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Recursive objective and subjective multiple priors

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Abstract

We provide an axiomatic characterization of recursive Maxmin preferences that stem from (possibly) incomplete preferences representing choices that are justified by hard evidence. The decision-maker disposes of objective probabilistic information that may induce dynamically inconsistent behavior. To ensure that her choices be informed by objective information, dynamically consistent, and ambiguity averse, she constructs her subjective set of priors as the rectangular hull of the objective information set. The characterization builds upon two axioms that naturally combine these three requirements in a behavioral way. Moreover, our main result suggests a principled justification for the use of recursive Maxmin preferences in applications to dynamic choice problems.

Keywords: Rectangularity, Rectangularization, Maxmin Expected Utility, Unanimity Rule, Dynamic Consistency, Prior-by-prior Updating, Objective and Subjective Rationality

JEL classification: D81

1 Introduction

Decision theory under uncertainty studies effective methods to evaluate uncertain alternatives when only imprecise (or partial) probabilistic information is available to the decision-maker. Since most decision processes take place over time, it is important that decision models provide sound explanations concerning how preferences evolve over time, for instance in response to the arrival of new information. In this regard, a key dynamic property of preferences is dynamic consistency, which roughly states that if an alternative is preferred to another one conditionally upon the realization of every event in some collection of exhaustive mutually exclusive events (a partition of the state space), it shall be preferred

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ex ante. Dynamic consistency is largely viewed as a rationality tenet, as it entails a number of desirable properties such as immunity to Dutch book operations and the positive value of information. Furthermore, it guarantees that value functions have a recursive structure, which ensures equality between ex ante optimal plans and backward induction solutions.

In this paper, we analyze preferences abiding to one of the most prominent models of decision theory under uncertainty, namely the Maxmin Expected Utility model of Gilboa and Schmeidler (1989). According to it, the decision-maker is characterized by a utility function and a set of prior probabilities, and the value she assigns to alternatives is given by their minimal expected utility, where the minimum is taken over the priors in the set. While Maxmin preferences (as most non-Bayesian decision theories) generally violate dynamic consistency[1], Epstein and Schneider (2003) show that they can satisfy dynamic consistency with respect to a given information structure[2], rather than with respect to all of them (see also Riedel (2004)). In particular, this requires the agent's set of priors to have a particular shape, i.e. to be rectangular. Broadly speaking, rectangularity means that the the set of priors is stable under pasting of conditional and marginal probabilities from different priors of the original set.

The fact that, for Maxmin preferences, dynamic consistency translates into a condition on the decision-maker's set of priors raises the question of how restrictive such an assumption is. Indeed, in most economic applications, where the information structure is fixed from the start and the decision-makers' priors are interpreted to represent evidential probabilistic information available to them, focusing on rectangular sets of priors may give the impression of overly limiting the scope of the economic behavior that one can model[3]. The objective of this paper is to provide a novel rationale for the emergence of dynamically consistent Maxmin preferences that is based on the evidential probabilistic information available to the decision-maker. We accomplish this task by following Gilboa et al. (2010) in endowing the decision-maker with two notions of rationality, "objective" and "subjective" rationality, that are meant to capture different aspects of a decision-maker's preferences. Objective rationality captures the partial ordering of alternatives that is justified by hard evidence, while subjective rationality describes the preference statements that the decision-maker cannot be convinced of being wrong in making. To extend Gilboa et al. (2010)'s framework to a dynamic setting, we suppose that the choice domain is the set of consumption processes proposed by Epstein and Schneider (2003). Additionally, we model objective rationality as an incomplete relation $\succsim^*$ that represents the objective information available to the decision-maker prior to any resolution of uncertainty, and subjective rationality as a collection of binary relations $\{\succsim^{\wedge}_{t,s},\ (t,s) \in \mathcal{T} \times \mathcal{S}\}$, representing subjectively rational preferences at each time-state pair.

Our first contribution, Theorem 1, consists in deriving a discounted unanimity represen-

[1] Canonical studies include Epstein and LeBreton (1993) and Sarin and Wakker (1998).

[2] By information structure we mean a partition or, more generally, a filtration of the state space representing the sequential resolution of the uncertainty faced by the decision-maker.

[3] As Al Najjar and Weinstein (2009) put it: "A sensible theory of updating should not selectively limit its scope to those situations where its desired implications seem to hold, while remain silent about what happens if slight perturbations to the information structure are introduced" (Al Najjar and Weinstein (2009), p. 271).

tation of objective rationality, according to which a consumption process f is objectively preferred to a consumption process g if and only if

$$\mathbb{E}_q\left[\sum_{\tau=0}^{T}\beta^\tau\ u\circ f(\tau,\cdot)\right]\ \geq\ \mathbb{E}_q\left[\sum_{\tau=0}^{T}\beta^\tau\ u\circ g(\tau,\cdot)\right]\quad\text{for every } q\in Q, \tag{1}$$

where $\beta > 0$ is the discount factor for future consumption, u is the decision-maker's utility and Q is the set of probability measures representing the objective probabilistic information. While this result is, we believe, of interest on its own, in the present setting it is meant to guarantee that the incompleteness of objective rationality is not due to any difficulty that the decision-maker might have about determining her preferences under risk or certainty. Under some relevant technical conditions, the above representation is derived by means of the well-known axiomatic requirement of "stationarity", plus a condition that roughly asserts that the decision-maker is neutral towards delaying the time of consumption.

Our second and main contribution, Theorem 2, consists in deriving Epstein and Schneider (2003)'s recursive Maxmin representation for subjective rationality, with the additional property that the subjective set of priors is the rectangular hull of the objective information set. Formally, and letting $\{[Q]_{t,s},\ (t,s)\in\mathcal{T}\times\mathcal{S}\}$ denote the rectangular hull of Q, each $\succsim_{t,s}$ is represented by the functional

$$V_{t,s}(f)=\min_{q\in[Q]_{t,s}}\ \mathbb{E}_q\left[\sum_{\tau=t}^{T}\beta^{\tau-t}\ u\circ f(\tau,\cdot)\right]. \tag{2}$$

Since the rectangular hull of a set is the smallest rectangular set containing it, the subjective set of priors typically contains measures that do not belong to the objective information set. However, such overselection is not arbitrary, as no overselection occurs if the objective information set is already rectangular. Such representation is based on two novel axiomatic requirements that are, in our opinion, rather natural dynamic extensions of the axioms used by Gilboa et al. (2010) to characterize the emergence of Maxmin preferences from objective rationality in a static setting. The first of such axioms, Consistency, roughly embodies two requirements: on one hand, it demands that preferences over consumption processes whose uncertainty resolves in the next time period[4] be fully determined by objective rationality. On the other, it imposes a specific form of dynamic consistency over this restricted domain of processes[5]. Such form of dynamic consistency roughly states that whenever it is objectively rational to prefer a process to another one conditionally upon observing every event that remains possible at some future date, such process shall subjectively be preferred at the present time. The rationale for the required restriction to processes whose uncertainty fully resolves in the next period is based on the intuition that these processes do not offer hedging opportunities across conditional preferences that are the source of

[4]Technically, these are the processes that are measurable with respect to the next period partition.

[5]We emphasize that we do not posit the standard notion of dynamic consistency, but rather obtain it as a consequence of our main result.

intertemporal preference reversal (i.e. dynamic inconsistencies). The second key axiom, Caution, roughly demands that if, for every posterior event that remains possible at some future period, it is not objectively rational to strictly prefer an ambiguous process to a conditionally unambiguous one, the latter shall be subjectively preferred at the present time-event pair. Representation 2 achieves the dynamic consistency of Maxmin preferences without imposing rectangularity constraints on the evidential information that is available to the decision-maker. It does so by proposing a specific methodology for constructing the decision-maker's prior probabilities that fusions the three behavioral desiderata of having choices being informed by objective information, dynamically consistent, and ambiguity averse. As such, we view it as offering a novel possible account by which Maxmin behavior can emerge from ambiguous evidential information. Moreover, it provides guidance for the use of Maxmin preferences in economic applications.

This article is organized as follows. In the next section, we illustrate the conflict between consistency with regard to objective information and dynamic consistency, as well as our axiomatic approach, by means of an example. Section 3 introduces the framework and relevant notation. The discounted unanimity representation of objective rationality is presented in Section 4, while the recursive Maxmin representation of subjective rationality appears in Section 5. Section 6 contains a discussion of our results in relation to the relevant literature and Section 7 concludes. All proofs are gathered in the Appendix.

2 Example

We begin by illustrating the incompatibility between dynamic consistency and consistency of preferences with respect to objective information by means of a simple example. We then outline this paper's proposal by introducing our key axioms and discussing their rationales in this simple environment.

Suppose that a ball is drawn from an urn containing 100 balls of four possible colors: red, blue, yellow and pink. A risk-neutral decision-maker is given the information that the number of red balls equals that of yellow ones and that the number of blue balls equals that of pink ones. This situation can be summarized by a state space $\mathcal{S} = \{r, b, y, p\}$ and the following set of priors capturing the objective information that is available to the decision-maker:

$$Q \;=\; \left\{ \;\left(q, \frac{1}{2} - q, q, \frac{1}{2} - q\right), \text{ for all } q \in \left[0, \frac{1}{2}\right] \;\right\}.$$

At date $t = 0$, a ball is drawn but the decision-maker does not observe its color. Hence, she only knows that the color of the extracted ball belongs to the set $\mathcal{S}$. At date $t = 1$, she is told whether the color obtained is red or blue (event $E := \{r, b\}$), or yellow or pink (event $F := E^c = \{y, p\}$). Finally, at date $t = 2$, the color is fully revealed to the decision-maker. This information structure can be summarized by the filtration on $\mathcal{S}$ defined by the sequence of partitions given by $\pi_0 = \{\mathcal{S}\}, \pi_1 = \{E, F\}$ and $\pi_2 = \{\{r\}, \{b\}, \{y\}, \{p\}\}$.

All the consumption processes that we consider are assumed to yield a default prize of 0 USD at dates $t = 0$ and $t = 1$. Hence, a consumption process is entirely defined by the state-contingent prize it yields at date $t = 2$. Let us consider the following consumption processes

$$h \;=\; (0,1,1,0) \quad \text{and} \quad k \;=\; \left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$$

We first suppose that the decision-maker exhibits **ambiguity aversion**. Concretely, this manifests through a subjective strict preference for k against h conditional on both E and F. We summarize this by writing $k \succ_E^\wedge h$ and $k \succ_F^\wedge h$. Indeed, conditional on each of E and F, the process k provides a sure amount of $1/2$, while h provides 1 or 0 with any probability in $[0,1]$. The two strict preferences are hence typical examples of ambiguity aversion.

Let us now apply the property of **dynamic consistency**, which was informally introduced in Section 1. Since ambiguity aversion manifests through the two rankings $k \succ_E^\wedge h$ and $k \succ_F^\wedge h$, dynamic consistency further requires an *ex ante* strict preference for k against h. We summarize this by writing $k \succ_0^\wedge h$.

Now, the property of consistency of preferences w.r.t. objective information is formally captured by the axiom of **consistency** of Gilboa et al. (2010). This requires subjective preferences to be partially determined by objective preferences in the sense that if a process f is objectively preferred to another process g, then f must also be subjectively preferred to g. This axiom being by now well-accepted in the literature, we refer to it as the "standard" notion of consistency. In our example, since h and k have exactly the same expectation under all priors in Q, they are objectively indifferent, which we denote by $h \sim^* k$. Consequently, the standard notion of consistency results in the indifference $k \sim_0^\wedge h$, in contradiction with the implication of dynamic consistency from the previous paragraph.

In our view, it is not the consistency requirement *per se* that must be blamed here. It is rather the fact that it applies to objective preferences that fail to detect the conditional ambiguity attached to h at E and F, and essentially treat this process as if it were an unambiguous one. Now, if the decision-maker insists on the ambiguous nature of h, he may be tempted to replace Q with its rectangular hull, which guarantees such acknowledgment of ambiguity. In the working example, this is given by:

$$[Q] \;=\; \left\{ \left(q,\frac{1}{2}-q,q',\frac{1}{2}-q'\right),\ \text{for all } q,q' \in \left[0,\frac{1}{2}\right] \right\}$$

Such replacement results in an obvious decoupling of the ambiguous evaluations of h conditional on the realization of E and F, which therefore cease to be a perfect hedge of each other as they do in the objective evaluation of h. By doing so, the decision-maker obtains a new preference relation that acknowledges the ambiguity of h. The objective of this paper is to provide an axiomatic foundation for this rectangularization strategy.

As outlined in the introduction, our approach builds upon a fair compromise between consistency, dynamic consistency, and ambiguity aversion. It relies on two axiomatic conditions that relate subjective to objective preferences across different time periods. In the simple informational environment of the above example, each axiom can be instructively decomposed into three requirements, which connect subjective preferences at time t with objective preferences at time $t' \geq t$, as we now illustrate. For the sake of clarity, when introducing each version of our axioms, we include the pair (t, t') to which it refers in its name. Now, in order to bridge consistency and dynamic consistency, we will replace the conjunction of these two axioms with the following condition:

01-Consistency. For all processes f and g, if $f_E g \succsim^* g$ and $f_F g \succsim^* g$, then $f \succsim_0^\wedge g$.

There is no need to enter the technical details at this stage. It is sufficient to understand $f_E g \succsim^* g$ as an objective preference for f against g conditional on E, and likewise for $f_F g \succsim^* g$. Our condition is structurally similar to dynamic consistency, with objective preferences playing the role of subjective ones in its antecedent. Meanwhile, it typically weakens the standard notion of consistency because objective preferences are usually (assumed to be) dynamically consistent: the antecedent in our condition is then weaker than that in the usual notion.

In the above example, 01-Consistency has no bite at all. Indeed, there can be no unanimous ranking between h and k conditional on E or F. At each of the two events, the decision-maker can find priors in Q that rank h above k, and also priors in Q that rank k above h. Hence, 01-Consistency can safely be assumed, but delivers little insight and must be complemented with other axioms.

We will complement 01-Consistency with other weak versions of the standard notion of consistency. First, note that the incompatibility between consistency and dynamic consistency illustrated in the example only arises because h is not π_1-measurable. If h were π_1-measurable, there would be no ambiguity attached to its outcomes conditional on any of E and F. So the only reason why the two strict preferences for k over h at E and F to hold would be that the outcomes of h are individually worse than 1/2, not the ambiguity they generate. This would be sufficient to obtain an *ex ante* strict preference for k over h, as dynamic consistency would require.

To illustrate, consider $h' = (x, x, y, y)$ for some $x, y \in \mathbb{R}$. Then, the rankings $k \succ_E^\wedge h'$ and $k \succ_F^\wedge h'$ essentially mean that $1/2 > x$ and $1/2 > y$. Hence, k is objectively strictly preferred to h', and this does not contradict dynamic consistency. Because of this remark, we will maintain the standard notion of consistency on the restricted domain of π_1-measurable processes. That is, we require the following:

00-Consistency. For all π_1-measurable processes f and g, if $f \succsim^* g$, then $f \succsim_0^\wedge g$.

