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On Investment and Cycles in Explicitely Solved Vintage Capital Models [*]

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Abstract

The purpose of this contribution is to consider a discrete time formulation that would allow for clarifying some salient features of a vintage based understanding of the capital stock. Three main lines of conclusions are established on an analytical basis. First and for an elementary configuration with linear utility, it is proved that the rate of growth of investment is prone to andoscillating—convergent, sustained or unstable—motions. Second and for an environment with a linear production technology and a AK setup, the dynamics of investment is explicitly solved and it is established that the rate of growth of investment may either converge to the steady growth solution in oscillating way, diverge from that solution in a oscillating way, or even undergo permanent sustained oscillations with a periodicity of two. Third, it is proved that no perennial fluctuations can emerge within a benchmark environment with strictly concave utilities and production technologies. On a methodological basis, few restrictions are superimposed, the arguments remain fairly general and the proofs are elementary.

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I. Introduction

This article reconsiders the role of capital and machines in capital accumulation through a minimal departure from the standard Ramsey-Cass-Koopmans for which the law of motion of the capital stock builds not only from *recent* but also from *old* investment. This may be understood as a raw and elementary form of the *vintage capital growth theory* literature. Whilst vintage capital growth models were core to growth theory by the glorious sixties, the technical complexity of their appraisal led to their progressive abandonment. They however and quite unexpectedly recently gained some traction that resulted in a flourishing literature: the development of optimal control techniques suited for their analysis together with their support by numerical tools, noticeably in the economic demography area, has even widened their relevance far beyond their earlier classical domains of capital theory. A comprehensive and thorough appraisal of this literature and its ongoing developments is available in Boucekkine, de La Croix & Licandro [13].

A limit to this literature however stems from the extreme sophistication of the entailed delay-difference equations structure and from the formally demanding nature of the tools that are involved. First, their need for closed forms solutions commonly requires a set of strong parametrical restrictions on the fundamentals of the economy. Second, even though many of the underlying ideas ought to improve and enrich the understanding of economic phenomena based upon complex aging and generations structures, the tools afforded by the modern vintage growth literature may at first not ease the direction of their use. The first purpose of this contribution is then to illustrate how some of its most stringent insights can be recovered through basic arguments and within a benchmark discrete time formulation. This will noticeably allow for clarifying some salient features of a vintage based understanding of the capital stock but also to generalize many of their conclusions beyond the reliance to parametric forms, standard in the delay-difference vintage capital theory literature. Finally, and in order to ensure tractability, this article specializes to a *putty-putty* acceptation of vintage capital that relates to a non-constant depreciation of the capital stock and a time to build assumption.

The approach proceeds from an augmented law of accumulation that embodies a memory for past investment. This introduction of a positive survival rate for past investments results in an augmented form of the Euler equation with extra dependencies of current investment with respect to not only *delayed* but also *advanced* investment. Having first completed some comparative statics on the enriched long-run definition of the capital stock, three main lines of conclusions are then established on an analytical basis. First and for an elementary configuration with linear utility, it is proved that the rate of growth of investment is prone to andoscillating—convergent, sustained or unstable—motions. This cycling dimension recovers one of the most striking features of vintage capital models that comes at odds with the standard monotone motions that derive from classical investment acceptions. It is further noticed that the capital stock being constant and the investment cyclical, the rate of depreciation of the capital stock is similarly cyclical. Second and for a configuration with a linear production technology and a AK unbounded growth argument setup, , the dynamics of investment is explicitly solved and it is established that the rate of growth of investment may either converge to the steady growth solution in oscillating way, diverge from that solution in a oscillating way, or even undergo permanent sustained oscillations with a periodicity of two. Third, the consideration of a general utility function and a strictly concave production technology clarifies the former results by allowing for the occurrence of complex roots while unequivocally ruling out any scope for the occurrence of a pair of imaginary roots and thus any area for self-sustained perennial fluctuations associated with the emergence of a limit cycle. All these lines of arguments do illustrate the methodological relevance of the current *discrete* approach to *vintage capital growth theory*: while few restrictions are superimposed, the arguments remain fairly general and the proofs are elementary.

The basic environment and an early characterization of steady states are available in Section II. A comprehensive analytical examination of echo effects will linear utility is detailed in Section III. A reconsideration of the dynamics of vintage environments with linear technologies is contemplated in Section IV. A generalised argument with strictly concave productions and technologies is considered in Section V. Some formal proofs are gathered in a final appendix.

II. Optimal Accumulation with Long-Standing Investment

Time is discrete. Consider a capital stock, denoted as K_t at period $t \geqslant 0$, that results from a *finite list* of past investments along $K_t = \iota \sum_{i=-\infty}^{t-1} \gamma^{t-i} I_i$. In its simplest form with current investment I_t and past investment I_{t-1}, this results in the following expression for the capital stock at date t:

(1) $$K_{t+1} = \iota \left(I_t + \gamma I_{t-1} \right),$$

for $\iota \in]0, 1]$ and $\gamma \in [0, 1/\iota]$ where $\iota\gamma$ corresponds to a *survival rate* for past investments. It is firstly obvious that the configuration $\iota = 1$ and $\gamma = 0$ is associated with the standard model with a unitary depreciation for the capital stock. For $\gamma \in]0, 1[$, the efficiency of past investment goods—*old machinery*—decreases over time and until their eventual decay after two periods. In opposition to this, along Arrow [3]-like *learning by using* ideas and for $\gamma \in]1, 1/\iota[$, the efficiency of investment goods increases between their actual installment and their eventual decay.

Remark 1 The configuration $\iota = 1$ and $\gamma \in]0, 1]$ pictures a vintage growth model with non constant depreciation of the capital stock and was analysed by Benhabib & Rustichini [10] and Benhabib & Rustichini [11] within a continuous time environment, these authors being the first to face mixed-type differential equations. It is worth emphasizing that, while Benhabib & Rustichini [10] briefly considered a setup with a two periods delay for investment, the current work is the first to provide an explicit solving of the model. On related concerns, it should be mentioned that, while Benhabib & Rustichini [11] provided a detailed continuous time appraisal, their solution is not analytically characterized, the difficult stability issue being only heuristically sketched.

Remark 2 The configuration with $\iota \in]0, 1[$ and $\gamma > 0$ relates to the extensive continuous time time-to-build literature. While this finds its roots in the celebrated Arrow [3] exposition, its discrete-time formal appraisal by Kydland & Prescott [23] led to numerous simulations in the Real Business Cycle literaure. While it was first investigated in a continuous time optimal growth environment by the contribution of El-Hodiri, Lehman and Whinston [18], more recent contributions in this hard to solve cotinuous time models include Asea & Zak [4], Bambi [5], Bambi, Gozzi and Fabbri [6] and d'Albis, Augeraud-Véron & Hupkes [1].

Having assumed that population—the labour force—was constant and normalized to 1, the description of the production process is complemented by the introduction of a function $F(K, 1)$. Letting C_t

denote the aggregate consumption amount at period $t \geqslant 0$, a set of sequences $(K_t, I_t)_{t\in\mathbb{N}}$ will be referred to as *feasible* from K_0, I_{-2} and I_{-1} if it satisfies:

(1a) $C_t + I_t \leqslant F\left[\iota\left(I_{t-1} + \gamma I_{t-2}\right), 1\right]$,

(1b) $K_t = \iota\left(I_{t-1} + \gamma I_{t-2}\right)$,

(1c) $I_{t-1} + \gamma I_{t-2} \geqslant 0$, $F\left[\iota\left(I_{t-1} + \gamma I_{t-2}\right), 1\right] - I_t \geqslant 0$ for all $t \geqslant 0$, $\iota\left(I_{-1} + \gamma I_{-2}\right) > 0$ given.

Remark 3 The depreciation rate η_t of this environment is *endogenouly determined* from the accumulation law of the capital stock according to: $K_{t+1} = \left(1 - \eta_t\right) K_t + \iota I_t$, whence, making use of (1), its availability as: $\eta_t = 1 - \iota\gamma I_{t-1}/K_t$. In other words, the greater the survival rate of past investments $\iota\gamma$ and the lower the depreciation rate η_t, that further happens to parallely decrease with the ratio I/K.

