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Michel Grabisch[1⋆], Alexis Poindron[2], and Agnieszka Rusinowska[3]

[1] Paris School of Economics, Université Paris I Panthéon-Sorbonne
Centre d'Economie de la Sorbonne, 106-112 Bd de l'Hôpital, 75647 Paris Cedex 13, France
michel.grabisch@univ-paris1.fr
[2] Université Paris I Panthéon-Sorbonne, Centre d'Economie de la Sorbonne
alexis.poindron@laposte.net
[3] CNRS, Paris School of Economics, Centre d'Economie de la Sorbonne
agnieszka.rusinowska@univ-paris1.fr

Abstract. We study a stochastic model of anonymous influence with conformist and anti-conformist individuals. Each agent with a 'yes' or 'no' initial opinion on a certain issue can change his opinion due to social influence. We consider anonymous influence, which depends on the number of agents having a certain opinion, but not on their identity. An individual is conformist/anti-conformist if his probability of saying 'yes' increases/decreases with the number of 'yes'-agents. We focus on three classes of aggregation rules (pure conformism, pure anti-conformism, and mixed aggregation rules) and examine two types of society (without, and with mixed agents). For both types we provide a complete qualitative analysis of convergence, i.e., identify all absorbing classes and conditions for their occurrence. Also the pure case with infinitely many individuals is studied. We show that, as expected, the presence of anti-conformists in a society brings polarization and instability: polarization in two groups, fuzzy polarization (i.e., with blurred frontiers), cycles, periodic classes, as well as more or less chaotic situations where at any time step the set of 'yes'-agents can be any subset of the society. Surprisingly, the presence of anti-conformists may also lead to opinion reversal: a majority group of conformists with a stable opinion can evolve by a cascade phenomenon towards the opposite opinion, and remains in this state.

JEL Classification: C7, D7, D85

Keywords: opinion dynamics, anonymity, anti-conformism, convergence, absorbing class

1 Introduction

This paper is devoted to anti-conformism in the framework of opinion formation with anonymous influence. Despite the fact that anti-conformism plays a crucial role in many social and economic situations, and can naturally explain human behavior and various dynamic processes, this phenomenon did not receive enough attention in the literature.

The seminal work of DeGroot (1974) and some of its extensions consider a non-anonymous positive influence in which agents update their opinions by using a weighted average of opinions of their neighbors. However, in many situations, like opinions and comments given on the internet, the identity of the agents is not known, or at least, there is no clue on the reliability or kind of personality of the agents. Therefore, agents can be considered as anonymous, and influence is merely due to the *number* of agents having a certain opinion, not their identity. Förster et al. (2013) investigate such an anonymous social influence, but restrict their attention to the conformist behavior. They depart from a general framework of influence based on aggregation functions (Grabisch and Rusinowska

⋆ Corresponding author.

(2013)), where every individual updates his opinion by aggregating the agents' opinions which determines the probability that his opinion will be 'yes' in the next period. Instead of allowing for arbitrary aggregation functions, Förster et al. (2013) consider anonymous aggregation. However, both frameworks of Grabisch and Rusinowska (2013) and Förster et al. (2013) cover only positive influence (imitation), since by definition aggregation functions are nondecreasing, and hence cannot model anti-conformism.

Our aim is to study opinion formation under anonymous influence in societies with conformist and anti-conformist individuals. We focus on three classes of aggregation rules that can be used by the agents when revising their opinions: pure conformism, pure anti-conformism, and mixed aggregation rules. As a consequence, we distinguish three types of agents: pure conformists who are more likely to say 'yes' when there are more agents who said 'yes' in the last period, pure anti-conformists who are less likely to do so, and mixed agents. All purely conformist agents are assumed to share the same minimum number of 'yes' for them to say 'yes' with positive chance and the same minimum number of 'yes' for them to say 'yes' with chance of 1. Similarly, all purely anti-conformist agents share the same minimum number of 'yes' for them to say 'no' with positive chance and the same minimum number of 'yes' for them to say 'no' with chance of 1. We call these four parameters the influenceability parameters. We consider two types of a society: without mixed agents, and containing mixed agents. For both cases we provide a complete qualitative analysis of convergence, i.e., we identify all absorbing classes and conditions for their occurrence. This full characterization of the absorbing classes is based on the size comparison among the four influenceability parameters and the number of conformists and anti-conformists in the society.

Our findings bring precise answers to the following fundamental questions: *What is the impact of the presence of anti-conformists on a society being mainly conformist? Is a chaotic or unstable situation possible? Is opinion reversal possible?*

The exact description of the impact is done through our main Theorems 1 and 2, giving all possible absorbing classes, that is, all possible states of the society in the long run, and conditions of their existence. The complexity of these results, however, asks for a further analysis which would extract the main trends. We have conducted such an analysis, supposing that the size of the society is very large, and considering several typical situations, e.g., the same influenceability parameters among the conformists and anti-conformists, a small proportion of anti-conformists, etc. This permits to answer the two other questions.

About the possibility of a chaotic or unstable situation, without much surprise, the answer is positive. We have distinguished however several types of unstable situations, ranked in increasing order of unstability: fuzzy polarization, cycle, fuzzy cycle and chaos. Polarization is the well-known phenomenon where a part of the society converges to 'yes' and the other part to 'no'. Obviously, this situation can arise here, with conformists and anti-conformists having opposite opinion, provided the proportion of the latter is not too high. Fuzzy polarization means that instead of having two stable groups, there is a kind of oscillation around the groups of conformists and anti-conformists. When the set of 'yes'-agents evolve according to a periodic sequence of subsets of the society, we speak of a cycle. For instance, it is easy to see that the cycle $N^a, N^c, N^a, \ldots$, where N^a, N^c are respectively the set of anti-conformists and the set of conformists, can arise, provided the number of anti-conformists is large enough. A fuzzy cycle is nothing other than a periodic class in the theory of Markov chains. That is, there is a cycle of *collections* of subsets of the society, and at each time step, a subset is picked at random in the collection under consideration. Finally, chaos means that at each time step, any subset of the society can be the set of 'yes'-agents.

Finally, is opinion reversal possible? Is misinformation possible? First, some explanation is necessary. In a purely conformist society, there is quick convergence to a consensus, either on 'yes' or on 'no'. Therefore, the opinion remains constant for ever, and no change can occur. Opinion reversal would mean that, starting from a state where a large majority of the society has a stable opinion (possibly, the ground truth), there would be an evolution leading for that majority group to the opposite opinion (cascade phenomenon). This is exactly what is most feared by, e.g., politicians during some election, or any leader governing some society of individuals. Surprisingly, such a phenomenon can occur, even with a small number of anti-conformists, under certain circumstances that we describe precisely. Hence, contrarily to the intuition which tells us that introducing anti-conformists is simply introducing unstability and chaos, we have shown that it is quite possible to manipulate, by a suitable choice of the influenceability parameters and the proportion of anti-conformists, the final opinion of the conformists, and therefore propagate misinformation. We consider this result as one of the main findings of the paper.

Our framework is suitable for many natural applications. It can explain various phenomena like stable and persistent shocks, large fluctuations, stylized facts in the industry of fashion, in particular its intrinsic dynamics, booms and burst in the frequency of surnames, etc. If fashion were only a matter of conformist imitation in an anonymous framework, there would be no trends over time. Anti-conformism and anti-coordination when individuals have an incentive to differ from what others do can also be detected, e.g., in organizational settings. For example, the choice of a firm to go compatible or not with other firms can be seen as a problem of anti-conformism. Anti-coordination can be optimal when adopting different roles or having complementary skills is necessary for a successful interaction or realization of a task in a team.

The rest of the paper is structured as follows. In Section 2 we deliver a brief overview of the related literature. In Section 3 we introduce the model of anonymous influence with anti-conformist agents and distinguish between two kinds of a society: pure case (containing only pure anti-conformists and pure conformists) and mixed case (including also mixed agents). The convergence analysis of the pure case, including the one with the number of agents tending to infinity, is provided in Section 4. In Section 5 we deliver the convergence analysis of the mixed case. Some concluding remarks are mentioned in Section 6. In Appendix A we explain how the present paper extends and differs from our previous work on conformism. The proof of our main results on the possible absorbing classes in the model is given in Appendix B.

2 Related literature

In this section we recall some related literature. The relation between the present paper and Förster et al. (2013) and Grabisch and Rusinowska (2013) is discussed in detail in Appendix A.

Opinion conformity has been studied widely in various fields and settings, and by using different approaches; for surveys, see e.g., Jackson (2008); Acemoglu and Ozdaglar (2011). A subset of this literature focuses on various extensions of the DeGroot model (DeGroot (1974)), see e.g., DeMarzo et al. (2003); Jackson (2008); Golub and Jackson (2010); Büchel et al. (2014, 2015); Grabisch et al. (2018), and for a survey, e.g., Golub and Sadler (2016). So far, the analysis of the anti-conformist behavior is much less common than the study devoted to the phenomenon of conformism.

Grabisch and Rusinowska (2010a,b) address the problem of measuring negative influence in a social network but only in one-step (static) settings. Büchel et al. (2015) study a dynamic model of opinion formation, where agents update their opinion by averaging over opinions of their neighbors, but might misrepresent their own opinion by conforming or counter-conforming with the neighbors. Although their model is related to DeGroot (1974), it is very different from our framework of anonymous influence with conformist and anti-conformist agents. Moreover, the authors focus on the relation between an agent's influence in the long run opinion and network centrality, and on wisdom of the society, while we determine all possible absorbing classes and conditions for their occurrence. In particular, these authors provide sufficient conditions for convergence and characterize the long-run (consensus) opinion. In a society with all agents being conformist or honest, the opinion dynamics converge. It is also shown that an agent's social influence on the group opinion is increasing in network centrality and decreasing in conformity.

Konishi et al. (1997) present a setting completely different from the present paper but related to our definition of anti-conformist agents. They consider a non-cooperative anonymous game with three assumptions on agents' preferences being independence of irrelevant choices, anonymity, and partial rivalry. The last assumption (partial rivalry) implies that the payoff of every player increases if the number of players who choose the same strategy declines. The authors examine the existence of strong Nash equilibrium in pure strategies for such a game with a finite set of players, and then with continuum of players. They show that if any of these three assumptions is violated, then Nash equilibrium may fail to exist.

There are several other works that study network formation and anti-coordination games, i.e., games where agents prefer to choose an action different from that chosen by their partners. Our approach is different from anti-coordination games, in particular, because we have an essential dissymmetry between agents. Bramoullé (2007) investigates anti-coordination games played on fixed networks. In his model, agents are embedded in a fixed network and play with each of their neighbors a symmetric anti-coordination game, like the chicken game. The author examines how social interactions interplay with the incentives to anti-coordinate, and how the social network affects choices in equilibrium. He shows that the network structure has a much stronger impact on the equilibria than in coordination games. Bramoullé et al. (2004) study anti-coordination games played on endogenous networks, where players choose partners as well as actions in games played with their partners, and prefer to choose actions other than the action chosen by their partners. The authors characterize (strict) Nash architectures and study the effects of network structure on agents' behavior. They show that both network structure and induced behavior depend crucially on the value of cost of forming links. The efficiency of different network structures are also examined. López-Pintado (2009) extends the model of Bramoullé et al. (2004) which is one-sided to a framework in which the cost of link formation is not necessarily distributed as in the one- or two-sided models, but is shared between the two players forming the link. She introduces an exogenous parameter specifying the partition of the cost and characterizes the Nash equilibria depending on the cost of link formation and the cost partition. In anti-coordination games, typically there is a conflict between efficiency and equilibrium requirements.

Kojima and Takahashi (2007) introduce the class of anti-coordination games and investigate the dynamic stability of the equilibrium in a one-population setting. They focus on the best response dynamic where agents in a large population take myopic best responses, and the perfect foresight dynamic where agents maximize total discounted payoffs from the present to the future. In particular, the authors show that for any initial distribution, the best response dynamics has a unique solution, and that no path escapes from the equilibrium in the perfect foresight dynamic once the path reaches the equilibrium.

Cao et al. (2013) consider the fashion game of pure competition and pure cooperation. It is a network game with conformists ('what is popular is fashionable') and rebels ('being different is the essence') that are located on social networks (a spatial cellular automata network and small-world networks). The authors run simulations showing that in most cases players can reach a very high level of cooperation through the best response dynamic. They define different indices (cooperation degree, average satisfaction degree, equilibrium ratio and complete ratio) and apply them to measure players' cooperation levels.

There is a branch of literature in (statistical) physics that studies the anti-conformist behavior. Galam (2004) examines the effects of contrarian choices on the dynamics of two-state opinion formation, where a contrarian is an individual who adopts the choice opposite to the choice of others; see also, e.g., Borghesi and Galam (2006) for a related work, and Galam and Vignes (2005) for the analysis of the clothes fashion market. In the absence of contrarians, the dynamics leads to a total polarization along the initial majority. The appearance of contrarians gives rise to new dynamics properties, e.g., a co-existence of both states, or a phase where agents keep shifting states but no global symmetry breaking, i.e., the appearance of a majority, takes place.

Inspired by social psychology literature, Nowak and Sznajd-Weron (2019) study some modified threshold models with a noise (nonconformity) on a complete graph, with independence or anti-conformity playing the role of the noise. They investigate the model via the mean-field approach, which gives the exact result in the case of a complete graph, and via Monte Carlo simulations. Nyczka and Sznajd-Weron (2013) examine how different types of social influence, introduced on the microscopic (individual) level, manifest on the macroscopic level, i.e. in the society.

Our setting can be applied to some existing models, like herd behavior and information cascades (Banerjee (1992); Bikhchandani et al. (1992)) which have been used to explain fads, investment patterns, etc.; see Anderson and Holt (2008) for a survey of experiments on cascade behavior. Although Bikhchandani et al. (1992) have already addressed the issue of fashion, the present model takes a different turn, since we assume no sequential choices and some agents are anti-conformists while others are conformists. In the model of herd behavior (Banerjee (1992)) agents play sequentially and wrong cascades can occur. Though it can be rational to follow the crowd, some anti-conformists may want to play a mixed-strategy: either following the crowd or not. This is particularly true under bounded rationality. Agents may not be able to know what is rational, for example because they lack information or do not have enough time or computational capacities. As a consequence, they may play according to rules of the thumb like counting how many people said 'yes' rather than computing bayesian probabilities. Chandrasekhar et al. (2015) show in a lab experiment that people tend to behave according to the DeGroot model rather than to Bayesian updating; see also Celen and Kariv (2004). This is also consistent with Anderson and Holt (1997) who show that counting is the most salient bias to explain departure from Bayesian updating.

3 The model

3.1 Description of the model

We consider a society N with $|N| = n$ agents who have to choose between two actions (e.g., in a yes/no decision, between two technologies, between being active or inactive, etc.). For the sake of uniformity in explanations, we assume throughout the paper that agents are faced with a yes/no decision on some issue. Initially, every agent has an opinion on that issue, but by knowing the opinion of the others or due to some social interaction, in each period the agent may modify his opinion due to mutual influence. In other words, there is an evolution in time of the opinion of the agents, which may or may not stop at some stable state of the society.

We define the *state* of the society at a given time as the set $S \subseteq N$ of agents whose opinion is 'yes'. As usual, the cardinality of a set is denoted by the corresponding lower case, e.g., $s = |S|$. Our fundamental assumption is that the evolution of the state is ruled by a homogeneous Markov chain, that is, the state evolves at discrete time steps, the new opinions depend only on the opinions among the society in the last period, and the transition matrix giving the probability of all possible transitions from one state to another is constant over time.

We study the opinion formation process of the society, where the updating of opinion relies only on how many agents said 'yes' and how many said 'no' in the previous period, disregarding who said 'yes' and who said 'no'. For this reason we call this opinion formation process *anonymous*. Both conformist and anti-conformist behaviors are allowed, i.e., agents can revise their opinions by following the trend as well as in a way contrary to the trend.

We now formalize the previous ideas. An *(anonymous) aggregation rule* describes for a given agent how the opinions of the other agents are aggregated in order to determine the updated opinion of this agent. Specifically, it is a mapping p from $\{0, 1, \ldots, n\}$ to $[0, 1]$, assigning to any $0 \leq s \leq n$, representing the number of agents saying 'yes', a quantity $p(s)$ which is the probability of saying 'yes' at next time step (and consequently, $1 - p(s)$ is the probability of saying 'no'). We focus on three classes of aggregation rules:

$$A^c = \{p \mid \text{if } s' > s \text{ then } p(s') \geq p(s),\ p(0) = 0 \text{ and } p(n) = 1\} \quad \textit{(pure conformism)} \quad (1)$$

$$A^a = \{p \mid \text{if } s' > s \text{ then } p(s') \leq p(s),\ p(0) = 1 \text{ and } p(n) = 0\} \quad \textit{(pure anti-conformism)} \quad (2)$$

and *mixed aggregation rules*:

$$A^m = \{p \mid p = \alpha q^c + (1 - \alpha) q^a \text{ with } \alpha \in \,]0, 1[\,, q^c \in A^c, q^a \in A^a\}. \quad (3)$$

Note that $p \in A^m$ is not necessarily monotone, and that $p(0) > 0$ and $p(n) < 1$.

