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Jean-Bernard Chatelain, Kirsten Ralf

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Jean-Bernard Chatelain* Kirsten Ralf†

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Abstract

A policy target (for example inflation) may depend on the persistent component of exogenous shocks, such as the cost-push shock of oil, energy or imported prices. The larger the persistence of these exogenous shocks, the larger the welfare losses and the larger the response of policy instrument to this exogenous shock in a feedback rule, in order to decrease the sensitivity of the policy target to this shock.

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1 Introduction

Ramsey optimal policy theoretically grounds Ashley, Tsang and Verbrugge's (2020) empirical evidence that the Fed funds rate response increases with the persistence of exogenous shocks. For example, imported inflation or oil prices changes do have a persistent component (Ashley and Tsang (2013))).

Firstly, the optimal response of the policy instrument to persistent shocks increases with the auto-correlation of shocks. Although the central bank cannot control the persistence of exogenous shocks, it can control the sensitivity of core inflation to these shocks with its policy rule response.

Secondly, optimal policy may even set this sensitivity to zero when responding to non-stationary shocks when the auto-correlation tends to one. This is one explanation among others for the observed smaller order of the dynamics of the policy target (a smaller number of lags) than the order of the dynamics (the number of lags) of the policy instrument. For example, this is the case for US inflation (one lag) and Fed funds rate (two lags) for quarterly data from 1982 to 2006.

Thirdly, a non-zero weight in the loss function of the variance of the policy instruments (which respond to persistent exogenous shocks) implies that the value function which evaluates welfare depends on the volatility of persistent exogenous shocks. This occurs even if the loss function has sets a zero weight on the volatility of persistent exogenous shocks.

*Paris School of Economics, Université Paris 1 Pantheon Sorbonne, PjSE, 48 Boulevard Jourdan, 75014 Paris. Email: jean-bernard.chatelain@univ-paris1.fr

†ESCE International Business School, INSEEC U. Research Center, 10 rue Sextius Michel, 75015 Paris, Email: Kirsten.Ralf@esce.fr.

2 Persistence-Dependent Optimal Policy Rule

The policy maker's optimal policy program is:

$$\max_{x_t} -\frac{1}{2}E_0 \sum_{t=0}^{t=+\infty} \beta^t \left[Q_{\pi^2}\pi_t^2 + 2Q_{\pi z}\pi_t z_t + Q_{z^2}z_t^2 + Rx_t^2\right]$$

with $R > 0$, $\mathbf{Q} \geq 0$, $0 < \beta \leq 1$, subject to:

$$\begin{pmatrix} \pi_{t+1} \\ z_{t+1} \end{pmatrix} = \begin{pmatrix} A & A_{\pi z} \\ 0 & \rho \end{pmatrix} \begin{pmatrix} \pi_t \\ z_t \end{pmatrix} + \begin{pmatrix} B \\ B_{zx} = 0 \end{pmatrix} x_t + \begin{pmatrix} 0 \\ \varepsilon_t \end{pmatrix}$$

with $A > 0$, $A_{\pi z} \neq 0$, $B \neq 0$ and $0 < \rho \leq 1$.

where E_t denotes the expectation operator, π_t denotes the rate of core inflation between periods $t-1$ and t. The persistent cost-push shock z_t may correspond to oil, energy or imported inflation due to foreign supply or demand shocks. The policy instrument is x_t. It may represent the welfare-relevant output gap, i.e. the deviation between (log) output and its efficient level (Gali (2015)).

The policy maker preferences includes a discount factor β and weights $\mathbf{Q}$ on the variance and covariance of inflation and of the persistent cost-push shock and R on the variance of the policy instrument. $\mathbf{Q}$ is a positive matrix. R is strictly positive.

The cost-push shock follows an autoregressive process of order one (AR(1)), $0 < \rho \leq 1$, with identically and independently normally distributed white noise disturbances ε_t of variance σ_ε^2. The policy maker's instrument cannot change the persistence of this shock ($B_{zx} = 0$). The full system of core inflation and imported inflation is of order two.

Our results do not depend on specific relations between the parameters $(\beta, A, A_{\pi z}, B)$ of the policy transmission mechanism. They hold for backward-looking models assuming inflation is predetermined or for Ramsey optimal policy where inflation is non-predetermined and optimally anchored by the policy maker. They can be extended to multiple policy targets, multiple persistent shocks and multiple policy instruments using Chatelain and Ralf (2019, 2020) algorithm. In this case, only numerical values are available. Closed form solutions can only be found for order two models with single target, single persistent shock and single policy instrument as follows.

We apply these results on Gali (2015, chapter 5) new-Keynesian Phillips curve (NKPC) transmission mechanism (table 1):

$$\pi_t = \kappa x_t + \beta E_t \pi_{t+1} + z_t \Leftrightarrow E_t \pi_{t+1} = \frac{1}{\beta}\pi_t - \frac{\kappa}{\beta}x_t - \frac{1}{\beta}z_t \tag{1}$$

Table 1: New-Keynesian Phillips curves parameters and welfare preferences, $\beta = 0.99$**,** $\kappa = 0.1275$, $\varepsilon = 6$.

Parameters	A	B	$A_{\pi z}$	Q_{π^2}	$Q_{\pi z}$	Q_{z^2}	R	ρ
NKPC	$\frac{1}{\beta} \geq 1$	$-\frac{\kappa}{\beta} < 0$	$-\frac{1}{\beta} = -A$	1	0	0	$\frac{\kappa}{\varepsilon}$	ρ
Calibration	$\frac{1}{0.99}$	$-\frac{0.1275}{0.99}$	$-\frac{1}{0.99}$	1	0	0	$\frac{0.1275}{6} = 2.125\%$	0.8

Gali's (2015) example has a large sensitivity (close to one in absolute value) between core inflation and cost-push shock: $A_{\pi z} = \frac{-1}{\beta} = \frac{-1}{0.99}$. One percent change of imported inflation leads to one percent change of core inflation ($A_{\pi z} = -\frac{1}{\beta} = -1.01$). Gali's (2015) calibration uses welfare computation for the weight on the variance of the policy instrument which is extremely small ($R/Q_{\pi^2} = 2\%$). This implies very large absolute values of optimal policy parameters consistent to an allowed large variance of the policy instrument.