Furthermore, the same logic must also apply at date $t = 1$. Since π_2 is the partition of $\mathcal{S}$ into singletons, all processes are π_2-measurable and we obtain the following unrestricted, conditional version of the standard notion of consistency

11-Consistency. For all processes f and g and $G \in \pi_1$, if $f_G g \succsim^* g$, then $f \succsim_G^\wedge g$.

As an aside, it becomes clear that, under 11-Consistency, 01-Consistency is weaker than the standard form of dynamic consistency.

Next, in order to bridge dynamic consistency with ambiguity aversion, we will also rely on an axiom of caution similar to that of Gilboa et al. (2010), which demands that an unambiguous process be subjectively preferred to a potentially ambiguous one, unless the opposite ranking is supported by objective preferences. Again, we refer to the notion of caution proposed by Gilboa et al. (2010) as the standard one. We will first suppose the following dynamic version of their axiom:

01-Caution. For all processes f and g with g π_1-measurable, if $f_E g \not\succsim^* g$ and $f_F g \not\succsim^* g$, then $g \succsim_0^\wedge f$.

This axiom says that the decision-maker only subjectively strictly prefers an arbitrary process to a π_1-measurable one when it is objectively rational to do so conditional on at least one of E or F. To illustrate its implications in the example, let h_α be the π_1-measurable process defined by $h_\alpha = (\alpha, \alpha, \alpha, \alpha)$ for all $\alpha \in [0, 1]$. It is then straightforward to see that for all $\alpha \in (0, 1]$

$$h_E h_\alpha = (0, 1, \alpha, \alpha) \not\succsim^* (\alpha, \alpha, \alpha, \alpha) = h_\alpha \quad \text{and} \quad h_F h_\alpha = (\alpha, \alpha, 1, 0) \not\succsim^* (\alpha, \alpha, \alpha, \alpha) = h_\alpha.$$

(To see why, take a prior in Q such that $q > (1-\alpha)/2$). An application of 01-Caution then yields $h_\alpha \succsim_0^\wedge h$ for all $\alpha \in (0, 1]$. A standard continuity argument provides $(0, 0, 0, 0) = h_0 \succsim_0^\wedge h$, and an equally standard monotonicity requirement finally leads to $k \succ_0^\wedge h$. The derivation of the latter strict preference from 01-Caution is of utter importance. It essentially establishes the validity of the dynamically consistent ranking $k \succ_0^\wedge h$ by merely positing 01-Caution and so, crucially, without explicitly requiring dynamic consistency.

We will nonetheless need to complement 01-Caution with the following intratemporal versions of the standard notion of caution:

00-Caution. For all process f and constant process x, if $f \not\succsim^* x$, then $x \succsim_0^\wedge f$.

11-Caution. For all processes f and g with g π_1-measurable and all $G \in \pi_1$, if $f_G g \not\succsim^* g$, then $g \succsim_G^\wedge f$.

The first axiom says that the decision-maker subjectively strictly prefers an arbitrary process to a riskless one *ex ante* only when it is objectively rational to do so. The second one extends the logic to time $t = 1$. They have no bite in the example because their antecedents fail.

The three axiomatic versions of each of consistency and caution presented in this section were only introduced for illustrative purposes. These axioms being instances of two more fundamental principles that are applied at different time periods, in what follows we will combine the three axioms of consistency into a single Consistency axiom and likewise the three axioms of caution into a single Caution axiom.

Finally, in the above example, dynamic consistency may give the impression of coming at quite a low price. Nevertheless, it is helpful to realize that our axiom of Consistency will essentially require that choice-relevant priors (i.e. the priors representing subjectively rational preferences) belong to the rectangular hull of Q. Additionally, Caution will require that the decision-maker be maximally ambiguity averse under the constraints imposed by Consistency. Consequently, the set of choice-relevant priors must be given by the rectangular hull of Q, and is, therefore, rectangular. As it is known since Epstein and Schneider (2003), this is then sufficient to secure dynamic consistency.

3 Framework

Let $\mathcal{S}$ be a set of states of the world. Time is finite and discrete and given by $\mathcal{T} = \{0, \ldots, T\}$. Let $\mathcal{T}^* = \{0, \ldots, T-1\}$. The information flow is given by a sequence of finite partitions $\{\pi_t,\, t \in \mathcal{T}\}$ on $\mathcal{S}$ where π_{t+1} refines π_t for every $t \in \mathcal{T}^*$. For a state $s \in \mathcal{S}$ and a time $t \in \mathcal{T}$, we denote by $\pi_t(s)$ the unique set in π_t which contains s. We assume that π_0 is the trivial partition. Thus, $\pi_0(s) = \mathcal{S}$ for all $s \in S$. For all $t \in \mathcal{T}$, $\mathcal{B}_t$ denote the Boolean algebra generated by π_t. $\mathcal{B}_T$ will also be denoted by $\mathcal{B}$.

Let X be an abstract set of consequences and $\mathcal{X}$ be the set of lotteries over X.

Let $\mathcal{F}$ represent the set of all functions f from $\mathcal{T} \times \mathcal{S}$ to $\mathcal{X}$ such that $f(\cdot, t)$ is π_t-measurable for every $t \in \mathcal{T}$. The objects in $\mathcal{F}$ are processes and represent distributions of outcomes across both states and dates. As customary, we sometimes abuse notation by writing x for the constant process that delivers lottery $x \in \mathcal{X}$ at every pair $(t, s) \in \mathcal{T} \times \mathcal{S}$.

For all $t \in \mathcal{T}$, let $\mathcal{A}_{\pi_t}$ denote the set of all π_t-measurable functions from $\mathcal{S}$ to $\mathcal{X}$. We will also denote $\mathcal{A}_{\pi_T}$ by $\mathcal{A}$.

The decision-maker has a collection $\{\succsim^{\wedge}_{t,s},\, (t, s) \in \mathcal{T} \times \mathcal{S}\}$ of binary relations on $\mathcal{F}$ representing her conditional preferences.

The decision-maker has a binary relation $\succsim^*$ on $\mathcal{F}$ representing objective rationality in the face of time and uncertainty.

The set $\mathcal{X}$ is endowed with the natural mixture operation. Hence, $\mathcal{F}$ inherits the induced mixture operation performed statewise. Furthermore, for all $(t, s) \in \mathcal{T} \times \mathcal{S}$ and $f, g \in \mathcal{F}$, let $f_{t,s}g$ be the function from $\mathcal{T} \times \mathcal{S}$ to $\mathcal{X}$ defined by $(f_{t,s}g)(t', s') = f(t', s')$ if $t' \geq t$ and $s' \in \pi_t(s)$ and $(f_{t,s}g)(t', s') = g(t', s')$ otherwise. Clearly, $f_{t,s}g$ is an element of $\mathcal{F}$ which yields the same consequence as f from t onward if the true state lies in $\pi_t(s)$ and g otherwise.

Furthermore, let $\mathcal{K}$ denote the set of all functions from $\mathcal{T}$ to $\mathcal{X}$. Let $\rho_{\mathcal{T}}$ be the function from $\mathcal{T} \times \mathcal{S}$ defined by $\rho_{\mathcal{T}}(t, s) = t$ for all $(t, s) \in \mathcal{T} \times \mathcal{S}$. Note that $k \circ \rho_{\mathcal{T}} \in \mathcal{F}$ for all $k \in \mathcal{K}$. Then, we define a binary relation $\succsim^*_{\mathcal{T}}$ on $\mathcal{K}$ by setting

$$k \ \succsim^*_{\mathcal{T}} \ l \iff k \circ \rho_{\mathcal{T}} \ \succsim^* \ l \circ \rho_{\mathcal{T}}$$

for all $k, l \in \mathcal{K}$. The binary relation $\succsim^*_{\mathcal{T}}$ represents the decision-maker's preferences over certain consumption processes.

Finally, fix $k \in \mathcal{K}$, $t \in \mathcal{T}$ and $x, x' \in \mathcal{X}$. Then, $x_t x'_{t+1} k$ denotes the element $l \in \mathcal{K}$ defined by $l(t') = k(t')$ for all $t' \neq t$ and $t' \neq t+1$, $l(t) = x$ and $l(t+1) = x'$. Similarly, $x_t k$ denotes the element $l \in \mathcal{F}$ defined by $l(t') = k(t')$ for all $t' \neq t$ and $l(t) = x$.

4 Discounted unanimity representation of objective rationality

We begin by imposing some standard axioms on $\succsim^*$ that are well-understood in the literature. We refer to Gilboa et al. (2010) and Faro and Lefort (2019) for their specific interpretation in the "objective and subjective rationality" framework. Note that A4 is a strong form of Nontriviality similar in spirit to Koopmans' (1960) "sensitivity" condition.

Preorder (A1). $\succsim^*$ is reflexive and transitive.

Monotonicity (A2). For every $f, g \in \mathcal{F}$, $f(t,s) \succsim^* g(t,s)$ for all $(t,s) \in \mathcal{T} \times \mathcal{S}$ implies $f \succsim^* g$.

Archimedean Continuity (A3). For all $f, g, h \in \mathcal{F}$, the sets $\{\alpha \in [0,1],\ \alpha f + (1-\alpha)g \succsim^* h\}$ and $\{\alpha \in [0,1],\ h \succsim^* \alpha f + (1-\alpha)g\}$ are closed in $[0,1]$.

Time Sensitivity (A4). There exist $x, y \in \mathcal{X}$ and $k \in \mathcal{K}$ such that $x_0 k \succ^*_{\mathcal{T}} y_0 k$.

Independence (A5). For all $f, g, h \in \mathcal{F}$ and all $\alpha \in (0,1]$, $f \succsim^* g$ if and only if $\alpha f + (1-\alpha)h \succsim^* \alpha g + (1-\alpha)h$.

In order to obtain the discounted unanimity representation 1, a richer set of axioms is needed. These additional axioms explicitly deal with the evaluation of *lottery processes* i.e. processes that are constantly equal to a lottery at every $t \in \mathcal{T}$. Since the consumption level of such processes depend on time and on the realization of each lottery, but not on the state of the world, these processes are the natural intertemporal counterpart of constant acts in the static Anscombe-Aumann setting: they involve risk but not ambiguity. This justifies the next axiom.

Time Completeness (A6). For $k, l \in \mathcal{K}$, $k \succsim^*_{\mathcal{T}} l$ or $l \succsim^*_{\mathcal{T}} m$.

Next axiom roughly demands that the decision-maker be indifferent to delaying consumption in order to obtain a statewise equivalent consumption level at the terminal date. To elaborate, suppose that $f, f' \in \mathcal{F}$ are such that $f'(T,s) \sim^*_{\mathcal{T}} f(\cdot,s)$ for all $s \in \mathcal{S}$. Then, f' yields a statewise equivalent of f at the terminal date. Put differently, all the intermediary consumption levels $f(t,s)$ for $t \in \mathcal{T}$ are plugged into $f'(T,s)$. But since consumption under f' at dates $t < T$ is unrestricted, it would be unreasonable to require the indifference between f and f'. Instead, suppose that the same logic relates two other processes g and g' in $\mathcal{F}$, and that additionally f' and g' equal to each other up to $T-1$, as required by A7. Then, f' and g' only differ from each other through the consumption levels that they yield at the terminal date, which are equivalent to the consumption levels delivered by f and g, respectively. Axiom A7 then requires the preference between f' and g' to be the same as the one between f and g.

Time Invariance (A7). For all $f, g, f', g' \in \mathcal{F}$ such that $f'(T,s) \sim^*_{\mathcal{T}} f(\cdot,s)$ and $g'(T,s) \sim^*_{\mathcal{T}} g(\cdot,s)$ for all $s \in \mathcal{S}$ and such that $f'(\tau,\cdot) = g'(\tau,\cdot)$ for all $\tau < T$, $f \succsim^* g$ if and only if $f' \succsim^* g'$.

Next axiom requires the trade-offs between consumption levels at two consecutive dates to be independent of the date of effective consumption. It is similar in spirit to Epstein and Schneider's (2003) "Risk Preference" axiom, and responsible for the exponential discounting component of the representation.

Stationarity (A8). For all $x, x', y, y' \in \mathcal{X}$, $t, t' \in \mathcal{T}^*$ and $k \in \mathcal{K}$, $x_t x'_{t+1} k \succsim^*_{\mathcal{T}} y_t y'_{t+1} k$ if and only if $x_{t'} x'_{t'+1} k \succsim^*_{\mathcal{T}} y_{t'} y'_{t'+1} k$.

Theorem 1 $\succsim^*$ satisfies A1—A8 if and only if there exist $\beta > 0$, a nonconstant and mixture-linear function u from $\mathcal{X}$ to $\mathbb{R}$ and a closed and convex set Q of probability measures on $\mathcal{B}$ such that, for all $f, g \in \mathcal{F}$,

$$f \succsim^* g \iff \mathbb{E}_q\left[\sum_{\tau=0}^{T} \beta^\tau \, u \circ f(\tau, \cdot)\right] \geq \mathbb{E}_q\left[\sum_{\tau=0}^{T} \beta^\tau \, u \circ g(\tau, \cdot)\right] \text{ for every } q \in Q.$$

Moreover, β and Q are unique, and u is unique up to a positive affine transformation.

Theorem 1 combines two influential models of decision-making: the unanimity rule of Bewley (2002) and Gilboa et al. (2010), which concerns static decision-making under uncertainty, and the discounted utility model, initially due to Samuelson (1937) and Koopmans (1960), that deals with dynamic decision-making under certainty. We believe that this result is interesting in its own right. For instance, if one adopts the traditional view that the binary relation $\succsim^*$ represents the choice behavior of the decision-maker, Theorem 1 opens the way to a dynamic theory of choice under ambiguity where dynamic consistency is achieved in its full force, i.e. with respect to all filtrations. However, in this paper Theorem 1 is a preliminary result that will prove instrumental to the study of the interplay between dynamic objective and subjective rationality. In this respect, the discounted unanimity representation of objective rationality is motivated by two different sorts of considerations. First, it guarantees that objectively rational preferences satisfy two forms of dynamic consistency, one with respect to time and the other with respect to uncertainty. And second, it ensures that the decision-maker encounters no difficulty in evaluating processes that are unambiguous, or, equivalently, that the incompleteness of $\succsim^*$ is only due to the presence of uncertainty.