Consider then the following planning program over the set of feasible sequences:

$$\max_{\{I_t\}_{t\geqslant 0}} \sum_{t=0}^{\infty} \delta^t u\left[F\left(\iota\left(I_{t-1} + \gamma I_{t-2}\right), 1\right) - I_t\right]$$
$$\text{s.t.} \quad -\gamma I_{t-1} \leqslant I_t \leqslant F\left(\iota\left(I_{t-1} + \gamma I_{t-2}\right), 1\right)$$
$$\delta \in]0, 1[, \quad \iota\left(I_{-1} + \gamma I_{-2}\right) > 0 \text{ given,}$$

where the following classical range of restrictions are superimposed on F(K, 1) and $u(\cdot)$:

Assumption T1 $F : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, for $F(\cdot, 1)$ that is twice continuously differentiable. Its derivatives satisfy $F'_K(K, 1) > 0$ and $F''_{KK}(K, 1) \leqslant 0$ for every $K \in \mathbb{R}^*_+$.

Assumption T2 $\lim_{K\to+\infty} F'_K(K, 1) = 0$, $\lim_{K\to 0} F'_K(K, 1) = +\infty$ when $F''_{KK}(K, 1) < 0$ for every $K \in \mathbb{R}^*_+$.

Assumption P1 $u : \mathbb{R}_+ \longrightarrow \mathbb{R}$, *for* $u(\cdot)$ *that is twice continuously differentiable and whose derivatives satisfy* $u'(C) > 0$ *and* $u''(C) \leqslant 0$ *for any* $C \in \mathbb{R}^*_+$.

Assumption P2 $\lim_{C\to 0} u'(C) = +\infty$ *when* $u''(C) < 0$ *for any* $C \in \mathbb{R}^*_+$.

Henceforward focusing on interior solutions and from the strict increasingness assumption P2 on $u(\cdot)$, the optimization problem conveniently reformulates as the optimal selection of *current* investment I_t for *given past* investments I_{t-1} and I_{t-2}. In reduced form, the optimization problem then reformulates to:

$$\max_{\{I_t\}_{t\geqslant 0}} \sum_{t=0}^{\infty} \delta^t \Psi_{u,\Gamma}\left(I_{t-2}, I_{t-1}, I_t\right)$$
$$\text{s.t.} \quad I_t \in \Gamma\left(I_{t-2}, I_{t-1}\right),$$
$$\delta \in]0, 1[, \quad \iota\left(I_{-1} + \gamma I_{-2}\right) > 0 \text{ given,}$$

for $\Psi_{u,\Gamma}\left(I_{t-2}, I_{t-1}, I_t\right) = u\left[F\left(\iota\left(I_{t-1} + \gamma I_{t-2}\right), 1\right) - I_t\right]$ and $\Gamma\left(I_{t-2}, I_{t-1}\right) := [-\gamma I_{t-1}, F(\iota(I_{t-1} + \gamma I_{t-2}), 1)]$.

Under the above list of assumptions, it is a standard argument to prove that $\Psi_{u,\Gamma}\left(I_{t-2}, I_{t-1}, I_t\right)$ is continuous with respect to the product topology on the compact set of feasible investment sequences and for a given pair (I_0, I_{-1}), that establishes the existence of an optimal solution for this optimization program. Likewise, it is straightforwardly established that an optimal selection of current investment I_t would then be to satisfy:

(2a) $\quad -u'\left(C_t\right) + \delta\iota\left\{F'_K\left[\iota\left(I_t + \gamma I_{t-1}\right), 1\right] u'\left(C_{t+1}\right) + \delta\gamma F'_K\left[\iota\left(I_{t+1} + \gamma I_t\right), 1\right] u'\left(C_{t+2}\right)\right\} = 0;$

(2b) $\quad C_t + I_t - F\left[\iota\left(I_{t-1} + \gamma I_{t-2}\right), 1\right] = 0;$

(2c) $\quad \lim_{t\longrightarrow+\infty} \delta^t u'\left(C_t\right)\cdot\iota\left(I_{t-1} + \gamma I_{t-2}\right) = 0,$

It is noticeable that the introduction of a positive survival rate for past investments—$\gamma > 0$—results in extra dependencies of current investment I_t with respect to not only *delayed* investment I_{t-2} but also *advanced* investment I_{t+2}. This[*] directly mimics the mixed type differential equations structure analysed in Benhabib & Rustichini [11], Bambi [5] and d'Albis, Augeraud-Véron and Hupkes [1].

Remark 4 While this article centrally focused on a centralized planning of the optimal growth problem, it is worthwhile mentioning that, for a competitive economy version, the inter temporal program of the firm would state as the determination of $(I_t, L_t)_{t\in\mathbb{N}}$ so as to maximise the discounted sum of its profits $\sum_{t=0}^{+\infty}\left[pF\left(K_t, L_t\right) \quad - pI_t - w_t L_t\right] / \prod_{i=0}^{t}\left(1 + r_i\right)$ subject to $K_t = \iota\left(I_{t-1} + \gamma I_{t-2}\right)$, for a given $K_0 > 0$. The optimality condition relative to labour demand uncovers a standard acceptation with $w_t/p = F'_L\left(K_t, L_t\right)$ whilst the condition that relates to investment now satisfies:

$$-1 + \frac{1}{1 + r_{t+1}}\left[F'_K\left(K_{t+1}, L_{t+1}\right) + \frac{\gamma}{1 + r_{t+2}} F'_K\left(K_{t+2}, L_{t+2}\right)\right] = 0.$$

It is worth emphasizing that the solving of such a program hence cannot come down to the one of a static optimization problem where the rate of interest at date t is defined by the capital stock at the same date.

A steady state position lists as a 3-uple $\left\{K^*, C^*, I^*\right\}$ that is a solution to:

(3a) $\quad F'_K\left(K^*, 1\right) = 1/\delta\iota(1 + \delta\gamma),$

(3b) $\quad I^* = K^*/\iota(1 + \gamma),$

(3c) $\quad C^* = F\left(K^*, 1\right) - K^*/\iota(1 + \gamma).$

Its main properties are gathered in the following statement:

Proposition II.1.— *Under Assumptions* P.1-2, T.1-2

(i) *the steady state exists and is unique;*

(ii) *the steady state depicts an under-accumulation configuration with respect to the golden rule of accumulation* $\hat{K}$ *defined from* $F_K(\hat{K}, 1) = 1/\iota(1 + \gamma)$;

[*] Likewise, the consideration of a generalised form of (1) and of a sequence of parameters $\gamma_1, \gamma_2, \ldots, \gamma_n$ that would respectively single out the dependency of K_t with respect to $I_{t-1}, I_{t-2}, \ldots, I_{t-n}$ would have resulted in extra dependencies of current investment I_t with respect to a list of *n advanced* investments $I_{t+2}, \ldots, I_{t+n+1}$.

(iii) *the greater the survival rate of past investments* γ *and the larger the steady state value of the capital stock;*

(iv) *letting* σ_F *and* π *respectively denote the elasticity of substitution and the share of profits in the production technology, the stationary value of investment* I^* *increases (decreases) as a function of past investments for* $\sigma_F \geqslant 1 - \pi$ *or when* $1 + \gamma\alpha \geqslant (1 - \alpha)/\delta$ *for a Cobb-Douglas production technology where* α *denotes the constant elasticity of the production with respect to the capital stock.*

While proposition II.1 provides some interesting insights about the qualitative properties of long-run vintage investment, it remains to wonder in which ways this can be imbedded within a more general picture involving the dynamics of investment. This issue will be tackled in the two subsequent sections with a look for explicit solutions for investment : while the first will specialize the argument to a linear utility function, the second will be interested in a homogeneous environment, popular in the vintage literature, that will open the road for steady growth rates of investement.

III. Echo Effects on Investment

Relaxing for now the strict concavity assumption P1 on the utility function, it is noticed that the retainment of a linear utility $u(C) = C$ implies that the Euler equation (2a) simplifies to a simple first-order equations along $-1 + \delta\iota\left[F'_K(K_{t+1}, 1) + \delta\gamma F'_K(K_{t+2}, 1)\right] = 0$. Its properties are then gathered in the following statement:

Proposition III.1.— *Consider equilibrium dynamics and the unique steady state for* $u(C) = C$*:*

(i) *for* $\iota = 1$ *and* $\gamma = 0$, *a jump of the investment onto the stationary solution takes place at* $t = 0$ *and capital assumes stationary values starting from date* $t = 1$ *;*

(ii) *for* $0 < \gamma \leqslant 1$, *the explicit dynamics of investment are given by:*

$$I_t = \frac{1}{\iota(1+\gamma)}\left[K^* + \left(-\gamma^{t+1}\right)\left[\iota(1+\gamma)I_{-1} - K^*\right]\right].$$

a/ *for* $0 < \gamma < 1$, *the dynamics of the capital stock follows a divergent cycling motion and* K^* *is locally unstable and investment converges in a cyclical way to its stationary value;*

b/ *for* $\gamma = 1$, *investment follows a cyclical motion by the infinite;*

(iii) *for* $1 < \gamma < 1/\iota$, *the dynamics of the capital stock follows a convergent cycling motion towards* K^*.