Following Chatelain and Ralf (2019, 2020) algorithm, the optimal policy rule is defined by endogenous policy rule parameters: F_π for the response of the policy instrument to core inflation, F_z for the response of the policy instrument to persistent shocks:

$$x_t = F_\pi \pi_t + F_z z_t \tag{2}$$

The results of optimal policy are summarized in the following propositions. Proofs are in the appendix.

Proposition 1 *Optimal core "intrinsic" inflation persistence is controllable by the policy maker and equal to $\lambda = A + BF_\pi$. The optimal response of the policy instrument to policy target F_π do not depend on the exogenous persistence (ρ) of cost-push shock nor on the sensitivity of core inflation to these cost-push variables ($A_{\pi z}$):*

$$0 < \lambda = \frac{1}{2}\left(A + \frac{1}{\beta A} + \frac{Q_{\pi^2}}{R}\frac{B^2}{A} - \sqrt{\left(A + \frac{1}{\beta A} + \frac{Q_{\pi^2}}{R}\frac{B^2}{A}\right)^2 - \frac{4}{\beta}}\right)$$

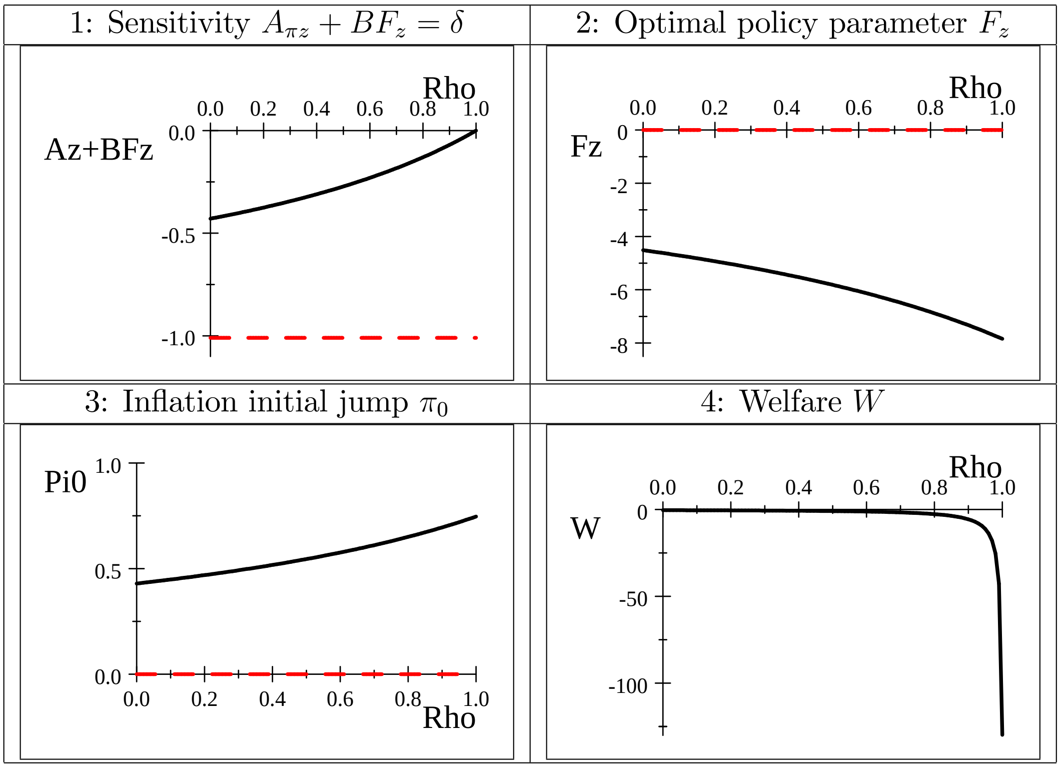
$$0 < \lambda \leq \min\left(A, \frac{1}{\beta A}\right) \text{ upper bound obtained for } Q_{\pi^2} = 0$$

$$F_\pi = \frac{\lambda - A}{B} \text{ and } BF_\pi < 0 \text{ so that } \lambda = A + BF \leq A.$$

Example 1 *For Gali's (2015) calibration, because the welfare weight on the variance of the policy instrument in the loss function is implausibly low ($R/Q_{\pi^2} = 2\%$), this implies a very low persistence of inflation ($\lambda = 0.4291604$) due to an implausibly large negative-feedback response of the policy instrument ($F_\pi = 4.51$) to deviations of inflation from its long run target.*

The comparison between policy rules which do not react to the persistent exogenous cost-push shocks versus optimal policy rules are presented in figures 1 to 4 corresponding to the following propositions. They highlight the dependence on the persistence of shocks (ρ) of the response of the policy instrument F_z to persistent shocks, of the resulting sensitivity of the policy target to the cost-push shock $A_{\pi z} + BF_z$, of the resulting impulse response functions of inflation, of the initial jump of non-predetermined inflation π_0 and of welfare W. Welfare is a particular case of Chatelain and Ralf (2020).

Figure 1 to 4: Key parameters functions of imported inflation persistence: continuous line $0 < \rho \leq 1$, dash line $\rho = 0$.