We now briefly sketch the proof of Theorem 1. Axioms A1—A5, as well as the restriction of A6 to $\mathcal{X}$, allow us to invoke Theorem 1 of Gilboa et al. (2010) and obtain a unanimity representation of $\succsim^*$ with respect to some set P of probability measures on (a specific algebra of subsets of) $\mathcal{T} \times \mathcal{S}$. The remaining of the proof closely resembles the one of Epstein and Schneider's (2003) Lemma A.1, which provides a discounted Maxmin representation of preferences. Roughly, A4, A6, A7 and A8 correspond to their "Full Support", "Independence", "Monotonicity" and "Risk Preference" axioms, respectively.
In particular, A6 allows us to show that the $\mathcal{T}$-marginals of measures in P all coincide. Meanwhile, A8 shows that they all emerge from a discount factor β according to the geometric law, and A4 provides the positivity of β, an important technical condition.

Finally, thanks to A6, it is always possible to construct $a, b \in \mathcal{A}$ such that $a(s) \sim^*_{\mathcal{T}} f(\cdot, s)$ and $b(s) \sim^*_{\mathcal{T}} g(\cdot, s)$ for all $s \in \mathcal{S}$ and $f, g \in \mathcal{F}$. From there, it is easy to construct $f', g' \in \mathcal{F}$ as in the antecedent of A7 and apply this axiom to establish the discounted unanimity representation of $\succsim^*$, where Q is obtained as the set of all updates of measure in P conditional on $\{T\} \times \mathcal{S}$.

5 Dynamically consistent Maxmin representation of subjective rationality

In this section, we present the axiomatic requirements to be imposed on subjective rationality. The first three conditions are basic rationality requirements applied to each of the conditional preferences $\succsim^{\wedge}_{t,s}$ separately, while the last two axioms, the core of this paper, impose restrictions on the connection between $\succsim^*$ and the collection $\{\succsim^{\wedge}_{t,s}, (t, s) \in \mathcal{T} \times \mathcal{S}\}$.

In what follows, we say that a pair $(t, s) \in \mathcal{T} \times \mathcal{S}$ is null if $f_{t,s}g \sim^* g$ for all $f, g \in \mathcal{F}$. If (β, u, Q) provides a discounted unanimity representation of $\succsim^*$ as in Theorem 1, then (t, s) is null if and only if $q(\pi_t(s)) = 0$ for all $q \in Q$. As per the basic rationality conditions, we require each $\succsim^{\wedge}_{t,s}$ to be a transitive and continuous relation whose restriction to $\mathcal{X}$ satisfies the von Neumann-Morgenstern independence axiom.

Transitivity (B1). For all nonnull $(t, s) \in \mathcal{T} \times \mathcal{S}$ and $f, g, h \in \mathcal{F}$, if $f \succsim^{\wedge}_{t,s} g$ and $g \succsim^{\wedge}_{t,s} h$, then $f \succsim^{\wedge}_{t,s} h$.

Archimedean Contintuity (B2). For all nonnull $(t, s) \in \mathcal{T} \times \mathcal{S}$ and $f, g, h \in \mathcal{F}$, $\{\alpha \in [0, 1],\ \alpha f + (1 - \alpha)g \succsim^{\wedge}_{t,s} h\}$ and $\{\alpha \in [0, 1],\ h \succsim^{\wedge}_{t,s} \alpha f + (1 - \alpha)g\}$ are closed in $[0, 1]$.

Lottery Independence (B3). For all nonnull $(t, s) \in \mathcal{T} \times \mathcal{S}$ and $x, y, z \in \mathcal{X}$ and $\alpha \in (0, 1]$, $x \succsim^{\wedge}_{t,s} y$ if and only if $\alpha x + (1 - \alpha)z \succsim^{\wedge}_{t,s} \alpha y + (1 - \alpha)z$.

The following two axioms are the cornerstone of this paper. To introduce them, for all $t \in \mathcal{T}$, let $\mathcal{F}_{\pi_t}$ denote the set of all processes $f \in \mathcal{F}$ such that $f(\tau, \cdot)$ is π_t-measurable for all $\tau \geq t$. The first condition asserts that, when evaluating processes at a time-event pair (t, s), if two processes f, g are measurable with respect to the partition induced by some future date $t' + 1$, and additionally they are equal up to date $t' - 1$, while at t', f is weakly objectively better than g conditionally on the occurrence of every s, then f should also be ranked higher by subjective preferences at (t, s). Formally:

Consistency (B4). For all $s \in \mathcal{S}$ and $t, t' \in \mathcal{T}^*$ with $t' \geq t$ and (t, s) nonnull, for all $f, g \in \mathcal{F}_{\pi_{t'+1}}$ such that $f(\tau, \cdot) = g(\tau, \cdot)$ for all $\tau \leq t' - 1$, if $f_{t',s'}g \succsim^* g$ for all $s' \in \mathcal{S}$, then $f \succsim^{\wedge}_{t,s} g$.

Consistency can be understood as a form of dynamic consistency of subjective preferences that also embodies a requirement of consistency with respect to objective probabilistic information. Indeed, the standard notion of dynamic consistency is obtained by letting $t' = t + 1$ in the above statement, replacing $\succsim^*$ with $\succsim^{\wedge}_{t',s'}$ in is antecedent and finally replacing $\mathcal{F}_{\pi_{t'+1}}$ with the unrestricted domain $\mathcal{F}$. On the other hand, consistency of subjective preferences with respect to objective probabilistic information derives from the use of the objective rationality relation $\succsim^*$ in the conditional comparisons of processes that appear in its antecedent. To illustrate, suppose that $f_{t',s'}g \succsim^* g$ for all $s' \in \mathcal{S}$. By applying Consistency at the rank $t = t'$, it follows that $f \succsim^{\wedge}_{t',s'} g$ for all $s' \in \mathcal{S}$. Then, Consistency demands the preference $f \succsim^{\wedge}_{t,s} g$ to hold as much as an iterated application of dynamic consistency (of subjective preferences) would. The most important feature of the above axiom is the measurability restriction on the processes that can be dynamically compared, i.e. the requirement that f and g belong to $\mathcal{F}_{\pi_{t'+1}}$ rather than $\mathcal{F}$. The rationale for this domain restriction is that the elements of $\mathcal{F}_{\pi_{t'+1}}$ are processes whose uncertainty is fully resolved in the next period and, as such, do not present hedging opportunities across $t'+1$-conditional preferences that are the source of dynamic inconsistencies for ambiguity-sensitive preferences. Crucially, *without* such restriction (i.e. when B3 is required to hold for arbitrary $f, g \in \mathcal{F}$) one would for instance obtain that the set of *ex ante* priors is some rectangular subset of the objective information set Q. But since some sets of measures are such that their only rectangular subsets are singletons, this would essentially force neutrality to ambiguity in these cases[6]. Of course, no neutrality towards ambiguity would result if the original set were already rectangular. However, restricting attention to such situations would essentially result in the imposition of admissibility restrictions on the objective information that is available to the decision-maker, an important *deminutio capitis* in the present setting. Axiom Consistency can be fruitfully compared to a number of conditions that appear in the literature. We postpone such discussion to Section 6. Here, we introduce the last building block of the analysis, which demands that when evaluating processes at a time-event pair (t, s), if two processes f and g that are equal up to some posterior date $t' - 1$, are such that, at date t', f is not objectively better than g conditionally on the occurrence of every s, and, additionally, g is constant on each event in $\pi_{t'}$, then subjective preferences at (t, s) should rank the conditionally unambiguous act g higher than the potentially ambiguous process f.

Caution (B5). For all $s \in \mathcal{S}$ and $t, t' \in \mathcal{T}^*$ with $t' \geq t$ and (t, s) nonnull, for all $f \in \mathcal{F}$ and $g \in \mathcal{F}_{\pi_{t'}}$ such that $f(\tau, \cdot) = g(\tau, \cdot)$ for all $\tau \leq t' - 1$, if $f_{t',s'}g \not\succ^* g$ for all $s' \in \mathcal{S}$, then $g \succsim^{\wedge}_{t,s} f$.

As suggested by its name, Caution implies that the decision-maker is rather averse to ambiguity. Indeed, it imposes a preference for (conditionally) unambiguous over ambiguous processes. But in its full force the axiom also incorporates a form of dynamic consistency of preferences. To see why, notice that by applying Caution at rank $t = t'$, $f_{t',s'}g \not\succ^* g$ for all $s' \in \mathcal{S}$ implies that $g \succsim^{\wedge}_{t',s'} f$ for all $s' \in \mathcal{S}$. Then, Caution requires that $g \succsim^{\wedge}_{t,s} f$ as much as an iterated application of dynamic consistency would. A further discussion of B5 relative to some existing conditions that appear in the literature can be found in Section 6.

We now present our main result, i.e. the recursive Maxmin representation of subjective

[6] In general, sets of priors contain a rectangular subset if and only if they have a nonempty interior in the appropriate parametrization given by marginal and conditional probabilities. We refer to Riedel et al. (2018) for a detailed analysis of this fact, further illustrated within the canonical dynamic extension of Ellsberg's single-urn experiment (Ellsberg (1961)).

rationality with the property that the decision-maker's subjective priors correspond to to the rectangular hull of the objective information set Q featured in the discounted unanimity representation of $\succsim^*$. We begin by formalizing such notion of priors.

Consider a probability measure q on $\mathcal{B}$. Fix $t \in \mathcal{T}$ and $s \in \mathcal{S}$. If $q(\pi_t(s)) > 0$, then, define $q_t(s) = q(\cdot\ |\pi_t(s))$, which is another probability measure. Moreover, for $t < T$, define $q_t^{+1}(s)$ as the restriction of $q_t(s)$ to the algebra generated by π_{t+1}.

Consider a set Q of probability measures on $\mathcal{B}$. We say that $(t, s) \in \mathcal{T} \times \mathcal{S}$ is Q-negligible if $q(\pi_t(s)) = 0$ for all $q \in Q$. Otherwise, we say that (t, s) is Q-nonnegligible.

Consider now a closed and convex set Q of probability measures on $\mathcal{B}$. Fix $t \in \mathcal{T}$ and $s \in \mathcal{S}$. If (t, s) is Q-nonnegligible, then we define:

$$Q_t(s) = \{q_t(s),\ q \in Q,\ q(\pi_t(s)) > 0\} \quad \text{and, if } t \in \mathcal{T}^*, \quad Q_t^{+1}(s) = \{q_t^{+1}(s),\ q \in Q_t(s)\}.$$

We now define recursively a collection of closed and convex sets $[Q]_{t,s}$ of probability measures on $\mathcal{S}$ for all $t \in \mathcal{T}$ and $s \in \mathcal{S}$ such that (t, s) is Q-nonnegligible in the following way:

$$[Q]_{T,s} \ := \ Q_T(s), \text{ and}$$

$$[Q]_{t,s} \ := \ \left\{ \int_S \xi(s') \cdot dm(s'), \text{ for all } m \in Q_t(s)^{+1} \text{ and } \xi \in \Xi_{t+1}^{Q,m} \right\},$$

where $\Xi_{t+1}^{Q,m}$ is the set of all π_{t+1}-measurable functions on $\mathcal{S}$ such that $\xi(s') \in [Q]_{t+1,s'}$ for all $s' \in \mathcal{S}$ such that $(t + 1, s')$ is Q-nonnegligible and $m(\pi_{t+1}(s')) > 0$.

Theorem 2 Suppose $\succsim^*$ satisfies A1—A8 and let (β, u, Q) be as in Theorem 1. Then, $\{\succsim^\wedge_{t,s},\ (t, s) \in \mathcal{T} \times \mathcal{S}\}$ satisfies B1—B5 if and only if, for all nonnull $(t, s) \in \mathcal{T} \times \mathcal{S}$ and all $f, g \in \mathcal{F}$,

$$f \succsim^\wedge_{t,s} g \iff \min_{q \in [Q]_{t,s}} \mathbb{E}_q \left[\sum_{\tau=t}^{T} \beta^{\tau-t}\ u \circ f(\tau, \cdot) \right] \ \geq \ \min_{q \in [Q]_{t,s}} \mathbb{E}_q \left[\sum_{\tau=t}^{T} \beta^{\tau-t}\ u \circ g(\tau, \cdot) \right].$$

Theorem 2 axiomatically characterizes the preferences of a Maxmin agent who, in order to guarantee the dynamic consistency of her behavior, constructs her subjective priors by "rectangularizing" the objective information she disposes of. Hence, whenever the objective information set Q is already rectangular with respect to $\mathcal{F}$, subjective priors coincide with it, and the decision-maker's subjectively rational preferences fully comply with objective rationality. In fact, they satisfy the stronger version of axiom Consistency in which no domain restriction is imposed. Conversely, when objective probabilistic information is not rectangular, the decision-maker's behavior selectively departs from compliance with objective rationality in order to ensure the dynamic consistency of her behavior. Finally, at a more conceptual level, the rectangularization of the objective information set that materializes in Theorem 2 may also be interpreted as a form of sophistication: by correctly anticipating potential discrepancies between her current and future preferences, the decision-maker incorporates her future preferences into the current ones via the logic of backward induction.

We view Theorem 2 as suggesting a principled justification for the use of recursive Maxmin preferences in economic applications to dynamic choice problems[7]. In applications, it is typically assumed that: (i) both the sequential information structure and the set of probabilistic information is known to the decision-maker (and coincides with the ones used by the decision-analyst); and (ii) the decision-maker's set of priors coincides with the objective information set. Then, the fact that standard dynamic consistency of Maxmin preferences results in the rectangularity of the the decision-maker's priors and, consequently, of the objective information set, has often been taken as a strong normative obstacle to the adoption of Maxmin preferences in dynamic applications (see for instance Al Najjar and Weinstein (2009), Etner et al. (2012) or Gilboa and Marinacci (2016)). Theorem 2 accomplishes a twofold agenda. It first disentangles (i) from (ii). Then, by replacing (ii) with the requirement that the decision-maker's set of priors coincides with the rectangular hull of the objective information set, it proposes a novel rationale for the emergence of dynamically consistent Maxmin preferences which is based on the agent's commitment to both the available objective information and to dynamically consistent behavior. Of course, it may still be objected that the type of rationality embodied in our axioms of Consistency and Caution is somewhat flawed, in the same way that the rationality content of axioms such as "uncertainty aversion" can be called into question. While we believe that Consistency and Caution are rather natural descriptions of the attitude that a Maxmin decision-maker can have in the face of uncertainty, the important point here is that Theorem 2 contains a possibility result that can be used to shift the existing debate to the question of whether Consistency and Caution are normatively acceptable principles of rationality. A final feature of the Maxmin representation obtained in Theorem 2 is that we cannot in general retrieve the objective information set by looking at $\{\succsim^{\wedge}_{t,s}, (t,s) \in \mathcal{T} \times \mathcal{S}\}$. Nevertheless, it may be objected that, at least in economic applications, objective information being objective, one may well assume that it is publicly available.