An interesting advance brought by this simple assessment of the linear utility case springs from the possibility of obtaining an explicit exact solution for investment. This allows for completing a graphical depiction of investment over time on Figure 1 and provides a tractable illustration of the so-called *echo effects*. More explicitly, and for a large initial value of investment I_{-1}, only a small investment I_0, *i.e.*, lower than its stationary value, will be completed at date 0. This will in its turn anew results in a relatively large investment ,*i.e.*, larger than its stationary value, at date 1 and as I_1. This process will converge according to the slope of the recurrence equation equation for investment in Proposition (ii).

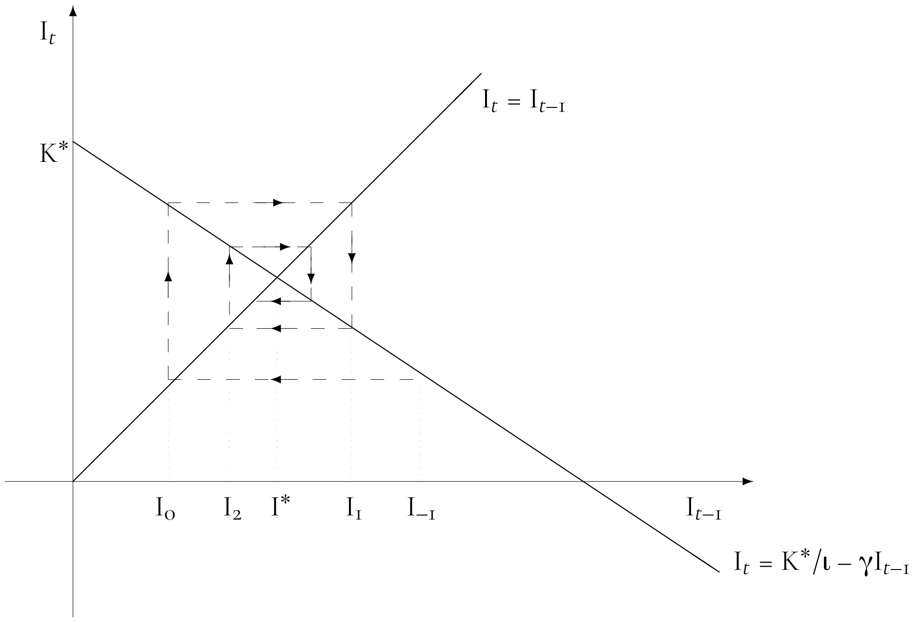

Figure 1: Oscillating convergence & echo effects

On Figure 2, the comparative statics properties of investment are first analyzed after a rise of δ that unsurprisingly translates into a raise of the stationary value of investment K^* but proceeds through the same above mentioned interesting echo effects with an initial over-investment at date 0. The impacts of a rise in ι and γ on the stationary investment are also of interest in that, while they unequivocally result in a higher value of K^*, they won't necessarily have the same class of impact on I^* and may very well translate in an immediate decrease in this latter value. The capital stock being constant and the investment cyclical, the rate of depreciation of the capital stock is similarly cyclical (though in the opposite sense).

REMARK 5 The upsurge of echo effects with linear utility was also assessed in Boucekkine, Germain & Licandro [16] that similarly emphasized the role of initial over-investment or under-investment. While the collection of results in Proposition (ii) compares closely to the ones of the recent vintage literature and more specifically Benhabib & Rustichini [10, 11], they are currently, as a result of the discrete time formulation and the allowance for explicit exact dynamics for investment, easier to derive in the current environment.

Interestingly, the retainment of an extra non-negativity constraint on investment brings some qualifications upon the above line of arguments:

COROLLARY III.1.— *Consider equilibrium dynamics and the unique steady state for* $u(C) = C$ *and further assume that the investment is non negative:*

(i) *for* $K^* < \gamma\iota I_{-1}$*, the optimal solution states as* $I_0 = 0$ *(that in turn implies* $K_1 = \gamma\iota I_{-1}$*),* $I_1 = K^*/\iota$ *(that implies* $K_2 = \iota K^*/\iota = K^*$*), followed by a cyclical convergence towards the stationary state at* I^* *;*

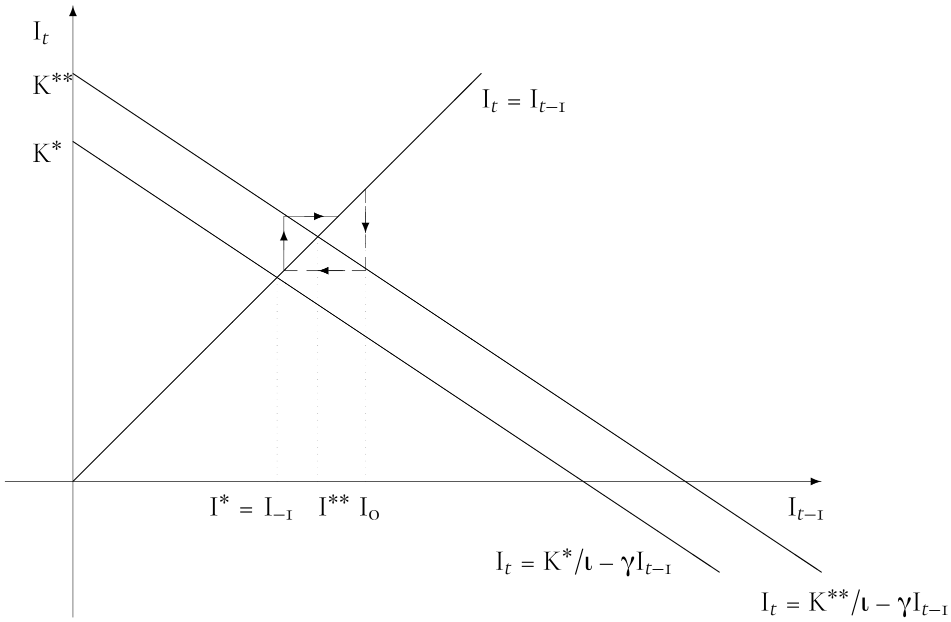

Figure 2: Adjustment of I_t after a rise in δ and associated echo effects.

(ii) *the exact solution for investment states, for any $t \geq 1$, as*

$$I_t = K^* \frac{\left[1 - (-\gamma)^t\right]}{\iota(1 + \gamma)}.$$

The non-negativity assumption on consumption is also to be taken into account, for it may delay the date at which the steady state is reached by the capital stock.

IV. Investment Growth Fluctuations

Relaxing now the strict concavity of Assumption T2 on the production technology but also the linearity assumption of Section III on the utility function, a configuration with $F(K_t, 1) = AK_t$ and $u(C_t) = (C_t)^{1-1/\sigma}/(1 - 1/\sigma)$, $\sigma \in]0, 1[\cup]1, +\infty[$ shall now be considered, noticeably because it had become, with Boucekkine, de la Croix & Licandro [15] and subsequent developments, the benchmark for numerous studies of the delay-difference vintage capital theory literature. The ressource constraint reformulates to:

(4a) $C_t = AK_t - I_t$

(4b) $\quad = -I_t + A\iota I_{t-1} + A\iota\gamma I_{t-2}$

The main complicatedness of this new environment with nonlinear iso-elastic utilities results from the involved formal complexity of the Euler equations with $\gamma \neq 0$, that was already clear from equation (2a). On the other hand, the linearity of (4a) and of the involved benchmark equation $C_t = AK_t - K_{t+1}$ was key to the popularity of this AK framework in the literature. While its appraisal is currently made more difficult by the retainment of $\gamma \neq 0$ in equation (4b), this paragraph will establish how the tractability of this environment is preserved, as enriched as it may be from the vintage structure.