Proposition 2 *The absolute value of the sensitivity of the policy target to the cost-push shock after policy rule response below the sensitivity before policy intervention* $|A_{\pi z} + BF_z| < |A_{\pi z}|$*. It is a non-linear decreasing function of the persistence of the of the cost-push shock measured by* ρ*. It is an affine function of the sensitivity* $A_{\pi z}$ *of the policy target to the persistent shock when the policy instrument does not respond to cost-push shock. We denote* $\delta = A_{\pi z} + BF_z$

$$\delta = A_{\pi z} + BF_z = \frac{\lambda}{A}\frac{1}{1-\lambda\beta\rho}\left(A_{\pi z}\left(1 - A\beta\rho\right) - Q_{\pi z}\frac{\beta B^2}{R}\rho\right)$$

One has:

$$|A_{\pi z} + BF_z| < |A_{\pi z}| \;\; \textit{because} \; \frac{BF_\pi}{A} = \frac{\lambda - A}{A} < 0$$

When cost-push shock tends to zero persistence, the sensitivity tends a lower sensitivity than the one obtained for $F_z = 0$.

$$\lim_{\rho\to 0} |A_{\pi z} + BF_z(\rho)| = \left| A_{\pi z}\frac{\lambda}{A}\right| \leq |A_{\pi z}|$$

When cost-push shock persistence tends to a unit root, the absolute value of the sensitivity of the policy instrument to the cost-push shock reaches its lowest value:

$$\lim_{\rho\to 1}\left(A_{\pi z} + BF_z\right) = \frac{\lambda}{A}\frac{1}{1-\beta\lambda}\left(A_{\pi z}\left(1 - A\beta\right) - Q_{\pi z}\frac{\beta B^2}{R}\right)$$

Because $\lambda \neq 0$ *and* $\beta \neq 0$*, this lowest value is equal to zero for* $A = 1/\beta$ *(which is a property of the new-Keynesian Phillips curve) and for a zero weight on the covariance of inflation and cost push shock* $Q_{\pi z} = 0$ *in the loss function.*

Example 2 *As seen in figure 1, for Gali (2015) preference* $Q_{\pi z} = 0$*,* $A = 1/\beta$ *and* *persistence $\rho = 0.8$ of the cost-push shock, because the welfare weight on the variance of the policy instrument in the loss function is extremely low (2%), the large response of the policy instrument to the cost-push shock implies that the sensitivity of the inflation to the cost push shock is 13% of what it would have been if the policy instrument would not have responded to the persistent cost-push shock ($F_z = 0$):*

$$A_{\pi z} + BF_z = \frac{-\lambda(1-\rho)}{1-\beta\lambda\rho} = \frac{-0.4291604 \cdot (1-\rho)}{1 - 0.4291604 \cdot 0.99 \cdot \rho} = -0.13$$

$$\lim_{\rho \to 1} (A_{\pi z} + BF_z) = 0 \text{ because } A = \frac{1}{\beta} \text{ and } Q_{\pi z} = 0.$$

$$\lim_{\rho \to 0} |A_{\pi z} + BF_z| = |-\lambda| = 0.429 < \left| -\frac{1}{\beta} \right| = 1.01$$

Proposition 3 *Core inflation overall persistence and impulse response function firstly depends on its "intrinsic" persistence (controllable root $\lambda = A + BF_\pi$) and secondly depends on the "extrinsic" non-controllable imported inflation persistence ρ of the cost push shock, which is itself attenuated by the policy instrument response F_z to the cost-push shock decreasing the sensitivity of core inflation to the cost-push shock $A_{\pi z} + BF_z$.*

$$E_t \pi_t \left(\underset{+}{A_{\pi z}}, \underset{+}{\rho} \right) = \lambda^t \pi_0 + \frac{\rho^t - \lambda^t}{\rho - \lambda} \delta z_0 \text{ if } \rho \neq \lambda$$

Under the condition $A = \frac{1}{\beta}$, if the cost-push shock autocorrelation tends to one, the order of the dynamics of the policy target is reduced to one (single eigenvalue λ):

$$\lim_{\rho \to 1} E_t \pi_t = \lambda^t \pi_0 \text{ if } A = \frac{1}{\beta} \text{ and } Q_{\pi z} = 0 \text{ so that } \delta \to 0$$

Example 3 *For Gali's (2015) calibration, the impulse response function of core inflation is lower when the policy rule responds to imported inflation ($F_z^* \neq 0$) than when the policy rule does not respond to imported inflation ($F_z = 0$). This is because the sensitivity of core inflation to imported inflation is 13% of $-\frac{1}{\beta} = -1.01$:*

$$E_t \pi_t = 0.429^t \pi_0 + \frac{0.8^t - 0.429^t}{0.8 - 0.429} (-0.13) z_0$$

$$\lim_{\rho \to 1} E_t \pi_t = 0.429^t \pi_0$$

In the limit case of unit root persistence for the cost push shock ($\rho \to 1$) such as a trend in oil price, the sensitivity of inflation to the cost push shock ($A_{\pi z} + BF_z$) tends to zero, so that inflation is an order one process (with single root λ) instead of an order two process depending on two roots (λ and ρ of the cost-push shock).

Proposition 4 *The policy instrument has a persistence-dependent rule parameter F_z increasing in absolute value with the persistence ρ of the cost-push shock. It increases in absolute value with the sensitivity $A_{\pi z}$ of the policy target to the cost-push shock:*

$$F_z \left(\underset{+}{A_{\pi z}}, \underset{+}{\rho} \right) = \frac{\delta - A_{\pi z}}{B} = F_z = \frac{A_{\pi z}}{A} \frac{1}{1-\beta\lambda\rho} \frac{\lambda - A}{B} - \frac{Q_{\pi z}}{A} \frac{B}{R} \frac{\beta\lambda\rho}{1-\beta\lambda\rho}$$

The limit are:

$$\lim_{\rho\to 0}|F_z(\rho)| = \left|\frac{A_{\pi z}}{A}\frac{\lambda - A}{B}\right| > 0 = F_z(\rho = 0)$$
$$\lim_{\rho\to 0}|F_z(\rho)| = 0 \text{ if } Q_{\pi^2} = 0 \text{ and if } \lambda = A < 1$$

The lack of response of the policy instrument when the persistence of the shock tends to zero is obtained only in an irrelevant case where welfare and the policy maker do not weight the volatility of the policy target in the loss function $Q_{\pi^2} = 0$ and if inflation is stationary for a fixed setting of the policy instrument at its long run equilibrium value ($\lambda = A < 1$).