Let p_i be agent i's (anonymous) aggregation rule. Supposing that the update of opinion is done independently across the agents, the probability of transition from a state S to a state T is

$$\lambda_{S,T} = \prod_{i \in T} p_i(s) \prod_{i \notin T} (1 - p_i(s)). \tag{4}$$

Observe that, as the aggregation rule is anonymous, the probability of transition to T is the same for all states S having the same size s.

We distinguish between three types of agents. We say that agent i is *purely conformist* if $p_i \in A^c$, *purely anti-conformist* if $p_i \in A^a$, and is a *mixed agent* if $p_i \in A^m$. The society of agents is partitioned into

$$N = N^c \cup N^a \cup N^m$$

where N^c is the set of purely conformist agents, N^a the set of purely anti-conformist agents, and N^m is the set of mixed agents. When $N^m = \emptyset$, we call it the *pure case.*

We make the following simplifying assumption: we suppose that all purely conformist agents, although having possibly different aggregation rules, have in common the same minimum number of 'yes' for them to say 'yes' with positive chance, and the same minimum number of 'yes' for them to say 'yes' with probability 1. Formally, we require that for each $i \neq j$ in N^c,

$$\min\{s \mid p_i(s) > 0\} = \min\{s \mid p_j(s) > 0\} =: l^c + 1 \tag{5}$$

$$\min\{s \mid p_i(s) = 1\} = \min\{s \mid p_j(s) = 1\} =: n - r^c. \tag{6}$$

l^c can be interpreted as the (maximum) number of 'yes' for which no effect on the probability of saying 'yes' arises, and is therefore called the *firing threshold.* On the other hand, r^c is the (maximum) number of 'no' for which no effect on this probability is visible (see Figure 1). In other words, $n - r^c$ is the *saturation threshold* beyond which there is no more change of opinion for the agent. This assumption permits to conduct a precise analysis of convergence while remaining reasonable (it can be considered as a mean field approximation).

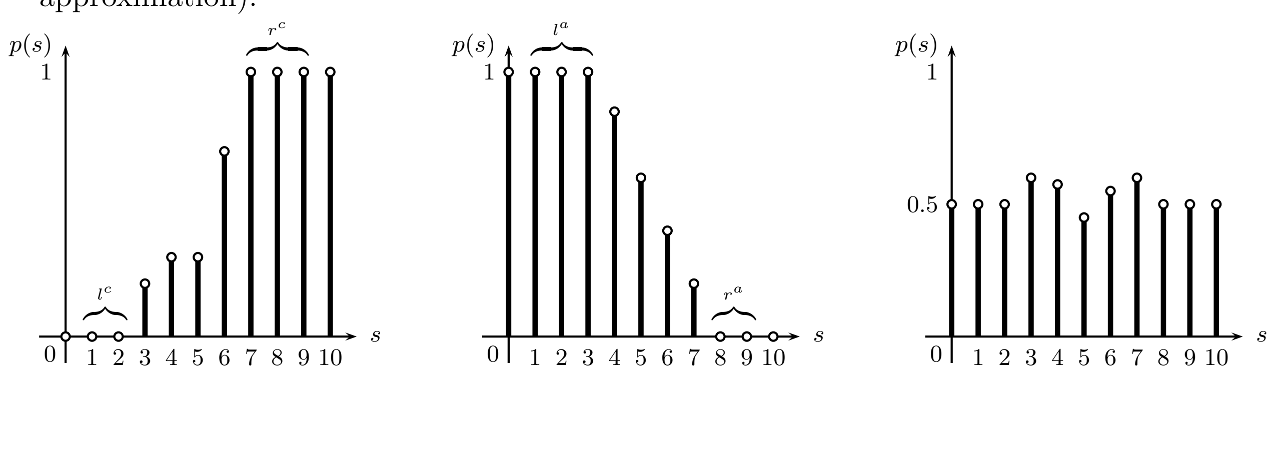

Fig. 1. Typical aggregation rules for conformist agents (left), anti-conformists (center), and mixed agents (right) with $n = 10$ agents. The latter is obtained by mixing the two first ones with $\alpha = 0.5$

Similarly, we assume that all purely anti-conformist agents share the same minimum number of 'yes' for them to say 'no' with positive chance and the same minimum number of 'yes' for them to say 'no' with probability 1, i.e., for each $i \neq j$ in N^a,

$$\min\{s \mid p_i(s) < 1\} = \min\{s \mid p_j(s) < 1\} =: l^a + 1 \tag{7}$$

$$\min\{s \mid p_i(s) = 0\} = \min\{s \mid p_j(s) = 0\} =: n - r^a. \tag{8}$$

As above, l^a can be interpreted as the firing threshold, while $n - r^a$ is the saturation threshold for anti-conformists (see Figure 1).

These assumptions carry over mixed agents: any mixed agent i has an aggregation rule p_i which is a convex combination with coefficient α_i of a conformist aggregation rule (with fixed l^c, r^c) and an anti-conformist aggregation rule (with fixed l^a, r^a). We allow, however, that two mixed agents have different convex combination coefficients α_i. Hence, a mixed agent i can be seen as an agent who does not have a fixed behavior, but who is conformist with probability α_i and anti-conformist with probability $1 - \alpha_i$. On Figure 1, we can see that a mixed agent with $\alpha_i = 1/2$ has a very indecisive behavior.

Based on the above assumptions, in this paper we fully characterize all possible absorbing classes based on size comparison among l^c, r^c, l^a, r^a and the number of conformist and anti-conformist agents in the society.

We summarize the main assumptions underlying our model:

- Every agent has a fixed type, which can be conformist, anti-conformist, or mixed.
- The update of opinions is made by all agents at each step.
- The update of opinions is made on the basis of the *number* of agents having a given opinion, and not on the basis of their name. In particular, their type is not taken into account (we may even consider that agents ignore the type of the other agents).
- Agents of a given type have the same firing and saturation thresholds, although their aggregation rules may differ.

3.2 An illustrative example

As said in the introduction, anti-conformism is the main mechanism underlying fashion phenomena. The example here is in this vein and consider as the two possible options (yes/no) the choice between two products A and B (e.g., Windows vs. Linux, Iphone vs. smartphone, electric car vs. gasoline car, etc.). In the sequel, we refer to it as the *technology example.*

We model the utility of agent i for the two products in the following way:

$$u_i(A) = \alpha_i e^+(s) - (1 - \alpha_i)e^-(s) - \beta_i c_A$$
$$u_i(B) = \alpha_i e^+(n - s) - (1 - \alpha_i)e^-(n - s) - \beta_i c_B,$$

where s is the number of agents having adopted product A, c_A, c_B are the respective costs of products A and B, and $\alpha_i \in [0, 1]$, $\beta_i \geq 0$. The function e^+ represents a positive externality expressing the fact that the more adopters of a product, the more available supporting services exist for that product, and the more help and assistance the agent can expect from his/her friends. Therefore, e^+ is an increasing function, with $e^+(0) = 0$ and we may fix $e^+(n) = 1$. On the other hand, e^- is a negative externality function, expressing the fact that when too many people use the same product, a phenomenon of "déjà vu" is getting present and a desire of being different and original arises. As e^- is counted negatively in the utility, we also assume that e^- is increasing with $e^-(0) = 0$ and $e^-(n) = 1$. The constant α_i is then interpreted as a trade-off between "comfort of use of the product" vs. "desire to be original". The constant β_i quantifies the importance of the price of the products in the utility. If $\beta_i = 0$, the price is not taken into account, while it becomes decisive when β_i increases.

Intuitively, e^+ should be a concave function (the increase in service tends to diminish as the number of adopters grows), and e^- a convex function (the desire to be original is very low with a low number of adopters, but then increases fast). To give more insight while keeping the computation simple, let us take a linear function for e^+ and a quadratic one for e^-:

$$e^+(s) = \frac{s}{n}; \quad e^-(s) = \left(\frac{s}{n}\right)^2,$$

and consider that product A, being currently more fashionable, is more expensive: $c_A > c_B$. The difference of utilities between the two products reads:

$$u_i(A) - u_i(B) = 2(2\alpha_i - 1)\frac{s}{n} - \beta_i(c_A - c_B) - 2\alpha_i + 1,$$

and the equality $u_i(A) = u_i(B)$ is attained for a number s_0 of adopters of product A given by

$$s_0 = \frac{n}{2}\left(\frac{\beta_i(c_A - c_B)}{2\alpha_i - 1} + 1\right), \text{ with the condition } s_0 \in [0, n].$$

Observe that $s_0 = n/2$ when $\beta_i = 0$, while there is no solution if $\beta_i > \frac{2\alpha_i - 1}{c_A - c_B}$ because this would imply $s_0 > n$. This value of β_i is therefore the threshold above which only price matters in the choice among the two products. Assuming β_i below the threshold, we conclude that, using a best response principle:

- When $\alpha_i > 1/2$, product A is chosen if $s > s_0$, and product B is chosen if $s < s_0$. This can be interpreted as conformism.
- When $\alpha_i < 1/2$, product A is chosen if $s < s_0$, otherwise product B is chosen. Such an attitude represents anti-conformism.

This is a particular case of our model: the aggregation rule p_i of agent i is then a threshold function, i.e., $l = n - r - 1$, increasing if $\alpha_i > 1/2$, decreasing if $\alpha_i < 1/2$, otherwise with $\alpha_i = 1/2$, p_i is the constant function $1/2$, i.e., i is of the mixed type.

Best response is an idealization of the behavior of a real agent, who will rather have a zone of uncertainty in the decision, due to hesitation or imperfect information. If the difference of utilities remains smaller than a quantity Δ, we may assume that the probability of choosing A is proportional to this perceived difference of utilities. Doing so, we recover aggregation rules with a linear part between l and $n - r$, either increasing ($\alpha_i > 1/2$) or decreasing ($\alpha_i < 1/2$) (see Figure 2).

In this example, the quantities l and $n - r$ comes directly from the uncertainty threshold Δ.

3.3 Basic properties of the transition matrix

We recall that (see, e.g., Kemeny and Snell (1976); Seneta (2006)) for a Markov chain with set of states E and transition matrix Λ and its associated digraph Γ, a *class* is a

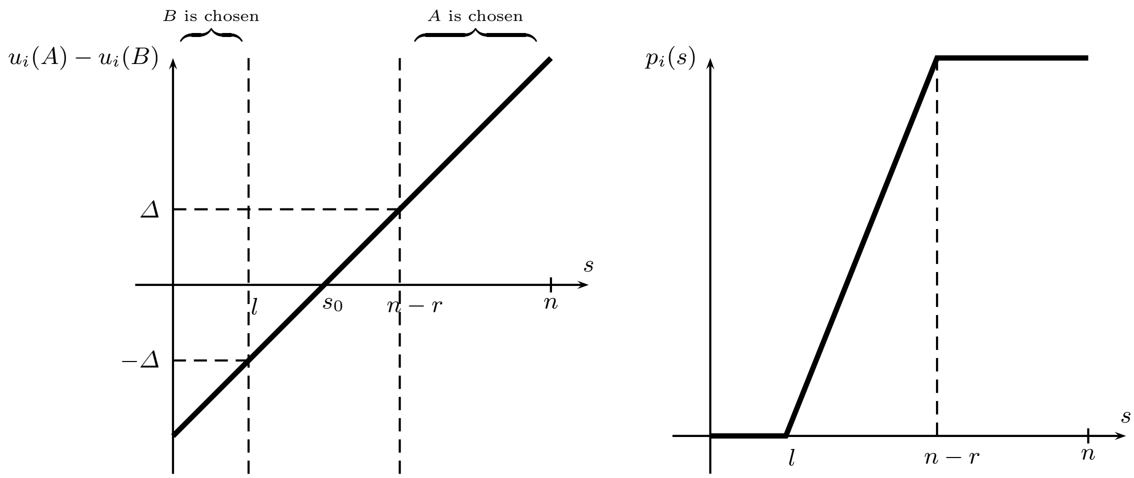

Fig. 2. Choice among two products A and B: difference of utility functions with $\alpha_i > 1/2$ (left) and the resulting aggregation rule p_i (right)

subset C of states such that for all states $e, f \in C$, there is a path in Γ from e to f, and C is maximal w.r.t. inclusion for this property. A class is *absorbing* if for every $e \in C$ there is no arc in Γ from e to a state outside C. An absorbing class C is *periodic of period* k if it can be partitioned in blocks $C_1, \ldots, C_k$ such that for $i = 1, \ldots, k$, every outgoing arc of every state $e \in C_i$ goes to some state in C_{i+1}, with the convention $k+1 = 1$. When each $C_1, \ldots, C_k$ reduces to a single state, one may speak of *cycle of length* k, by analogy with graph theory.

In our framework, states are subsets of agents and therefore classes are collections of sets, which we denote by calligraphic letters, like $\mathcal{C}, \mathcal{B}$, etc. By definition, an absorbing class indicates the final state of opinion of the society. For instance, if an absorbing class reduces to a single state S, it means that in the long run, the society is dichotomous (unless $S = N$ or $S = \emptyset$, in which case consensus is reached): there is a set of agents S who say 'yes' forever, while the other ones say 'no' forever. Otherwise, there are endless transitions with some probability from one set $S \in \mathcal{C}$ to another one $S' \in \mathcal{C}$.

We now study the properties of the transition matrix Λ, with entries $\lambda_{S,T}$, $S, T \in 2^N$, where $\lambda_{S,T}$ is given by (4). Our aim is to find under which conditions one has a possible transition from S to T, i.e., $\lambda_{S,T} > 0$. From (4), we have:

$$\lambda_{S,T} > 0 \Leftrightarrow [p_i(s) > 0 \quad \forall i \in T] \quad \& \quad [p_i(s) < 1 \quad \forall i \notin T].$$

The pure case. We start with the pure case, i.e., $N^m = \emptyset$. We first observe that $p_i(0) = 1$ if $i \in N^a$ and 0 otherwise, and $p_i(n) = 1$ if $i \in N^c$ and 0 otherwise. Therefore, we have in any case the sure transitions

$$\lambda_{\emptyset, N^a} = 1, \quad \lambda_{N, N^c} = 1.$$

Moreover, we get immediately from (5) to (8) for any $s \neq 0, n$,

$$(i \in N^c) \qquad p_i(s) > 0 \qquad \Leftrightarrow \qquad s > l^c \tag{9}$$

$$p_i(s) < 1 \qquad \Leftrightarrow \qquad s < n - r^c \tag{10}$$

$$(i \in N^a) \qquad p_i(s) > 0 \qquad \Leftrightarrow \qquad s < n - r^a \tag{11}$$

$$p_i(s) < 1 \qquad \Leftrightarrow \qquad s > l^a. \tag{12}$$

From these conditions, the transition from S to T depends only on the cardinality s of S and its position in the curves $p_i(s)$ for conformists and anti-conformists:

- If $s \leq l^a$ and $s \leq l^c$, then $p_i(s) = 1$ if $i \in N^a$ and $p_i(s) = 0$ if $i \in N^c$. Therefore $T = N^a$ with certainty. Similarly, if $s \geq n - r^a$ and $s \geq n - r^c$, we have with certainty $T = N^c$.
- If $s \leq l^a$ but $s > l^c$, we still have $p_i(s) = 1$ for all anti-conformists agents, but now $p_i(s) > 0$ for the conformists, and we have $p_i(s) = 1$ if $s \geq n - r^c$. Then T contains for sure N^a and may contain also a subset of N^c (N^c with certainty if $s \geq n - r^c$). In summary, any T such that $N^a \subseteq T \subseteq N$ is possible, which we denote simply by $T \in [N^a, N]$, using the interval notation.
- A similar analysis can be done for the other cases as well. Note that in the case $s \in]l^a, n - r^a[\cap]l^c, n - r^c[$, i.e., when s falls into the non-flat zone of the curves p_i for conformists and anti-conformists, T is the union of any subset of N^a and any subset of N^c, therefore it can be any subset of N.

Table 1 summarizes all possible transitions.

	$0 \leq s \leq l^c$	$l^c < s < n - r^c$	$n - r^c \leq s \leq n$
$0 \leq s \leq l^a$	N^a	$T \in [N^a, N]$	N
$l^a < s < n - r^a$	$T \in [\emptyset, N^a]$	$T \in 2^N$	$T \in [N^c, N]$
$n - r^a \leq s \leq n$	$\emptyset$	$T \in [\emptyset, N^c]$	N^c

Table 1. Possible transitions from $S \in 2^N$ in the pure case

We end this section by some considerations of symmetry. To this end, we introduce $Z = (l^c, r^c, l^a, r^a)$ and write p_i^Z to emphasize the dependency of the aggregation rule p_i on these parameters (and similarly for $\lambda_{S,T}$). Equations (9) to (12) show some interesting symmetries, among which the symmetry between l and r. Z being given, we introduce the reversal of Z, $Z^\partial := (r^c, l^c, r^a, l^a)$ which amounts to interchanging the l and r parameters. We observe that

$$p_i^Z(s) > 0 \text{ for } i \in N^c \quad \Leftrightarrow \quad p_i^{Z^\partial}(n-s) < 1 \text{ for } i \in N^c$$
$$p_i^Z(s) < 1 \text{ for } i \in N^c \quad \Leftrightarrow \quad p_i^{Z^\partial}(n-s) > 0 \text{ for } i \in N^c$$

(idem with N^a, N^c exchanged) The following lemma shows that the reversal of Z amounts to taking set complement.