Because of rectangularity of the subjective set of priors, it is straightforward to verify that the representation of $\{\succsim^{\wedge}_{t,s}, (t,s) \in \mathcal{T} \times \mathcal{S}\}$ obtained in Theorem 2 satisfies the following list of normative desiderata that are commonly imposed on dynamic preferences (for instance, see Epstein and Schneider (2003) and Riedel et al. (2018)):

Adaptedness. For all $s, s' \in \mathcal{S}$ and $t \in \mathcal{T}$ such that (t,s) and (t,s') are nonnull, if $\pi_t(s) = \pi_t(s')$, then $\succsim_{t,s} = \succsim_{t,s'}$.

Dynamic Consistency. For all nonnull $(t,s) \in \mathcal{T}^* \times \mathcal{S}$ and all $f, g \in \mathcal{F}$ such that $f(\tau,\cdot) = g(\tau,\cdot)$ for all $\tau \leq t$, if $f_{t+1,s'}g \succsim^* g$ for all $s' \in \mathcal{S}$, then $f \succsim^{\wedge}_{t,s} g$.

Consequentialism. For all $t \in \mathcal{T}$ and $s \in \mathcal{S}$ such that (t,s) is nonnull and all $f, g \in \mathcal{F}$, if $f(\tau, s') = g(\tau, s')$ for all $\tau \geq t$ and $s' \in \pi_t(s)$, then $f \sim^{\wedge}_{t,s} g$.

[7] A conceptually related point has recently been made by Hill (2020).

Here, these properties are delivered by the representation rather than imposed from the start[8]. More importantly, since the subjectively rational preferences $\{\succsim^{\wedge}_{t,s},\ (t,s) \in \mathcal{T} \times \mathcal{S}\}$ characterized in Theorem 2 belong to the class of recursive Maxmin preferences axiomatized by Epstein and Schneider (2003), their representation also satisfies the properties of recursivity of the value functions $\{V_{t,s},\ (t,s) \in \mathcal{T} \times \mathcal{S}\}$, and prior-by-prior Bayesian updating of the subjective sets of priors. We will not expand on these features and simply refer to Epstein and Schneider (2003) and Appendix B for further details.

We now briefly sketch the proof of Theorem 2. Consider first $t, t' \in \mathcal{T}^*$ with $t' \geq t$ and $s \in \mathcal{S}$ such that (t,s) is nonnull. Consider also any $f \in \mathcal{F}_{\pi_{t'+1}}$. We first construct two processes $f^{t'}_-, f^{t'}_+ \in \mathcal{F}_{\pi_{t'}}$ with the following properties:

(i) $f^{t'}_-(\tau,\cdot) = f^{t'}_+(\tau,\cdot) = f(\tau,\cdot)$ for all $\tau \leq t'-1$,

(ii) $f_{t',s'} f^{t'}_- \succsim^* f^{t'}_-$ and $f^{t'}_{+\,t',s'} f \succsim^* f$ for all $s' \in \mathcal{S}$,

(iii) $V_{t,s}(f^{t'}_-) = V_{t,s}(f)$ and $W_{t,s}(f^{t'}_+) = W_{t,s}(f)$,

where $V_{t,s}$ is defined by Formula (2) and $W_{t,s}$ is the dual functional obtained by replacing the `min` operator with the `max` one in this formula. In other words, $f^{t'}_-$ and $f^{t'}_+$ represent respectively the worst and best versions of f inside $\mathcal{F}_{\pi_{t'}}$. Now, Axiom B4 implies the preference ordering $f^{t'}_+ \succsim_{t,s} f \succsim_{t,s} f^{t'}_-$. Then, B2 and B5 allows us to further show that f is in fact indifferent to its worst version inside $\mathcal{F}_{\pi_{t'}}$; that is, $f \sim_{t,s} f^{t'}_-$. Next, using an induction argument, B1 and Item (iii) provide the Maxmin representation of $\succsim^{\wedge}_{t,s}$ on the limited domain $\mathcal{F}_{\pi_{t'+1}}$, for all $t' \in \mathcal{T}^*$. Taking $t' = T-1$ provides the Maxmin representation of $\succsim^{\wedge}_{t,s}$ on all of $\mathcal{F}$.

A slight difficulty emerges in establishing the representation of the terminal preferences $\succsim^{\wedge}_{T,s}$ for $s \in \mathcal{S}$ such that (T,s) is nonnull. This is because B4 does not inform us at all about these terminal preferences. Hence, the proof of their representation is different and actually crucially relies on B3 and B5. Rougly, B5 allows us to show that $f \sim^{\wedge}_{T,s} f(s)$ for all $f \in \mathcal{F}$ and $s \in \mathcal{S}$ such that (T,s) is nonnull. Meanwhile, B3 entails that $\succsim^{\wedge}_{T,s}$ and $\succsim^*$ agree on constant processes.

6 Discussion and related literature

As explained earlier, Consistency and Caution can be each understood as incorporating a form of dynamic consistency into their respective "pointwise" notions, obtained by letting $t = t'$ in their statements. That is, Pointwise Consistency demands that at every date t, objective preference instances amongst processes whose uncertainty fully resolves in

[8]Proof: Under item (ii), Adaptedness follows from the remark that, by the definition of the rectangular hull and the MEU representation, $\succsim^{\wedge}_{t,s}$ only depends on s through $Q_t(s)$. Hence, it only depends on s through $\pi_t(s)$. Dynamic Consistency is a straightforward consequence of Lemma 4 (in the Appendix) and the monotonicity properties of the integral and the min operator. Consequentialism is a straightforward consequence of Lemma 1(ii) (in the Appendix).

the next period be maintained by subjective rationality, while POINTWISE CAUTION is the requirement that, at every date, a conditionally unambiguous process be subjectively preferred to a potentially unambiguous one, unless the contrary preference is supported by conditional objective preferences. Formally:

POINTWISE CONSISTENCY. For all nonnull $(t,s) \in \mathcal{T}^* \times \mathcal{S}$ and all $f, g \in \mathcal{F}_{\pi_{t+1}}$, if $f_{t,s}g \succsim^* g$, then $f \succsim^{\wedge}_{t,s} g$.

POINTWISE CAUTION. For all nonnull $(t,s) \in \mathcal{T} \times \mathcal{S}$, all $f \in \mathcal{F}$ and all $g \in \mathcal{F}_{\pi_t}$, if $f_{t,s}g \not\succsim^* g$, then $g \succsim^{\wedge}_{t,s} f$.

As an aside, note that POINTWISE CONSISTENCY and POINTWISE CAUTION are strictly speaking slightly stronger than the exact pointwise versions of CONSISTENCY and CAUTION. This is because the former do not feature the restriction that f and g be equal to each other up to time t. In fact, it is sufficient in CONSISTENCY and CAUTION to consider that f and g agree with each other at all dates $\tau \in \mathcal{T}$ with $t \leq \tau \leq t' - 1$. Then, in the pointwise versions of the two axioms where $t = t'$, this agreement trivially holds for all consumption processes. Hence, the specific pointwise versions of the two axioms that we propose.

POINTWISE CONSISTENCY is a version of the axiom of "Local Dominance" that was used by Riedel et al. (2018) in order to derive a recursive representation of another class of multiple priors preferences, the imprecision averse preferences of Gajdos et al. (2008). The authors work in a framework with a single primitive relation, that they assume to admit an imprecision averse Maxmin representation, and then employ the standard notion of dynamic consistency to obtain that the decision-maker's priors are a rectangular subset of the rectangular hull of objective information. This result is more general than ours, since it can accommodate less cautious behavior than our model. On the other hand, we do not assume a representation, but rather derive it from natural axiomatic restrictions on the interplay between objective and subjective rationality. The fact that we obtain that choice-relevant priors are given by the rectangular hull of the information set, instead of being an arbitrary rectangular subset thereof, is of course due to CAUTION. Another difference with the result of Riedel et al. (2018) concerns the domain of preferences. While we work on the Epstein and Schneider (2003)'s domain of consumption processes, which accounts for consumption at intermediary dates, they define preferences on Anscombe-Aumann acts, in which consumption only takes place at the terminal date. Note that it is possible to obtain a version of our results in their setting by replacing the discounted utility representation with a standard von Neumann-Morgenstern utility function. It is also possible to obtain a version of our results adapted to the class of imprecision averse preferences, that account for variable objective information sets.

The most important feature of POINWISE CONSISTENCY is that t-conditional objective preferences are only binding if they compare processes whose uncertainty fully resolves in the next period. To further elaborate on this domain restriction, it is useful to consider its unrestricted version, obtained by replacing $\mathcal{F}_{\pi_{t'+1}}$ with $\mathcal{F}$ in its statement. Formally:

Unrestricted Pointwise Consistency. For all nonnull $(t, s) \in \mathcal{T}^* \times \mathcal{S}$ and all $f, g \in \mathcal{F}$, if $f_{t,s} g \succsim^* g$, then $f \succsim^{\wedge}_{t,s} g$.

Combining Unrestricted Pointwise Consistency and Pointwise Caution, one obtains a Maxmin representation of subjective rationality with the property that the decision-maker's subjective conditional priors at every time-event pair (t, s) are obtained by prior-by-prior updating from the set Q representing $\succsim^*$. Formally:

Theorem 3 Suppose $\succsim^*$ satisfies A1—A8 and let (β, u, Q) be as in Theorem 1. Then, $\{\succsim^{\wedge}_{t,s},\ (t, s) \in \mathcal{T} \times \mathcal{S}\}$ satisfies B1—B3, Unrestricted Pointwise Consistency and Pointwise Caution if and only if, for all nonnull $(t, s) \in \mathcal{T} \times \mathcal{S}$ and all $f, g \in \mathcal{F}$,

$$f \succsim^{\wedge}_{t,s} g \iff \min_{q \in Q_t(s)} \mathbb{E}_q \left[\sum_{\tau=t}^{T} \beta^{\tau-t}\ u \circ f(\tau, \cdot) \right] \geq \min_{q \in Q_t(s)} \mathbb{E}_q \left[\sum_{\tau=t}^{T} \beta^{\tau-t}\ u \circ g(\tau, \cdot) \right].$$

Theorem 3 reformulates the main result of Faro and Lefort (2019) to a setting in which preferences are defined over consumption processes rather than Anscombe-Aumann acts. It replaces their unanimity representation of objective rationality with the discounted unanimity representation. Consequently, it replaces their von Neumann-Morgenstern utility index in the representation of conditional subjective preferences with a discounted utility defined, at every (t, s), over the t-continuations of alternatives. Unrestricted Pointwise Consistency and Pointwise Caution are essentially the axioms used by Faro and Lefort (2019)[9]. A quick comparison between Theorems 2 and 3 can also be instructive. The representation of Theorem 3 achieves a greater level of consistency of subjective preferences with respect to objective information (the consistency captured by Unrestricted Pointwise Consistency), but at the price of generally failing recursivity and dynamic consistency. On the other hand, the agent modelled in Theorem 2 ensures dynamic consistency by requiring consistency of subjective preferences with respect to objective information to hold only on the restricted domain $\mathcal{F}_{\pi_t}$ of processes in which no dynamic inconsistency can occur.

Pointwise Consistency and Pointwise Caution are iterated versions of the axioms proposed by Gilboa et al. (2010) to derive a Maxmin representation of subjective rationality with the property that the subjective set of priors coincide with the objective information set representing objective rationality[10]. We will now revisit their result and explain how Theorem 2 can be interpreted as a natural dynamic extension of the Gilboa et al. (2010) static model.

[9]For the sake of precision, they adopt a slightly different notion of Pointwise Caution that, however, as shown by Theorem 3, is equivalent to ours.

[10]The modeling approach of objective and subjective rationality proposed by Gilboa et al. (2010) inspired a number of recent papers that therefore, despite indirectly, are related to ours (for instance, see Kopylov (2009), Faro (2015), Cerreia-Vioglio (2016), and Cerreia-Vioglio et al. (2020)). The study of pairs of interwoven binary relations to model an agent's behavior is also not new (recent contributions include, for example, Ghirardato et al. (2004), Lehrer and Teper (2011), and Giarlotta and Greco (2013)).

Let $\mathcal{S}$ be a set equipped with a Boolean algebra $\mathcal{B}$ of subsets. Let X be another set, and $\mathcal{X}$ denote the set of all lotteries on X. Let $\mathcal{A}$ denote the set of all finitely-valued and $\mathcal{B}$-measurable functions from $\mathcal{S}$ to $\mathcal{X}$. Consider two binary relations $\succsim^*$ and $\succsim^\wedge$ on $\mathcal{A}$, as before interpreted as representing a decision-maker's objective and subjective rationality. We suppose that objective rationality admits a unanimity representation, i.e. there exist a nonconstant, mixture-linear function u from $\mathcal{X}$ to $\mathbb{R}$ and a closed convex set Q of probability measures on $\mathcal{B}$ such that for all $a, b \in \mathcal{A}$,

$$a \succsim^* b \iff \mathbb{E}_q[u \circ a] \geq \mathbb{E}_q[u \circ b] \text{ for all } q \in Q.$$

As per $\succsim^\wedge$, we suppose that it satisfies the following conditions:

(C1). For all $a, b, c \in \mathcal{A}$, if $a \succsim^\wedge b$ and $b \succsim^\wedge c$, then $a \succsim^\wedge c$.

(C2). For all $a, b, c \in \mathcal{A}$, $\{\alpha \in [0,1], \alpha a + (1-\alpha)b \succsim^\wedge c\}$ and $\{\alpha \in [0,1], c \succsim^\wedge \alpha a + (1-\alpha)b\}$ are closed subsets in $[0,1]$.

(C3). For all $x, y, z \in \mathcal{X}$ and $\lambda \in (0,1]$, $x \succsim^\wedge y$ if and only if $\lambda x + (1-\lambda)z \succsim^\wedge \lambda y + (1-\lambda)z$.

(C4)[11]. For all $a, b \in \mathcal{A}$, if $a \succsim^* b$, then $a \succsim^\wedge b$.

(C5). For all $a \in \mathcal{A}$ and $x \in \mathcal{X}$, if $a \not\succsim^* x$, then $x \succsim^\wedge a$.

Theorem 4 Let (u, Q) provide a unanimity representation of $\succsim^*$. Then, $\succsim^\wedge$ satisfies C1—C5 if and only if, for all $a, b \in \mathcal{A}$,

$$a \succsim^\wedge b \iff \min_{q \in Q} \mathbb{E}_q[u \circ a] \geq \min_{q \in Q} \mathbb{E}_q[u \circ b].$$

Theorem 4 is the static version of our Theorem 2. Meanwhile, it reformulates the core insight of Theorem 3 of Gilboa et al. (2010). Indeed, supposing a unanimity representation of objectively rational preferences with respect to some set of measures, it obtains a Maxmin representation of subjectively rational preferences with respect to the same set. A first difference with the Gilboa et al. (2010) result is that Theorem 4 relies on "minimal" requirements, as it does not invoke the completeness, monotonicity, nontriviality of $\succsim^\wedge$ and weakens their C-Independence into C3. Another apparent difference lies in the specific form of (C5) that we assume. But this difference is inessential as, under (C4), our (C5) is equivalent to the axiom of Caution of Gilboa et al. (2010).