Example 4 *As seen in figure 1, for Gali (2015) persistence $\rho = 0.8$ of the cost-push shock, because the welfare weight on the variance of the policy instrument in the loss function is extremely low (2%), this allows large variations of the policy instrument with a large response of the policy instrument to the cost-push shock (-6.86):*

$$F_z = \frac{1}{\kappa}\left(-\frac{1-\beta\lambda}{1-\beta\lambda\rho}\right) = \frac{1}{0.1275}\left(-\frac{1-0.4291604\cdot 0.99}{1-0.4291604\cdot 0.99\cdot\rho}\right) = -6.86$$
$$\lim_{\rho\to 0}|F_z(\rho)| = |F_\pi| = 4.52 > |F_z(\rho = 0)| = 0 \text{ and } \lim_{\rho\to 1}|F_z(\rho)| = 7.84$$

Proposition 5 *If core inflation π_0 is non-predetermined, in Ramsey optimal policy, it is optimally anchored on the policy instrument which is itself anchored on the predetermined cost-push shock. The first order conditions sets the marginal value of the loss function with respect to inflation at the initial date (itself equal to it costate variable, the Lagrange multiplier μ_0) equal to zero. Initial core inflation π_0 increases with imported inflation persistence ρ.*

$$\frac{\partial L}{\partial \pi_0} = \mu_0 = 2\left(P_{\pi^2}\pi_0 + P_z z_0\right) = 0 \Rightarrow \pi_0 = -P_{\pi^2}^{-1}P_{\pi z}\cdot z_0$$
$$\pi_0\left(\underset{+}{A_{\pi z}}, \underset{+}{\rho}\right) = \frac{-1}{1-\beta\rho\lambda}\left(A_{\pi z}\beta\lambda + \frac{Q_{\pi z}}{Q_{\pi^2}}(1-\beta\lambda A)\right)$$
$$\lim_{\rho\to 0}\pi_0 = -A_{\pi z}\beta\lambda - \frac{Q_{\pi z}}{Q_{\pi^2}}(1-\beta\lambda A)$$

Example 5 *As seen in figure 3, with Gali's (2015) calibration ($\rho = 0.8$):*

$$\pi_0 = \frac{\lambda}{1-\lambda\beta\rho} = \frac{0.4291604}{1-0.4291604\cdot 0.99\cdot\rho} = 0.64$$
$$\lim_{\rho\to 0}\pi_0 = \lambda = 0.42274,\ \lim_{\rho\to 1}\pi_0 = \frac{\lambda}{1-\beta\lambda} = 0.73$$

Proposition 6 *Welfare: Even if households and/or Central Bank preferences set a zero weight on the covariance between core inflation and cost-push shock ($Q_{\pi z} = 0$) and the variance of cost-push shock ($Q_{zz} = 0$), the variance of the policy instrument depends on this variance. Therefore, the optimal value of welfare depends on the covariance of core inflation and of the cost-push shock ($P_{\pi z} \neq 0$) and of the variance of the cost-push shock ($P_{zz} \neq 0$). Both weights ($P_{\pi z}, P_{zz}$) increases with the persistence of the cost-push shock ρ and with the sensitivity of core inflation to imported inflation $A_{\pi z}$. Overall welfare* *(the opposite of the optimal value of the loss function) decreases in a non-linear fashion with the persistence of imported inflation ρ and with the sensitivity of core inflation to imported inflation $A_{\pi z}$ with no response of the policy instrument to the cost-push shock. Welfare can be computed for a given initial predetermined inflation π_0 as follows. For non-predetermined inflation, we take into account the optimal initial anchor of inflation for the welfare of Ramsey optimal policy, using $\pi_0 = -P_{\pi^2}^{-1} P_{\pi z} z_0$:*

$$W\left(\underset{-}{A_{\pi z}}, \underset{-}{\rho}\right) = -\begin{pmatrix} -\frac{P_{\pi z}}{P_{\pi^2}} z_0 & z_0 \end{pmatrix} \begin{pmatrix} P_{\pi^2} & P_{\pi z} \\ P_{\pi z} & P_{z^2} \end{pmatrix} \begin{pmatrix} -\frac{P_{\pi z}}{P_{\pi^2}} z_0 \\ z_0 \end{pmatrix} = -\left(P_{z^2} - \frac{P_{\pi z}^2}{P_{\pi^2}}\right) z_0^2 > 0$$

Example 6 *For welfare dependence on the persistence of the cost-push shock, with Gali's (2015) calibration with $\rho = 0.8$ (figure 4):*

$$\frac{W}{z_0^2} = \frac{-\lambda}{(1-\beta\lambda\rho)^2 (1-\beta\rho^2)} = \frac{-0.4291604}{(1-0.99\cdot 0.4291604\cdot\rho)^2} \frac{1}{1-0.99\cdot\rho^2} = -2.688$$

$$\lim_{\rho\to 0} \frac{W}{z_0^2} = -\lambda = -0.429, \ \lim_{\rho\to 1} \frac{W}{z_0^2} = \frac{-\lambda}{(1-\beta\lambda)^2 (1-\beta)} = -130$$

3 Conclusion

Optimal policy facing persistent exogenous cost-push shock, for example, imported inflation or deflation due to oil or energy shocks implies a dependence to the persistence of the cost-push shock firstly of the policy instrument in the policy rule, secondly of the policy target persistence and of its impulse responses function through a change of its sensitivity to the cost push shock and of the initial jump of inflation and thirdly of welfare.