For that purpose, the argument shall proceed through the consideration of a change of variables that will allow for recovering a structure formally related to the benchmark environment. Let indeed:

$$(5) \qquad Z_t = I_t - \mu I_{t-1}.$$

Replacing the entailed expression of I_t in Equation (4b), it derives that:

$$(6) \qquad C_t = - Z_t + \left(A\iota - \mu\right) Z_{t-1} + \left[\mu \left(A\iota - \mu\right) + A\iota\gamma\right] I_{t-2}.$$

Now, in order for equation (6) to replicate the analytical tractability of the benchmark structure $C_t = AK_t - K_{t+1}$, one is to seek a value of μ that will allow for reformulating C_t as a mere function of Z_t and Z_{t-1} in this last expression. It is thus to solve: $-\mu^2 + \mu A\iota + A\iota\gamma = 0$, its roots being available as:

$$\mu_{1,2} = \frac{A\iota}{2}\left[1 \mp \left(1 + \frac{4\gamma}{A\iota}\right)^{1/2}\right].$$

Selecting the largest root, the entailed expression of C_t hence derives as:

$$(7) \qquad C_t = -Z_t + \frac{A\iota}{2}\,(1 + \zeta)\, Z_{t-1},$$

for $\zeta := (1 + 4\gamma/A\iota)^{1/2}$. Noticing that the value of Z_0 is unknown while the one of Z_{-1} is taken as a given, the optimisation problem formulates as:

$$(8) \qquad \max_{\{Z_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \delta^t \frac{1}{1 - 1/\sigma}\left[-Z_t + \frac{A\iota}{2}(1 + \zeta)Z_{t-1}\right]^{1-1/\sigma}, \; Z_{-1} \text{ given.}$$

The inter-temporal optimality conditions are available as:

$$(9a) \qquad -\left[-Z_t + \frac{A\iota}{2}(1 + \zeta)Z_{t-1}\right]^{-1/\sigma} + \delta\frac{A\iota}{2}(1 + \zeta)\left[-Z_{t+1} + \frac{A\iota}{2}(1 + \zeta)Z_t\right]^{-1/\sigma} = 0,$$

$$(9b) \qquad \lim_{t\longrightarrow+\infty} \delta^t \left[-Z_t + \frac{A\iota}{2}(1 + \zeta)Z_{t-1}\right]^{-1/\sigma} \cdot K_t = 0,$$

It is worth emphasizing that, from (7), while equation (9a) will deliver the explicit rate of growth of consumption, solely this largest root for μ is admissible in this regard.† This inter-optimaility optimality condition on Z also favorably compares to equation (2a): even though it allows for $\gamma \neq 0$, it also provides the analysis with a much simpler structure that mimics the one of the benchmark AK model. One thus eventually obtains an explicit solution for the constant rate of growth of consumption as:

$$(10) \qquad 1 + g^C := \frac{C_{t+1}}{C_t} = \left\{\delta\frac{A\iota}{2}(1 + \zeta)\right\}^{\sigma}.$$

It is worth emphasizing that, had the more complex Euler equation (2a) been considered, this would have resulted in a *dynamic formulation* for the growth rate of consumption. While its constant formulation would have assumed two solution candidates, only the one displayed in equation (10) would have been admissible.

In order now to look for then entailed expression of investment, one is first to characterize the involved structure of Z_t. These are summarised in the following statement:

† Had he other candidate solution for μ been retained, this optimality condition would have listed as $-(C_t)^{-1/\sigma} + \delta A\iota[1 - (1 + 4\gamma/A\iota)^{1/2}](C_{t+1})^{-1/\sigma}/2 = 0$, *i.e.*, an inconsistent solution as a result of the satisfaction of $1 - (1 + 4\gamma/A\iota)^{1/2} < 0$ for $\gamma > 0$.

Lemma IV.1 *Let* $F(K_t, 1) = AK_t$, $u(C_t) = (C_t)^{1-1/\sigma}/(1 - 1/\sigma)$, $\sigma \in]0,1[\cup]1,+\infty[$, $Z_t = I_t - \mu I_{t-1}$, *for* $\mu = A\iota\zeta/2$, *and consider the homogenous optimization program (8). Then the optimal sequence of* $(Z_t)_{t\geqslant 0}$ *assumes the following properties:*

(i) *Its constant growth rate is available as:*

$$1 + g^Z := \frac{Z_{t+1}}{Z_t} = \left[\delta \frac{A\iota}{2}(1+\zeta)\right]^{\sigma}.$$

(ii) *Its expression at date t reformulates to:*

$$Z_t = \left[\delta \frac{A\iota}{2}(1+\zeta)\right]^{\sigma(t+1)} \left[I_{-1} - \frac{A\iota}{2}(1+\zeta)I_{-2}\right].$$

This will in turn enable for the obtention of the first main result of this section, namely an explicit expression for the optimal investment and the establishment of its oscillating motion:

Proposition IV.1 *Let* $F(K_t, 1) = AK_t$, $u(C_t) = (C_t)^{1-1/\sigma}/(1 - 1/\sigma)$, $\sigma \in]0,1[\cup]1,+\infty[$, $Z_t = I_t - \mu I_{t-1}$ *and consider the homogenous optimization program (8). Then the optimal value of* I_t *is available at any date* $t \geqslant 0$ *as:*

$$I_t = I_{-1}\sum_{j=0}^{t+1}\left[\frac{A\iota}{2}(1-\zeta)\right]^j \left(\delta\frac{A\iota}{2}(1+\zeta)\right)^{\sigma(t+1-j)} - I_{-2}\sum_{j=0}^{t}\left[\frac{A\iota}{2}(1-\zeta)\right]^{j+1}\left(\delta\frac{A\iota}{2}(1+\zeta)\right)^{\sigma(t+1-j)}$$

where the scope for oscillations emerges from the satisfaction of $\zeta > 1$, *and thus of* $(1-\zeta)^j$ *that alternatively assumes negative and positive values.*

This homogenous structure in the formulation of I_t unravels the actual scope for the growth dynamics of investment:

Lemma IV.2 *Let* $F(K_t, 1) = AK_t$, $u(C_t) = (C_t)^{1-1/\sigma}/(1 - 1/\sigma)$, $\sigma \in]0,1[\cup]1,+\infty[$, $Z_t = I_t - \mu I_{t-1}$ *and consider the homogenous optimization program (8).*

(i) *Letting* $1 + g_t^I := I_{t+1}/I_t$, *the dynamics of the growth rate for investment is available from:*

$$1 + g_{t+1}^I = \left[1 - \frac{\iota A}{2}(1-\zeta)\frac{1}{1+g_t^I}\right]\left(1+g^C\right) + \frac{\iota A}{2}(1-\zeta).$$

(ii) *The unique admissible steady growth solution is such that:*

a/ *It is characterized by a constant ratio* I/Z *and satisfies:*

$$1 + g^I = 1 + g^Z = 1 + g^C = 1 + g^K = \left[\delta\frac{A\iota}{2}(1+\zeta)\right]^{\sigma}.$$

b/ *It corresponds to an interior steady growth solution for investment that ensures positive consumption, ensures a bounded value for the objective and fulfills the transversality condition for:*

$$\delta^{-1} < \frac{A\iota}{2}(1+\zeta) < \delta^{1/(1-1/\sigma)}.$$

With respect to the benchmark solution with $\gamma = 0$ and $\zeta = 1$, this hence differs in allowing for an explicit dynamics for the growth rate of investment in place of the requisite for an immediate jump on an unstable growth ray. Again and mimicking in this regard proposition IV.1, the satisfaction of $\zeta > 1$ for $\gamma \neq 0$ will result in lemma IV.2(i) in an oscillation motion for the growth rate of investment. Its long run convergence properties are eventually gathered in the following statement:

Proposition IV.2 *The long-run steady growth rate of investment is such that:*

(i) *It is locally stable (unstable) if and only the following condition is satisfied:*

$$\frac{A\iota}{2}(1+\zeta) > (<) \frac{1}{\delta}\left[-\frac{\iota A}{2}(1-\zeta)\right]^{1/\sigma}.$$

(ii) *Assume that none of the conditions of* (i) *is fulfilled. Then:*

a/ *For* $\sigma \leq 1$, *there exists a unique critical value* γ^c *such that permanent sustained cycles occurs in its neighbourhood.*

b/ *For* $\sigma = 1$,

$$\gamma^c = \frac{A\iota}{4}\left[\left(\frac{1+\delta}{1-\delta}\right)^2 - 1\right]$$

and permanent sustained cycles occur in its neighbourhood.

c/ *For* $\sigma > 1$, *there does not exist permanent sustained cycles in its neighbourhood if* $\sigma\delta^\sigma(A\iota)^{\sigma-1} > 1$.