Lemma 1 (symmetry principle) *Let $S, T \in 2^N$ and $Z = (l^c, r^c, l^a, r^a)$. The following equivalence holds:*

$$\lambda_{S,T}^Z > 0 \Leftrightarrow \lambda_{N\setminus S, N\setminus T}^{Z^\partial} > 0.$$

Proof. Letting $\lambda_{S,T}^Z > 0$ means that for every $i \in N \setminus T$, $0 \leq p_i^Z(s) < 1$, and for every $i \in T$, $0 < p_i^Z(s) \leq 1$. Using the equivalences in (ii), we find that for every $i \in N \setminus T$, $0 < p_i^{Z^\partial}(n-s) \leq 1$ and for every $i \in T$, $0 \leq p_i^{Z^\partial}(n-s) < 1$. But this means that $\lambda_{N\setminus S, N\setminus T}^{Z^\partial} > 0$. □

The mixed case. Next we consider the mixed case, assuming $p_i = \alpha_i q^c + (1-\alpha_i)q^a$ with $q^c \in A^c$, $q^a \in A^a$ and fixed l^c, r^c, l^a, r^a. We can easily derive the conditions for $p_i(s)$ to be 0 or 1, using (9) to (12). For any $0 < s < n$, we obtain

$$(i \in N^m) \qquad p_i(s) = 0 \quad \Leftrightarrow \quad n - r^a \leq s \leq l^c \tag{13}$$
$$p_i(s) = 1 \quad \Leftrightarrow \quad n - r^c \leq s \leq l^a, \tag{14}$$

and $0 < p_i(s) < 1$ for all other cases. Observe that having $p_i(s) = 0$ or 1 is a very special situation for the values of l and r, and having both is impossible. Moreover, the domain where $p_i(s) = 0$ (or 1) is in the interior of $\{0, \dots, n\}$, meaning that p_i cannot be zero (or 1) on the boundaries $s = 0$, $s = n$. This depicts a rather strange behavior for mixed agents. Finally, if $S = \emptyset$, then $\lambda_{S,T} > 0$ for every $T \in [N^a, N^a \cup N^m]$, and if $S = N$, then $\lambda_{S,T} > 0$ for every $T \in [N^c, N^c \cup N^m]$. Table 2 presents all possible transitions from $S \in 2^N$ in the mixed case. Compared to Table 1, one can see that N^a and N^c are

	$0 \leq s \leq l^c$	$l^c < s < n - r^c$	$n - r^c \leq s \leq n$
$0 \leq s \leq l^a$	$T \in [N^a, N^a \cup N^m]$	$T \in [N^a, N]$	N
$l^a < s < n - r^a$	$T \in [\emptyset, N^a \cup N^m]$	$T \in 2^N$	$T \in [N^c, N]$
$n - r^a \leq s \leq n$	$\emptyset$	$T \in [\emptyset, N^c \cup N^m]$	$T \in [N^c, N^c \cup N^m]$

Table 2. Possible transitions from $S \in 2^N$ in the mixed case

"blurred" by the adjonction of N^m, i.e., thery are replaced by the intervals $[N^a, N^a \cup N^m]$ and $[N^c, N^c \cup N^m]$ respectively. Also, N^m does not appear alone. This shows the indecisive behavior of mixed agents.

4 Convergence in the pure case

This section is devoted to the study of absorbing classes in the model introduced in Section 3.1, supposing for the moment that no mixed agent exists ($N^m = \emptyset$). Unlike the case of a model with only conformist agents, their study appears to be extremely complex.

We start by the simple cases where there is no anti-conformist or no conformist agents. Then we establish the main result (Theorem 1) giving all possible absorbing classes in the pure case. These results are valid without any restriction on the number of agents n, nor on any parameter describing the society. They give an exact description of all possible absorbing classes (there are 20 classes), with their conditions of existence. In order to make the results more readable, we then consider particular cases of parameters, as well as the case where the number of agents tends to infinity, proposing some rewriting of the parameters describing the society. Based on that, we provide a clear analysis of the convergence in three typical types of society.

Throughout, we will use the following notation: we write $S \to T$ if a transition from S to T is possible, i.e., $\lambda_{S,T} > 0$, and $S \xrightarrow{1} T$ if $\lambda_{S,T} = 1$ (sure transition). We extend the latter notation to collections of sets: letting $\mathcal{S}, \mathcal{T}$ be two nonempty collections of sets in 2^N, we write

$$\mathcal{S} \xrightarrow{1} \mathcal{T} \quad \Leftrightarrow \quad \forall T \in \mathcal{T}, \exists S \in \mathcal{S} \text{ s.t. } \lambda_{S,T} > 0 \text{ and } \forall S \in \mathcal{S}, \forall T \notin \mathcal{T}, \lambda_{S,T} = 0.$$

This means that every set of $\mathcal{T}$ can be attained from some set in $\mathcal{S}$, and no set in $\mathcal{S}$ can lead to a set outside $\mathcal{T}$.

4.1 Cases with no anti-conformists ($N^a = N^m = \emptyset$) or no conformists ($N^c = N^m = \emptyset$)

In Appendix A we recall the anonymous model of conformity and elaborate on the relation between that model and the present framework. In particular, for the anonymous model of conformism, any state could be absorbing, and other absorbing classes are intervals of sets. However, the case with $N^a = \emptyset$ is more specific as every agent has the same l, r, and there are very few possibilities left, as shown below. To see this, it suffices to rewrite Table 1 which becomes:

$0 \leq s \leq l^c$	$l^c < s < n - r^c$	$n - r^c \leq s \leq n$
$\emptyset$	2^N	N

It follows that only $\emptyset, N$ are absorbing states and neither 2^N nor any of its subcollections is an absorbing class, because a transition from any $S \neq \emptyset, N$ to $\emptyset$ or N is possible.

Both $\emptyset, N$ are possible, regardless of the values of r^c, l^c. This means that the society converges to a consensus, which depends on the initial state S. If $s \leq l^c$, there is convergence to 'no' in one step, and if $s \geq n - r^c$, there is convergence to 'yes' in one step.

The analysis of the extreme case with $N^c = \emptyset$ is also done by rewriting Table 1 which becomes now:

$0 \leq s \leq l^a$	$l^a < s < n - r^a$	$n - r^a \leq s \leq n$
N	2^N	$\emptyset$

Then there is only one absorbing class which is the cycle $\emptyset \xrightarrow{1} N \xrightarrow{1} \emptyset$. We see that, without much surprise, a society of anti-conformist agents can never reach a stable state.

4.2 The general pure case ($N^m = \emptyset$)

Since $N^m = \emptyset$, we have $n^a = n - n^c$, where $n^a = |N^a|$ and $n^c = |N^c|$. Hence, the model is entirely determined by $l^c, r^c, l^a, r^a, n^c, n$. We recall that these parameters must satisfy the following constraints:

$$0 \leq l^a + r^a < n$$
$$0 \leq l^c + r^c < n$$
$$0 < n^c < n.$$

Below is the main result of the whole paper.

Theorem 1 *Assume that $N^m = \emptyset$, $N^a \neq \emptyset$ and $N^c \neq \emptyset$. There are twenty possible absorbing classes, grouped in the following categories*[1]*:*

(i) **Polarization:**

(1) N^a if and only if $n^c \geq (n - l^c) \vee (n - l^a)$;

[1] We use the standard notation $\vee$ and $\wedge$ to denote max and min, respectively.

(2) N^c *if and only if* $n^c \geq (n - r^c) \vee (n - r^a)$*;*

(ii) **Cycles:**

(3) $N^a \xrightarrow{1} \emptyset \xrightarrow{1} N^a$ *if and only if* $n - l^c \leq n^c \leq r^a$*;*

(4) $N^c \xrightarrow{1} N \xrightarrow{1} N^c$ *if and only if* $n - r^c \leq n^c \leq l^a$*;*

(5) $N^a \xrightarrow{1} N^c \xrightarrow{1} N^a$ *if and only if* $n^c \leq l^c \wedge l^a \wedge r^c \wedge r^a$*;*

(6) $\emptyset \xrightarrow{1} N^a \xrightarrow{1} N^c \xrightarrow{1} \emptyset$ *if and only if* $n^c \leq r^c \wedge r^a \wedge l^c$ *and* $n^c \geq n - r^a$*;*

(7) $N^a \xrightarrow{1} N \xrightarrow{1} N^c \xrightarrow{1} N^a$ *if and only if* $n^c \leq l^c \wedge l^a \wedge r^c$ *and* $n^c \geq n - l^a$*;*

(iii) **Fuzzy cycles:**

(8) $N^a \xrightarrow{1} [\emptyset, N^c] \xrightarrow{1} N^a$ *if and only if* $n^c \leq l^c \wedge l^a \wedge r^a$ *and* $r^c < n^c < n - l^c$*;*

(9) $N^c \xrightarrow{1} [N^a, N] \xrightarrow{1} N^c$ *if and only if* $n^c \leq r^c \wedge r^a \wedge l^a$ *and* $l^c < n^c < n - r^c$*;*

(10) $[\emptyset, N^c] \xrightarrow{1} [N^a, N] \xrightarrow{1} [\emptyset, N^c]$ *if and only if* $r^c \vee l^c < n^c \leq r^a \wedge l^a \wedge (n - l^c - 1) \wedge (n - r^c - 1)$*;*

(iv) **Fuzzy polarization:**

(11) $[\emptyset, N^a]$ *if and only if* $(n - l^c) \vee (r^a + 1) \leq n^c < n - l^a$*;*

(12) $[N^c, N]$ *if and only if* $(n - r^c) \vee (l^a + 1) \leq n^c < n - r^a$*;*

(v) **Chaotic polarization:**

(13) $[\emptyset, N^a] \cup [\emptyset, N^c]$ *if and only if* $l^c \geq n - r^a$ *and* $n^c \in \big( ]r^c, n - l^c[\cap]l^a, n - r^c[\big) \cup \big(\big( ]l^a, n - r^a[\cup]l^c, n - r^c[\big) \cap]0, r^c] \big)$*;*

(14) $[N^a, N] \cup [N^c, N]$ *if and only if* $l^a \geq n - r^c$ *and* $n^c \in \big( ]l^c, n - r^c[\cap]r^a, n - l^c[\big) \cup \big(\big( ]r^a, n - l^a[\cup]r^c, n - l^c[\big) \cap]0, l^c] \big)$*;*

(15) $[\emptyset, N^a] \cup \{N^c\}$ *if and only if* $l^c + r^c = n - 1$, $r^a \geq r^c$, $l^c > l^a$ *and* $l^a < n^c < (n - r^a) \wedge (n - l^c)$*;*

(16) $[N^c, N] \cup \{N^a\}$ *if and only if* $l^c + r^c = n - 1$, $l^a \geq l^c$, $r^c > r^a$ *and* $r^a < n^c < (n - l^a) \wedge (n - r^c)$*;*

(17) $[\emptyset, N^c] \cup \{N^a\}$ *if and only if* $l^a + r^a = n - 1$, $l^c \geq l^a$, $n^c < n - r^c$ *and* $n^c \in]r^c, n - l^c[\cup]l^c, r^c]$*;*

(18) $[N^a, N] \cup \{N^c\}$ *if and only if* $l^a + r^a = n - 1$, $r^c \geq r^a$, $n^c < n - l^c$ *and* $n^c \in]l^c, n - r^c[\cup]r^c, l^c]$.

(19) $[\emptyset, N^a] \cup [N^c, N]$ *if and only if* $l^c + r^c = n - 1$ *and* $l^a \vee r^a < n^c \leq l^c \wedge r^c$*;*

(vi) **Chaos:**

(20) 2^N *otherwise.*

The proof of Theorem 1 can be found in Appendix B.

We begin by explaining the different categories of classes introduced in the theorem:

- **Polarization:** the society of agents is divided in two groups, one with opinion ‘yes’, the other with opinion ‘no’. As it can be seen, the two possible groups are N^a and N^c, and therefore consensus (N or $\emptyset$) never occurs.
- **Cycles:** sequences of states made of the infinite repetition of a pattern. The states in the pattern are limited to the two consensus states $N, \emptyset$ and the two groups N^a, N^c.
- **Fuzzy cycles:** the pattern contains states but also intervals of states. This means that there is no exact repetition of the same pattern, but at each repetition a state is picked at random in the interval. For example, in case (8), at some step in the long run all anti-conformists say ‘yes’, in the next time step they all say ‘no’, but a fraction of conformists says ‘yes’, then in the next step again all anti-conformists say ‘yes’, etc.

- **Fuzzy polarization:** the polarization is defined by an interval, which means that at each time step, a state is picked at random in the interval, representing the set of agents with opinion 'yes'. For example in case (11), in the long run all conformists say 'no', while the anti-conformists have a chaotic behavior, in the sense that at each time step a (different) fraction of them say 'yes', while the others say 'no'.
- **Chaotic polarization:** it is similar to the previous case but more complex as several intervals are involved.
- **Chaos[2]:** at each time step a state is picked at random among all possible states. Therefore, at any time step in the long run, the set of agents with opinion 'yes' can be any set of agents. In terms of Markov chain theory, it means that the transition matrix is irreducible.

We make several observations before addressing important particular cases, which will permit a deeper analysis.

- First of all, these results are given without any simplifying assumption and are valid whatever the values of the parameters describing the society are. Once they are known, Theorem 1 gives the possible absorbing classes.
- Cases (1) to (20) are not exclusive. This can immediately be seen by considering cases (1) and (2). Indeed, for a given society, which is represented by the set of parameters n, n^c, l^a, r^a, l^c and r^c, under some conditions both cases (1) and (2) are possible, and therefore two different absorbing classes might occur, N^a and N^c. However, the process will end up in only one of them, with some probability.
- A necessary condition for the existence of cycles and periodic classes (5) to (10) is that there is strictly more anticonformists than conformists.
- Lastly, the analysis for conformists and anti-conformists is not symmetric. For example, $[\emptyset, N^a]$ is a possible absorbing class but not $[\emptyset, N^c]$. However, while there is no symmetry between "a" and "c" in this framework, there exists symmetry between S and $N \setminus S$ as pointed out in Lemma 1. This can be readily seen in the classes obtained, as they can be paired by taking complement of the sets: (1) and (2), (3) and (4), (6) and (7), (8) and (9), (11) and (12), (13) and (14), (15) and (16), (17) and (18). The remaining classes are complement of themselves: (5), (10), (19) and (20).

4.3 Analysis of the pure case for extreme values of l^a, l^c, r^a, r^c

There are two such extreme cases: when l^a, l^c, r^a, r^c are zero, and when they sum up to $n - 1$, making the aggregation rule a threshold function.

We begin by supposing $l^a = l^c = r^a = r^c = 0$. Then the aggregation rule p_i of each agent has a firing threshold equal to zero and no saturation effect. Referring to the technology example (Section 3.2), this amounts to have a large Δ (uncertainty zone), i.e., agents are very unsure about their choice. Under this assumption, Table 1 reduces to its central box 2^N, which means that *among all the 20 possible absorbing classes, only the case of chaos (class (20)) remains.* As a conclusion, special phenomena like polarization, cycles and their more or less fuzzy versions are only due to the presence of thresholds in the aggregation rule.

[2] The term "chaos" is evidently taken in the usual language sense and means "erratic" or "unpredictable". It has nothing to do with chaos in physics.

We suppose now that either $l^a + r^a = n - 1$ or $l^c + r^c = n - 1$, or both, which means that aggregation rules are threshold functions. Again referring to the technology example, this means that agents apply the best response strategy ($\Delta = 0$).

Suppose that anti-conformists have a threshold function. Looking at Table 1, we see that the middle row ($l^a < s < n - r^a$) disappears because $l^a = n - r^a - 1$. Therefore, any class involving $[\emptyset, N^a]$, 2^N or $[N^c, N]$ becomes impossible. We conclude (and this can be double-checked with the help of Theorem 1) that classes from (11) to (16), (19) and (20) are impossible. On the other hand, by Theorem 1, we see that classes (17) and (18) are possible and exist only in this case.

Suppose now that the conformists have a threshold function. By Table 1, we see that the middle column disappears, so that any class involving $[N^a, N]$, 2^N or $[\emptyset, N^c]$ becomes impossible. Therefore, classes (8), (9), (10), (13), (14), (17), (18) and (20) disappear, and by Theorem 1, we see that (16) and (19) become possible, and exist only in this case.

Finally, suppose that all agents have a threshold function. Then there remain only the following classes: the two polarization cases (1) and (2), and the cycles (3) to (7). As a conclusion, considering threshold functions make the analysis much simpler, and any fuzzy or chaotic behavior disappears.

4.4 Analysis of the pure case when n tends to infinity

We make the assumption that the number of agents is very large and approximate this situation by making n tend to infinity. For notational convenience, each of the previous parameters n^a, l^a, r^a, l^c, r^c is now divided by n, keeping (with some abuse) the same notation for these parameters, so that now these are real numbers in $[0, 1]$. It follows that

$$\begin{aligned} n^c &= 1 - n^a \\ l^a + r^a &< 1 \\ l^c + r^c &< 1. \end{aligned}$$

Note that the particular cases $l^a + r^a = n - 1$, $l^c + r^c = n - 1$ appearing in classes (15) to (19) become limit cases $l^a + r^a \to 1$, $l^c + r^c \to 1$, making the latter classes appearing only as limit cases.