A further remark concerns the conceptual implications of our Consistency in the framework of objective and subjective rationality. In the interpretation of objective rationality proposed by Gilboa et al. (2010), a ranking $f \succsim^* g$ means that the decision-maker can convince others of being right in preferring f to g. It must then be *de facto* the case that the decision-maker prefers f to g. According to this view, the requirement that $\succsim^*$ be a

[11] A version of C4 was originally proposed by Nehring (2001) (see also Nehring (2009)) under the name of "compatibility".

subrelation of $\succsim^\wedge$ - i.e. Axioms C4 or Unrestricted Pointwise Consistency - is of a tautological nature, and not something optional. Accordingly, we acknowledge that our interpretation of the objective preference relation somewhat differs from the one of Gilboa et al. (2010). Here, a ranking $f \succsim^* g$ more simply means that f dominates g under every possible probabilistic scenario. In our setting, this probabilistic dominance is necessary, but not sufficient for convincing others. Indeed, probabilistic dominance does not secure dynamic consistency and hence fails to convince whoever values dynamic consistency as an inescapable property of rationality. The decision context modelled here involves two forms of objective information, the set Q and the filtration $\{\mathcal{B}_t,\, t \in \mathcal{T}\}$. Accordingly, two forms of consistency with respect to information arise. Given that the two may be incompatible with ambiguity-sensitive behavior, our Consistency requires that the subjective set of priors be a function of the set of objective priors, rather than to coincide with it.

Finally, the idea of characterizing the preferences of a Maxmin agent whose choice-relevant priors correspond to the rectangular hull of objective information within the model of Gilboa et al. (2010) is built upon the paper of Bastianello et al. (2020). The authors suppose that the sequential resolution of uncertainty is modelled by a single partition of the state space and consumption occurs at the terminal date. They further assume that the decision-maker is endowed with a second preference relation $\succsim^{**}$ that admits a unanimity representation and identify axiomatic conditions on the interplay between objective rationality and $\succsim^{**}$ that guarantee that the set of priors representing $\succsim^{**}$ is the rectangular hull of the one representing objective rationality. These axiomatic conditions essentially demand that $\succsim^{**}$ be the maximal restriction of objective rationality such that the set of priors representing $\succsim^{**}$ is contained in the rectangular hull of the objective information set. They then establish the Maxmin representation of subjective rationality with respect to this rectangular hull by applying the Gilboa et al. (2010) completion technique to $\succsim^{**}$. In contrast, the rectangularization procedure of objective information that is described in Theorem 2 solely emerges from the interplay between objective and subjective preferences, and hence does not invoke the intermediary preference relation $\succsim^{**}$. Nonetheless, following Ghirardato et al. (2004), it is possible to retrieve $\succsim^{**}$ in our model by looking at the "unambiguous restriction" of subjectively rational preferences (i.e. the maxmimal restriction of $\succsim^\wedge$ that satisfies the axiom of independence).

7 Conclusions

We axiomatize dynamically consistent Maxmin preferences that emerge from an incomplete preference relation that encompasses the (possibly partial) objective probabilistic information available to the decision-maker. The decision-maker builds her subjective priors by adopting a recursive procedure of rectangularization of objective information. The latter is justified by two natural conditions on preferences that combine the requirements that preferences be informed by objective information, as well as dynamically consistent. Consequently, our result seems to suggest a principled justification for the adoption of recursive Maxmin preferences in applications.

Appendix A: Proof of Theorems 3 and 4

Proof of Theorem 4. Sufficiency of the axioms. Fix $a \in \mathcal{A}$. Let $x_a, y_a \in \mathcal{X}$ be such that

$$u(x_a) \;=\; \min_{q \in Q} \mathbb{E}_q[u \circ a] \quad \text{and} \quad u(y_a) \;=\; \max_{q \in Q} \mathbb{E}_q[u \circ a].$$

Then, by construction, we have $y_a \succsim^* a \succsim^* x_a$. C4 yields $y_a \succsim^\wedge a \succsim^\wedge x_a$.

We will show that in fact $a \sim^\wedge x_a$. For all $\alpha \in [0,1]$, let $x^a_\alpha \in \mathcal{X}$ be defined by $x^a_\alpha = \alpha y_a + (1-\alpha) x_a$. We first show $a \not\succ^* x^a_\alpha$ for all $\alpha \in (0,1]$. Fix $\alpha \in (0,1]$ and consider the following two cases:

Case 1: $u(y_a) = u(x_a)$. Then, $a \sim^* x^a_\alpha$ and, therefore, $a \not\succ^* x^a_\alpha$.

Case 2: $u(y_a) > u(x_a)$. Then, let $q_0 \in Q$ be such that $u(x_a) = \mathbb{E}_{q_0}[u \circ a]$. Since $\alpha > 0$, we have $u(x^a_\alpha) > \mathbb{E}_{q_0}[u \circ a]$. Therefore, $a \not\succsim^* x^a_\alpha$ and $a \not\succ^* x^a_\alpha$.

C5 yields $x^a_\alpha \succsim^\wedge a$ for all $\alpha \in (0,1]$. Now, let $I \subseteq [0,1]$ be the set of all $\alpha \in [0,1]$ such that $x^a_\alpha \succsim^\wedge a$. By C2, I is a closed subset, and it must contain $(0,1]$. Therefore, $I = [0,1]$. In particular, we obtain $x_a \succsim^\wedge a$ and finally $a \sim^\wedge x_a$.

Now, fix $a, b \in \mathcal{A}$. By C1, we obtain $a \succsim^\wedge b$ if and only if $x_a \succsim^\wedge x_b$. But, by the unanimity representation of $\succsim^*$ and C4, $\succsim^\wedge$ is complete on $\mathcal{X}$. By C1—C3, on $\mathcal{X}$, it is further transitive and continuous, and satisfies independence. Then, it satisfies all the von Neumann and Morgenstern (1947) axioms. Hence, there exists a nonconstant and mixture-linear function $u^\wedge$ from $\mathcal{X}$ to $\mathbb{R}$ such that $x \succsim^\wedge y$ if and only if $u^\wedge(x) \geq u^\wedge(y)$ for all $x, y \in \mathcal{X}$. Still by C4 we have $u(x) \geq u(y)$ implies $u^\wedge(x) \geq u^\wedge(y)$ for all $x, y \in \mathcal{X}$. By Corollary B.3 from Ghirardato et al. (2004), u and $u^\wedge$ are positive affine transformations of each other. Hence, we obtain

$$a \succsim^\wedge b \quad \Longleftrightarrow \quad u(x_a) \;\geq\; u(x_b) \quad \Longleftrightarrow \quad \min_{q \in Q} \mathbb{E}_q[u \circ a] \;\geq \min_{q \in Q} \mathbb{E}_q[u \circ b].$$

Necessity of the axioms. The necessity of C1—C4 is standard. See for instance Gilboa et al. (2010). As for C5, suppose $a \in \mathcal{A}$ and $x \in \mathcal{X}$ are such that $a \not\succ^* x$. Then, either $a \not\succsim^* x$ or $a \sim^* x$. In the former case, we get $q \in Q$ such that $u(x) > \mathbb{E}_q[u \circ a]$ and hence $x \succ^\wedge a$. In the latter case, we get $u(x) = \mathbb{E}_q[u \circ a]$ for all $q \in Q$ and hence $x \sim^\wedge a$. □

Proof of Theorem 3. Some preliminary remarks are in order. Fix a nonnull $(t,s) \in \mathcal{T} \times \mathcal{S}$. Define a binary relation $\succsim^*_{t,s}$ on $\mathcal{F}$ by setting $f \succsim^*_{t,s} g$ if and only if $f_{t,s} g \succsim^* g$ for all $f, g \in \mathcal{F}$. Furthermore, $\mathcal{F}$ can be seen as the set of all functions from $\mathcal{T} \times \mathcal{S}$ that are measurable with respect to the algebra Σ generated by all $\{\tau\} \times E$ with $\tau \in \mathcal{T}$ and

$E \in \mathcal{B}_\tau$. Finally, we have the following discounted unanimity representation of $\succsim^*_{t,s}$: For all $f, g \in \mathcal{F}$,

$$f \succsim^*_{t,s} g \iff \mathbb{E}_q\left[\sum_{\tau=t}^{T} \beta^{\tau-t}\ u \circ f(\tau,\cdot)\right] \ \geq\ \mathbb{E}_q\left[\sum_{\tau=t}^{T} \beta^{\tau-t}\ u \circ g(\tau,\cdot)\right] \quad \text{for all } q \in Q_t(s).$$

Equivalently, $\succsim^*_{t,s}$ has a unanimity representation with respect to u and the set $P_t(s)$ of measures p on Σ_t such that

$$p(\{\tau\} \times E) = \begin{cases} \frac{1-\beta}{1-\beta^{T-t+1}} \beta^{\tau-t} \cdot q(E) & \text{if } \tau \geq t, \\ 0 & \text{otherwise,} \end{cases}$$

for all $\{\tau\} \times E \in \Sigma_t$ and some $q \in Q_t(s)$.

Sufficiency of the axioms. For all nonnull $(t,s) \in \mathcal{T} \times \mathcal{S}$, the pair $(\succsim^*_{t,s}, \succsim^\wedge_{t,s})$ satisfies C1—C3. It is further straightforward to see that it satisfies C4 and C5. Then, by Theorem 4, $\succsim^\wedge_{t,s}$ has a Maxmin representation with respect to $(u, P_t(s))$, which delivers the discounted Maxmin representation with respect to $(u, \beta, Q_t(s))$.

Necessity of the axioms. The necessity of B1—B3 is standard. As for Unrestricted Pointwise Consistency, fix a nonnull $(t,s) \in \mathcal{T}^* \times \mathcal{S}$ and $f, g \in \mathcal{F}$ such that $f_{t,s}g \succsim^* g$. Then, by the discounted unanimity representation of $\succsim^*_{t,s}$, we have

$$\mathbb{E}_q\left[\sum_{\tau=t}^{T} \beta^{\tau-t}\ u \circ f(\tau,\cdot)\right] \ \geq\ \mathbb{E}_q\left[\sum_{\tau=t}^{T} \beta^{\tau-t}\ u \circ g(\tau,\cdot)\right] \quad \text{for all } q \in Q_t(s).$$

It is then sufficient to take the minima on both sides and invoke the Maxmin representation of $\succsim^\wedge_{t,s}$ to conclude.

As for Pointwise Caution, fix a nonnull $(t,s) \in \mathcal{T}^* \times \mathcal{S}$, $f \in \mathcal{F}$ and $g \in \mathcal{F}_{\pi_t}$ such that $f_{t,s}g \not\succsim^* g$. Then, the discounted unanimity representation of $\succsim^*$ yields the existence of $q \in Q_t(s)$ such that

$$\sum_{\tau=t}^{T} \beta^{\tau-t}\ u \circ g(\tau,s) \ \geq\ \mathbb{E}_q\left[\sum_{\tau=t}^{T} \beta^{\tau-t}\ u \circ f(\tau,\cdot)\right].$$

Hence, we obtain

$$\min_{q \in Q_t(s)} \mathbb{E}_q\left[\sum_{\tau=t}^{T} \beta^{\tau-t}\ u \circ g(\tau,\cdot)\right] \ =\ \sum_{\tau=t}^{T} \beta^{\tau-t}\ u \circ g(\tau,s) \ \geq\ \min_{q \in Q_t(s)} \mathbb{E}_q\left[\sum_{\tau=t}^{T} \beta^{\tau-t}\ u \circ g(\tau,\cdot)\right].$$

The Maxmin representation of $\succsim^\wedge_{t,s}$ then provides $g \succsim^\wedge_{t,s} f$. □

Appendix B: Useful lemmata

Consider a given set Q of probability measures on $\mathcal{S}$. For all $t, t' \in \mathcal{T}$ with $t' \geq t$ and all $s \in \mathcal{S}$, let $\pi_t^{t'}(s)$ denote the collection of all $s' \in \pi_t(s)$ such that (t', s') is Q-nonnegligible.

Lemma 1 Let Q be a closed convex set of probability measures on $\mathcal{S}$. The following hold:

(i) For all Q-nonnegligible $(t, s) \in \mathcal{T}^* \times \mathcal{S}$ and $m \in Q_t^{+1}(s)$, we have $m(\pi_t^{t+1}(s)) = 1$,

(ii) For all Q-nonnegligible $(t, s) \in \mathcal{T} \times \mathcal{S}$ and $q \in [Q]_{t,s}$, we have $q(\pi_t(s)) = 1$,

(iii) For all Q-nonnegligible $(t, s) \in \mathcal{T} \times \mathcal{S}$, $t' \geq t$ and $q \in [Q]_{t,s}$, we have $q(\pi_t^{t'}(s)) = 1$.

Proof. (i) Fix a Q-nonnegligible $(t, s) \in \mathcal{T}^* \times \mathcal{S}$ and $m \in Q_t^{+1}(s)$. Let $p \in Q_t(s)$ be such that m is the restriction of p to the algebra generated by π_{t+1}. Then, $m(\pi_t(s)) = p(\pi_t(s)) = 1$. Let $s' \in \pi_t(s)$ be such that $(t+1, s')$ is Q-negligible. Then, $m(\pi_{t+1}(s')) = p(\pi_{t+1}(s')) = 0$. But $\pi_t(s) \setminus \pi_t^{t+1}(s)$ is contained in a finite disjoint union of subsets $\pi_{t+1}(s')$ for such $s' \in \pi_t(s)$. Therefore, $m(\pi_t^{t+1}(s)) = m(\pi_t(s)) = 1$.

(ii) We proceed by backward induction on t. Suppose first $t = T$. Then, by definition, $[Q]_{T,s} = Q_T(s)$, and the result is obvious. Suppose now the result at rank $t+1$. Fix a Q-nonnegligible $(t, s) \in \mathcal{T} \times \mathcal{S}$ and $q \in [Q]_{t,s}$. Then, by definition, there exist $m \in Q_t^{+1}(s)$ and a π_{t+1}-measurable function ξ on $\mathcal{S}$ with $\xi(s') \in [Q]_{t+1,s'}$ for all $s' \in \mathcal{S}$ such that $(t+1, s')$ is Q-nonnegligible and $m(\pi_{t+1}(s')) > 0$ satisfying

$$q \;=\; \int_S \xi(s') \cdot dm(s'). \tag{3}$$

By the induction assumption, for all Q-nonnegligible $(t+1, s')$ such that $m(\pi_{t+1}(s')) > 0$, we have $\xi(s')(\pi_{t+1}(s')) = 1$. In particular, for all $s' \in \pi_t^{t+1}(s)$ such that $m(\pi_{t+1}(s')) > 0$, we have $\pi_{t+1}(s') \subseteq \pi_t(s)$ and therefore $\xi(s')(\pi_t(s)) = 1$. Furthermore, and by (i), $\pi_t^{t+1}(s)$ is of measure 1 under m. Likewise, the set of $s' \in \mathcal{S}$ such that $m(\pi_{t+1}(s')) > 0$ is trivially of measure 1 under m. The intersection of these two sets is also of measure 1 under m, and Formula (3) then yields the result.