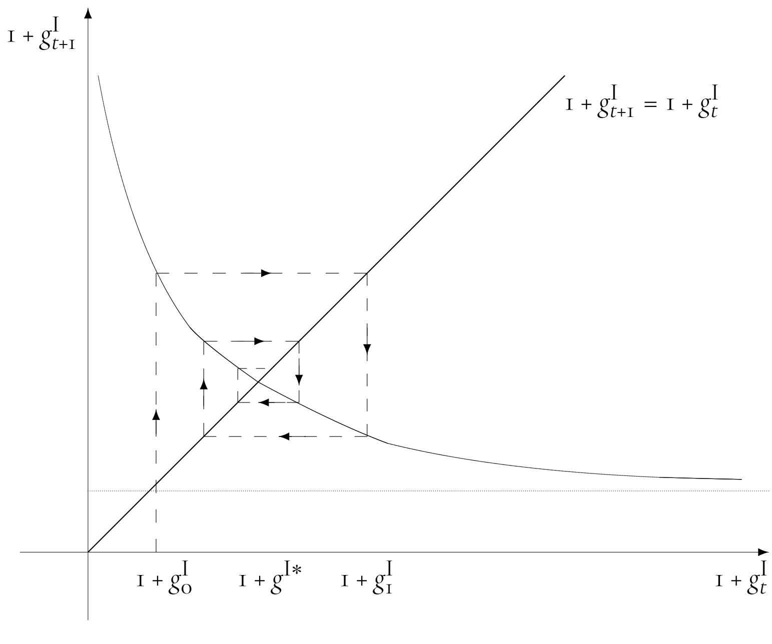

Figure 3: Convergent oscillating dynamics for the investment growth rate

As unsurprising at it may appear at first sight on Figure 3, the convergence result to the steady growth ray is worth emphasising in three regards. First, and contrasting in this regard with the benchmark environment with $\gamma = 0$ and $\zeta = 1$ that is associated with an immediate jump to the long-run solution, the achievement of a steady growth solution here proceeds from an explicit convergence process. Second, this convergence is not monotonic but oscillatory and hence conveys the most stringent feature of dynamics with vintage capital. Third, and this is presumably the most striking feature of this environment, such a convergence process is conditional to the satisfaction of the condition in Proposition IV.2(i).

Its violation results either in a divergent motion, in which case solely the obtention of $1 + g_0^I = 1 + g^{I*}$ would guarantee the obtention of a steady growth solution, or, more interestingly and as this is emphasized in Proposition IV.2(ii) and pictured in figure 4, in an optimal sustained cycling motion $(1 + g^{I_1^*}, 1 + g^{I_2^*})$ for the steady growth rate. It is worth emphasising that figure 4 pictures the so-called *supra-critical* configuration where optimal 2-period cycles emerge in a right neighbourdhood of the critical value γ^c, *i.e.*, a configuration for which $1 + g^{I^*}$ is locally unstable while the steady growth cycle is locally stable. To the best knowledge of the authors, this cycles configuration has hitherto only been reached in benchmark optimal multi-sectoral environments or in sub-optimal models of overlapping generations with strong income effects.

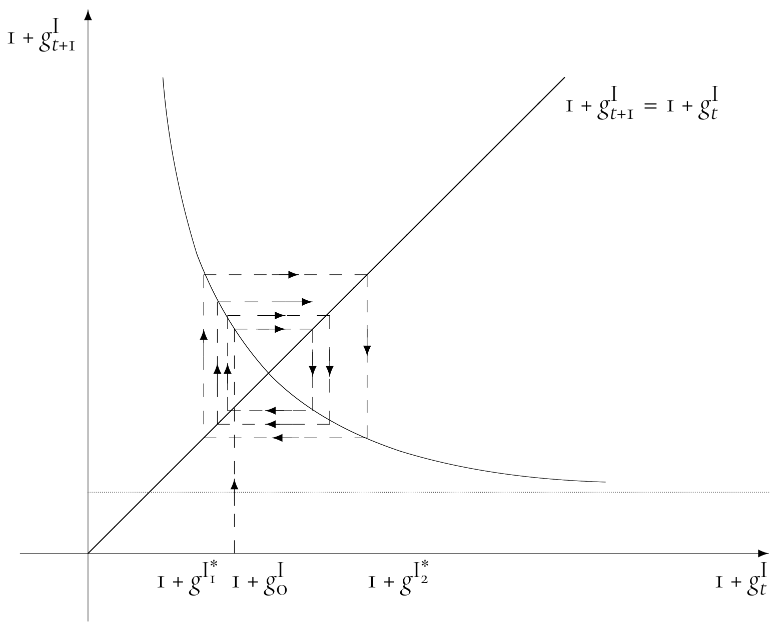

Figure 4: Stable sustained cycling dynamics for the investment growth rate

Remark 6 The benchmark study of endogenous growth with a vintage capital formulation is due to Bouccekine, Licandro, Puch & del Rio [14]. While these authors provide a new theoretical approach to stability, in contradistinction with the current analysis, they are rely upon numerical techniques in order to appraise dynamics. Mention should also be made of Bambi [5] and Bambi, Gozzi & Fabbri [6] who complete a methodologically distinct approach of a time-to-build vintage environment. Finally, and in the interesting contribution of Bambi, Gozzi & Licandro [7], fluctuations spring from the time that is required after some innovation is order for the latter to get incorporated into the productive system.

V. Generalized Understandings

V.1 A Generalized Investment Lag Structure

Consider now a straightforwardly generalized vintage structure with a two periods lag:

$$K_{t+1} = I_t + \gamma I_{t-1} + \gamma^2 I_{t-2}.$$

The associated optimization problem formulates as:

$$\max_{\{I_t\}_{t\geqslant 0}} \sum_{t=0}^{\infty} \delta^t u\left[F\left[\iota\left(I_{t-1} + \gamma I_{t-2} + \gamma^2 I_{t-3}\right), 1\right] - I_t\right]$$
$$\text{s.t. } I_t \in \Gamma\left(I_{t-1}, I_{t-2}, I_{t-3}\right),$$
$$\delta \in]0, 1[, \quad \iota\left(I_{-1} + \gamma I_{-2} + \gamma^2 I_{-3}\right) > 0 \text{ given},$$

for $\Gamma\left(I_{t-2}, I_{t-1}\right) := \left[\gamma I_{t-1}+, F\left(I_{t-1} + \gamma I_{t-2} + \gamma^2 I_{-3}, 1\right)\right]$.

The system of first-order conditions is available as:

(11a) $\quad -u'\left(C_t\right) + \delta F'_K\left(K_{t+1}, 1\right) u'\left(C_{t+1}\right) + \delta^2 \gamma F'_K\left(K_{t+2}, 1\right) u'\left(C_{t+2}\right) + \delta^3 \gamma^2 F'_K\left(K_{t+3}, 1\right) u'\left(C_{t+3}\right) = 0;$

(11b) $\quad C_t + I_t - F\left[\iota\left(I_{t-1} + \gamma I_{t-2} + \gamma^2 I_{t-3}\right), 1\right] = 0;$

(11c) $\quad \lim_{t\longrightarrow+\infty} \delta^t u'\left(C_t\right) \cdot \left(I_{t-1} + \gamma I_{t-2} + \gamma^2 I_{t-3}\right) = 0.$

Consider then the configuration where $u(C) = \zeta C$, for $\zeta \in \mathbb{R}_+$. Letting $T = 3$ and $x_t := \zeta \delta F'_K(K_t, 1)$, it is immediate that equation (11a) reformulates to:

$$-\frac{1}{\zeta} + \sum_{s=1}^{T} (\delta\gamma)^{s-1} x_{t+s} = 0.$$

This features a T-order linear difference equation in the variable x, a related system having noticeably recently been analyzed by Gauthier [19]. It is readily checked that the above equation assumes a pair of complex values when $T \geqslant 3$, an extensive characterization of the case $T > 3$ being left for future research.

V.2 General Utilities and the Absence of Permanent Fluctuations in Investment Growth

For the unrestricted configuration with $\gamma \neq 0$ and for $u(\cdot)$ that is strictly concave, the dynamical system may be restated according to a fourth-order recurrence equation such as $Z\left(I_{t-2}, I_{t-1}, I_t, I_{t+1}, I_{t+2}\right) = 0$, that will locally lead to a fourth-order polynomial $Q(\lambda) = \left(\lambda - \lambda_1\right)\left(\lambda - \lambda_2\right)\left(\lambda - \lambda_3\right)\left(\lambda - \lambda_4\right) = 0$ in the neighborhood of the steady state. From the appendix, consider a steady state of the unrestricted dynamical system with $u(\cdot)$ strictly concave, the eigenvalues associated with the unrestricted system in the neighbourhood of the steady state are $\lambda_1, 1/\delta\lambda_1, \lambda_2, 1/\delta\lambda_2$. Letting $\mu_i = \lambda_i + 1/\delta\lambda_i$, it is further shown that:

$$
\begin{aligned}
S &:= \mu_1 + \mu_2 \\
&= \iota\delta^{-1}\left(-1 + \frac{\gamma}{1+\delta\gamma}\right) + \gamma\iota^2 \frac{1-\pi}{\sigma_F \chi_c}\left(\frac{C^*}{K^*}\right), \\
P &:= \mu_1\mu_2 - 2/\delta; \\
&= -\left\{\delta^{-1}\iota\left(1+\delta\gamma\right) + \left[\frac{1}{\delta\iota(1+\delta)} + \frac{(1-\pi)}{\sigma_F\chi_c}\left(\frac{C^*}{K^*}\right)\right]\delta^{-1}\iota^2\left(1+\delta\gamma^2\right)\right\} - 2/\delta; \\
&\mu_1, \mu_2 \in \mathbb{R}
\end{aligned}
$$

where σ_F denotes the elasticity of substitution in the production technology and χ_c denotes the inter temporal elasticity of substitution in consumption.