We study in details in the rest of this section some specific situations. Observe that from the results of Section 4.1, the "model" is not continuous in n^a at 0, in the sense that if $n^a > 0$, we have proved that 2^N can be an absorbing class (it suffices to take $l^c = r^c = l^a = r^a = 0$). Similarly, it is not continuous in n^a at 1.

Given an aggregation rule with l, r specified, we introduce the new parameter

$$\gamma = \frac{1}{1 - r - l},$$

which is the average slope of the function giving the probability p to say 'yes' given the number of agents saying 'yes'. l indicates how much an agent needs agents saying 'yes' before starting to change his mind (firing threshold), while γ measures the *reactiveness* once he has started to change his mind. Note that $\gamma \in \left[\frac{1}{1-l}, \infty\right[$. On the other hand, as already mentioned, $1 - r$ is the saturation threshold.

The pair of parameters (l, γ) may be easier to interpret than the pair (l, r), and so we will use it in the sequel whenever convenient. We have $r = 1 - l - \frac{1}{\gamma}$.

In what follows, we make a detailed analysis and interpretation of the convergence in several typical situations. We assume throughout $N^m = \emptyset$.

Situation 1: $l^a = l^c = l$ and $r^a = r^c = r$. This depicts a society where all agents, conformists or anti-conformists, have the same influenceability characteristics. Among the initially 15 possible absorbing classes, only the following ones remain possible:

- N^a if and only if $n^a \leq l$
- N^c if and only if $n^a \leq r$
- cycle $N^a \xrightarrow{1} N^c \xrightarrow{1} N^a$ if and only if $n^a \geq 1 - l$ and $n^a \geq 1 - r$
- 2^N otherwise.

Let us translate this with the pair (l, γ). We obtain:

(i) N^a if and only if $n^a \leq l$

(ii) N^c if and only if $n^a \leq 1 - l - \frac{1}{\gamma}$

(iii) cycle $N^a \xrightarrow{1} N^c \xrightarrow{1} N^a$ if and only if $n^a \geq 1 - l$ and $n^a \geq \frac{1}{\gamma} + l$

(iv) 2^N otherwise.

We make a "phase diagram" with the three parameters n^a, l, γ showing the possible absorbing classes, keeping in mind that $\gamma \geq \frac{1}{1-l}$.

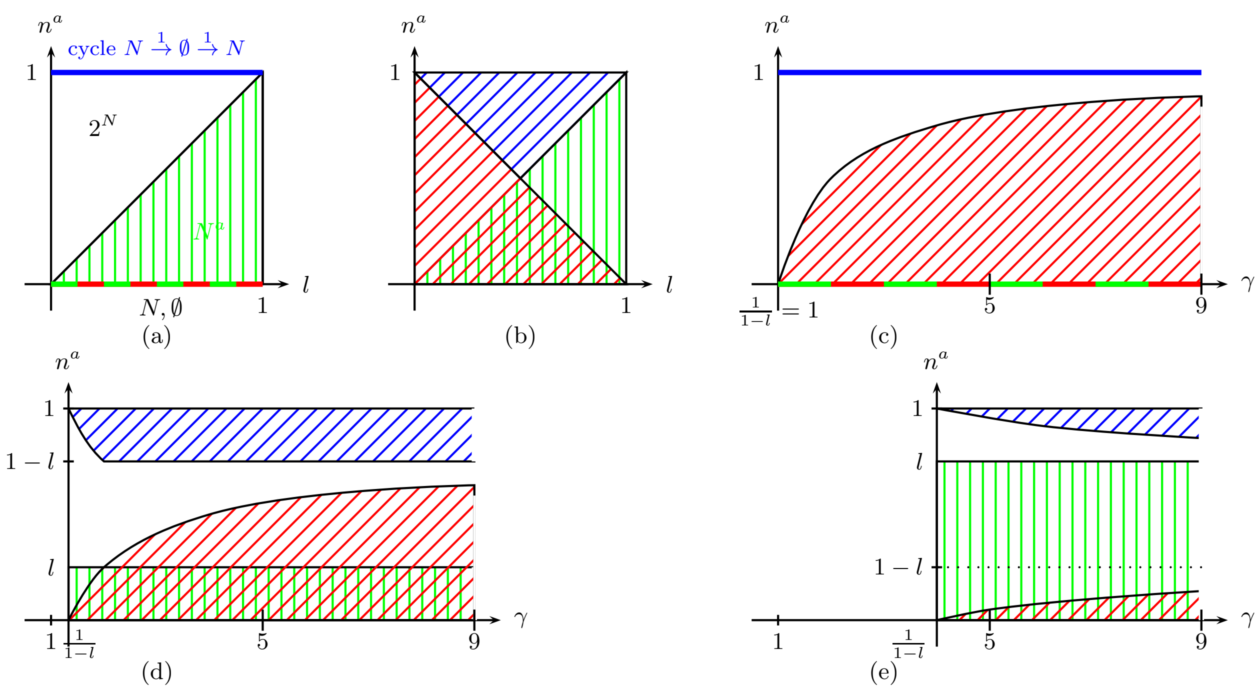

Fig. 3. Phase diagram for Situation 1: (a) γ has minimum value $\frac{1}{1-l}$ $(r = 0)$; (b) $\gamma \to \infty$; (c) $l = 0$; (d) $l \in [0, 1/2]$; (e) $l \in [1/2, 1[$. Color code: white=2^N, blue=cycle, red=N^c, green=N^a

We comment on these phase diagrams.

– The two first cases (i) and (ii) of absorbing classes are "polarization", the last case (iv) is "chaos" (no convergence). The extreme cases ($n^a = 0$ or 1) are already commented (see Section 4.1). Also, it can be checked that when γ tends to infinity, the limit cases (15) to (19) do not appear as the existence conditions become contradictory.
– When n^a increases from 0 to 1, we go from consensus, next to polarization, next to chaos, and finally to a cycle.
– When the firing threshold l is very low (c), there is a cascade effect leading to a polarization where all conformist agents say 'yes', which increases with reactiveness. Indeed, suppose that all agents say 'no'. Then, as l is very low, all anti-conformists start to say 'yes', which make gradually the conformists saying 'yes'. If n^a is not too large, the conformists rapidly reach the consensus 'yes'. Otherwise, as anti-conformists will say 'no' again, the non-negligible proportion of 'no' causes trouble in the convergence and a chaotic situation may appear. As the reactiveness increases, the chaotic behavior is less and less probable.
– Similarly, when the firing threshold is high (e), there is a cascade effect leading all conformists to say 'no', if the proportion of anti-conformists is not too small but less than the firing threshold. Indeed, suppose that all agents say 'yes' at some time. Then all anti-conformists will say 'no'. As the firing threshold is high, some conformists will start to say 'no', and there will be more and more. At the same time, as the number of 'yes' in the society is decreasing, the anti-conformists will gradually change to 'yes'. The situation of polarization remains stable unless the number of anti-conformists exceeds the firing threshold, in which case a chaotic situation (or even a cycle) occurs.
– (d) shows an intermediary situation where both cascades can occur. The higher the firing threshold, the higher the probability to have a cascade of 'no' among the conformists. The two cases (c) and (e) show how, in a society of conformists, the opinion can be manipulated by introducing a relatively small proportion of anti-conformists. The final opinion depends essentially on the firing threshold.

Situation 2: $l^a = r^a$ and $l^c = r^c$. Here, there is a symmetry between l and r. As mentioned before, this means that agents treat in the same way 'yes' and 'no' opinions. This assumption might be relevant for instance when voting for two candidates. However, it might not be relevant when saying 'yes' means 'adopting a new technology', where a bias towards a status quo or a bias towards technology adoption makes sense. Taking the technology example (Section 3.2), $l = r$ can arise if and only if $s_o = n/2$, which means that the term depending on price in s_o is zero: this happens when $\beta_i = 0$ (the price is not taken into account by the agent), or when products A and B have the same price. Under this assumption, the possible absorbing classes are:

– N^a, N^c iff $n^a \leq l^a$ and $n^a \leq l^c$ (referred hereafter as "polarization")
– cycle $N^a \xrightarrow{1} N^c \xrightarrow{1} N^a$ iff $n^a \geq 1 - l^a$ and $n^a \geq 1 - l^c$ (referred hereafter as "cycle")
– periodic class $[\emptyset, N^c] \xrightarrow{1} [N^a, N] \xrightarrow{1} [\emptyset, N^c]$ iff $n^a \geq 1 - l^a$ and $l^c < n^a < 1 - l^c$ (referred hereafter as "fuzzy cycle")
– $[\emptyset, N^a], [N^c, N]$ iff $n^a \leq l^c$ and $l^a < n^a < 1 - l^a$ (referred hereafter as "fuzzy polarization")
– 2^N (referred hereafter as "chaos") otherwise.

It can be checked that in the limiting case where γ^a, γ^c tend to infinity, classes (15) to (19) are not possible since then $l^a = l^c = 1/2$ which makes the conditions of existence contradictory.

Figure 4 gives four cuts of the phase diagram with the three parameters n^a, l^a, l^c. Recall that here l^a, l^c vary in $[0, 1/2[$, and $\gamma^a = \frac{1}{1-2l^a}$, $\gamma^c = \frac{1}{1-2l^c}$. Note that the polarization at

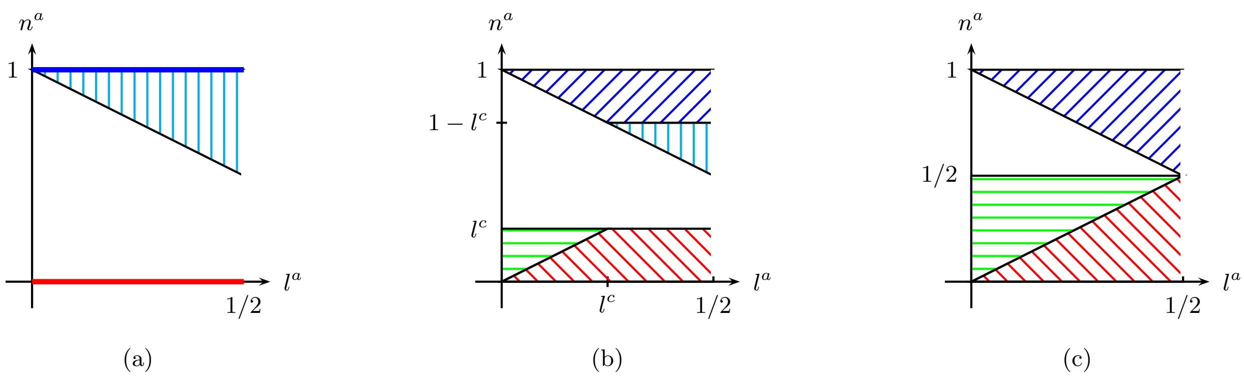

Fig. 4. Phase diagram for Situation 2: (a) $l^c = 0$; (b) $l^c \in]0, 1/2[$; (c) $l^c \to 1/2$. Color code: white=chaos, blue=cycle, cyan=fuzzy cycle, red=polarization, green=fuzzy polarization

$n^a = 0$ becomes a consensus (either N or $\emptyset$). As before, the cycle at $n^a = 1$ is $N \xrightarrow{1} \emptyset \xrightarrow{1} N$.

Some comments about these phase diagrams:

- Compared to Situation 1, the chaos case takes a relatively large area, which grows as l^c or l^a tend to 0 (agents have a low firing threshold, but a low reactiveness). In particular, it can be observed that when conformist agents have a low reactivity, a very small proportion of anti-conformists in the society suffices to make it chaotic.
- Contrarily to Situation 1, there is no cascade effect. Indeed, the absorbing states N^a and N^c always appear together, hence both are possible with some probability, or both are impossible. This polarization effect happens if the anti-conformists are not "seen" by the conformists (their number stay below the firing threshold), and all the more since the anti-conformists are reactive. Less reactive anti-conformists have a tendency to provoke fuzzy polarization.
- As for Situation 1, cycles and fuzzy cycles happen all the more since the number of anti-conformists is growing. A limit phenomenon happens when l^a, l^c, n^a tend all together to 1/2: a kind of "triple point" appears (see (c)), in the sense that the three types of behavior (polarization, fuzzy polarization and cycle) happen together, which is also visible for Situation 1 (Figure 3(b)). Observe that the mix of polarization and fuzzy polarization are nothing else than the limit classes (15) and (16). According to Theorem 1, they happen iff $l^c, l^a \to 1/2$ and $n^a = 1/2$, which is exactly the locus of this triple point.

Situation 3: The case where n^a tends to 0. Let us put $n^a = \epsilon > 0$, arbitrarily small. Therefore, $n^c = 1-\epsilon$. This case is the most plausible in real situations, as anti-conformists can be reasonably thought of forming a tiny part of the society. The crucial question is however to know whether this tiny part can have a non-negligible effect on the opinion of the society.

The first task is to see which of the 15 classes remain possible. One can check that:

- (1) N^a iff $l^c \wedge l^a \geq \epsilon$;
- (2) N^c iff $r^c \wedge r^a \geq \epsilon$;
- (3) $N^a \xrightarrow{1} \emptyset \xrightarrow{1} N^a$ iff $l^c \geq \epsilon$ and $r^a \geq 1-\epsilon$;
- (4) $N^c \xrightarrow{1} N \xrightarrow{1} N^c$ iff $r^c \geq \epsilon$ and $l^a \geq 1-\epsilon$
- Classes (5) to (10) are impossible;
- (11) $[\emptyset, N^a]$ iff $l^a < \epsilon$, $l^c \geq \epsilon$ and $r^a < 1-\epsilon$;
- (12) $[N^c, N]$ iff $r^a < \epsilon$, $r^c \geq \epsilon$ and $l^a < 1-\epsilon$;
- (13) $[\emptyset, N^a] \cup [\emptyset, N^c]$ iff $l^c < \epsilon$, $r^c < \epsilon$ and $r^a > 1-\epsilon$;
- (14) $[N^a, N] \cup [N^c, N]$ iff $l^c < \epsilon$, $r^c < \epsilon$ and $l^a > 1-\epsilon$;
- (20) 2^N otherwise.

It can be checked that classes (15) to (19) lead to contradictory conditions. Keeping in mind that ϵ is small, we can provide the following interpretation of the above absorbing classes: (1) and (2) are consensus to 'no' and 'yes', respectively, up to the negligible fraction of anti-conformists. (3) is almost the same as (1), while (4) is almost the same as (2). Also, (11) and (12) are almost the same as (1) and (2), respectively. (13) is a chaotic situation with mainly a tendency to 'no' for the society, while (14) is also a chaotic situation, but with a tendency of 'yes'.

From this analysis, we can draw the following conclusions:

- Suppose that the conformists have $l^c, r^c > \epsilon$: this means that they cannot "see" the anti-conformists. Then (1), (2) are together possible as soon as $r^a, l^a > \epsilon$ (the anti-conformists do not react to themselves). On the border area where l^a or r^a is smaller than ϵ, classes (3) and (11) (almost consensus 'no') or classes (4) and (12) (almost consensus 'yes') appear. The situation is made clear by looking at Figure 5 (recall that $l^a + r^a < 1$). Observe that in all parts of the triangle, both consensus (and almost

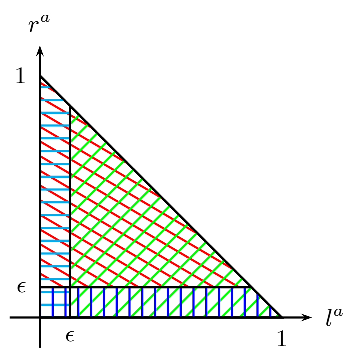

Fig. 5. Phase diagram for Situation 3, with $l^c, r^c > \epsilon$. Color code: green=N^a, red=N^c, cyan=almost consensus 'no' ((3) or (11)), blue=almost consensus 'yes' (4) or (12)

consensus) 'yes' and 'no' coexist. Therefore, no cascades of 'yes' or 'no' may occur. Also, no cycle nor chaotic behavior is possible, and we conclude that this situation is almost identical to the situation where no anti-conformist is present.
- Suppose that the conformists have very small l^c, r^c ($< \epsilon$), which means that they react to the anti-conformists. Then most of the classes become impossible, in particular

N^a, N^c, and only (13) (if r^a is large enough), and (14) (if l^a is large enough) remain. Otherwise, we get 2^N (see Figure 6). In this case, no consensus is possible, even

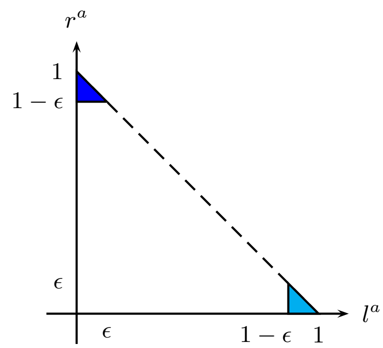

Fig. 6. Phase diagram for Situation 3, with $l^c, r^c < \epsilon$. Color code: blue=chaotic 'no' (13), cyan=chaotic 'yes' (14), white=chaos (20)

in a weak sense, and only chaotic situations arise. This is in accordance with the conclusions of Section 4.3.