(iii) We proceed by backward induction on t. Suppose first $t = t'$. Then, we have $\pi_t^{t'}(s) = \pi_t(s)$, and the result follows from (ii). Suppose now the result at rank $t+1$. Fix $q \in [Q]_{t,s}$. Then, by definition, there exist $m \in Q_t^{+1}(s)$ and a π_{t+1}-measurable function ξ on $\mathcal{S}$ with $\xi(s') \in [Q]_{t+1,s'}$ for all $s' \in \mathcal{S}$ such that $(t+1, s')$ is Q-nonnegligible and $m(\pi_{t+1}(s')) > 0$ satisfying Formula (3). For all such s', the induction assumption yields $\xi(s')(\pi_{t+1}^{t'}(s)) = 1$. Now, note that $\pi_{t+1}^{t'}(s) \subseteq \pi_t^{t'}(s)$. Hence $\xi(s')(\pi_t^{t'}(s)) = 1$ for all $s' \in \mathcal{S}$ such that $(t+1, s')$ is Q-nonnegligible and and $m(\pi_{t+1}(s')) > 0$. Furthermore, and by (i), $\pi_t^{t'}(s)$ is of measure 1 under m. Likewise, the set of $s' \in \mathcal{S}$ such that $m(\pi_{t'}(s')) > 0$ is trivially of measure 1 under m. The intersection of these two sets is also of measure 1 under m, and Formula (3) then yields the result. □

For all set Q of probability measures on $\mathcal{S}$ and all $E \subseteq \mathcal{S}$ such that $q(E) > 0$ for some $q \in Q$, let $Q|E$ denote the set of measures $q \in Q$ such that $q(E) > 0$ conditionalized on E.

Lemma 2 Let Q be a closed convex set of probability measures on $\mathcal{S}$. The following hold: For all Q-nonnegligible $(t, s) \in \mathcal{T} \times \mathcal{S}$,

(i) $[Q]_{t+1,s} = [Q]_{t,s}|\pi_{t+1}(s)$ whenever $t \in \mathcal{T}^*$ and $(t+1, s)$ is Q-nonnegligible,

(ii) $[Q]_{t,s} = [Q]_{0,s}|\pi_t(s)$,

(iii) $Q_t(s) \subseteq [Q]_{t,s}$.

Proof. (i) Consider a Q-nonnegligible $(t, s) \in \mathcal{T}^* \times \mathcal{S}$ such that $(t+1, s)$ is Q-nonnegligible. Suppose first $q \in [Q]_{t+1,s}$. Consider any $m \in Q_t(s)^{+1}$ and any π_{t+1}-measurable function ξ on $\mathcal{S}$ with $\xi(s') \in [Q]_{t+1,s'}$ for all $s' \in \mathcal{S}$ such that $(t+1, s')$ is Q-nonnegligible and $\xi(s) = q$. Then, let q' be defined by

$$q' \;=\; \int_S \xi(s') \cdot dm(s'). \tag{4}$$

By definition, q' is an element of $[Q]_{t,s}$. Since $(t+1, s)$ is Q-nonnegligible, we may suppose $m(\pi_{t+1}(s)) > 0$. Then, $q'(\pi_{t+1}(s)) > 0$ and $q'(\cdot|\pi_{t+1}(s)) = q \in [Q]_{t+1,s}$. Hence, the inclusion $[Q]_{t+1,s} \subseteq [Q]_{t,s}|\pi_{t+1}(s)$.

Suppose now $p' \in [Q]_{t,s}|\pi_{t+1}(s)$ and let $q' \in [Q]_{t,s}$ be such that $p' = q'(\cdot|\pi_{t+1}(s))$. By definition, there exist $m \in Q_t(s)^{+1}$ and a π_{t+1}-measurable function ξ on $\mathcal{S}$ with $\xi(s') \in [Q]_{t+1,s'}$ for all $s' \in \mathcal{S}$ such that $(t+1, s')$ is Q-nonnegligible and $m(\pi_{t+1}(s')) > 0$ satisfying Formula (4). But clearly $(t+1, s)$ is Q-nonnegligible and $m(\pi_{t+1}(s)) > 0$. Then, it must be that $p' = \xi(s) \in [Q]_{t+1,s}$. Hence, the inclusion $[Q]_{t,s}|\pi_{t+1}(s) \subseteq [Q]_{t+1,s}$.

(ii) We proceed by forward induction. Suppose first $t = 0$. Then, since $\pi_0(s) = \mathcal{S}$, the result is obvious. Suppose now the result at rank $t < T$ and let $s \in \mathcal{S}$ be such that $(t+1, s)$ is Q-nonnegligible. Then, (t, s) is Q-nonnegligible and, by (i), we have $[Q]_{t+1,s} = [Q]_{t,s}|\pi_{t+1}(s)$. By the induction assumption, $[Q]_{t,s}$ is the set of all measures in $[Q]_{0,s}$ conditionalized on $\pi_t(s)$. As a consequence, $[Q]_{t+1,s} = [Q]_{0,s}|\pi_{t+1}(s)$.

(iii) We proceed by backward induction. Suppose first $t = T$. Then, $Q_T(s) = [Q]_{T,s}$ by definition. Suppose now the result at rank $t+1$ and let $q' \in Q_t(s)$. Let $q_0 \in Q$ be such that $q' = q_0(\cdot|\pi_t(s))$. Let also m be the restriction of q' to π_{t+1} and let ξ be a π_{t+1}-measurable function on $\mathcal{S}$ with $\xi(s') \in Q_{t+1}(s')$ for all $s' \in \mathcal{S}$ such that $(t+1, s')$ is Q-nonnegligible and additionally satisfying $\xi(s') = q_0(\cdot|\pi_{t+1}(s'))$ for all $s' \in \mathcal{S}$ such that $q_0(\pi_{t+1}(s')) > 0$. By the induction assumption $\xi(s') \in [Q]_{t+1,s'}$ for all $s' \in \mathcal{S}$ such that $q_0(\pi_{t+1}(s')) > 0$. Finally, Formula (4) holds by construction. Then, by definition of $[Q]_{t,s}$, $q' \in [Q]_{t,s}$. □

Lemma 3 Consider a closed convex set Q of probability measures on $\mathcal{S}$. Then, for all Q-nonnegligible $(t,s) \in \mathcal{T} \times \mathcal{S}$ and all $t' \in \mathcal{T}$ with $t' \geq t$,

$$[Q]_{t,s} \ := \ \left\{ \int_S \xi(s') \cdot dm(s'),\ m \in [Q]_{t,s},\ \xi \in \Xi_{t'}^{Q,m} \right\}.$$

Proof. Consider a Q-nonnegligible $(t,s) \in \mathcal{T} \times \mathcal{S}$ and $t' \in \mathcal{T}$ such that $t' \geq t$,. Suppose first $t = t'$. The result is then obvious because, by Lemma 1(ii), all measures in $[Q]_{t,s}$ assign a unit probability on $\pi_t(s)$, while at the same time every ξ in some $\Xi_t^{Q,m}$ is constantly equal to some element of $[Q]_{t,s}$ on $\pi_t(s)$. Suppose now $t' > t$. Let $q \in [Q]_{t,s}$. Let $m = q$ and ξ be a $\pi_{t'}$-measurable function on $\mathcal{S}$ satisfying $\xi(s') = q(\cdot|\pi_{t'}(s'))$ for all $s' \in \mathcal{S}$ such that $m(\pi_{t'}(s')) > 0$. By construction,

$$q \quad = \quad \int_S \xi(s') \cdot dm(s').$$

Iterated applications of Lemma 2(i) yield $\xi(s') \in [Q]_{t',s'}$ for all $s' \in \mathcal{S}$ such that (t', s') is Q-nonnegligible and $m(\pi_{t'}(s')) > 0$. Hence, ξ belongs to $\Xi_{t'}^{Q,m}$. This shows the one inclusion. As for the other one, we proceed by backward induction on t for all $t \leq t'$. We have already established the result in the case $t = t'$. Suppose now the result at rank $t+1 \leq t'$. Let $m \in [Q]_{t,s}$ and $\xi \in \Xi_{t'}^{Q,m}$. Define a π_{t+1}-measurable function ξ' satisfying $\xi'(s') \in [Q]_{t+1,s'}$ for all $s' \in \mathcal{S}$ such that $(t+1, s')$ is nonnull and also satisfying

$$\xi'(s') \quad = \quad \int_{\mathcal{S}} \xi(s'') \cdot dm(s''|\pi_{t+1}(s')),$$

for all $s' \in \mathcal{S}$ such that $m(\pi_{t+1}(s')) > 0$. The two requirements are consistent with each other. Indeed, suppose $s' \in \mathcal{S}$ such that $(t+1, s')$ is nonnull and $m(\pi_{t+1}(s')) > 0$. By Lemma 2(i), $m(\cdot|\pi_{t+1}(s')) \in [Q]_{t+1,s'}$. But furthermore note that $\xi \in \Xi_{t'}^{Q,m}$. So the induction assumption yields that $\xi'(s')$ as defined by the previous formula is indeed an element of $[Q]_{t+1,s'}$. Finally, we have

$$\int_S \xi(s'') \cdot dm(s'') \quad = \quad \int_S \xi'(s') \cdot dm(s') \ \in \ [Q]_{t,s},$$

where the inclusion in $[Q]_{t,s}$ is by the definition of $[Q]_{t,s}$. Indeed, ξ' is π_{t+1}-measurable, and its integral on $\mathcal{S}$ with respect to m only depends on the restriction of m to π_{t+1}, which lies in $Q_t(s)^{+1}$. ☐

In the next result, we will adopt the following convention: a finite sum on an empty set is equal to 0.

Lemma 4 Consider a nonconstant mixture-linear function u from $\mathcal{X}$ to $\mathbb{R}$ and a closed convex set Q of probability measures on $\mathcal{S}$. For all for all Q-nonnegligible $(t,s) \in \mathcal{T} \times \mathcal{S}$, let $V_{t,s}$ be the function from $\mathcal{F}$ to $\mathbb{R}$ defined for all $f \in \mathcal{F}$ by

$$V_{t,s}(f) \;=\; \min_{q \in [Q]_{t,s}} \mathbb{E}_q \left[\sum_{\tau=t}^{T} \beta^{\tau-t}\; u \circ f(\tau, \cdot) \right].$$

Then, for all Q-nonnegligible $(t,s) \in \mathcal{T} \times \mathcal{S}$, all $t' \in \mathcal{T}$ with $t' \geq t$, and all $f \in \mathcal{A}$,

$$V_{t,s}(f) \;=\; \min_{m \in [Q]_{t,s}} \int_{\mathcal{S}} \left[\sum_{\tau=t}^{t'-1} \beta^{\tau-t}\; u[f(\tau, s')] \;+\; \beta^{t'-t}\; V_{t',s'}(f) \right] \cdot dm(s').$$

Proof. Consider a Q-nonnegligible $(t,s) \in \mathcal{T} \times \mathcal{S}$ and $t' \in \mathcal{T}$ such that $t' \geq t$. Consider any $q \in [Q]_{t,s}$. Then, by Lemma 3, there exist $m \in [Q]_{t,s}$ and a $\pi_{t'}$-measurable function ξ on $\mathcal{S}$ with $\xi(s') \in [Q]_{t',s'}$ for all $s' \in \mathcal{S}$ such that $m(\pi_{t'}(s')) > 0$, which satisfies

$$q \;=\; \int_S \xi(s') \cdot dm(s'). \tag{5}$$

Then, for all $f \in \mathcal{F}$,

$$A_q \;:=\; \mathbb{E}_q \left[\sum_{\tau=t}^{T} \beta^{\tau-t}\; u \circ f(\tau, \cdot) \right] \;=\; \int_{\mathcal{S}} \mathbb{E}_{\xi(s')} \left[\sum_{\tau=t}^{T} \beta^{\tau-t}\; u \circ f(\tau, \cdot) \right] \cdot dm(s')$$

Hence, we obtain

$$\begin{aligned} A_q \;&=\; \int_{\mathcal{S}} \left[\sum_{\tau=t}^{t'-1} \beta^{\tau-t}\; u[f(\tau, s')] \;+\; \mathbb{E}_{\xi(s')} \left[\sum_{\tau=t'}^{T} \beta^{\tau-t}\; u \circ f(\tau, \cdot) \right] \right] \cdot dm(s') \\ &\geq\; \min_{m \in [Q]_{t,s}} \int_{\mathcal{S}} \left[\sum_{\tau=t}^{t'-1} \beta^{\tau-t}\; u[f(\tau, s')] \;+\; \beta^{t'-t}\; V_{t',s'}(f) \right] \cdot dm(s'). \end{aligned}$$

As this holds for all $q \in [Q]_{t,s}$, we obtain the one inequality. As for the other one, consider any $m \in [Q]_{t,s}$ and a $\pi_{t'}$-measurable function ξ on $\mathcal{S}$ with $\xi(s') \in [Q]_{t',s'}$ for all $s' \in \mathcal{S}$ such that $m(\pi_{t'}(s')) > 0$. Let q be defined by Formula (5). By Lemma 3, q belongs to $[Q]_{t,s}$. Then, for all $f \in \mathcal{F}$,

$$B_{m,\xi} \;=\; \int_{\mathcal{S}} \mathbb{E}_{\xi(s')} \left[\sum_{\tau=t}^{T} \beta^{\tau-t}\; u \circ f(\tau, \cdot) \right] \cdot dm(s'),$$

where $B_{m,\xi}$ is defined by

$$B_{m,\xi} \;:=\; \int_{\mathcal{S}} \left[\sum_{\tau=t}^{t'-1} \beta^{\tau-t}\; u[f(\tau, s')] \;+\; \mathbb{E}_{\xi(s')} \left[\sum_{\tau=t'}^{T} \beta^{\tau-t}\; u \circ f(\tau, \cdot) \right] \right] \cdot dm(s').$$

Hence, we obtain

$$B_{m,\xi} \;=\; \mathbb{E}_q\left[\sum_{\tau=t}^{T} \beta^{\tau-t}\; u \circ f(\tau,\cdot)\right] \;\geq\; V_{t,s}(f).$$

As this holds for all m and ξ, we obtain the second inequality. □

Appendix C: Proof of Theorem 1

Sufficiency of the axioms. Let Σ be the Boolean algebra of subsets of $\mathcal{T} \times \mathcal{S}$ generated by the collection $\{t\} \times E$ for all $t \in \mathcal{T}$ and $E \in \mathcal{B}_t$. Note that $\mathcal{F}$ can be seen as the set of all finitely-valued and Σ-measurable functions from $\mathcal{T} \times \mathcal{S}$ to $\mathcal{X}$. Then, by A1—A6, we can invoke GMMS' Theorem 1 and obtain a nonconstant and mixture-linear function u from $\mathcal{X}$ to $\mathbb{R}$ and a convex and closed set P of probability measures on Σ such that, for all $f, g \in \mathcal{F}$,

$$f \;\succsim^*\; g \quad\Longleftrightarrow\quad \mathbb{E}_p\,[u \circ f] \;\geq\; \mathbb{E}_p\,[u \circ g] \quad \text{for every } p \in P. \tag{6}$$

Next, let $P_{\mathcal{T}}$ denote the set of all $\mathcal{T}$-marginals of measures in P. Clearly, $\succsim^*_{\mathcal{T}}$ has a unanimity representation à la GMMS on $\mathcal{K}$ with respect to $(u, P_{\mathcal{T}})$. By A6 and the Anscombe and Aumann (1963) theorem, $P_{\mathcal{T}}$ must be a singleton. Let $p_{\mathcal{T}}$ be the one probability measure on $\mathcal{T}$ in $P_{\mathcal{T}}$.

