludes the proof.

Appendix F. Proof of Property 1 of CRRA utility function

Let us show that, for any CRRA utility function $u(\cdot)$

$$r_u^{(n+1)} = r_u^{(n)} + 1\,.$$

Let us recall that

$$r_u^{(n)}(x) := -x \cdot \frac{u^{(n+1)}(x)}{u^{(n)}(x)}\,.$$

For all $n \geq 0$, with u given by eq. (8)

$$\begin{aligned} u^{(n)}(x) &= (-1)^{n+1}\,\gamma\,(\gamma+1)\,(\gamma+2)\cdots(\gamma+n-2)\cdot x^{-\gamma-n+1}\,,\\ &= (-1)^{n+1}\left[\prod_{j=0}^{n-2}(\gamma+j)\right]\cdot x^{-\gamma-n+1}\,. \end{aligned}$$

Hence

$$\begin{aligned} -x\cdot\frac{u^{(n+1)}(x)}{u^{(n)}(x)} &= -x\cdot\frac{(-1)^{n+2}\left[\prod_{j=0}^{n-1}(\gamma+j)\right]\cdot x^{-\gamma-n}}{(-1)^{n+1}\left[\prod_{j=0}^{n-2}(\gamma+j)\right]\cdot x^{-\gamma-n+1}}\,,\\ &= \gamma+n-1\,. \end{aligned}$$

So

$$\begin{aligned} -x\cdot\frac{u^{(n+2)}(x)}{u^{(n+1)}(x)} &= \gamma+n-1+1\,,\\ &= -x\cdot\frac{u^{(n+1)}(x)}{u^{(n)}(x)}+1\,, \end{aligned}$$

that is

$$r_u^{(n+1)} = r_u^{(n)} + 1\,,$$

which concludes the proof.

Appendix G. Proof of Theorem 3

From Properties 1 and 2, we immediately conclude that $r_u^{(k)} = \gamma - 1 + k$ for a CRRA utility function with coefficient of relative risk aversion $\gamma > 0$. The CRRA utility functions are obviously MRA so that we can apply Corollary 2 to characterize multiplicative risk apportionment.

Based on Corollary 2, an MRA DM from 1 to $s_2 = \max(s_1, s_2)$ exhibits preference for multiplicative risk apportionment of orders (s_1, s_2) if, and only if, his utility function u satisfies

$$(-1)^{k_1}\sum_{k=0}^{k_1}\frac{(-k_1)_k}{k!\,(k_2-k_1+1)_k}\cdot\left(\prod_{i=0}^{k-1}(\gamma-1+k_2+i)\right)\geq 0\,,\qquad \forall x\geq 0\,,$$

for all $k_1 = 1, 2, \ldots, s_1$, and $k_2 = k_1, k_1+1, \ldots, s_2$.

Accounting for the definition of Pochhammer's symbol (NIST Digital Library of Mathematical Functions, 2020, Eq. 5.2.4), the relation above reads

$$(-1)^{k_1}\cdot\sum_{k=0}^{k_1}\frac{(-k_1)_k\cdot(\gamma-1+k_2)_k}{k!\,(k_2-k_1+1)_k}\geq 0\,,\tag{G.1}$$

for all $k_1 = 1, 2, \ldots, s_1$ and $k_2 = k_1, k_1 + 1, \ldots, s_2$ and, by use of Chu-Vandermonde identity (NIST Digital Library of Mathematical Functions, 2020, Eq. 15.4.24), the left-hand side simplifies:

$$(-1)^{k_1} \cdot \sum_{k=0}^{k_1} \frac{(-k_1)_k \cdot (\gamma - 1 + k_2)_k}{k! \, (k_2 - k_1 + 1)_k} = \frac{(k_2 - k_1)!}{k_2!} \cdot (\gamma - 1)_{k_1} \; .$$

Hence the condition in Corollary 2 holds if, and only if, $\gamma - 1 \geq 0$.

Appendix H. Proof of Lemma 1

Defining

$$Q_u^{(k_1,k_2)}(x) \coloneqq (-1)^{k_1+k_2+1} \sum_{k=0}^{\min(k_1,k_2)} \frac{k_1! \cdot k_2!}{k! \, (k_1-k)! \, (k_2-k)!} \cdot x^{k_1+k_2-k} u^{(k_1+k_2-k)}(x) \; ,$$

Lemma 1 states that

$$Q_u^{(k_1,k_2+1)} = Q_u^{(k_1+1,k_2)} + (k_2 - k_1) \cdot Q_u^{(k_1,k_2)}$$

whatever k_1 and k_2.