4.5 Examples

We study two situations of interest: cascades (already mentioned above) seen in a context of misinformation, and shocks.

Misinformation and cascades. Cascade effects and herd behavior are well-known observed phenomena in many areas (sociology, marketing, stock markets, etc.), and various studies have been published on this topic (see, e.g., Banerjee (1992); Bikhchandani et al. (1992), who investigate how a crowd may or may not choose massively a given option). It is interesting to assume in this context that a ground truth exists (let us say that in our study 'yes' is the ground truth), and to determine under which conditions a "wrong" cascade occurs. In other words, starting from a state of the society where a large majority of agents has the correct opinion 'yes', how does misinformation spread and provoke an ineluctable convergence of the society into the wrong opinion 'no'?

We give several examples from Situation 1 above, that is, when $l^a = l^c = l$ and $r^a = r^c = r$. As $\emptyset$ is not an absorbing state, a wrong cascade amounts to ending in state N^a if $n_a < n_c$ (green zones in Figure 3), or state N^c if $n_a > n_c$ (red zones). We suppose hereafter than there are less anti-conformists than conformists, so that green zones correspond to wrong cascades. Similar considerations can be made in the other case. Throughout this section we assume that $n = 20$ and simulate the mechanism of opinion dynamics by a computer program. In all simulations, the functions $p_i(s)$ are linear between $s = l$ and $s = n - r$.

We begin by illustrating Figure 3 (a), taking $l = 1$ and $r = 0$. Suppose that the present state is N (all agents say 'yes'). If there is no anti-conformist, this is an absorbing state, and the correct opinion remains for ever. Suppose now that one agent becomes anti-conformist ($n_a = 1$). This is enough to be in the green zone of Figure 3 (a), and convergence to the wrong opinion is ineluctable. The corresponding simulation is shown on Figure 7 (left).

Starting from a state S with $|S| = 19$ evidently does not change anything when $n_a = 1$ (wrong cascade). The question is the following: if there is no anti-conformists, is a wrong cascade possible and with which probability? Indeed, $\emptyset$ and N are both possible when starting from S with $|S| = 19$ (or smaller). An exact computation of the probability of reaching the absorbing classes N and $\emptyset$ reveals to be very difficult. Simulations done over 30,000 realizations show that the probability to have a wrong cascade is only about 0.15, while it becomes a sure event with only one anti-conformist introduced in the society.

There is also a green zone on Figure 3 (b), however this is in the case where $l+r = n-1$, i.e., when all agents have a threshold function which implies a deterministic behavior. The convergence to the wrong opinion is therefore extremely fast. Taking for example $l = 12$, $r = 7$, and $n_a = 9$, we have the sure transitions $N \xrightarrow{1} N^c \xrightarrow{1} N^a$ with N^a absorbing.

Let us study case (d) of Figure 3. There, except on a tiny part, the green zone coexists with the red zone: there is some probability to reach one state or the other (i.e., right or wrong cascade), depending on the initial state S. As said before, an exact calculation of these probabilities seems to be extremely difficult, but simulations can be done. Let us take $n_a = 4$, $l = r = 5$, which yields $\gamma = 2$. If the initial state S is such that $s \leq 5$, then $p_i(s) = 0$ for conformists and $p_i(s) = 1$ for anti-conformists, hence in one shot the process converges to N^a (wrong cascade). Similarly, if $s \geq 15$, the process converges to N^c (right cascade). Figure 8 shows the probability to attain N^c for all values of $s \in [5, 15]$, obtained by simulation of 30,000 realizations. Equal chances for right and wrong cascades are obtained for $s = 10$.

Lastly, we illustrate case (e), taking $n_a = 4$, $l = 12$ and $r = 0$. There is fast convergence to N^a (wrong cascade), as shown on Figure 7 (right).

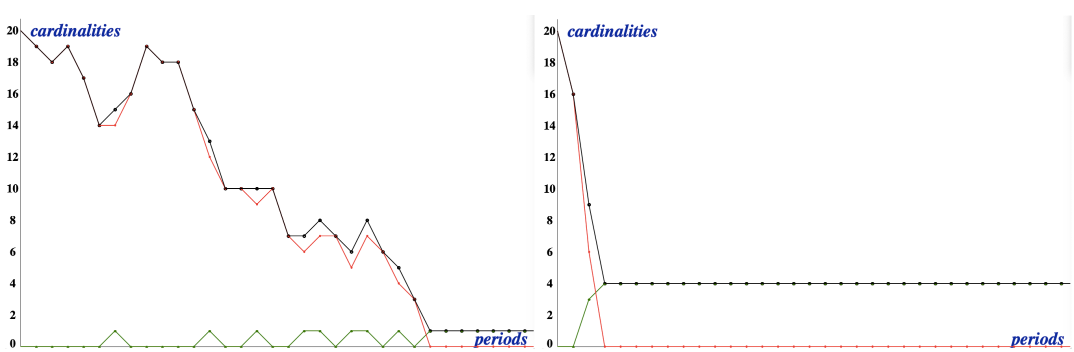

Fig. 7. (left) Evolution of the number of 'yes' (red: conformists, green: anticonformists, black: total) with 19 conformists and 1 anti-conformist, $l = 1$, $r = 0$. (right) Evolution of the number of 'yes' with 16 conformists, 4 anti-conformists, $l = 12$, $r = 0$

Shocks. Predicting shocks, in particular, shocks in demand for a product when referring to our technology example, is one of the major concerns of economists. Some of the absorbing classes that we have identified hide such a behavior, for example class (19),

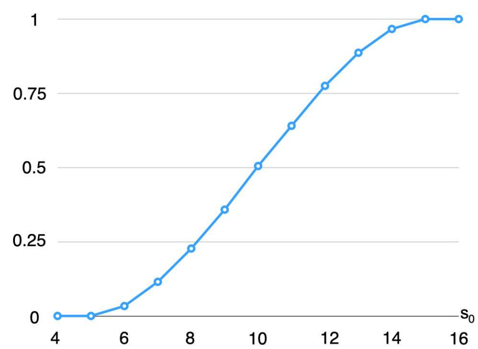

Fig. 8. Probability of a right cascade, with 16 conformists, 4 anti-conformists, $l = r = 5$.

and more generally, all those involving threshold functions for the agents (classes (15) to (19)). Figure 9, which can be interpreted in the framework of the technology example, shows an evidence of such a behavior corresponding to class (19), obtained with $n^c = 8$, $n^a = 12$, $l^a = r^a = 0$ and $l^c + 1 = r^c = 10$. Shocks are visible for conformists, and are triggered by the fluctuations of the anti-conformists.

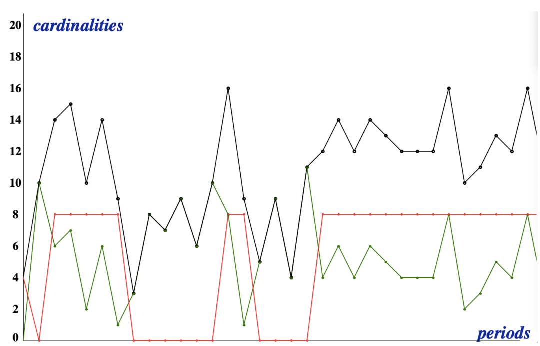

Fig. 9. Evolution of the number of adopters of product A (red: conformists, green: anticonformists, black: total) illustrating the existence of shocks for conformists

5 Convergence in the mixed case

We now address the mixed case ($N^m \neq \emptyset$). The analysis goes in a similar way, therefore we omit details and point out only differences with the pure case.

5.1 The case of only mixed agents ($N^m = N$)

Prior to the study of the mixed case, we consider the situation where the society is formed only by mixed agents: $N = N^m$. It is easily done by using Eqs. (13) and (14):

$$p_i(s) = 0 \Leftrightarrow n - r^a \leq s \leq l^c$$
$$p_i(s) = 1 \Leftrightarrow n - r^c \leq s \leq l^a$$

which permits to rewrite Table 1:

	$0 \leq s < n - r^c$	$n - r^c \leq s \leq l^a$	$l^a < s \leq n$
$0 \leq s < n - r^a$	2^N	N	2^N
$n - r^a \leq s \leq l^c$	$\emptyset$	does not occur	$\emptyset$
$l^c < s \leq n$	2^N	N	2^N

We can see that $\emptyset, N$ are not absorbing states, hence the only absorbing class is 2^N.

5.2 The general mixed case ($N^m \neq \emptyset$)

Next, we consider a society in which pure conformists, pure anti-conformists, and mixed agents co-exist, i.e., $n = n^a + n^c + n^m$, with $n^a > 0$, $n^c > 0$ and $n^m > 0$. We get the following result.

Theorem 2 *Assume that $N^m \neq \emptyset$, $N^a \neq \emptyset$ and $N^c \neq \emptyset$. Let $\overline{N}^a = N^a \cup N^m$ and $\overline{N}^c = N^c \cup N^m$. There are twenty possible absorbing classes which are:*

(i) **Fuzzy polarization:**

(1) $[N^a, \overline{N}^a]$ *if and only if* $n^c \geq (n - l^c) \vee (n - l^a)$;

(2) $[N^c, \overline{N}^c]$ *if and only if* $n^c \geq (n - r^c) \vee (n - r^a)$;

(3) $[\emptyset, \overline{N}^a]$ *if and only if* $(n - l^c) \vee (r^a + 1) \leq n^c < n - l^a$;

(4) $[N^c, N]$ *if and only if* $(n - r^c) \vee (l^a + 1) \leq n^c < n - r^a$;

(ii) **Fuzzy cycles:**

(5) $[N^a, \overline{N}^a] \xrightarrow{1} \emptyset \xrightarrow{1} [N^a, \overline{N}^a]$ *if and only* $n - l^c \leq n^c \leq r^a - n^m$;

(6) $[N^c, \overline{N}^c] \xrightarrow{1} N \xrightarrow{1} [N^c, \overline{N}^c]$ *if and only if* $n - r^c \leq n^c \leq l^a - n^m$;

(7) $[N^a, \overline{N}^a] \xrightarrow{1} [N^c, \overline{N}^c] \xrightarrow{1} [N^a, \overline{N}^a]$ *if and only if* $n^c + n^m \leq l^c \wedge l^a \wedge r^c \wedge r^a$;

(8) $\emptyset \xrightarrow{1} [N^a, \overline{N}^a] \xrightarrow{1} [N^c, \overline{N}^c] \xrightarrow{1} \emptyset$ *if and only if* $n^c + n^m \leq r^c \wedge r^a \wedge l^c$ *and* $n^c \geq n - r^a$;

(9) $[N^a, \overline{N}^a] \xrightarrow{1} N \xrightarrow{1} [N^c, \overline{N}^c] \xrightarrow{1} [N^a, \overline{N}^a]$ *if and only if* $n^c + n^m \leq l^c \wedge l^a \wedge r^c$ *and* $n^c \geq n - l^a$;

(10) $[N^a, \overline{N}^a] \xrightarrow{1} [\emptyset, \overline{N}^c] \xrightarrow{1} [N^a, \overline{N}^a]$ *if and only if* $n^c + n^m \leq l^c \wedge l^a \wedge r^a$ *and* $r^c - n^m < n^c < n - l^c$;

(11) $[N^c, \overline{N}^c] \xrightarrow{1} [N^a, N] \xrightarrow{1} [N^c, \overline{N}^c]$ *if and only if* $n^c + n^m \leq r^c \wedge r^a \wedge l^a$ *and* $l^c - n^m < n^c < n - r^c$;

(12) $[\emptyset, \overline{N}^c] \xrightarrow{1} [N^a, N] \xrightarrow{1} [\emptyset, \overline{N}^c]$ *if and only if* $l^c \vee r^c < n^c + n^m \leq r^a \wedge l^a \wedge (n - (l^c + 1)) \wedge (n - (r^c + 1))$;

(iii) **Chaotic polarization:**

(13) $[\emptyset, \overline{N}^a] \cup [\emptyset, \overline{N}^c]$ *if and only if* $l^c \geq n - r^a$ *and*

$$n^c \in \big(]r^c - n^m, n - l^c[\cap]l^a, n - r^c[\big) \cup \big(\big(]l^a - n^m, n - r^a[\cup]l^c - n^m, n - r^c[\big) \cap]0, r^c - n^m]\big);$$

(14) $[N^a, N] \cup [N^c, N]$ *if and only if* $r^c \geq n - l^a$ *and*

$$n^c \in \big(]l^c - n^m, n - r^c[\cap]r^a, n - l^c[\big) \cup \big(\big(]r^a - n^m, n - l^a[\cup]r^c - n^m, n - l^c[\big) \cap]0, l^c - n^m]\big);$$

(15) $[\emptyset, \overline{N}^a] \cup [N^c, \overline{N}^c]$ *if and only if* $l^c + r^c = n - 1$, $r^c \leq r^a$, $l^c > l^a$, $n^c < n - l^c$ *and* $l^a < n^c + n^m < n - r^a$;

(16) $[N^c, N] \cup [N^a, \overline{N}^a]$ *if and only if* $l^c + r^c = n-1$, $l^c \leq l^a$, $r^a < r^c$, $n^c < n - r^c$ *and* $r^a < n^c + n^m < n - l^a$;

(17) $[\emptyset, \overline{N}^c] \cup [N^a, \overline{N}^a]$ *if and only if* $l^a + r^a = n-1$, $l^c \geq l^a$, $n^c < n - r^c$ *and* $n^c \in]r^c - n^m, n - l^c[\cup]l^c - n^m, r^c - n^m[$;

(18) $[N^a, N] \cup [N^c, \overline{N}^c]$ *if and only if* $l^a + r^a = n-1$, $r^c \geq r^a$, $n^c < n - l^c$ *and* $n^c \in]l^c - n^m, n - r^c[\cup]r^c - n^m, l^c - n^m[$;

(19) $[\emptyset, \overline{N}^a] \cup [N^c, N]$ *if and only if* $l^c + r^c = n-1$ *and* $l^a \vee r^a < n^c \leq l^c \wedge r^c$;

(iv) **Chaos:**

(20) 2^N *otherwise.*

The proof is similar to the one of Theorem 1 and is omitted here, but is available upon request.

Theorems 1 and 2 lead to clear conclusions concerning the comparison of absorbing classes in the pure and mixed cases. Comparing the two results shows that, while the classes remain similar in the two cases (same number and roughly same structure), the addition of mixed agents "blurs" the classes obtained in the pure case, in the sense that the sets N^a and N^c are replaced by the intervals $[N^a, N^a \cup N^m]$ and $[N^c, N^c \cup N^m]$, respectively.

We now describe more precisely the results. First of all, when mixed agents exist in a society, a polarization into two groups (one saying 'yes' and another one saying 'no', which was the case under $N^m = \emptyset$) is not possible anymore. However, under the same conditions as before (see cases (1) and (2) in Theorem 1), N^a and N^c are now replaced by $[N^a, \overline{N}^a]$ and $[N^c, \overline{N}^c]$. In other words, while anti-conformists (conformists, respectively) continue saying 'yes' and conformists (anti-conformists, respectively) say 'no' forever, the new type of individuals – mixed agents – oscillate between 'yes' and 'no'. In the pure case, two (simple) intervals $[\emptyset, N^a]$ and $[N^c, N]$ (cases (11) and (12) in Theorem 1) are possible absorbing classes. With the presence of mixed agents, we have the corresponding intervals $[\emptyset, \overline{N}^a]$ and $[N^c, N]$ (cases (3) and (4) in Theorem 2) under the same conditions as in the pure case. This means that while conformists do not change their behavior when mixed agents join the society and say either 'no' (absorbing class $[\emptyset, \overline{N}^a]$) or 'yes' (absorbing class $[N^c, N]$) forever, now besides anti-conformists also mixed agents oscillate. Another consequence of the presence of mixed agents on possible absorbing classes is that cycles (i.e., periodic classes with only single states, cases (3) through (7) in Theorem 1) are not possible anymore. Instead, we have eight periodic classes with mixed agents oscillating (cases (5) till (12) in Theorem 2) that correspond to absorbing classes (3) - (10) of Theorem 1. The conditions for the existence of these periodic classes in the mixed case are the same as the ones for the corresponding 'pure' cases, but adjusted by the presence of n^m mixed agents. Finally, the unions of intervals in the mixed case (absorbing classes (13) till (19) in Theorem 2) correspond to the unions of intervals (13) till (19) in the pure case (Theorem 1), but again with mixed agents oscillating and the conditions taking into account n^m.

6 Concluding remarks

Clearly, the present paper has taken a different road than the references mentioned in Section 2. We analyze a process of opinion formation in a society with different types of agents: pure conformists, pure anti-conformists, and mixed agents. We focus on anonymous influence, where a change of an agent's opinion depends on the number of agents with a certain opinion and not on their identities. We determine all possible absorbing classes and conditions for their occurrence for the society without mixed agents as well as for the mixed case. Moreover, the analysis of a very large society in different types of situations is provided.