Fix $k \in \mathcal{K}$ and $t \in \mathcal{T}$. Define a binary relation $\succsim^\sharp_{t,k}$ on $\mathcal{X}^2$ by setting $(x, x') \succsim^\sharp_{t,k} (y, y')$ if and only if $x_t x'_{t+1} k \succsim^*_{\mathcal{T}} y_t y'_{t+1} k$ for all $x, x', y, y' \in \mathcal{X}$. Then, we have

$$(x, x') \succsim^\sharp_{t,k} (y, y') \iff p_{\mathcal{T}}(\{t\}){\cdot}u(x) + p_{\mathcal{T}}(\{t{+}1\}){\cdot}u(x') \geq p_{\mathcal{T}}(\{t\}){\cdot}u(y) + p_{\mathcal{T}}(\{t{+}1\}){\cdot}u(y'),$$

for all $x, x', y, y' \in \mathcal{X}$. Now, fix $t' \in \mathcal{T}$. By A8, we also have

$$(x, x') \succsim^\sharp_{t',k} (y, y') \iff p_{\mathcal{T}}(\{t'\}){\cdot}u(x) + p_{\mathcal{T}}(\{t'{+}1\}){\cdot}u(x') \geq p_{\mathcal{T}}(\{t'\}){\cdot}u(y) + p_{\mathcal{T}}(\{t'{+}1\}){\cdot}u(y'),$$

for all $x, x', y, y' \in \mathcal{X}$. That is, $\succsim^\sharp_{t,k}$ has two Subjective Expected Utility representations à la Anscombe and Aumann (1963). By the uniqueness part of their theorem, we obtain that the quantity

$$C := \frac{p_{\mathcal{T}}(\{t\})}{p_{\mathcal{T}}(\{t\}) + p_{\mathcal{T}}(\{t+1\})}$$

is independent of $t \in \mathcal{T}$. Moreover, by A4, C lies in $(0, 1)$. Hence, $p_{\mathcal{T}}(\{t+1\}) = p_{\mathcal{T}}(\{t\}) \cdot (1 - C)/C$ for all $t \in \mathcal{T}^*$. Then, we have for all $k, l \in \mathcal{K}$,

$$k \;\succsim^*_{\mathcal{T}}\; l \quad\Longleftrightarrow\quad \sum_{\tau=0}^{T} \beta^\tau\; u[k(t)] \;\geq\; \sum_{\tau=0}^{T} \beta^\tau\; u[l(t)], \tag{7}$$

where $\beta > 0$ is defined by $\beta = (1 - C)/C$.

For all $a \in \mathcal{A}$ and all $h \in \mathcal{F}$, let $a_T h$ denote the element of $\mathcal{F}$ defined by $a_T h(t, s) = h(t, s)$ if $t < T$ and $a_T h(T, s) = a(s)$ for all $(t, s) \in \mathcal{T} \times \mathcal{S}$.

Now, fix $f, g \in \mathcal{F}$ and let a, b be functions from $\mathcal{S}$ to $\mathcal{X}$ satisfying

$$\frac{1-\beta^{T+1}}{1-\beta}\ u[a(s)] \ = \ \sum_{\tau=0}^{T} \beta^{\tau}\ u[f(\tau, s)] \quad \text{and} \quad \frac{1-\beta^{T+1}}{1-\beta}\ u[b(s)] \ = \ \sum_{\tau=0}^{T} \beta^{\tau}\ u[g(\tau, s)]$$

for all $s \in \mathcal{S}$. Since each $f(t, \cdot)$ and $f(t, \cdot)$ are π_T-measurable, we can assume without loss of generality that a and b are π_T-measurable, hence elements of $\mathcal{A}$. By Formula (7), we have $a(s) \sim^*_{\mathcal{T}} f(\cdot, s)$ and $b(s) \sim^*_{\mathcal{T}} g(\cdot, s)$ for all $s \in \mathcal{S}$. Then, A7 implies $f \succsim^* g$ if and only if $a_T h \succsim^* b_T h$ for all $h \in \mathcal{F}$. Let Q be the closed and convex set of probability measures q on $\mathcal{B}$ such that $q(E) = p[\{T\} \times E]/p[\{T\} \times \mathcal{S}]$ for all $E \in \mathcal{B}$ and for some $p \in P$. Then, Formula (6) yields

$$f \succsim^* g \quad \Longleftrightarrow \quad \mathbb{E}_q[u \circ a] \ \geq \ \mathbb{E}_q[u \circ b] \ \text{ for all } q \in Q.$$

Finally, combining the latter finding to the formulas defining a and b yields the discounted unanimity representation.

Necessity of the axioms. The necessity of A1—A6 and A8 is standard. As for A7, consider $f, g, f', g' \in \mathcal{F}$ such that $f'(T, s) \sim^*_{\mathcal{T}} f(\cdot, s)$ and $g'(T, s) \sim^*_{\mathcal{T}} g(\cdot, s)$ for all $s \in \mathcal{S}$ and such that $f'(\tau, \cdot) = g'(\tau, \cdot)$ for all $\tau < T$, $f \succsim^* g$ if and only if $f' \succsim^* g'$. Let $a, b \in \mathcal{A}$ be defined by $a = f'(T, \cdot)$ and $b = g'(T, \cdot)$. Let also $h \in \mathcal{F}$ be such that $h(\tau, \cdot) = f'(\tau, \cdot) = g'(\tau, \cdot)$ for all $\tau < T$. Then,

$$u \circ a \ = \ \sum_{\tau=0}^{T} \ \beta^t\ u \circ f(\tau, \cdot) \quad \text{and} \quad u \circ b \ = \ \sum_{\tau=0}^{T} \ \beta^t\ u \circ g(\tau, \cdot). \tag{8}$$

Furthermore, by the discounted unanimity representation, we have

$$a_T h \succsim^* b_T h \iff \mathbb{E}_q[u \circ a] \ \geq \ \mathbb{E}_q[u \circ b] \text{ for all } q \in Q.$$

Using again the discounted unanimity representation and Formula (8), we obtain $f \succsim^* g$ if and only if $a_T h \succsim^* b_T h$. Finally, A7 follows by noting that $f' = a_T h$ and $g' = b_T h$.

Uniqueness. Suppose (β, u, Q) and (β', u', Q') both provide a discounted unanimity representation of $\succsim^*$. Then, the pairs (β, u) and (β', u') provide Subjective Expected Utility representations of $\succsim^*_{\mathcal{T}}$ à la Anscombe and Aumann (1963). By the uniqueness claim in their theorem, we obtain $\beta = \beta'$, and u and u' are positive affine transformation of each other.

Moreover, define a binary relation $\succsim^*_{\mathcal{A}}$ on $\mathcal{A}$ by setting $a \succsim^*_{\mathcal{A}} b$ if and only if $a_T h \succsim^* b_T h$ for some $h \in \mathcal{F}$. Then, the pairs (u, Q) and (u', Q') provide unanimity representations of $\succsim^*_{\mathcal{A}}$ à la Gilboa et al. (2010). By the uniqueness claim in their Theorem 1, we obtain $Q = Q'$.

Appendix D: Proof of Theorem 2

Sufficiency of the axioms. Let (β, u, Q) be as in Theorem 1. To simplify the notation, we define for all $f \in \mathcal{F}$ and nonnull $(t,s) \in \mathcal{T} \times \mathcal{S}$

$$V_{t,s}(f) \;=\; \min_{q \in [Q]_{t,s}} \mathbb{E}_q \left[\sum_{\tau=t}^{T} \beta^{\tau-t} \; u \circ f(\tau,\cdot) \right] \text{ and } W_{t,s}(f) \;=\; \max_{q \in [Q]_{t,s}} \mathbb{E}_q \left[\sum_{\tau=t}^{T} \beta^{\tau-t} \; u \circ f(\tau,\cdot) \right].$$

Fix $s \in \mathcal{S}$ and $t \in \mathcal{T}^*$ such that (t,s) is nonnull and $t' \in \{t, \ldots, T-1\}$. For all $f \in \mathcal{F}_{\pi_{t'+1}}$, consider a function $f_-^{t'}$ from $\mathcal{T} \times \mathcal{S}$ to $\mathcal{X}$ satisfying, for all $\tau \in \mathcal{T}$ and $s' \in \mathcal{S}$,

$$u[f_-^{t'}(\tau,s')] = \begin{cases} V_{t',s'}(f) & \text{if } \tau \geq t' \text{ and } (t',s') \text{ is nonnull,} \\ u[f(\tau,s')] & \text{otherwise.} \end{cases}$$

Without loss of generality, we can assume that $f_-^{t'}(\tau,\cdot) = f(\tau,\cdot)$ for all $\tau \leq t'-1$ and that also $f_-^{t'}(\tau,\cdot)$ is $\pi_{t'}$-measurable for all $\tau \geq t'$. Hence, $f_-^{t'}$ lies in $\mathcal{F}$ and in fact in $\mathcal{F}_{\pi_{t'}}$.

We will show that $V_{t,s}(f_-^{t'}) = V_{t,s}(f)$. Indeed, we have

$$V_{t,s}(f_-^{t'}) \;=\; \min_{m \in [Q]_{t,s}} \int_{\mathcal{S}} \left[\sum_{\tau=t}^{t'-1} \beta^{\tau-t} \; u[f(\tau,s')] \;+\; \beta^{t'-t} \; V_{t',s'}(f) \right] \cdot dm(s') \;=\; V_{t,s}(f),$$

where the first equality is by construction of $f_-^{t'}$ and the second one is by Lemma 4. Now, fix $q \in Q$. Suppose first $q(\pi_{t'}(s')) > 0$. Then, $q(\cdot|\pi_{t'}(s')) \in Q_{t'}(s') \subseteq [Q]_{t',s'}$ by Lemma 2(iii). Therefore,

$$\mathbb{E}_q \left[\sum_{\tau=t'}^{T} \beta^{\tau-t'} \; u \circ f(\tau,\cdot)|\pi_{t'}(s') \right] \quad \geq \quad V_{t',s'}(f) \quad = \quad \mathbb{E}_q \left[\sum_{\tau=t'}^{T} \beta^{\tau-t'} \; u \circ f_-^{t'}(\tau,\cdot)|\pi_{t'}(s') \right].$$

Here, the equality is because $f_-^{t'} \in \mathcal{F}_{\pi_{t'}}$. Indeed, since $f_-^{t'}(\tau,\cdot)$ is $\pi_{t'}$-measurable for all $\tau \geq t'$, the function $\sum_{\tau=t'}^{T} \beta^{\tau-t'} \; u \circ f_-^{t'}(\tau,\cdot)$ is constant on $\pi_t(s)$.

Hence, we obtain

$$\mathbb{E}_q \left[\sum_{\tau=t'}^{T} \beta^{\tau-t'} \; u \circ (f_{t',s'} f_-^{t'})(\tau,\cdot) \right] \quad \geq \quad \mathbb{E}_q \left[\sum_{\tau=t'}^{T} \beta^{\tau-t'} \; u \circ f_-^{t'}(\tau,\cdot) \right].$$

Note that this also holds true if $q(\pi_{t'}(s')) = 0$. Hence, we have $f_{t',s'} f_-^{t'} \succsim^* f_-^{t'}$ and this holds for all $s' \in \mathcal{S}$. Then, B4 implies $f \succsim_{t,s}^{\wedge} f_-^{t'}$.

Similarly, consider a function $f_+^{t'} \in \mathcal{F}_{\pi_{t'}}$ satisfying, for all $\tau \in \mathcal{T}$ and $s' \in \mathcal{S}$,

$$u[f_+^{t'}(\tau,s')] = \begin{cases} W_{t',s'}(f) & \text{if } \tau \geq t' \text{ and } (t',s') \text{ is nonnull,} \\ u[f(\tau,s')] & \text{otherwise.} \end{cases}$$

By a symmetric argument, B4 also implies $f_+^{t'} \succsim_{t,s}^{\wedge} f$.

Now, for all $\alpha \in [0,1]$, let $f_\alpha^{t'} = \alpha f_+^{t'} + (1-\alpha) f_-^{t'}$. Furthermore, let I be the set of all $\alpha \in [0,1]$ such that $f_\alpha^{t'} \succsim_{t,s}^{\wedge} f$. Let us show that I contains $(0,1]$. Since, by B2, I is a closed set, this will then imply that I is in fact equal to $[0,1]$.

Fix $\alpha > 0$ and $s' \in \mathcal{S}$. Suppose first (t', s') is null. Then, $q(\pi_{t'}(s')) = 0$ and

$$\mathbb{E}_q\left[\sum_{\tau=0}^{T} \beta^\tau \ u \circ (f_{t',s'} f_\alpha^{t'})(\tau,\cdot)\right] \quad = \quad \mathbb{E}_q\left[\sum_{\tau=0}^{T} \beta^\tau \ u \circ f_\alpha^{t'}(\tau,\cdot)\right] \ \text{ for all } q \in Q.$$

By the discounted unanimity representation of $\succsim^*$, we obtain $f_{t',s'} f_\alpha^{t'} \sim^* f_\alpha^{t'}$

Suppose now that (t', s') is nonnull and consider the following two cases:

Case 1: $V_{t',s'}(f) = W_{t',s'}(f)$. Fix $q \in Q$ such that $q(\pi_{t'}(s')) > 0$. By Lemma 2(iii), we have $q(\cdot|\pi_{t'}(s')) \in [Q]_{t',s'}$. Therefore,

$$\mathbb{E}_q\left[\sum_{\tau=0}^{T} \beta^\tau \ u \circ (f_{t',s'} f_\alpha^{t'})(\tau,\cdot)\right] \quad = \quad \mathbb{E}_q\left[\sum_{\tau=0}^{T} \beta^\tau \ u \circ f_\alpha^{t'}(\tau,\cdot)\right].$$

(Here, we use the fact that f and $f_-^{t'}$ are equal to each other up to $t'-1$.) If $q \in Q$ is such that $q(\pi_{t'}(s')) = 0$, the previous equality also holds true. Hence, the unanimity representation of $\succsim^*$ shows that $f_{t',s'} f_\alpha^{t'} \sim^* f_\alpha^{t'}$ for every $\alpha \in [0,1]$.