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4 Appendix

Stable subspace of the Hamiltonian system

Following Chatelain and Ralf (2020), we form the Lagrangian by attaching a sequence of Lagrange multipliers $\beta^{t+1}\mu_{t+1}$ and $\beta^{t+1}\nu_{t+1}$ to the sequence of constraints of the policy transmission mechanism:

$$L = -\sum_{t=0}^{t=+\infty} \beta^t \begin{bmatrix} \frac{1}{2}Q_{\pi^2}\pi_t^2 + Q_{\pi z}\pi_t z_t + \frac{1}{2}Q_{z^2}z_t^2 + \frac{1}{2}Rx_t^2 \\ +\beta\mu_{t+1}\left(A\pi_t + Bx_t + A_{\pi z}z_t - \pi_{t+1}\right) \\ +\beta\nu_{t+1}\left(\rho z_t - z_{t+1}\right) \end{bmatrix}$$

The first order necessary conditions are:

$$\begin{aligned}
\frac{\partial L}{\partial x_t} &= Rx_t + \beta B\mu_{t+1} = 0 \Rightarrow x_t = \frac{-\beta B}{R}\mu_{t+1} \text{ or } \mu_{t+1} = -\frac{R}{\beta B}x_t \\
\frac{\partial L}{\partial \pi_t} &= Q_{\pi^2}\pi_t + Q_{\pi z}z_t + \beta A\mu_{t+1} - \mu_t = 0 \\
\frac{\partial L}{\partial z_t} &= Q_{\pi z}\pi_{tt} + Q_{z^2}z_t + \beta A_{\pi z}\mu_{t+1} + \beta\rho v_{t+1} - v_t = 0
\end{aligned}$$

The Hamiltonian system is:

$$\begin{pmatrix} 1 & 0 & \frac{\beta B^2}{R} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \beta A & 0 \\ 0 & 0 & \beta A_{\pi z} & \beta\rho \end{pmatrix} \begin{pmatrix} \pi_{t+1} \\ z_{t+1} \\ \gamma_{t+1} \\ \mu_{t+1} \end{pmatrix} = \begin{pmatrix} A & A_{\pi z} & 0 & 0 \\ 0 & \rho & 0 & 0 \\ -Q_{\pi^2} & -Q_{\pi z} & 1 & 0 \\ -Q_{z\pi} & -Q_{z^2} & 0 & 1 \end{pmatrix} \begin{pmatrix} \pi_t \\ z_t \\ \gamma_t \\ \mu_t \end{pmatrix}$$

We seek the elements of the value function (welfare) matrix: P_{π^2}, $P_{\pi z}$ and P_{z^2}, which are the unknown parameters of eigenvectors of the stable subspace of the Hamiltonian system:

$$\begin{pmatrix} \pi_t \\ z_t \\ \gamma_t \\ \mu_t \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ P_{\pi^2} & P_{\pi z} & 0 & 0 \\ P_{z\pi} & P_{z^2} & 0 & 0 \end{pmatrix} \begin{pmatrix} \pi_t \\ z_t \\ \gamma_t \\ \mu_t \end{pmatrix}$$

It follows:

$$\begin{pmatrix} \beta\frac{B^2}{R}P_{\pi^2} + 1 & \beta\frac{B^2}{R}P_{\pi z} \\ 0 & 1 \\ \beta AP_{\pi^2} & \beta AP_{\pi z} \\ \beta\rho P_{\pi z} + \beta A_{\pi z}P_{\pi^2} & \beta\rho P_{z^2} + \beta A_{\pi z}P_{\pi z} \end{pmatrix} \begin{pmatrix} \pi_{t+1} \\ z_{t+1} \end{pmatrix} = \begin{pmatrix} A & A_{\pi z} \\ 0 & \rho \\ P_{\pi^2} - Q_{\pi^2} & P_{\pi z} - Q_{\pi z} \\ P_{\pi z} - Q_{\pi z} & P_{z^2} - Q_{z^2} \end{pmatrix} \begin{pmatrix} \pi_t \\ z_t \end{pmatrix}$$

We eliminate z_{t+1} by ρz_t. *Solving the model amounts to use the following three key equations for finding the three unknown parameters* P_{π^2}, $P_{\pi z}$ and P_{z^2} of the welfare matrix:

$$\begin{pmatrix} 1 + \beta\frac{B^2}{R}P_{\pi^2} \\ \beta AP_{\pi^2} \\ \beta\rho P_{\pi z} + \beta A_{\pi z}P_{\pi^2} \end{pmatrix} (\pi_{t+1}) = \begin{pmatrix} A & A_{\pi z} - \beta\rho\frac{B^2}{R}P_{\pi z} \\ P_{\pi^2} - Q_{\pi^2} & (1 - \beta\rho A)P_{\pi z} - Q_{\pi z} \\ P_{\pi z} - Q_{\pi z} & (1 - \beta\rho^2)P_{z^2} - Q_{z^2} - \beta\rho A_{\pi z}P_{\pi z} \end{pmatrix} \begin{pmatrix} \pi_t \\ z_t \end{pmatrix} \tag{3}$$

The first terms of the three equations implies three formulas for the intrinsic persistence of the policy target, $\lambda = A + BF_\pi$:

$$\lambda = A + BF_\pi = \frac{A}{1 + \frac{\beta B^2}{R} P_{\pi^2}} = \frac{P_{\pi^2} - Q_{\pi^2}}{A\beta P_{\pi^2}} = \frac{P_{\pi z} - Q_{\pi z}}{\beta\rho P_{\pi z} + \beta A_{\pi z} P_{\pi^2}}.$$

The second terms of the three equations implies three formulas for the sensitivity of the policy target to the persistent shock, $\delta = A_{\pi z} + BF_{\pi z}$, with $F_{\pi z} = \frac{\delta - A_{\pi z}}{B}$.