First of all, our study confirms and puts in precise terms what the intuition says to us: the introduction of anti-conformists in a society, even in a very small proportion, prevents from reaching a consensus and causes either polarization or various instabilities: cycles, chaotic behavior, etc. Our study has shown that, even under some simplifying assumptions (the parameters l^c, r^c, l^a, r^a are supposed to be the same for every agent in a category), the convergence issue is very complex and many (up to 20) different situations can occur. Despite this apparent complexity, we have managed to draw some general and instructive conclusions which are valid in different typical situations. We summarize below our main findings of Section 3, established in the pure case (that is, when agents are either purely conformist or purely anti-conformist) and with a society of large size:

- In a society where all agents have the same influenceability characteristics, a cascade effect leading to a polarization is likely to occur. The type of polarization depends on the firing threshold, i.e., the proportion of 'yes' which is necessary to start being influenced. If the firing threshold is low, then all conformist agents will finally say 'yes', while if the firing threshold is high, the cascade effect leads all conformists to say 'no'. This cascade phenomena happen even with a very small number of anti-conformists, and tend to be blurred by a chaotic behavior if the proportion of anti-conformists becomes large. It shows a very important fact: *the opinion of a society can be manipulated by introducing a small proportion of anti-conformists in it (opinion reversal, spread of misinformation).* Hence, anti-conformists do not only introduce chaotic behavior, they can steer the opinion in some direction.
- When agents have a symmetric behavior w.r.t. 'yes' or 'no' in terms of influenceability, no cascade phenomenon can occur, and a chaotic behavior is very likely. However, a polarization can occur if the anti-conformists are not "seen" by the conformists (i.e., their number stays below the firing threshold), and all the more since the anti-conformists are reactive.
- Special phenomena like polarization, cycles and their more or less fuzzy versions are only due to the presence of thresholds (firing threshold and saturation threshold) in the aggregation rule. If on the contrary $l^a = r^a = l^c = r^c = 0$, there is only one class containing all states (chaos). In our technology example, we have shown that this corresponds to agents very unsure about their decision. By contrast, when agents apply a best response strategy, which corresponds to aggregation rules of the threshold type, any fuzzy/chaotic behavior disappear, and it only remains polarization and cycles.
- Lastly we mention a special situation similar to a triple point in physics: the three types of behavior (polarization, fuzzy polarization and cycle) coexist. This can happen if and only if half of the population is anti-conformist and the firing threshold of conformists and anti-conformists is equal to 1/2.

The introduction of mixed agents has clear effects on opinion formation. Mixed agents do not change the number of possible absorbing classes, but their presence blurs them, because the opinion of mixed agents always oscillates between conformism and anti-conformism. This means that neither N^a nor N^c can appear as absorbing states or be a constituent of an absorbing class, but they are replaced by their blurred version, where any subset of mixed agents can be present. As a consequence, polarization *stricto sensu* cannot appear anymore.

7 Acknowledgments

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A Relation with the anonymous model of conformity

We recall the anonymous model of conformism (Förster et al. (2013)) and show that it is equivalent to our class of conformist aggregation rules. By doing this we show that the present model is a natural extension of Förster et al. (2013).

The assumption that agents modify their opinions in a Markovian way is basically that underlying Grabisch and Rusinowska (2013). As the number of states is 2^n, the size of the transition matrix is $2^n \times 2^n$. In order to avoid this exponential complexity, like in the present paper the latter reference uses a simple mechanism to generate the transition matrix, based on aggregation functions, that is, nondecreasing mappings $A : [0,1]^n \to [0,1]$ satisfying $A(0,\ldots,0) = 0$ and $A(1,\ldots,1) = 1$. Specifically, supposing that agent i aggregates opinions by the function A_i, the probability that agent says 'yes' at next time step, given that S is the set of agents saying 'yes' at present, is given by

$$p_i(S) = A_i(1_S), \tag{15}$$

where 1_S is the indicator function of S, i.e., $1_S(j) = 1$ iff $j \in S$ and 0 otherwise. This model is presented and studied in general in Grabisch and Rusinowska (2013).

The most common example of aggregation function, used, e.g., in DeGroot (1974), is the weighted arithmetic mean

$$A_i(x) = \sum_{j=1}^{n} w^i_j x_j, \tag{16}$$

where $x = (x_1, \dots, x_n)$ and the w^i_j's are weights on the entries, satisfying $w^i_j \geq 0$ and $\sum_{j=1}^{n} w^i_j = 1$. The weight w^i_j represents to which extent agent i puts confidence on the opinion x_j of agent j. It depicts a situation where every agent knows the identity of every other agent, and is able to assess to which extent he trusts or agrees with the opinion or personal tastes of others.

Förster et al. (2013) investigate the model of conformism with anonymous social influence, which depends only on the *number* of agents with a certain opinion, not on their identity. They use the *ordered weighted averages* (commonly called OWA operators, Yager (1988)), which are the unique anonymous aggregation functions:

$$\mathrm{OWA}_w(x) = \sum_{j=1}^{n} w_j x_{(j)}, \tag{17}$$

where the entries $x_1, \dots, x_n$ are rearranged in decreasing order: $x_{(1)} \geq x_{(2)} \geq \cdots \geq x_{(n)}$, and $w = (w_1, \dots, w_n)$ is the weight vector defined as above. Hence, the weight w_j is not assigned to agent j but to rank j, and thus permits to model quantifiers. For example, taking $w_1 = 1$ and all other weights being 0 models the quantifier "there exists". Indeed, it is enough to have one of the entries being equal to 1 to get 1 as output. In our context, it means that only one agent saying 'yes' is enough to make your opinion being 'yes' for sure. Similarly, "for all" is modeled by $w_n = 1$ and all other weights being 0, and means that you need that all agents (including you) say 'yes' to be sure to continue to say 'yes'. Intermediate situations can be modeled as well.

For the anonymous model of conformism, there exist two types of absorbing classes (Förster et al. (2013)):

(i) any single state $S \in 2^N$ (including the consensus states $\emptyset$ and N);
(ii) union of intervals of the type $[S, S \cup K]$, where $S, K \neq \emptyset, N$ (recall that $[S, S \cup K] = \{T \in 2^N \mid S \subseteq T \subseteq S \cup K\}$).

For the second case, when the absorbing class is reduced to a single interval $[S, S \cup K]$, it depicts a situation in the long run where agents in S say 'yes', those outside $S \cup K$ say 'no', and those in K oscillate between 'yes' and 'no' forever. Interestingly, no periodic class can occur, although in general for arbitrary aggregation functions cycles can occur (Grabisch and Rusinowska (2013)).

We now establish the relation with our framework. Supposing that agent i aggregates opinions anonymously by OWA_{w^i} with weight vector w^i, we have from (15) and (17), for any $S \subseteq N$:

$$p_i(S) = \mathrm{OWA}_{w^i}(1_S) = \sum_{j=1}^{s} w^i_j =: p_i(s).$$

We see that $p_i(s)$ is an aggregation rule in the sense of Section 3.1, and that, due to the properties of w^i, $p_i \in A^c$. Conversely, for any $p \in A^c$, we can define $w \in \mathbb{R}^n$ by

$w_j = p(j) - p(j-1)$, $j = 1, \dots, n$, and by properties of $p \in A^c$, w is a well-defined weight vector s.t. $p \equiv \text{OWA}_w$. The introduction of the classes of aggregation rules A^a, A^m are therefore natural generalizations of the conformist model of Förster et al. (2013).

Note that the numbers l^c, r^c, l^a, r^a defined in (5) - (8) correspond to the numbers of left and right zeroes in the weight vectors of an OWA operator, for conformists and anti-conformists, respectively. Moreover, the assumptions (5) - (8) simply mean that while agents in N^c (N^a, respectively) may have different weight vectors, the number of left and right zeroes is the same for all of them. The number of left/right zeroes indicates how many people the agent needs in order to start being influenced towards the yes/no opinion. In particular, a non symmetrical weight vector w.r.t. the number of left and right zeroes means that the agent is biased towards the 'yes' or 'no' answer, i.e., he needs a different number of people to start being convinced to say 'yes' or 'no'.

B Proof of Theorem 1

As preliminaries, we observe the following basic facts:

(F0) $\emptyset \xrightarrow{1} N^a$, $N \xrightarrow{1} N^c$ (as already observed).

(F1) If $\mathcal{S} \xrightarrow{1} \mathcal{T}$, $\mathcal{S}' \xrightarrow{1} \mathcal{T}'$ and $\mathcal{S} \subset \mathcal{S}'$, then $\mathcal{T} \subseteq \mathcal{T}'$.

(F2) Applying (F0) and (F1), we find that in a transition $\mathcal{S} \to \mathcal{T}$, $\emptyset \in \mathcal{S}$ implies $N^a \in \mathcal{T}$ and $N \in \mathcal{S}$ implies $N^c \in \mathcal{T}$.

(F3) Consider $\mathcal{S} \xrightarrow{1} \mathcal{T}_1 \xrightarrow{1} \cdots \xrightarrow{1} \mathcal{T}_p$, with $p \geq 2$. If $\mathcal{S} \subseteq \mathcal{T}_1$, then $\mathcal{S} \subseteq \mathcal{T}_1 \subseteq \cdots \subseteq \mathcal{T}_p$.

(F4) 2^N is a possible absorbing class. Indeed, take $l^c = r^c = l^a = r^a = 0$. From Table 1 we immediately see that for any $S \neq \emptyset, N$ we have $S \xrightarrow{1} 2^N$. Since the power set of the set of states is the "default" absorbing class when no other can exist, we exclude it from our study and do not consider transitions to 2^N.

(F5) From Table 1, we see that we have to deal only with the sets $\emptyset, N^a, N^c, N$ and the intervals $[\emptyset, N^c], [\emptyset, N^a], [N^a, N], [N^c, N]$ (2^N being excluded by (F4)), i.e., only these can be constituents of an absorbing class. We put

$$\mathbb{B} = \{\{\emptyset\}, \{N^a\}, \{N^c\}, \{N\}, [\emptyset, N^c], [\emptyset, N^a], [N^a, N], [N^c, N]\}$$

the set of collections relevant to our study. Intervals not reduced to a singleton are called *nontrivial intervals*.

(F6) $\mathcal{S} \subseteq 2^N$ is an absorbing class if and only if $\mathcal{S} \xrightarrow{1} \mathcal{S}$ and $\mathcal{S}$ is *connected* (i.e., there is a path (chain of transitions) from S to T for any $S, T \in \mathcal{S}$).

Our strategy is based on (F6): aperiodic absorbing classes are connected collections $\mathcal{S}$ such that $\mathcal{S} \xrightarrow{1} \mathcal{S}$. Periodic absorbing classes are of the form $\mathcal{S}_1 \xrightarrow{1} \cdots \xrightarrow{1} \mathcal{S}_p$ with all $\mathcal{S}_i$ pairwise incomparable, and $\mathcal{S}_1 \cup \cdots \cup \mathcal{S}_p$ is connected. Consequently, we study all possible kinds of transition $\mathcal{S} \xrightarrow{1} \mathcal{T}$, and check connectedness for each candidate. We distinguish between "simple" transitions of the type $\mathcal{B} \xrightarrow{1} \mathcal{B}'$ with $\mathcal{B}, \mathcal{B}' \in \mathbb{B}$, and "multiple" transitions $\mathcal{S} \xrightarrow{1} \mathcal{T}$, where $\mathcal{S}, \mathcal{T}$ are composed with several elements of $\mathbb{B}$, e.g., $[\emptyset, N^a] \cup [\emptyset, N^c]$.

B.1 Simple transitions

We focus on transitions of the type $\mathcal{B} \xrightarrow{1} \mathcal{B}'$, with $\mathcal{B}, \mathcal{B}' \in \mathbb{B}$, and look for conditions on the parameters of the model to obtain such transitions.

Observe that if $\mathcal{B}'$ is a nontrivial interval, it cannot be the union of other elements of $\mathcal{B}$. Therefore, $\mathcal{B} \xrightarrow{1} \mathcal{B}'$ if and only if for any $S \in \mathcal{B}$, $S \xrightarrow{1} \mathcal{B}''$ with $\mathcal{B}'' \in \mathbb{B}$ and $\mathcal{B}'' \subseteq \mathcal{B}'$, and there is at least one $S \in \mathcal{B}$ s.t. $S \xrightarrow{1} \mathcal{B}'$. Let us denote by $\mathcal{C}[\mathcal{B}]$ the conditions on $s = |S|$ to have a sure transition from S to $\mathcal{B}$, as given in Table 1. All these conditions are intervals.

Observe that all $\mathcal{B} \in \mathbb{B}$ are either singletons $\{B\}$ or nontrivial intervals $[\underline{B}, \overline{B}]$, and $\mathcal{B} \subset \mathcal{B}'$ if and only if $\mathcal{B} = \{\underline{B'}\}$ or $\{\overline{B'}\}$, with $\mathcal{B}' = [\underline{B'}, \overline{B'}]$. Hence:

$$\mathcal{B} \xrightarrow{1} \mathcal{B}' \Leftrightarrow \begin{cases} [\underline{b}, \overline{b}] \subseteq \mathcal{C}[\mathcal{B}'] \cup \mathcal{C}[\{\underline{B'}\}] \cup \mathcal{C}[\{\overline{B'}\}] \\ [\underline{b}, \overline{b}] \cap \mathcal{C}[\mathcal{B}'] \neq \emptyset, \end{cases} \tag{18}$$

with $\underline{b}, \overline{b}$ the cardinalities of $\underline{B}, \overline{B}$. Let us apply (18) to all possibilities. When $\{\mathcal{B}'\}$ is a singleton, the above condition reduces to $[\underline{b}, \overline{b}] \subseteq \mathcal{C}[\mathcal{B}']$, as given in Table 1. Otherwise,

(i) with $\mathcal{B}' = [\emptyset, N^a]$, we obtain $[\underline{b}, \overline{b}] \subseteq [0, l^c]$ and $[\underline{b}, \overline{b}] \cap]l^a, n - r^a[\cap [0, l^c] \neq \emptyset$, which simplifies to

$$[\underline{b}, \overline{b}] \subseteq [0, l^c] \text{ and } [\underline{b}, \overline{b}] \cap]l^a, n - r^a[\neq \emptyset; \tag{19}$$

(ii) with $\mathcal{B}' = [\emptyset, N^c]$, we obtain

$$[\underline{b}, \overline{b}] \subseteq [n - r^a, n] \text{ and } [\underline{b}, \overline{b}] \cap]l^c, n - r^c[\neq \emptyset; \tag{20}$$

(iii) with $\mathcal{B}' = [N^c, N]$, we obtain

$$[\underline{b}, \overline{b}] \subseteq [n - r^c, n] \text{ and } [\underline{b}, \overline{b}] \cap]l^a, n - r^a[\neq \emptyset; \tag{21}$$

(iv) with $\mathcal{B}' = [N^a, N]$, we obtain

$$[\underline{b}, \overline{b}] \subseteq [0, l^a] \text{ and } [\underline{b}, \overline{b}] \cap]l^c, n - r^c[\neq \emptyset. \tag{22}$$

This yields Table 3. Observe that the table is symmetric w.r.t. its center by the symmetry principle (Lemma 1): just exchange r with l. The transitions being sure, all cases on each line are exclusive.

From Table 3, we can deduce absorbing classes reduced to singletons or intervals: they correspond to transitions $\mathcal{S} \xrightarrow{1} \mathcal{S}$ in the table, provided they are connected. We obtain:

(i) N^a, under the condition $n^c \geq (n - l^c) \vee (n - l^a)$;
(ii) N^c, under the condition $n^c \geq (n - r^c) \vee (n - r^a)$;
(iii) $[\emptyset, N^a]$, under the condition $n - l^c \leq n^c < n - l^a$;
(iv) $[N^c, N]$, under the condition $n - r^c \leq n^c < n - r^a$.

We check connectedness for (iii) ((iv) follows by symmetry). We see from Table 1 that every $S \in [\emptyset, N^a]$ with $s \leq l^a$ has a sure transition to N^a, while the other ones go to every set in the interval. Therefore, the interval is connected if and only if N^a has a possible transition to every set in the interval, i.e., we need $l^a < n^a < n - r^a$ and $n^a \leq l^c$, so the additional condition $n^a < n - r^a$ is needed. In summary:

(i) N^a is an absorbing class if and only if $n^c \geq (n - l^c) \vee (n - l^a)$;
(ii) N^c is an absorbing class if and only if $n^c \geq (n - r^c) \vee (n - r^a)$;
(iii) $[\emptyset, N^a]$ is an absorbing class if and only if $(n - l^c) \vee (r^a + 1) \leq n^c < n - l^a$;
(iv) $[N^c, N]$ is an absorbing class if and only if $(n - r^c) \vee (l^a + 1) \leq n^c < n - r^a$.

In order to get (absorbing) cycles and periodic classes, we study chains of sure transitions of length 2: $\mathcal{S}_1 \xrightarrow{1} \mathcal{S}_2 \xrightarrow{1} \mathcal{S}_3$, with $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$ being pairwise disjoint, except possibly $\mathcal{S}_1 = \mathcal{S}_3$. An inspection of Table 3 yields all such possible chains of length 2, summarized in Table 4. A second table can be obtained by symmetry.