Case 2: $V_{t',s'}(f) \neq W_{t',s'}(f)$. Then, since f belongs to $\mathcal{F}_{\pi_{t'+1}}$, we have

$$\max_{q \in Q_{t'}(s')} \mathbb{E}_q\left[\sum_{\tau=t'}^{T} \beta^{\tau-t'} \ u \circ f(\tau,\cdot)\right] \ = \ W_{t',s'}(f) \ > \ V_{t',s'}(f) \ = \ \min_{q \in Q_{t'}(s')} \mathbb{E}_q\left[\sum_{\tau=t'}^{T} \beta^{\tau-t'} \ u \circ f(\tau,\cdot)\right].$$

Let $q^* \in Q_{t'}(s')$ be a measure achieving the minimal discounted expected utility of f throughout $Q_{t'}(s')$ and let $p^* \in Q$ be such that $q^* = p^*(\cdot|\pi_{t'}(s'))$ (hence, in particular, $p^*(\pi_{t'}(s')) > 0$). Then, using the definitions of $f_-^{t'}$, $f_+^{t'}$ and $f_\alpha^{t'}$, we have:

$$\sum_{\tau=t'}^{T} \beta^{\tau-t'} \ u[f_\alpha^{t'}(\tau,s')] \ = \ \alpha \cdot W_{t',s'}(f) \ + \ (1-\alpha) \cdot V_{t',s'}(f).$$

But since $\alpha > 0$ and $W_{t',s'}(f) > V_{t',s'}(f)$, the definition of p^* further yields:

$$\sum_{\tau=t'}^{T} \beta^{\tau-t'} \ u[f_\alpha^{t'}(\tau,s')] \ > \ V_{t',s'}(f) \ = \ \mathbb{E}_{p^*}\left[\sum_{\tau=t'}^{T} \beta^{\tau-t'} \ u \circ f(\tau,\cdot)|\pi_{t'}(s')\right].$$

Now, for all $\tau \geq t'$, $f_\alpha^{t'}(\tau,\cdot)$ is $\pi_{t'}$-measurable and, on $\pi_{t'}(s')$, equal to $f_\alpha^{t'}(\tau,s')$. Hence,

$$\mathbb{E}_{p^*}\left[\sum_{\tau=t'}^{T} \beta^{\tau-t'} \ u \circ f_\alpha^{t'}(\tau,\cdot)|\pi_{t'}(s')\right] \ > \ \mathbb{E}_{p^*}\left[\sum_{\tau=t'}^{T} \beta^{\tau-t'} \ u \circ f(\tau,\cdot)|\pi_{t'}(s')\right].$$

Finally, since $f(\tau,\cdot)$ and $f_\alpha^{t'}(\tau,\cdot)$ are equal to each other up to $t'-1$, we obtain

$$\mathbb{E}_{p^*}\left[\sum_{\tau=0}^{T}\beta^\tau\ u\circ f_\alpha^{t'}(\tau,\cdot)\right]\ >\ \mathbb{E}_{p^*}\left[\sum_{\tau=0}^{T}\beta^\tau\ u\circ(f_{t',s'}f_\alpha^{t'})(\tau,\cdot)\right].$$

Combining the conclusions of the previous paragraphs, we obtain $f_{t',s'}f_\alpha^{t'}\not\succ^* f_\alpha^{t'}$ for all $s'\in\mathcal{S}$ and $\alpha\in(0,1]$. Then, B5 yields $f_\alpha^{t'}\succsim_{t,s}^\wedge f$ for all $\alpha\in(0,1]$ and hence for all $\alpha\in[0,1]$. Finally, we obtain $f_-^{t'}\succsim_{t,s}^\wedge f$ and therefore, $f\sim_{t,s}^\wedge f_-^{t'}$.

By the previous paragraph, we obtain the following: For all $t\in\mathcal{T}^*$, $t'\in\{t,\dots,T-1\}$, $s\in\mathcal{S}$ and $\pi_{t'+1}$-measurable $f,g\in\mathcal{A}$,

$$f\ \succsim_{t,s}^\wedge\ g\quad\Longleftrightarrow\quad f_-^{t'}\ \succsim_{t,s}^\wedge\ g_-^{t'}. \tag{9}$$

We now show the following by forward induction on t': for all $t\in\mathcal{T}^*$, $t'\in\{t,\dots,T-1\}$ and $s\in\mathcal{S}$ such that (t,s) is nonnull, for all $f,g\in\mathcal{F}_{\pi_{t'+1}}$,

$$f\ \succsim_{t,s}^\wedge\ g\quad\Longleftrightarrow\quad V_{t,s}(f)\ \geq\ V_{t,s}(g).$$

Suppose first $t'=t$. For all $f\in\mathcal{F}_{\pi_{t+1}}$, $f(\tau,\cdot)$ is π_{t+1}-measurable for all $\tau\geq t+1$. So it is sufficient to show that, for all $f,g\in\mathcal{F}_{\pi_{t+1}}$,

$$f\ \succsim_{t,s}^\wedge\ g\quad\Longleftrightarrow\quad\min_{q\in Q_t(s)}\mathbb{E}_q\left[\sum_{\tau=t}^{T}\beta^{\tau-t}\ u\circ f(\tau,\cdot)\right]\ \geq\ \min_{q\in Q_t(s)}\mathbb{E}_q\left[\sum_{\tau=t}^{T}\beta^{\tau-t}\ u\circ g(\tau,\cdot)\right]. \tag{10}$$

Define a binary relation $\succsim_{t,s}^*$ on $\mathcal{F}$ by setting $f\succsim_{t,s}^* g$ if and only if $f_{t,s}g\succsim^* g$ for all $f,g\in\mathcal{F}$. Since (β,u,Q) provides a discounted unanimity representation of $\succsim^*$, $(\beta,u,Q_t(s))$ provides a discounted unanimity representation of $\succsim_{t,s}^*$ in the following sense: for all $f,g\in\mathcal{F}$,

$$f\ \succsim_{t,s}^*\ g\iff\mathbb{E}_q\left[\sum_{\tau=t}^{T}\beta^\tau\ u\circ f(\tau,\cdot)\right]\ \geq\ \mathbb{E}_q\left[\sum_{\tau=t}^{T}\beta^\tau\ u\circ g(\tau,\cdot)\right]\ \text{ for all } q\in Q_t(s).$$

Note that $\mathcal{F}_{\pi_{t+1}}$ can be seen as the set of all finitely-valued Σ_t-measurable functions from $\mathcal{T}\times\mathcal{S}$ to $\mathcal{X}$, where Σ_t is the Boolean algebra generated by all subsets $\{\tau\}\times E$ with $\tau\leq t$ and $E\in\mathcal{B}_\tau$, or $\tau\geq t+1$ and $E\in\pi_{t+1}$. On this restricted domain $\mathcal{F}_{\pi_{t+1}}$, the pair $(\succsim_{t,s}^*,\succsim_{t,s}^\wedge)$ satisfies all the axioms of Theorem 4 and $\succsim_{t,s}^*$ has a unanimity representation with respect to u and the set $P_t(s)$ of measures p on Σ_t such that

$$p(\{\tau\}\times E)=\begin{cases}\frac{1-\beta}{1-\beta^{T-t+1}}\beta^{\tau-t}\cdot q(E) & \text{if } \tau\geq t,\\ 0 & \text{otherwise,}\end{cases}$$

for all $\{\tau\}\times E\in\Sigma_t$ and some $q\in Q_t(s)$. Then, by Theorem 4, for all $f,g\in\mathcal{F}_{\pi_{t'+1}}$,

$$f\ \succsim_{t,s}^\wedge\ g\quad\Longleftrightarrow\quad\min_{p\in P_t(s)}\mathbb{E}_p[u\circ f]\ \geq\ \min_{p\in P_t(s)}\mathbb{E}_p[u\circ g].$$

Given the definition of $P_t(s)$, the latter representation simplifies into Formula (10).

Suppose now the result at rank $t'-1$ with $t'>t$. Let $f,g\in\mathcal{F}_{\pi_{t'+1}}$. As already noted, $f_-^{t'}$ and $g_-^{t'}$ lie in $\mathcal{F}_{\pi_{t'}}$. Hence, by Formula (9) and the induction assumption, we obtain

$$f \ \succsim_{t,s}^{\wedge} \ g \quad \Longleftrightarrow \quad V_{t,s}(f_-^{t'}) \ \geq \ V_{t,s}(g_-^{t'}).$$

But we have already proved $V_{t,s}(f_-^{t'})=V_{t,s}(f)$ and $V_{t,s}(g_-^{t'})=V_{t,s}(g)$, and the intermediary result follows.

Finally, note that for $t'=T-1$, we have $\mathcal{F}_{\pi_{t'+1}}=\mathcal{F}$, and we obtain that $V_{t,s}$ provides a representation of $\succsim_{t,s}^{\wedge}$ on all of $\mathcal{F}$.

To complete the proof, we still need to establish the representation of $\succsim_{T,s}^{\wedge}$ for all $s\in\mathcal{S}$ such that (T,s) is nonnull. Consider any $f\in\mathcal{F}$. Then, clearly $f(s)_{T,s}f \not\succ^* f$ and, by the discounted unanimity representation of $\succsim^*$, $f_{T,s}f(s)\not\succ^* f(s)$. A double application of Caution leads to $f\sim_{T,s}^{\wedge} f(s)$. Hence, it is sufficient to show that $\succsim^*$ and $\succsim_{T,s}^{\wedge}$ agree on $\mathcal{X}$ and invoke the unanimity representation of $\succsim^*$ to conclude. Now, consider $x,y\in\mathcal{X}$ such that $x\succsim^* y$. Then, $x\succsim^* y_{T,s}x$ and $y_{T,s}x\not\succ^* x$. Hence, B5 implies $x\succsim_{T,s}^{\wedge} y$. The discounted unanimity representation of $\succsim^*$ then implies that $\succsim_{T,s}^{\wedge}$ is complete on $\mathcal{X}$. By the von Neumann and Morgenstern (1947) theorem, there exists a nonconstant and mixture-linear function $u_{T,s}$ representing $\succsim_{T,s}^{*}$ on $\mathcal{X}$, and we must have $u(x)\geq u(y)$ implies $u_{T,s}(x)\geq u_{T,s}(y)$ for all $x,y\in\mathcal{X}$. By Corollary B.3 of Ghirardato et al. (2004), $u_{T,s}$ and u are positive affine transformations of each other. Hence, $\succsim^*$ and $\succsim_{T,s}^{\wedge}$ agree with each other on $\mathcal{X}$.

Necessity of the axioms. The necessity of B1—B3 is standard. To show B4, suppose $s\in\mathcal{S}$ and $t,t'\in\mathcal{T}^*$ are such that $t'\geq t$ and (t,s) nonnull. Consider $f,g\in\mathcal{F}_{\pi_{t'+1}}$ such that $f(\tau,\cdot)=g(\tau,\cdot)$ for all $\tau\leq t'-1$ and $f_{t',s'}g\succsim^* g$ for all $s'\in\mathcal{S}$.

Fix $s'\in\mathcal{S}$ such that (t',s') is nonnull. Let $q\in[Q]_{t',s'}$ and let p be the restriction of q to $\mathcal{B}_{t'+1}$. Then, $p\in Q_{t'}^{+1}(s')$. Let $m\in Q_{t'}(s')$ be such that p is the restriction of m to $\mathcal{B}_{t'+1}$. We obtain:

$$\begin{aligned}\mathbb{E}_q\left[\sum_{\tau=t'}^{T}\ \beta^{\tau-t'}\ u\circ f(\tau,\cdot)\right] \ &= \ \mathbb{E}_m\left[\sum_{\tau=t'}^{T}\ \beta^{\tau-t'}\ u\circ f(\tau,\cdot)\right] \\ &\geq \ \mathbb{E}_m\left[\sum_{\tau=t'}^{T}\ \beta^{\tau-t'}\ u\circ g(\tau,\cdot)\right] \quad = \quad \mathbb{E}_q\left[\sum_{\tau=t'}^{T}\ \beta^{\tau-t'}\ u\circ g(\tau,\cdot)\right],\end{aligned}$$

where the equalities are because $f,g\in\mathcal{F}_{\pi_{t'+1}}$ and the inequality is by $f_{t',s'}g\succsim^* g$. Then, we further obtain $V_{t',s'}(f)\geq V_{t',s'}(g)$. This inequality holds in particular for all $s'\in\pi_t^{t'}(s)$. By Lemma 1(i), it then holds almost surely under every probability measure in $[Q]_{t,s}$. Finally, since $f(\tau,\cdot)=g(\tau,\cdot)$ for all $\tau\leq t'-1$, Lemma 4 shows $V_{t,s}(f)\geq V_{t,s}(g)$ and hence $f\succsim_{t,s}^{\wedge} g$.

As for B5, suppose $s\in\mathcal{S}$ and $t,t'\in\mathcal{T}^*$ are such that $t'\geq t$ and (t,s) nonnull. Consider $f\in\mathcal{F}$ and $g\in\mathcal{F}_{\pi_{t'}}$ such that $f(\tau,\cdot)=g(\tau,\cdot)$ for all $\tau\leq t'-1$ and $f_{t',s'}g\not\succ^* g$ for all $s'\in\mathcal{S}$.

Fix $s' \in \mathcal{S}$ such that (t', s') is nonnull. Since $f_{\pi_{t'}(s')} g \not\succsim^* g$, the discounted unanimity representation of $\succsim^*$ yields $p \in Q_{t'}(s')$ such that

$$V_{t',s'}(g) \; = \; \sum_{\tau=t'}^{T} \; \beta^{\tau-t'} \; u(g(\tau, s')) \; \geq \; \mathbb{E}_p \left[\sum_{\tau=t'}^{T} \; \beta^{\tau-t'} u \circ f(\tau, \cdot) \right] \; \geq \; V_{t',s'}(f),$$

where the latter inequality is by Lemma 2(iii). This inequality holds in particular for all $s' \in \pi_t^{t'}(s)$. By Lemma 1(i), it then holds almost surely under every probability measure in $[Q]_{t,s}$. Finally, since $f(\tau, \cdot) = g(\tau, \cdot)$ for all $\tau \leq t' - 1$, Lemma 4 shows $V_{t,s}(g) \geq V_{t,s}(f)$ and hence $g \succsim_{t,s}^{\wedge} f$.

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