$$\delta = \frac{A_{\pi z} - \rho\frac{\beta B^2}{R} P_{\pi z}}{1 + \frac{\beta B^2}{R} P_{\pi^2}} = \frac{(1 - \beta\rho A) P_{\pi z} - Q_{\pi z}}{\beta A P_{\pi^2}} = \frac{(1 - \beta\rho^2) P_{z^2} - Q_{z^2} - \beta\rho A_{\pi z} P_{\pi z}}{\beta\rho P_{\pi z} + \beta A_{\pi z} P_{\pi^2}}.$$

Proof of proposition 1. Compute the optimal root $\lambda = A + BF_\pi$, a first element of the welfare matrix P_{π^2} (proposition 5a) and the policy rule parameter F_π:

The first term of each of the three equation has the same value:

$$\lambda = \frac{A}{1 + \frac{\beta B^2}{R} P_{\pi^2}} = \frac{P_{\pi^2} - Q_{\pi^2}}{A\beta P_{\pi^2}} = \frac{P_{\pi z} - Q_{\pi z}}{\beta\rho P_{\pi z} + \beta A_{\pi z} P_{\pi^2}}$$

Using the first equality:

$$1 + \frac{\beta B^2}{R} P_{\pi^2} = \frac{A}{\lambda} \Leftrightarrow P_{\pi^2} = \frac{R}{\beta B^2}\left(\frac{A - \lambda}{\lambda}\right) > 0$$

Using the second equality:

$$P_{\pi^2} - Q_{\pi^2} = \lambda A\beta P_{\pi^2} \Leftrightarrow P_{\pi^2} = \frac{Q_{\pi^2}}{1 - \beta A\lambda}$$

Using the first and second equality leads to a characteristic polynomial for solving λ:

$$\lambda\left(1 + \frac{\beta B^2}{R} P_{\pi^2}\right) - A = 0$$
$$\lambda\left(1 + \frac{B^2}{R}\beta\frac{Q_{\pi^2}}{1 - A\beta\lambda}\right) - A = 0$$
$$\lambda^2 - \left(A + \frac{1}{A\beta} + \frac{B^2 Q_{\pi^2}}{AR}\right)\lambda + \frac{1}{\beta} = 0$$

Optimal persistence is the stable root of this characteristic polynomial:

$$0 < \lambda = \frac{1}{2}\left(A + \frac{1}{\beta A} + \frac{Q_{\pi^2}}{R}\frac{B^2}{A} - \sqrt{\left(A + \frac{1}{\beta A} + \frac{Q_{\pi^2}}{R}\frac{B^2}{A}\right)^2 - \frac{4}{\beta}}\right)$$
$$0 < \lambda \leq \min\left(A, \frac{1}{\beta A}\right) \text{ for } Q_{\pi^2} = 0$$

The policy rule parameter is a function of the optimal persistence λ:

$$F_\pi = \frac{\lambda - A}{B} \text{ and } BF_\pi < 0 \text{ so that } \lambda = A + BF_\pi \leq A.$$

Proof of proposition 5b: Second element of the welfare matrix $P_{\pi z}$

We use the third equality for the first term:

$$\lambda = \frac{A}{1 + \frac{\beta B^2}{R} P_{\pi^2}} = \frac{P_{\pi^2} - Q_{\pi^2}}{A\beta P_{\pi^2}} = \frac{P_{\pi z} - Q_{\pi z}}{\beta\rho P_{\pi z} + \beta A_{\pi z} P_{\pi^2}}$$

$P_{\pi z}$ is an increasing function of the two characteristics of the forcing variable $A_{\pi z}$ and ρ:

$$P_{\pi z} - Q_{\pi z} = \lambda\beta\rho P_{\pi z} + \lambda\beta A_{\pi z} P_{\pi^2} \Rightarrow$$
$$P_{\pi z} = \frac{\beta\lambda A_{\pi z} P_{\pi^2} + Q_{\pi z}}{1 - \beta\rho\lambda}$$

With:

$$P_{\pi^2} = \frac{R}{\beta B^2}\left(\frac{A-\lambda}{\lambda}\right) = \frac{Q_{\pi^2}}{1 - A\beta\lambda}$$

$P_{\pi z}$ can be written as a function of λ:

$$P_{\pi z}(A_{\pi z}, \rho) = \frac{1}{1-\beta\lambda\rho}\left(A_{\pi z}\frac{R}{\beta B^2}\beta\lambda\left(\frac{A-\lambda}{\lambda}\right) + Q_{\pi z}\right)$$
$$P_{\pi z}(A_{\pi z}, \rho) = \frac{1}{1-\beta\lambda\rho}\left(A_{\pi z}\frac{R}{B^2}(A-\lambda) + Q_{\pi z}\right) \text{ or}$$
$$P_{\pi z}(A_{\pi z}, \rho) = \frac{1}{1-\beta\lambda\rho}\left(A_{\pi z} Q_{\pi^2}\frac{\beta\lambda}{1-A\beta\lambda} + Q_{\pi z}\right)$$

Proof of proposition 2: Compute the sensitivity $\delta = A_{\pi z} + BF_z$**:**

One has:

$$\delta = \frac{A_{\pi z} - \beta\rho\frac{B^2}{R}P_{\pi z}}{\frac{A}{\lambda}} = \frac{(1 - A\beta\rho)P_{\pi z} - Q_{\pi z}}{\beta A P_{\pi^2}} = \frac{(1-\beta\rho^2)P_{z^2} - Q_{z^2} - \beta\rho A_{\pi z} P_{\pi z}}{\beta\rho P_{\pi z} + \beta A_{\pi z} P_{\pi^2}}.$$