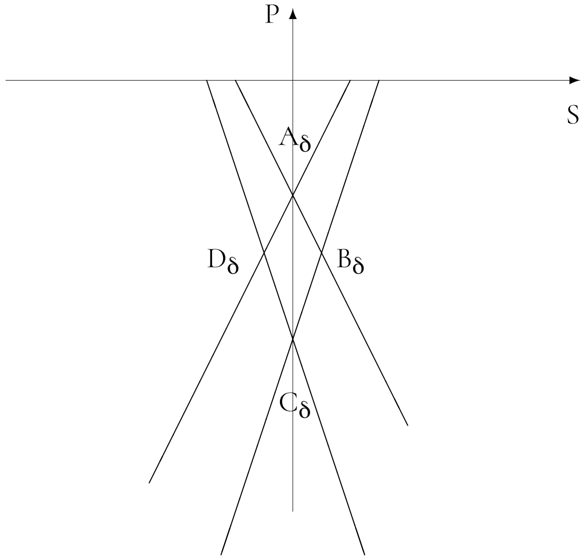

Figure 5: Stability and complexity for Paired Roots Systems with negative P

Following then the parallel approach of Drugeon [17], the consideration of the parameterized coefficients P and D instead of the roots $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ will allow for a simple appraisal of the stability properties of the 4th-dimensional dynamical system that is associated with a vintage environment with a strictly concave utility function. Consider indeed the Figure 5 that is drawn for negative values of P.

The stability properties will organize upon the localization of the system with respect to the four points A_δ, B_δ, C_δ and D_δ that correspond to the intercrossing of some critical lines over the plane (S, P). As this is detailed in Drugeon [17], the triangular area above the point A_δ and, more generally, the lines $(A_\delta B_\delta)$ and $(A_\delta D_\delta)$, do correspond to the occurrence of two pairs of complex eigenvalues with moduli that are greater than one, whence a complex rooted unstable configuration. In contradiction with this, the area below the point C_δ and, more generally, the lines $(B_\delta C_\delta)$ and $(C_\delta D_\delta)$, is associated with the occurrence of four eigenvalues, of witch two assume an absolute value that is lower than one while the two remaining ones assume an absolute value larger than one, whence a *saddle-point configuration* in a system with two predetermined variables. From another perspective, it is worth emphasizing that, while a downward crossing of both of the lines $(A_\delta B_\delta)$ and $(A_\delta D_\delta)$ corresponds to a transition from complex to real for a pair of eigenvalues, their modulus (absolute value) with respect to one being unchanged, a downward crossing of the lines $(B_\delta C_\delta)$ and $(C_\delta D_\delta)$ would correspond to the transition from an unstable area towards a stable area for a unique real eigenvalue, the complex/real status of the eigenvalues being unmodified.

Following now the classical approach inaugurated by Grandmont, Pintus & de Vilder [21] that recently got extended by Barinci & Drugeon [8], and in parallel with the companion article by Drugeon [17], this article will consider a family of parameterized economies that is described by a straight-line over the plane (S, P) and sketch an elementary stability analysis based upon over the geometrical properties of that straight-line. Letting indeed σ_F vary between 0 and $+\infty$ allows for describing a straight-line $_{1/\sigma_F}\Delta$ in the space (S, P) that assumes the following equation:

$$P = -\frac{1+\delta\gamma^2}{\delta\gamma}S - \delta^{-1}\left\{\iota(1+\delta\gamma) + \frac{1}{\delta}\left[\frac{\gamma}{1+\delta\gamma} + \iota\left(1 - \frac{\gamma}{1+\delta\gamma}\right)\right]\iota^2\left(1+\delta\gamma^2\right) + 2\right\}.$$

A first remarkable property is that the straight-line is south-east orientated on the plane (S, P) and, as $\sigma_F \in]0, +\infty[$ spans its interval, the coefficient S being indeed increasing as a function of σ_F while P is a decreasing one. It is also noticed that the locus of the origins of the straight-line, defined for $\sigma^F = 0$, depicts a concave decreasing line. The properties of this economy are illustrated on Figure 6 and for the configurations $\gamma \in]0, 1[$, $\gamma = 1$ and $\gamma > 1$:

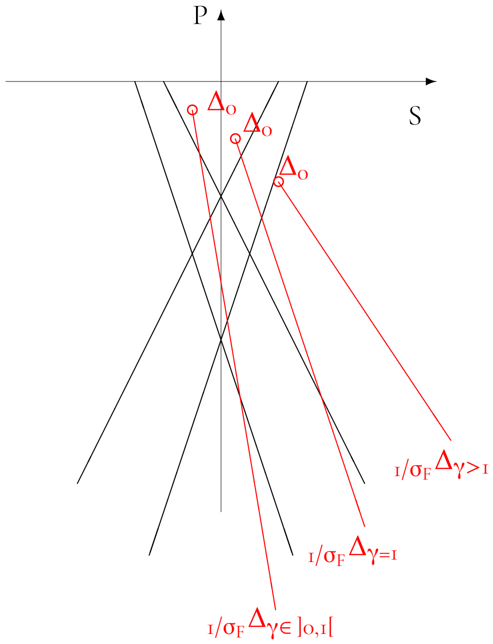

Figure 6: Stability for $\gamma \in]0, 1[$, $\gamma = 1$ and $\gamma > 1$

While leaving unaffected its south-east orientation, the consideration of milder vintage effects $\gamma \in]0, 1[$ results in a straight-line $_{1/\sigma}\Delta$ that is steeper than for the strong vintage effects configuration $\gamma > 1$. A remarkable, though unsurprising, result is thus that, for the benchmark case of the vintage capital theory literature, *i.e.*, $\gamma \in]0, 1[$, the straightline $_{1/\sigma}\Delta$ asymptotically locates within an area where the eigenvalues are real and the saddlepoint property is satisfied. One is also to emphasize that, on Figure 6, the consideration of a general strictly concave utility function clarifies the former results of this article by allowing for the occurrence of complex roots. However, and as a direct corollary of the uniformly negative values for the coefficient P, the current setting rules out any scope for the occurrence of a pair of imaginary roots and thus any area for self-sustained fluctuations associated with the emergence of a limit cycle.

VI. Concluding Comments

While the study of this environment has provided numerous clarification and new insights, it is to be noticed that this simple putty-putty acception of vintage capital could be adapted to the more general putty-clay ones that got analyzed by Phelps [1962] and Bliss [1968]. These structures are of interest in providing a clear-cut formal account of *embodied technical change* and will be considered in future work.

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A. Proofs for Section II

A.1 Proof of Proposition II.1

(i) The retainment of Inada and strict concavity restrictions on $F(\cdot, 1)$ through Assumption T1-2 is sufficient for the existence and the uniqueness of a stationary state K^*, that in turn univocally determines C^*, I^* and η^*.

(ii) The stationary golden rule value of the capital stock maximizing consumption, denoted as $\hat{K}$, is available from the following equation:

$$F'_K\left(\hat{K}, 1\right) = \frac{1}{\iota\left(1+\gamma\right)}.$$

The steady-state value of the capital stock K^* being however increasing as a function of δ, the stationary state is necessarily in an *under-accumulation* configuration with respect to the golden rule. (iii) It hence does increase with both ι and γ : *the weaker its depreciation rate, the greater the value of the capital stock.* (iv) The stationary value of investment relates to the survival rate according to:

$$\frac{dI^*}{d\iota\gamma} = I^*\left[-\frac{F'_K\left(K^*, 1\right)}{F''_K\left(K^*, 1\right)K^*}\frac{\delta}{\delta\iota(1+\gamma)} - \frac{1}{\iota(1+\gamma)}\right].$$

The sign of this formulation remains indeterminate in its general acception. For a Cobb-Douglas technology where the share of capital summarizes to a parameter α, it is however obtained that

$$\frac{dI^*}{d\gamma} \geq 0 \Leftrightarrow 1 + \gamma\alpha \geq \frac{1-\alpha}{\delta}.$$

This result mimics the one obtained through a constant depreciation rate for the capital stock.Q.E.D

B. Proofs for Section III

B.1 Proof of Proposition III.1

(i) Within the configuration for which $\iota = 1$ and $\gamma = 0$, one recovers the standard discontinuous solution.