Assuming $k_1 < k_2$, without loss of generality, we have

$$(-1)^{k_1+k_2+1} \cdot Q_u^{(k_1,k_2)} = \sum_{k=0}^{k_1} \frac{k_1! \cdot k_2!}{k! \, (k_1-k)! \, (k_2-k)!} \cdot x^{k_1+k_2-k} u^{(k_1+k_2-k)}(x)$$

and

$$\begin{aligned}
(-1)^{k_1+k_2} \cdot Q_u^{(k_1+1,k_2)} &= \sum_{k=0}^{k_1+1} \frac{(k_1+1)! \cdot k_2!}{k! \, (k_1+1-k)! \, (k_2-k)!} \cdot x^{k_1+k_2+1-k} u^{(k_1+k_2+1-k)}(x) \; , \\
&= x^{k_1+k_2+1} u^{(k_1+k_2+1)}(x) \\
&\quad + \sum_{k=1}^{k_1+1} \frac{(k_1+1)! \cdot k_2!}{k! \, (k_1+1-k)! \, (k_2-k)!} \cdot x^{k_1+k_2+1-k} u^{(k_1+k_2+1-k)}(x) \; , \\
&\overset{k'=k-1}{=} x^{k_1+k_2+1} u^{(k_1+k_2+1)}(x) \\
&\quad + \sum_{k'=0}^{k_1} \frac{(k_1+1)! \cdot k_2!}{(k'+1)! \, (k_1-k')! \, (k_2-k'-1)!} \cdot x^{k_1+k_2-k'} u^{\left(k_1+k_2-k'\right)}(x) \; .
\end{aligned}$$

Let us evaluate

$$\begin{aligned}
&(-1)^{k_1+k_2} \cdot Q_u^{(k_1+1,k_2)} - (k_2 - k_1) \cdot (-1)^{k_1+k_2+1} \cdot Q_u^{(k_1,k_2)} \; , \\
=& x^{k_1+k_2+1} u^{(k_1+k_2+1)}(x) \\
&+ \sum_{k=0}^{k_1} \left[\frac{(k_1+1)! \cdot k_2!}{(k+1)! \, (k_1-k)! \, (k_2-k-1)!} - \frac{(k_2-k_1) \, k_1! \cdot k_2!}{k! \, (k_1-k)! \, (k_2-k)!} \right] \cdot x^{k_1+k_2-k} u^{(k_1+k_2-k)}(x) \; , \\
=& x^{k_1+k_2+1} u^{(k_1+k_2+1)}(x) \\
&+ \sum_{k=0}^{k_1-1} \frac{k_1! \cdot k_2!}{k! \, (k_1-k)! \, (k_2-k-1)!} \left[\frac{(k_1-k)\,(k_2+1)}{(k_2-k)\,(k+1)} \right] \cdot x^{k_1+k_2-k} u^{(k_1+k_2-k)}(x) \; .
\end{aligned}$$

The upper bound of the sum is $k_1 - 1$ and not k_1 because the term of order k_1 vanishes. Hence

$$\begin{aligned}
&(-1)^{k_1+k_2} \cdot Q_u^{(k_1+1,k_2)} - (k_2 - k_1) \cdot (-1)^{k_1+k_2+1} \cdot Q_u^{(k_1,k_2)} , \\
=&x^{k_1+k_2+1} u^{(k_1+k_2+1)}(x) \\
&+ \sum_{k=0}^{k_1-1} \frac{k_1! \cdot (k_2+1)!}{(k+1)!\,(k_1-k-1)!\,(k_2-k)!} \cdot x^{k_1+k_2-k} u^{(k_1+k_2-k)}(x) , \\
=&\sum_{k=-1}^{k_1-1} \frac{k_1! \cdot (k_2+1)!}{(k+1)!\,(k_1-k-1)!\,(k_2-k)!} \cdot x^{k_1+k_2-k} u^{(k_1+k_2-k)}(x) , \\
\overset{k'=k+1}{=}&\sum_{k'=0}^{k_1} \frac{k_1! \cdot (k_2+1)!}{k'!\,(k_1-k')!\,(k_2+1-k')!} \cdot x^{k_1+k_2+1-k'} u^{(k_1+k_2+1-k')}(x) , \\
=&(-1)^{k_1+k_2} \cdot Q_u^{(k_1,k_2+1)} .
\end{aligned}$$

To sum up, we have

$$Q_u^{(k_1,k_2+1)} = Q_u^{(k_1+1,k_2)} + (k_2 - k_1) \cdot Q_u^{(k_1,k_2)}$$

whenever $k_1 < k_2$.

Since $Q_u^{(k_1,k_2)} = Q_u^{(k_2,k_1)}$, the previous derivations also hold whenever $k_1 > k_2$, while the result is obvious when $k_1 = k_2$. It concludes the proof.

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