In the first equality, substitute $P_{\pi z}$ and the denominator by A/λ leads to:

$$\delta = \frac{\lambda}{A}\left(A_{\pi z} - \frac{\beta\rho}{1-\beta\lambda\rho}\frac{B^2}{R}\left(A_{\pi z}\frac{R}{B^2}(A-\lambda) + Q_{\pi z}\right)\right)$$
$$\delta = \frac{\lambda}{A}\left(A_{\pi z}\left(1 - \frac{\beta\rho}{1-\beta\lambda\rho}(A-\lambda)\right) - Q_{\pi z}\frac{B^2}{R}\frac{\beta\rho}{1-\beta\lambda\rho}\right)$$
$$\delta = \frac{\lambda}{A}\left(A_{\pi z}\frac{1-A\beta\rho}{1-\lambda\beta\rho} - Q_{\pi z}\frac{\beta B^2}{R}\frac{\rho}{1-\beta\lambda\rho}\right)$$

Proof of example 2: For Gali's example: $Q_{\pi z}=0$ and $A=-A_{\pi z}=1/\beta$

$$\delta=\frac{1}{\beta}\left(1-\frac{1-\beta\lambda}{1-\beta\lambda\rho}\right)=(1-\rho)\frac{-\lambda}{1-\beta\lambda\rho}$$

Proof of proposition 3: The impulse response function of inflation
The following result can be found using at least three methods:

$$\pi_t=\lambda^t\left(\pi_0-\frac{(A_{\pi z}+BF_z)\,z_0}{\rho-\lambda}\right)+\rho^t\frac{(A_{\pi z}+BF_z)\,z_0}{\rho-\lambda}$$

(a) Solve the sum of two geometric sequences or geometric progressions using the homogeneous solution related to the geometric sequence with common ratio λ and a particular solution proportional to the forcing variable following a geometric sequence with common ratio ρ;

(b) compute the power of the matrix $\begin{pmatrix}\lambda & A_{\pi z}+BF_z\\ 0 & \rho\end{pmatrix}$ using its Jordan decomposition;

(c) prove it by mathematical induction.

Proof of proposition 4: Policy rule parameter F_z:

$$F_z=\frac{\delta-A_{\pi z}}{B}=\frac{A_{\pi z}}{B}\left(\frac{\lambda-A\lambda\beta\rho}{A-A\lambda\beta\rho}-1\right)-Q_{\pi z}\frac{B}{R}\frac{\beta\lambda\rho}{A-A\beta\lambda\rho}$$
$$F_z=-\frac{A_{\pi z}}{AB}\left(\frac{A-\lambda}{1-\lambda\beta\rho}\right)-Q_{\pi z}\frac{B}{R}\frac{\beta\lambda\rho}{A-A\beta\lambda\rho}$$
$$F_z=\frac{A_{\pi z}}{A}\frac{1}{1-\beta\lambda\rho}\frac{\lambda-A}{B}-\frac{Q_{\pi z}}{A}\frac{B}{R}\frac{\beta\lambda\rho}{1-\beta\lambda\rho}$$

Proof of example 4: For Gali's example: $Q_{\pi z}=0$ and $A=-A_{\pi z}=1/\beta$

$$F_z=\frac{1}{\kappa}\left(-\frac{1-\beta\lambda}{1-\beta\lambda\rho}\right)$$

Proof of proposition 5c: Optimal jump of inflation π_0:
The first order conditions sets the marginal value of the loss function with respect to inflation at the initial date (itself equal to it costate variable, the Lagrange multiplier μ_0) equal to zero.

$$\frac{\partial L}{\partial\pi_0}=\mu_0=2\left(P_{\pi^2}\pi_0+P_z z_0\right)=0\Rightarrow\pi_0\left(\underset{+}{A_{\pi z}},\underset{+}{\rho}\right)=-\frac{P_{\pi z}}{P_{\pi^2}}\cdot z_0$$

One has:

$$-\frac{P_{\pi z}}{P_{\pi^2}}=\frac{-1}{P_{\pi^2}}\frac{Q_{\pi z}+A_{\pi z}\beta\lambda P_{\pi^2}}{1-\beta\rho\lambda}\text{ and }P_{\pi^2}=\frac{Q_{\pi^2}}{1-\beta\lambda A}$$
$$-\frac{P_{\pi z}}{P_{\pi^2}}=\frac{-1}{1-\beta\rho\lambda}\left(A_{\pi z}\beta\lambda+\frac{Q_{\pi z}}{Q_{\pi^2}}(1-\beta\lambda A)\right)$$

Proof of proposition 6: Welfare parameter P_{z^2}
The third equation related to the sensitivity of the policy target δ to the persistent

shock determines P_{z^2}:

$$\delta = \frac{\lambda}{A}\left(A_{\pi z} - \rho\frac{\beta B^2}{R}P_{\pi z}\right) = \frac{(1-A\beta\rho)\,P_{\pi z} - Q_{\pi z}}{\beta A P_{\pi^2}} = \frac{(1-\beta\rho^2)\,P_{z^2} - Q_{z^2} - \beta\rho A_{\pi z}P_{\pi z}}{\beta\rho P_{\pi z} + \beta A_{\pi z}P_{\pi^2}}$$