(ii)-(iii) Within the configuration for which $0 < \iota_- < \iota$, the dynamics of the capital stock follows a divergent cycling motion along

$$\begin{aligned}\frac{dK_{t+2}}{dK_{t+1}}\Big|_{K^*} &= -\frac{1}{\delta\gamma}\\ &< -1,\end{aligned}$$

that gives rise to instability for $\gamma < 1$. For a given K_0, I_0 adjusts so that $K_1 = K^*$, whence, rearranging,

$$I_t = \left(K^* - \gamma\iota I_{t-1}\right)/\iota \text{ for } I_{-1} > 0 \quad \text{given.}$$

Its solving gives the expression of the statement. Investment converges in a cyclical way to its stationary value but oppositely follows a cyclical motion by the infinite if $\gamma = 1$.

(iv) In opposition to this and within the *learning-by-using* configuration for which $1 < \gamma < 1/\iota$, the dynamics of the capital stock follows a *convergent cycling motion* towards K^* if $1 < \gamma\delta$. Q.E.D.

C. Proofs for Section IV

C.1 Proof of Lemma IV.1

Taking advantage of the expression of C_t as a function of Z_t and Z_{t-1} in Equation (7), it is obtained that:

$$\left\{-Z_t + \frac{A\iota}{2}(1+\zeta)Z_{t-1}\right\}^{-1}\left\{-Z_{t+1} + \frac{A\iota}{2}(1+\zeta)Z_t\right\} = \left\{\delta\frac{A\iota}{2}(1+\zeta)\right\}^{\sigma}$$

that becomes:

$$-\frac{Z_{t+1}}{Z_t} + \frac{A\iota}{2}(1+\zeta) = \left\{\delta\frac{A\iota}{2}(1+\zeta)\right\}^{\sigma}\left\{-1 + \frac{A\iota}{2}(1+\zeta)\frac{Z_{t-1}}{Z_t}\right\}.$$

This equation assumes two obvious candidate solutions:

$$\frac{Z_{t+1}}{Z_t} = \left\{\delta\frac{A\iota}{2}(1+\zeta)\right\}^{\sigma},$$
$$\frac{Z_{t+1}}{Z_t} = \frac{A\iota}{2}(1+\zeta).$$

Firstly considering the first root, the ensued solution of Z_t is available as :

$$\begin{aligned} Z_t &= \left\{\delta\frac{A\iota}{2}(1+\zeta)\right\}^{\sigma(t+1)} Z_{-1} \\ &= \left\{\delta\frac{A\iota}{2}(1+\zeta)\right\}^{\sigma(t+1)}\left\{I_{-1} - \frac{A\iota}{2}(1-\zeta)I_{-2}\right\}, \end{aligned}$$

where the eventual line springs from the expression of Z_t as a function of I_t and I_{t-1} in Equation (5).

C.2 Proof of Proposition IV.1

Reformulating this in terms of I_t and I_{t-1} and from Equation (5), this gives:

$$\begin{aligned} I_t &= Z_t + \frac{A\iota}{2}(1-\zeta)I_{t-1} \\ &= \left\{\delta\frac{A\iota}{2}(1+\zeta)\right\}^{\sigma(t+1)}\left\{I_{-1} - \frac{A\iota}{2}(1-\zeta)I_{-2}\right\} + \frac{A\iota}{2}(1-\zeta)I_{t-1}. \end{aligned}$$

It is noticed that, for $\gamma = 0$, this last expression writes down as: $I_t = (\delta A\iota)^{\sigma(t+1)} I_{-1}$. Computing then, and for the case $\gamma \geq 0$, the exact expression of I_t, from the above, its equation is to assume the following form :

$$I_t = B^{\sigma(t+1)}C + DI_{t-1}$$

where B, C, D denote some constants. It derives:

$$\begin{aligned}
I_t &= B^{\sigma(t+1)}C + D\left[B^{\sigma t}C + DI_{t-2}\right] \\
&= B^{\sigma(t+1)}C + DB^{\sigma t}C + D^2\left[B^{\sigma(t-1)}C + DI_{t-3}\right] \\
&= C\sum_{j=0}^{i-1} D^j B^{\sigma(t+1-j)} + D^i I_{t-i} \\
&= C\sum_{j=0}^{t} D^j B^{\sigma(t+1-j)} + D^{t+1} I_{-1}
\end{aligned}$$

The actual expression of I_t is consequently derived as:

$$I_t = \left[I_{-1} - \frac{A\iota}{2}(1-\zeta)I_{-2}\right]\sum_{j=0}^{t}\left[\frac{A\iota}{2}(1-\zeta)\right]^j\left(\delta\frac{A\iota}{2}(1+\zeta)\right)^{\sigma(t+1-j)} + \left[\frac{A\iota}{2}(1-\zeta)\right]^{t+1} I_{-1}$$

or, restating:

$$I_t = I_{-1}\sum_{j=0}^{t+1}\left[\frac{A\iota}{2}(1-\zeta)\right]^j\left(\delta\frac{A\iota}{2}(1+\zeta)\right)^{\sigma(t+1-j)} - I_{-2}\sum_{j=0}^{t}\left[\frac{A\iota}{2}(1-\zeta)\right]^{j+1}\left(\delta\frac{A\iota}{2}(1+\zeta)\right)^{\sigma(t+1-j)}$$

where oscillations emerge from the satisfaction of $\zeta > 1$, and thus of $(1-\zeta)^j$ that alternatively assumes negative and positive values, whence the unambiguous emergence of oscillations.

As for the second candidate solution, it is first inferred that the Z_t solution is, for every $t = 0, 1, 2, \ldots$:

$$\begin{aligned}
Z_t &= \left(\frac{A\iota}{2}(1+\zeta)\right)^{\sigma(t+1)} Z_{-1} \\
&= \left(\frac{A\iota}{2}(1+\zeta)\right)^{\sigma(t+1)}\left[I_{-1} - \frac{A\iota}{2}(1-\zeta)I_{-2}\right]
\end{aligned}$$

and thus that

$$\begin{aligned}
I_t &= Z_t + \frac{A\iota}{2}(1-\zeta)I_{t-1} \\
&= \left(\frac{A\iota}{2}(1+\zeta)\right)^{\sigma(t+1)}\left[I_{-1} - \frac{A\iota}{2}(1-\zeta)I_{-2}\right] + \frac{A\iota}{2}(1-\zeta)I_{t-1}
\end{aligned}$$

Noticing that for $\gamma = 0$, this last expression reformulates to: $I_t = (A\iota)^{(t+1)} I_{-1}$. For the more general case $\gamma \geq 0$, the actual expression of investment is available as:

$$I_t = I_{-1}\sum_{j=0}^{t+1}\left[\frac{A\iota}{2}(1-\zeta)\right]^j\left(\frac{A\iota}{2}(1+\zeta)\right)^{\sigma(t+1-j)} - I_{-2}\sum_{j=0}^{t}\left[\frac{A\iota}{2}(1-\zeta)\right]^{j+1}\left(\frac{A\iota}{2}(1+\zeta)\right)^{\sigma(t+1-j)}.$$

The statement follows.

C.3 Proof of Proposition IV.2

(i) As for the growth rate of investment, recall that, having incorporated the solution value for μ, the investment states as:

$$I_t = Z_t + \frac{\iota A}{2}(1-\zeta)I_{t-1}$$

Restating Z_t as a function of I_t and I_{t-1}

$$\frac{Z_{t+1}}{Z_t} = \left[I_t - \frac{\iota A}{2}(1-\zeta)I_{t-1}\right]^{-1}\left[I_{t+1} - \frac{\iota A}{2}(1-\zeta)I_t\right]$$

or

$$\left[I_t - \frac{\iota A}{2}(1-\zeta)I_{t-1}\right]\left\{\delta\frac{A\iota}{2}(1+\zeta)\right\}^{\sigma} = I_{t+1} - \frac{\iota A}{2}(1-\zeta)I_t$$

once integrated the expression of the growth rate Z_{t+1}/Z_t. This rearranges to an expression that makes explicit the dynamics of the growth rate of I_t:

$$\frac{I_{t+1}}{I_t} = \left[1 - \frac{\iota A}{2}(1-\zeta)\frac{I_{t-1}}{I_t}\right]\left\{\delta\frac{A\iota}{2}(1+\zeta)\right\}^{\sigma} + \frac{\iota A}{2}(1-\zeta)]$$

that restates as:

$$(12) \qquad 1 + g^I_{t+1} = \left[1 - \frac{\iota A}{2}(1-\zeta)\frac{1}{1+g^I_t}\right]\left(1+g^C\right) + \frac{\iota A}{2}(1-\zeta).$$