From Table 4, we obtain the following candidates for absorbing cycles and periodic classes, after eliminating double occurrences and using symmetry:

(i) $N^a \xrightarrow{1} \emptyset \xrightarrow{1} N^a$, under the condition $n - l^c \leq n^c \leq r^a$;
(ii) $N^c \xrightarrow{1} N \xrightarrow{1} N^c$, under the condition $n - r^c \leq n^c \leq l^a$;
(iii) $N^c \xrightarrow{1} N^a \xrightarrow{1} N^c$, under the condition $n^c \leq l^c \wedge l^a \wedge r^c \wedge r^a$;
(iv) $[\emptyset, N^c] \xrightarrow{1} N^a \xrightarrow{1} [\emptyset, N^c]$, under the condition $n^c \leq l^c \wedge l^a \wedge r^a, r^c < n^c < n - l^c$
(v) $[N^a, N] \xrightarrow{1} N^c \xrightarrow{1} [N^a, N]$, under the condition $n^c \leq r^c \wedge r^a \wedge l^a, l^c < n^c < n - r^c$
(vi) $[N^a, N] \xrightarrow{1} [\emptyset, N^c] \xrightarrow{1} [N^a, N]$, under the condition $r^c \vee l^c < n^c \leq r^a \wedge l^a$.

It remains to check connectedness of (iv) and (vi) ((v) is obtained by symmetry). For (iv), we must check that N^a has a possible transition to every set in $[\emptyset, N^c]$. By Table 1, we must have $n^a \geq n - r^a$ and $l^c < n^a < n - r^c$, which is true by the conditions in (iv). We address (vi). We claim that under the conditions in (vi) $[N^a, N] \cup [\emptyset, N^c]$ is connected if and only if $N^a \xrightarrow{1} [\emptyset, N^c]$ and $N^c \xrightarrow{1} [N^a, N]$. Take any $S \in [\emptyset, N^c]$. Then S goes either to any set T in $[N^a, N]$ or only to N^a or only to N. In the first case, similarly, T goes either to any set $S' \in [\emptyset, N^c]$ (and we are done) or only to $\emptyset$ or only to N^c. If $T \xrightarrow{1} \emptyset$, then we have $T \xrightarrow{1} \emptyset \xrightarrow{1} N^a \xrightarrow{1} [\emptyset, N^c]$ and we are done. Otherwise we have $T \xrightarrow{1} N^c \rightarrow N^a \xrightarrow{1} [\emptyset, N^c]$. Suppose now that $S \xrightarrow{1} N^a$, then N^a goes to any $S' \in [\emptyset, N^c]$ and we are done. Otherwise, $S \xrightarrow{1} N \xrightarrow{1} N^c \rightarrow N^a \xrightarrow{1} [\emptyset, N^c]$ and we are done. This proves sufficiency. Now suppose the condition is not fulfilled. This means that N^a goes to either $\emptyset$ or N^c (or similar condition for N^c). In fact, due to the conditions in (vi) and Table 1, we have that $N^a \xrightarrow{1} \emptyset$, but this yields the cycle $N^a \xrightarrow{1} \emptyset \xrightarrow{1} N^a$.

So in summary, candidates from (i) to (v) are all periodic classes under the specified conditions, and for (vi), the additional condition that $N^a \xrightarrow{1} [\emptyset, N^c]$ and $N^c \xrightarrow{1} [N^a, N]$ yields:

(vi') $[N^a, N] \xrightarrow{1} [\emptyset, N^c] \xrightarrow{1} [N^a, N]$ under the condition $r^c \vee l^c < n^c \leq r^a \wedge l^a \wedge (n - l^c - 1) \wedge (n - r^c - 1)$.

For cycles and periodic classes of length 3, by combining the possible chains of length 2 of Table 4 with possible transitions of Table 3, we have only one candidate, all other being eliminated because the collections are not disjoint:

$$N^c \xrightarrow{1} \emptyset \xrightarrow{1} N^a \xrightarrow{1} N^c.$$

Hence we find, taking into account the symmetry, two additional cycles:

(i) $N^c \xrightarrow{1} \emptyset \xrightarrow{1} N^a \xrightarrow{1} N^c$, under the condition $n^c \leq r^c \wedge r^a \wedge l^c, n^c \geq n - r^a$;
(ii) $N^a \xrightarrow{1} N \xrightarrow{1} N^c \xrightarrow{1} N^a$, under the condition $n^c \leq l^c \wedge l^a \wedge r^c, n^c \geq n - l^a$.

We now show that periodic classes of period greater than three cannot exist, which finishes the study of simple transitions.

Lemma 2 *There exists no periodic class of period $k \geq 4$.*

Proof. Let $\mathcal{S}$ be a periodic class. First, observe that if $\emptyset, N$ are not elements of $\mathcal{S}$, it is not possible to choose four distinct elements of $\mathbb{B} \setminus \{\{\emptyset\}, \{N\}\}$ such that these elements are pairwise disjoint. Hence, we suppose that there are transitions $\mathcal{B} \xrightarrow{1} \emptyset$ and/or $\mathcal{B} \xrightarrow{1} N$ in $\mathcal{S}$. From Table 3, we see that $\mathcal{B}$ is necessarily $\{N^a\}$ or $\{N^c\}$.

We claim that the cycle $\emptyset \xrightarrow{1} N^a \xrightarrow{1} N \xrightarrow{1} N^c \xrightarrow{1} \emptyset$ is impossible. Indeed, by Table 4, we have $\emptyset \xrightarrow{1} N^a \xrightarrow{1} N$ iff $n - l^a \leq n^c \leq r^c$ and $N \xrightarrow{1} N^c \xrightarrow{1} \emptyset$ (its symmetric) iff $n - r^a \leq n^c \leq l^c$. This yields, respectively,

$$\begin{aligned} 2n^c &\geq 2n - l^a - r^a > n \\ 2n^c &\leq r^c + l^c < n, \end{aligned}$$

a contradiction.

Assume that we have a transition to $\emptyset$ (the case for N is obtained by symmetry). We have either $N^a \xrightarrow{1} \emptyset$ (which is discarded because it leads to the cycle $N^a \xrightarrow{1} \emptyset \xrightarrow{1} N^a$) or $N^c \xrightarrow{1} \emptyset$. Then, the only possible absorbing class of the form $N^c \xrightarrow{1} \emptyset \xrightarrow{1} N^a \xrightarrow{1} \mathcal{B}_1 \xrightarrow{1} \cdots \xrightarrow{1} \mathcal{B}_p \xrightarrow{1} N^c$ is the cycle $\emptyset \xrightarrow{1} N^a \xrightarrow{1} N^c \xrightarrow{1} \emptyset$, for, either $\mathcal{B}_1 = N$, and we obtain the impossible cycle in the claim above, or $\mathcal{B}_1$ contains N^a or N^c, which is impossible since elements in $\mathcal{S}$ should be pairwise disjoint. □

B.2 Multiple transitions

We examine the case of transitions of the form $\mathcal{S} \xrightarrow{1} \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_p$, with $p \geq 2$, $\mathcal{S} \in 2^N$ and formed only from sets in $\mathbb{B}$, $\mathcal{B}_1, \ldots, \mathcal{B}_p \in \mathbb{B}$, and all $\mathcal{B}_1, \ldots, \mathcal{B}_p$ are pairwise incomparable by inclusion[3]. The analysis is done in the same way as for simple transitions: the above transition exists if and only if for every $S \in \mathcal{S}$, $S \xrightarrow{1} \mathcal{B}'$ with $\mathcal{B}' \in \mathbb{B}$ and $\mathcal{B}' \subseteq \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_p$ and there exist distinct $S_1, \ldots, S_p \in \mathcal{S}$ such that $S_j \xrightarrow{1} \mathcal{B}_j$ for $j = 1, \ldots, p$, which readily shows that $\mathcal{S}$ cannot be a singleton. More explicitly, using previous notation and denoting by $\mathrm{supp}(\mathcal{S}) = \{|S| \ : \ S \in \mathcal{S}\}$ the support of $\mathcal{S}$, we get:

$$\mathcal{S} \xrightarrow{1} \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_p \Leftrightarrow \begin{cases} \mathrm{supp}(\mathcal{S}) \subseteq \bigcup_{j=1}^{p} \mathcal{C}[\mathcal{B}_j] \cup \bigcup_{j=1}^{p} \mathcal{C}[\{\underline{B}_j\}] \cup \bigcup_{j=1}^{p} \mathcal{C}[\{\overline{B}_j\}] \\ \mathrm{supp}(\mathcal{S}) \cap \mathcal{C}[\mathcal{B}_j] \neq \emptyset, \quad j = 1, \ldots, p. \end{cases} \tag{23}$$

Let us investigate what the possible candidates for $\mathcal{B}_1 \cup \cdots \cup \mathcal{B}_p$ are. We begin by restricting to nontrivial intervals and $p = 2$. From Table 1, we find:

[3] The "$\cup$" is understood at the level of collections of sets, i.e., $\mathcal{B}_1 \cup \mathcal{B}_2 = \{S \in 2^N \mid S \in \mathcal{B}_1 \text{ or } S \in \mathcal{B}_2\}$.

(i) $\mathcal{S} \xrightarrow{1} [\emptyset, N^a] \cup [\emptyset, N^c]$ if and only if

$$\mathrm{supp}(\mathcal{S}) \subseteq [0, l^c] \cup [n - r^a, n] \text{ and } \begin{cases} \mathrm{supp}(\mathcal{S}) \cap]l^a, n - r^a[\cap [0, l^c] \neq \emptyset \\ \mathrm{supp}(\mathcal{S}) \cap]l^c, n - r^c[\cap [n - r^a, n] \neq \emptyset \end{cases} ; \quad (24)$$

(ii) $\mathcal{S} \xrightarrow{1} [\emptyset, N^a] \cup [N^c, N]$ if and only if

$$\mathrm{supp}(\mathcal{S}) \subseteq [0, l^c] \cup [n - r^c, n] \text{ and } \begin{cases} \mathrm{supp}(\mathcal{S}) \cap]l^a, n - r^a[\cap [0, l^c] \neq \emptyset \\ \mathrm{supp}(\mathcal{S}) \cap]l^a, n - r^a[\cap [n - r^c, n] \neq \emptyset \end{cases} ; \quad (25)$$

(iii) $\mathcal{S} \xrightarrow{1} [N^a, N] \cup [\emptyset, N^c]$ if and only if

$$\mathrm{supp}(\mathcal{S}) \subseteq [0, l^a] \cup [n - r^a, n] \text{ and } \begin{cases} \mathrm{supp}(\mathcal{S}) \cap]l^c, n - r^c[\cap [0, l^a] \neq \emptyset \\ \mathrm{supp}(\mathcal{S}) \cap]l^c, n - r^c[\cap [n - r^a, n] \neq \emptyset \end{cases} ; \quad (26)$$

(iv) $\mathcal{S} \xrightarrow{1} [N^a, N] \cup [N^c, N]$ if and only if

$$\mathrm{supp}(\mathcal{S}) \subseteq [0, l^a] \cup [n - r^c, n] \text{ and } \begin{cases} \mathrm{supp}(\mathcal{S}) \cap]l^c, n - r^c[\cap [0, l^a] \neq \emptyset \\ \mathrm{supp}(\mathcal{S}) \cap]l^a, n - r^a[\cap [n - r^c, n] \neq \emptyset \end{cases} , \quad (27)$$

the other combinations $[\emptyset, N^a] \cup [N^a, N]$ and $[\emptyset, N^c] \cup [N^c, N]$ being impossible as it can be checked. This readily shows that $p > 2$ with nontrivial intervals is impossible since a forbidden combination would appear in the list.

We consider now that singletons may appear. We begin by noticing that there is no absorbing class of the form $\{S_1, \dots, S_p\}$ with $S_j \in \{\emptyset, N, N^a, N^c\}$ for all j and $p \geq 2$. Indeed, Table 3 shows that transitions from a set S can only lead to a single T, with no possibility of multiple transition. Hence, such collections would never be connected.

Let us examine the case $\mathcal{S} \xrightarrow{1} \mathcal{B}_1 \cup \{S\}$, where $\mathcal{B}_1$ is a nontrivial interval. With $[\emptyset, N^a] \cup \{N\}$ we obtain:

$$\mathrm{supp}(\mathcal{S}) \subseteq [0, l^c] \cup ([0, l^a] \cap [n - r^c, n]) \text{ and } \begin{cases} \mathrm{supp}(\mathcal{S}) \cap [0, l^c] \cap]l^a, n - r^a[\neq \emptyset \\ \mathrm{supp}(\mathcal{S}) \cap [0, l^a] \cap [n - r^c, n] \neq \emptyset \end{cases} ,$$

which is impossible. With $[\emptyset, N^a] \cup \{N^c\}$ we obtain

$$\mathrm{supp}(\mathcal{S}) \subseteq [0, l^c] \cup ([n - r^a, n] \cap [n - r^c, n]) \text{ and } \begin{cases} \mathrm{supp}(\mathcal{S}) \cap [0, l^c] \cap]l^a, n - r^a[\neq \emptyset \\ \mathrm{supp}(\mathcal{S}) \cap [n - r^c, n] \cap [n - r^a, n] \neq \emptyset \end{cases} , \quad (28)$$

which is possible. Similarly, we find that $[\emptyset, N^c] \cup \{N\}$, $[N^a, N] \cup \{\emptyset\}$ and $[N^c, N] \cup \{\emptyset\}$ are impossible, while the following are possible:

(i) $\mathcal{S} \xrightarrow{1} [\emptyset, N^c] \cup \{N^a\}$ iff

$$\mathrm{supp}(\mathcal{S}) \subseteq [n - r^a, n] \cup ([0, l^a] \cap [0, l^c]) \text{ and } \begin{cases} \mathrm{supp}(\mathcal{S}) \cap [n - r^a, n] \cap]l^c, n - r^c[\neq \emptyset \\ \mathrm{supp}(\mathcal{S}) \cap [0, l^a] \cap [0, l^c] \neq \emptyset \end{cases} , \quad (29)$$

(ii) $\mathcal{S} \xrightarrow{1} [N^a, N] \cup \{N^c\}$ iff

$$\mathrm{supp}(\mathcal{S}) \subseteq [0, l^a] \cup ([n-r^a, n] \cap [n-r^c, n]) \text{ and } \begin{cases} \mathrm{supp}(\mathcal{S}) \cap [0, l^a] \cap]l^c, n-r^c[\neq \emptyset \\ \mathrm{supp}(\mathcal{S}) \cap [n-r^a, n] \cap [n-r^c, n] \neq \emptyset \end{cases}, \quad (30)$$

(iii) $\mathcal{S} \xrightarrow{1} [N^c, N] \cup \{N^a\}$ iff

$$\mathrm{supp}(\mathcal{S}) \subseteq [n-r^c, n] \cup ([0, l^a] \cap [0, l^c]) \text{ and } \begin{cases} \mathrm{supp}(\mathcal{S}) \cap [n-r^c, n] \cap]l^a, n-r^a[\neq \emptyset \\ \mathrm{supp}(\mathcal{S}) \cap [0, l^a] \cap [0, l^c] \neq \emptyset \end{cases}. \quad (31)$$

This shows that transitions of the form $\mathcal{S} \xrightarrow{1} \mathcal{B} \cup \{S_1\} \cup \{S_2\}$ are not possible since a forbidden configuration would appear.

We are now in position to study aperiodic absorbing classes.

(i) With $\mathcal{S} = [\emptyset, N^a] \cup [\emptyset, N^c]$, we find from (24) that

$$[0, n^a \vee n^c] \subseteq [0, l^c] \cup [n-r^a, n] \text{ and } \begin{cases} [0, n^a \vee n^c] \cap]l^a, n-r^a[\cap [0, l^c] \neq \emptyset \\ [0, n^a \vee n^c] \cap]l^c, n-r^c[\cap [n-r^a, n] \neq \emptyset \end{cases}$$

which is equivalent to

$$n^a \vee n^c > l^c \geq n - r^a. \quad (32)$$

We check connectedness. We begin by a simple observation. We have $\emptyset \xrightarrow{1} N^a$, therefore we must forbid the transitions $N^a \xrightarrow{1} \emptyset$ and $N^a \xrightarrow{1} N^a$. Using Table 1 and (32), we find that $n^a \in]l^a, n-r^a[\cup]l^c, n[$. Suppose that $n^a \in]l^a, n-r^a[$. From Table 1, we obtain that $N^a \xrightarrow{1} [\emptyset, N^a] \xrightarrow{1} [\emptyset, N^a]$, hence no connection to $[\emptyset, N^c]$ is obtained. Therefore we are forced to consider $n^a \in]l^c, n[$, which with (32) leads to

$$n^a > l^c \geq n - r^a. \quad (33)$$

From Table 1 again, this implies $N^a \xrightarrow{1} [\emptyset, N^c]$ when $n^a \in]l^c, n-r^c[$, or $N^a \xrightarrow{1} N^c$ when $n^a \in [n-r^c, n[$. We distinguish the two cases.