The third equality leads to:

$$\left(1-\beta\rho^2\right)P_{z^2} = Q_{z^2} + (A_{\pi z}+\delta)\,\beta\rho P_{\pi z} + \delta\beta A_{\pi z}P_{\pi^2}$$

with:

$$P_{\pi z} = \frac{1}{1-\beta\lambda\rho}\left(A_{\pi z}\beta\lambda\frac{Q_{\pi^2}}{1-\lambda\beta A} + Q_{\pi z}\right) \text{ and } P_{\pi^2} = \frac{Q_{\pi^2}}{1-\lambda\beta A}$$
$$A_{\pi z} + \delta = A_{\pi z} + \frac{(1-A\beta\rho)\,P_{\pi z} - Q_{\pi z}}{A\beta P_{\pi^2}}$$

so that:

$$\begin{pmatrix} P_{\pi^2} & P_{\pi z} \\ P_{\pi z} & P_{z^2} \end{pmatrix} = \begin{pmatrix} \frac{Q_{\pi^2}}{1-\beta\lambda A} & \frac{Q_{\pi z}}{1-\beta\lambda\rho} \\ \frac{Q_{\pi z}}{1-\beta\lambda\rho} & \frac{Q_{z^2}}{1-\beta\rho^2} \end{pmatrix} + \begin{pmatrix} 0 & \frac{1}{1-\beta\lambda\rho}A_{\pi z}\beta\lambda P_{\pi^2} \\ \frac{1}{1-\beta\lambda\rho}A_{\pi z}\beta\lambda P_{\pi^2} & \frac{1}{1-\beta\rho^2}\left((A_{\pi z}+\delta)\,\beta\rho P_{\pi z} + \delta\beta A_{\pi z}P_{\pi^2}\right) \end{pmatrix}$$

Proof of welfare, example 6 (Gali): Computation of P_{z^2}:

$$\left(1-\beta\rho^2\right)P_{z^2} = Q_{z^2} + \delta\left(\beta\rho P_{\pi z} + \beta A_{\pi z}P_{\pi^2}\right) + \beta A_{\pi z}\rho P_{\pi z}$$

with:

$$P_{\pi z} = \frac{-\lambda}{1-\beta\rho\lambda}\frac{1}{1-\lambda} \text{ and } P_{\pi^2} = \frac{1}{1-\lambda}$$
$$\delta = (1-\rho)\left(\frac{-\lambda}{1-\beta\lambda\rho}\right),\ Q_{z^2} = 0,\ \beta A_{\pi z} = -1$$

One has:

$$\left(1-\beta\rho^2\right)P_{z^2} = \frac{-\lambda}{1-\beta\lambda\rho}\left((1-\rho)\left(\beta\rho\left(\frac{-\lambda}{1-\beta\rho\lambda}\frac{1}{1-\lambda}\right) - \frac{1}{1-\lambda}\right) - \rho\frac{1}{1-\lambda}\right)$$
$$\left(1-\beta\rho^2\right)P_{z^2} = \frac{\lambda}{1-\lambda}\frac{-1}{1-\beta\lambda\rho}\left(-\frac{\beta\lambda\rho^2-1}{\beta\lambda\rho-1}\right)$$
$$P_{z^2} = \frac{1}{1-\lambda}\frac{\lambda}{(1-\lambda\beta\rho)^2}\frac{1-\lambda\beta\rho^2}{1-\beta\rho^2}$$

so that:

$$\mathbf{P} = \begin{pmatrix} \frac{1}{1-\lambda} & -\frac{1}{1-\lambda}\frac{\lambda}{1-\beta\rho\lambda} \\ -\frac{1}{1-\lambda}\frac{\lambda}{1-\beta\rho\lambda} & \frac{1}{1-\lambda}\frac{\lambda}{(1-\beta\rho\lambda)^2}\frac{1-\beta\lambda\rho^2}{1-\beta\rho^2} \end{pmatrix} = \begin{pmatrix} 1.7518055 & -1.1389181 \\ -1.1389181 & 3.4285107 \end{pmatrix}$$

For welfare dependence on the persistence of the cost-push shock, with Gali's (2015) calibration:

$$P_{z^2} = \frac{1}{1-\lambda}\frac{\lambda}{(1-\beta\rho\lambda)^2}\frac{1-\lambda\beta\rho^2}{1-\beta\rho^2} = 3.4285$$
$$-\frac{P_{\pi z}^2}{P_{\pi^2}} = -\frac{1}{1-\lambda}\frac{\lambda^2}{(1-\beta\lambda\rho)^2}$$

Welfare has this form:

$$\frac{W}{z_0^2} = -P_{z^2} + \frac{P_{\pi z}^2}{P_{\pi^2}} = \frac{1}{1-\lambda}\frac{\lambda}{(1-\beta\rho\lambda)^2}\left(\lambda - \frac{1-\beta\lambda\rho^2}{1-\beta\rho^2}\right)$$
$$\frac{W}{z_0^2} = \frac{\lambda}{1-\lambda}\frac{-1}{(1-\beta\rho\lambda)^2}\frac{1-\lambda}{1-\beta\rho^2}$$
$$\frac{W}{z_0^2} = \frac{-\lambda}{(1-\lambda\beta\rho)^2}\frac{1}{1-\beta\rho^2} = \frac{-0.4291604}{(1-0.99\cdot\rho\cdot 0.4291604)^2}\frac{1}{1-0.99\cdot\rho^2}$$
$$\frac{W}{z_0^2} = \frac{-0.4291604}{(1-0.99\cdot 0.8\cdot 0.4291604)^2}\frac{1}{1-0.99\cdot 0.8^2} = 2.688$$

QED.

SCILAB Code for numerical solutions:

```
beta1=0.99; eps=6; kappa=0.1275; rho=0.8;
Qpi=1; Qz=0 ; Qzpi=0; R=kappa/eps;
A1=[1/beta1 -1/beta1 ; 0 rho] ;
A=sqrt(beta1)*A1;
B1=[-kappa/beta1 ; 0];
B=sqrt(beta1)*B1;
Q=[Qpi Qzpi ;Qzpi Qz ];
Big=sysdiag(Q,R);
[w,wp]=fullrf(Big);
C1=wp(:,1:2);
D12=wp(:,3:$);
M=syslin('d',A,B,C1,D12);
[Fy,Py]=lqr(M)
A1+B1*Fy
-inv(Py(1,1))*Py(1,2)
Py(2,2)-Py(1,2)*inv(Py(1,1))*Py(1,2)
```