1. Suppose $n^a \in]l^c, n-r^c[$, so we have $\emptyset \xrightarrow{1} N^a \xrightarrow{1} [\emptyset, N^c]$. In order to connect $\emptyset, N^a$ to any set in $]\emptyset, N^a[$, there must exist $S \in [\emptyset, N^c]$ such that $S \xrightarrow{1} [\emptyset, N^a]$, i.e., $s \in]l^a, n-r^a[\cap [0, l^c] =]l^a, n-r^a[$ by (33). This is possible iff $n^c > l^a$. Let us check whether N^c is connected to any set in the class. From Table 1 and the condition $n^c > l^a$, we see that there is a possible transition to $\emptyset$, which suffices to prove that N^c is connected to any set in the class, except if $n^c \in [n-r^c, n]$ in which case $N^c \xrightarrow{1} N^c$. Therefore, we must ensure the following condition:

$$n^c \in]l^a, n-r^c[\,. \quad (34)$$

We check similarly whether any other set in the class is connected with the rest. Take $S \in]\emptyset, N^a[$. If $s \leq l^c$, there will be either a possible transition to $\emptyset$ or to N^a, so that S is connected to any set in the class. If $s > l^c$, S behaves like N^a and we are done. Take now $S \in]\emptyset, N^c[$. If $s \leq l^a$, then $S \xrightarrow{1} N^a$ and we are done. If $s \in]l^a, l^c]$, S has a possible transition to $\emptyset$ and we are done. Finally, if $s \in]l^c, n-r^c[$, S behaves like N^c. In conclusion, (34) summarizes the condition for connectedness in Case 1.

2. Suppose $n^a \in [n-r^c, n[$, so we have $\emptyset \xrightarrow{1} N^a \xrightarrow{1} N^c$. We must ensure that N^c is connected to any set in the class. In order to avoid $N^c \xrightarrow{1} N^c$ and the transitions $N^c \xrightarrow{1} N^a$ and $N^c \xrightarrow{1} \emptyset$ which would lead to cycles, we are left with the cases $n^c \in]l^a, n-r^a[$ (yielding $N^c \xrightarrow{1} [\emptyset, N^a]$) and $n^c \in]l^c, n-r^c[$ (yielding $N^c \xrightarrow{1} [\emptyset, N^c]$). We examine both cases.

2.1. Suppose $n^c \in]l^a, n-r^a[$, then we have $N^c \xrightarrow{1} [\emptyset, N^a]$. It remains to ensure that there exists $S \in]\emptyset, N^a[$ which is connected with $[\emptyset, N^c]$. We must have $s \in]l^c, n-r^c[$, always possible under Case 2. So we have established that $\emptyset, N^a, N^c$ are connected with the rest of the class. It remains to check if this is true for the other sets in the class. Take $S \in]\emptyset, N^a[$. If $s \leq l^c$, a transition to $\emptyset$ of N^a is possible, and so we are done. If $s \in]l^c, n[$, then $S \rightarrow N^c$, and we are done. Take now $S \in]\emptyset, N^c[$. Then $s \in]0, n-r^a[$, so that $S \rightarrow N^a$ and we are done. As a conclusion, connectedness holds when $n^c \in]l^a, n-r^a[$.

2.2. Suppose $n^c \in]l^c, n-r^c[$, then $N^c \xrightarrow{1} [\emptyset, N^c]$. It remains to connect some set S in $]\emptyset, N^c[$ to $[\emptyset, N^a]$, which is possible iff $s \in]l^a, n-r^a[$. This is possible under Case 2, so N^c is connected to any set in the class. We check for the remaining sets. Take $S \in]\emptyset, N^a[$. If $s \leq l^c$, a connection is possible to N^a or $\emptyset$ so we are done. Otherwise, a connection to N^c is possible and we are done. For $S \in]\emptyset, N^c[$, it works exactly the same.

In conclusion of Case 2, connectedness is ensured iff $n^c \in]l^a, n-r^a[\cup]l^c, n-r^c[$.

There does not seem to be a simple way to write the final condition. Here is one possible: connectedness holds iff $l^c \geq n-r^a$ and

$$n^c \in \big( ]r^c, n-l^c[\cap]l^a, n-r^c[\big) \cup \Big(\big( ]l^a, n-r^a[\cup]l^c, n-r^c[\big) \cap]0, r^c] \Big).$$

(ii) Similarly, using (27), $\mathcal{S} = [N^a, N] \cup [N^c, N]$ is an absorbing class if and only if $l^a \geq n-r^c$ and $n^c \in \big( ]l^c, n-r^c[\cap]r^a, n-l^c[\big) \cup \Big(\big( ]r^a, n-l^a[\cup]r^c, n-l^c[\big) \cap]0, l^c] \Big)$.

(iii) With $\mathcal{S} = [\emptyset, N^c] \cup [N^a, N]$ we find from (26) the condition $l^c \vee r^c < n^c \leq l^a \wedge r^a$. Let us check connectedness. Starting from $\emptyset$, we have $\emptyset \xrightarrow{1} N^a$, and by Table 1 and the above condition we have $N^a \xrightarrow{1} [\emptyset, N^c]$ if $n^a > l^c$, and $N^a \xrightarrow{1} \emptyset$ otherwise. Clearly, the latter must be forbidden otherwise a cycle occurs. Therefore, we must have $n^a > l^c$. Moreover, we have $N^c \xrightarrow{1} [N^a, N]$ if $n^c < n-r^c$ and $N^c \xrightarrow{1} N$ otherwise. Since $N \xrightarrow{1} N^c$, the latter must be forbidden to avoid a cycle. Therefore, we must have $n^c < n-r^c$. Under these condition, from $\emptyset$ or N^a or N^c, any set can be attained. Now, taking $S \in]\emptyset, N^c[$, we have $S \xrightarrow{1} N^a$ or $S \xrightarrow{1} [N^a, N]$ so that $S \rightarrow N^a$ and we are done. Lastly, taking $S \in]N^a, N[$, we have $S \xrightarrow{1} N^c$ or $[\emptyset, N^c]$ and we are done. As a conclusion, the condition is $l^c \vee r^c < n^c \leq l^a \wedge r^a$ and $n^c < (n-l^c) \wedge (n-r^c)$, but then we obtain the periodic absorbing class studied before. Indeed, we see from the proof that we have necessarily $[\emptyset, N^c] \xrightarrow{1} [N^a, N] \xrightarrow{1} [\emptyset, N^c]$.

(iv) With $\mathcal{S} = [\emptyset, N^a] \cup [N^c, N]$, using (25), we find that $l^a \vee r^a < n^a \leq l^c \wedge r^c$. Suppose first $l^c + r^c < n-1$. Then $\mathcal{S}$ cannot be connected. Indeed, starting from N^a, we have from Table 1 that for any set $S \in [\emptyset, N^a]$, we have either $S \xrightarrow{1} N^a$, or $S \xrightarrow{1} [\emptyset, N^a]$ or $S \xrightarrow{1} \emptyset$. Therefore, $[\emptyset, N^a]$ is not connected with every set in $\mathcal{S}$.

Suppose now that $l^c + r^c = n - 1$. The first condition in (25) reduces to the void condition $\text{supp}(\mathcal{S}) \subseteq [0, n]$. By the second condition we deduce $l^a < l^c$ and $n - r^c < n - r^a$. We check connectedness by using Table 1. We must have $N^a \xrightarrow{1} [N^c, N]$, which happens iff $n - r^c \leq n^a < n - r^a$. Now, observe that for any $S \in [\emptyset, N^a]$, the transition is either in $[\emptyset, N^a]$ or N^a (when $s \leq l^c$), or in $[N^c, N]$. To ensure that $[N^c, N]$ is connected to $[\emptyset, N^a]$, we must have $N^c \xrightarrow{1} [\emptyset, N^a]$, which happens iff $l^a < n^c \leq l^c$. Then any set $S \in [N^c, N]$ has a transition to either $[\emptyset, N^a]$ or N^a (if $s \leq l^c$) or to $[N^c, N]$ or N^c. In summary, this class exists iff $l^c + r^c = n - 1$, $n - r^c \leq n^a < n - r^a$ and $l^a < n^c \leq l^c$.

(v) We show that $[\emptyset, N^a] \cup \{N^c\}$ cannot be connected when $l^c + r^c \neq n - 1$. Indeed, we must have $N^c \xrightarrow{1} [\emptyset, N^a]$ or $N^c \xrightarrow{1} N^a$, which implies by Table 1 the condition $n^c \leq l^c$. However, by (28) and the condition $l^c + r^c \neq n - 1$, $\text{supp}(\mathcal{S})$ must be in two disjoint intervals, implying that $[0, n^a] \subseteq [0, l^c]$ and $n^c \in [n - r^c, n]$, a contradiction.

We suppose now $l^c + r^c = n - 1$ and $n - r^a \leq n - r^c$, so that in (28) the first condition reduces to the void condition $\text{supp}(\mathcal{S}) \subseteq [0, n]$. Observe that the second condition implies $l^c > l^a$. To ensure connectedness, we must have a transition from N^a to N^c, which happens iff $n^a \geq n - r^c$. Also, we must ensure $N^c \xrightarrow{1} [\emptyset, N^a]$, which happens iff $l^a < n^c < n - r^a$. Finally, we must ensure that any $S \in [\emptyset, N^a]$ is such that either $S \xrightarrow{1} N^c$ or $S \xrightarrow{1} [\emptyset, N^a]$ or $S \xrightarrow{1} N^a$. The two latter transitions arise when $s \leq l^c$, while the former transition arises when $s \geq n - r^c$. Since $n - r^c = l^c + 1$, no other transition can happen. Connectedness is then proved. Finally, it can be checked that the second condition in (28) is satisfied. In summary, this class exists iff $l^c + r^c = n - 1$, $n - r^a \leq n - r^c$, $l^c > l^a$, $n^a \geq n - r^c$ and $l^a < n^c < n - r^a$.

(vi) With $[\emptyset, N^c] \cup \{N^a\}$, we find from (29) and the assumption $l^a + r^a \neq n - 1$ that $\text{supp}(\mathcal{S})$ must be in two disjoint intervals, which forces $n - r^a \leq n^a < n - r^c$ and $n^c \leq l^a \wedge l^c$. We know already that $[\emptyset, N^c] \xrightarrow{1} N^a \xrightarrow{1} [\emptyset, N^c]$ is a periodic class. Let us show that this is the only possibility. Indeed, otherwise there should exist $S \in [\emptyset, N^c]$ such that $S \xrightarrow{1} [\emptyset, N^c]$. This would imply that $l^c < s < n - r^c$, which is impossible by the condition $n^c \leq l^c$.

Let us consider now that $l^a + r^a = n - 1$ and $l^c \geq l^a$, so that in (29) the first condition simply reduces to the void condition $\text{supp}(\mathcal{S}) \subseteq [0, n]$, while the second becomes: either $n^a \in]l^c, n - r^c[$ or $n^c > l^c$. Let us check connectedness. We must have $N^a \xrightarrow{1} [\emptyset, N^c]$ or $N^a \xrightarrow{1} N^c$. The first case happens iff $n^a \in]l^c, n - r^c[$. Then observe that without further condition on n^c, any set in $[\emptyset, N^c]$ is connected to either N^a, $\emptyset$, $[\emptyset, N^c]$ or N^c. It suffices then to forbid the transition $N^c \xrightarrow{1} N^c$, i.e., $n^c < n - r^c$. The second case happens iff $n^a \geq n - r^c$, which forces $n^c > l^c$. To ensure that N^c is connected to $[\emptyset, N^c]$, we must have $l^c < n^c < n - r^c$. Then any set in $[\emptyset, N^c]$ has a transition to either $N^a, \emptyset$ or $[\emptyset, N^c]$. In summary, this class exists iff $l^a + r^a = n - 1$, $l^c \geq l^a$, and either $n^a \in]l^c, n - r^c[$ and $n^c < n - r^c$, or $n^a \geq n - r^c$ and $l^c < n^c < n - r^c$.

(vii) The case of $[N^a, N] \cup \{N^c\}$ is similar to its symmetric $[\emptyset, N^c] \cup \{N^a\}$. The class exists iff $l^a + r^a = n - 1$, $n - r^c \leq n - r^a$, and either $n^c \in]l^c, n - r^c[$ and $n^a > l^c$, or $n^c \leq l^c$ and $l^c < n^a < n - r^c$.

(viii) The case of $[N^c, N] \cup \{N^a\}$ is similar to its symmetric $[\emptyset, N^a] \cup \{N^c\}$. The class exists iff $l^c + r^c = n - 1$, $n - l^a \leq n - l^c$, $r^c > r^a$, $n^a \geq n - l^c$ and $r^a < n^c < n - l^a$.

It remains to study the existence of periodic classes. Since the collections must be pairwise disjoint, the only possibility is the periodic class $[\emptyset, N^a] \cup [\emptyset, N^c] \xrightarrow{1} N \xrightarrow{1} [\emptyset, N^a] \cup [\emptyset, N^c]$. But we know that the second transition is impossible since a singleton cannot lead to a multiple transition. Hence, there are no such periodic absorbing classes.

$\mathcal{S} \backslash \mathcal{T}$	$\emptyset$	N^a	$[\emptyset, N^c]$	$[\emptyset, N^a]$	$[N^c, N]$	$[N^a, N]$	N^c	N
$\emptyset$	$\times$	always	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$
N^a	$n-l^c \leq n^c \leq r^a$	$n^c \geq n-l^c$ $n^c \geq n-l^a$	$n^c \leq r^a$ $r^c < n^c < n-l^c$	$n^c \geq n-l^c$ $r^a < n^c < n-l^a$	$n^c \leq r^c$ $r^a < n^c < n-l^a$	$n^c \geq n-l^a$ $r^c < n^c < n-l^c$	$n^c \leq r^c \wedge r^a$	$n-l^a \leq n^c \leq r^c$
$[\emptyset, N^c]$	$\times$	$n^c \leq l^c \wedge l^a$	$\times$	$l^a < n^c \leq l^c$	$\times$	$l^c < n^c \leq l^a$	$\times$	$\times$
$[\emptyset, N^a]$	$\times$	$n^c \geq n-l^c$ $n^c \geq n-l^a$	$\times$	$n-l^c \leq n^c$ $n^c < n-l^a$	$\times$	$n-l^a \leq n^c$ $n^c < n-l^c$	$\times$	$\times$
$[N^c, N]$	$\times$	$\times$	$n-r^a \leq n^c$ $n^c < n-r^c$	$\times$	$n-r^c \leq n^c$ $n^c < n-r^a$	$\times$	$n^c \geq n-r^c$ $n^c \geq n-r^a$	$\times$
$[N^a, N]$	$\times$	$\times$	$r^c < n^c \leq r^a$	$\times$	$r^a < n^c \leq r^c$	$\times$	$n^c \leq r^c \wedge r^a$	$\times$
N^c	$n-r^a \leq n^c \leq l^c$	$n^c \leq l^c \wedge l^a$	$n^c \geq n-r^a$ $l^c < n^c < n-r^c$	$n^c \leq l^c$ $l^a < n^c < n-r^a$	$n^c \geq n-r^c$ $l^a < n^c < n-r^a$	$n^c \leq l^a$ $l^c < n^c < n-r^c$	$n^c \geq n-r^c$ $n^c \geq n-r^a$	$n-r^c \leq n^c \leq l^a$
N	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	always	$\times$

Table 3. Conditions for sure transitions $\mathcal{S}$ to $\mathcal{T}$

$N^a \xrightarrow{1} \emptyset \xrightarrow{1} N^a$	$n - l^c \leq n^c \leq r^a$
$N^c \xrightarrow{1} \emptyset \xrightarrow{1} N^a$	$n - r^a \leq n^c \leq l^c$
$\emptyset \xrightarrow{1} N^a \xrightarrow{1} [N^c, N]$	$n^c \leq r^c$ $r^a < n^c < n - l^a$
$\emptyset \xrightarrow{1} N^a \xrightarrow{1} N^c$	$n^c \leq r^c \wedge r^a$
$\emptyset \xrightarrow{1} N^a \xrightarrow{1} N$	$n - l^a \leq n^c \leq r^c$
$N^c \xrightarrow{1} N^a \xrightarrow{1} \emptyset$	$n - l^c \leq n^c \leq l^c \wedge l^a \wedge r^a$
$[\emptyset, N^c] \xrightarrow{1} N^a \xrightarrow{1} [\emptyset, N^c]$	$n^c \leq l^c \wedge l^a \wedge r^a$ $r^c < n^c < n - l^c$
$N^c \xrightarrow{1} N^a \xrightarrow{1} N^c$	$n^c \leq l^c \wedge l^a \wedge r^c \wedge r^a$
N^c or $[\emptyset, N^c] \xrightarrow{1} N^a \xrightarrow{1} N$	$n - l^a \leq n^c \leq l^a \wedge l^c \wedge r^c$
$N^a \xrightarrow{1} [\emptyset, N^c] \xrightarrow{1} N^a$	$n^c \leq l^a \wedge l^c \wedge r^a$ $r^c < n^c < n - l^c$
$[N^a, N] \xrightarrow{1} [\emptyset, N^c] \xrightarrow{1} [N^a, N]$	$l^c \vee r^c < n^c \leq r^a \wedge l^a$

Table 4. Conditions for chains of length 2 potentially yielding periodic classes