Now, when considered in the plane $(1+g^I_t, 1+g^I_{t+1})$, it is immediate that the RHS member describes a decreasing convex function, or $1 - g^I_{t+1} = A/\left(1+g^I_t\right) + B$, where $A > 0$ and $B < 0$, the steady growth rate $1+g^I = 1+g^C$ being available as the obvious root of the above equation. Noticing, from the definition of Z_t and with the solution value for μ, that:

$$\frac{I_t}{I_{t-1}} = \frac{Z_t}{I_t}\frac{It}{I_{t-1}} + \frac{A\iota}{2}(1-\zeta) \Longrightarrow \frac{Z}{I} = 1 - \frac{A\iota}{2}(1-\zeta)\frac{1}{1+g^I},$$

that expresses the constancy of the ratio Z/I along the steady growth solution where $1+g^I = 1+g^Z$ share the same constant value. It is further noticed from the RHS of equation (12) leaves room for an asymptote in the positive orthant for $1+g^I_t \to \infty$ when

$$1+g^C > -\frac{A\iota}{2}(1-\zeta).$$

Finally, and as for its stability properties, they are obtained through the slope of the curve in the neighbourhood of the steady growth rate:

$$\left.\frac{d\left(1+g^I_{t+1}\right)}{d\left(1+g^I_t\right)}\right|_{1+g^I_t=1+g^C} = \frac{\iota A}{2}(1-\zeta)\frac{1}{\left(1+g^C\right)}$$

and thus assumes a negative sign.

It will further be locally stable and greater than -1 for

$$(13) \qquad \left\{\delta\frac{A\iota}{2}(1+\zeta)\right\}^{\sigma} > -\frac{\iota A}{2}(1-\zeta)$$

This condition can further be recovered in the $(1+g^I_t, 1+g^I_{t+1})$ plane and ensures that the decreasing curve (12) locally describes a stable configuration.

Summarizing, the stability condition (13) plus the aforementioned positiveness of consumption do jointly impose bounds for the growth rate of investment:

$$(14)\quad -\frac{\iota A}{2}(1-\zeta) < \left\{\delta\frac{A\iota}{2}(1+\zeta)\right\}^{\sigma} < \frac{A\iota}{2}(1+\zeta)$$

where it is recalled that an interior growth solution is also associated with a productivity requirement:

$$(15)\quad \frac{A\iota}{2}(1+\zeta) > \frac{1}{\delta}.$$

Interestingly, the conditions in equation (14) are to be paralleled with the condition that ensures the finiteness of the objective function in the optimization problem (this also relates to the transversality condition):

$$\delta\left(1+g^C\right)^{(1-1/\sigma)} < 1$$

$$\Longleftrightarrow \delta\left[\left\{\delta\frac{A\iota}{2}(1+\zeta)\right\}^{\sigma}\right]^{\sigma-1} < 1$$

$$(16)\quad \Longleftrightarrow \frac{A\iota}{2}(1+\zeta) < \delta^{1/(1-1/\sigma)}.$$

Interestingly, the positiveness condition on consumption, that corresponds to the satisfaction of the second inequality in equation (14), hence happens to be fufilled as as simple corollary of the finite objective condition (16). In opposition to this, the requesites for the stability of the steady growth solution, that correspond to the first inequality in equation (16), are not necessarily satisfied.

(ii) The critical value γ^c is to implicitly defined as the solution of:

$$\frac{A\iota}{2}\left[1+\left(1+\frac{4\gamma^c}{A\iota}\right)^{1/2}\right] = \frac{1}{\delta}\left\{-\frac{\iota A}{2}\left[1-\left(1+\frac{4\gamma^c}{\iota A}\right)^{1/2}\right]\right\}^{1/\sigma}.$$

For $\sigma = 1$, it is straightforwardly derived as:

$$\gamma^c = \frac{A\iota}{4}\left[\left(\frac{1+\delta}{1-\delta}\right)^2 - 1\right].$$

For $\sigma \neq 1$, restate the defining equation of γ^c as:

$$\left\{\delta\frac{A\iota}{2}\left[1+\left(1+\frac{4\gamma^c}{A\iota}\right)^{1/2}\right]\right\}^{\sigma} = -\frac{\iota A}{2}\left[1-\left(1+\frac{4\gamma^c}{\iota A}\right)^{1/2}\right].$$

Making again use of the notation ζ, this boils down to the look after a value of $\zeta \geq 1$ that solves:

$$\zeta = 1 + \delta^{\sigma}\left(\frac{A\iota}{2}\right)^{\sigma-1}(1+\zeta)^{\sigma}$$

or $\Phi(\zeta) = \Psi(\zeta)$. First notice that, for $\zeta = 1$, $\Psi(1) > \Phi(1)$. Then notice that the derivative with respect to ζ of the RHS component is available as:

$$\Psi'(\zeta) = \sigma\delta^{\sigma}\left(\frac{A\iota}{2}\right)^{\sigma-1}(1+\zeta)^{\sigma-1}.$$

Further remark that, while for $\zeta = 1$, $\Psi'(1) = \sigma\delta^{\sigma}(A\iota)^{\sigma-1}$, for $\zeta \to \infty$, that sums up to 0 for $\sigma < 1$ but to $+\infty$ for $\sigma > 1$. Finally, the second-order derivative of the aforementioned RHS lists as

$$\Psi''(\zeta) = \sigma(\sigma-1)\,\delta^{\sigma}\left(\frac{A\iota}{2}\right)^{\sigma-1}(1+\zeta)^{\sigma-1}$$

and is hence of negative sign for $\sigma < 1$ but of positive sign for $\sigma > 1$. To sum up, and while, for $\sigma < 1$, there does exist a unique critical value γ^c, for $\sigma > 1$, there oppositely does not exist any critical γ^c as soon as $\sigma\delta^{\sigma}(A\iota)^{\sigma-1} > 1$.

D. Proofs for Section V

D.1 The Characteristic Polynomial

Consider a linearization of the degree four equation $Z\left(I_{t-2}, I_{t-1}, I_t, I_{t+1}, I_{t+2}\right) = 0$ in the neighbourhood of a stationary value for I^*:

$$\begin{aligned}
&\left\{(-1)u''\delta^2F'_K\iota\gamma\right\}\Delta I_{t+2} + \left\{(-1)\delta u''F'_K\iota + F'_K u''\delta^2F'_K\iota^2\gamma + u'\delta^2F''_{KK}\iota^2\gamma\right\}\Delta I_{t+1}\\
&+\left\{-1(-1)u'' + F'_K u''\delta F'_K\iota^2 + u'\delta F''_{KK}\iota^2 + \iota\gamma F'_K u''\delta^2F'_K\iota\gamma + u'\delta^2\iota\gamma F'_{KK}\iota\gamma\right\}\Delta I_t\\
&+\left\{F'_K(-1)\iota u'' + \delta\iota^2\gamma F'_K u''F'_K + u'\iota^2\gamma F''_{KK}\right\}\Delta I_{t-1} + \left\{\iota\gamma F'_K(-1)u''\right\}\Delta I_{t-2} = 0
\end{aligned}$$

where it is noticed that the coefficients associated with ΔI_{t-2} and ΔI_{t+2} both cancel out to zero for either $\gamma = 0$ or $u(C) = C$. In order to prove that the real and complex roots of the fourth-order polynomial equation $Q(\lambda) = \left(\lambda-\lambda_1\right)\left(\lambda-\lambda_2\right)\left(\lambda-\lambda_3\right)\left(\lambda-\lambda_4\right) = 0$ do actually correspond to the ones of the statement, it suffices to establish that if λ is a root of $Q(\lambda) = 0$, then $1/\delta\lambda$ is similarly a root of $Q(\lambda) = 0$.

First assuming that $Q(\lambda) = 0$, a direct computation gives $Q(1/\delta\lambda) - Q(\lambda) = 0$, that in turn implies $Q(1/\delta\lambda) = 0$.

The roots of $Q(\lambda) = 0$ thus exhibit a paired root structure along $\lambda_i, 1/\delta\lambda_i, i = 1, 2$. Denoting $\mu_i := \lambda_i + 1/\delta\lambda_i$, $Q(\lambda)$ reformulates as :

$$\begin{aligned}
Q(\lambda) &= \left(\lambda-\lambda_1\right)\left(\lambda-1/\delta\lambda_1\right)\left(\lambda-\lambda_2\right)\left(\lambda-1/\delta\lambda_2\right)\\
&= \lambda^4 - \left(\mu_1+\mu_2\right)\lambda^3 + \left(2/\delta+\mu_1\mu_2\right)\lambda^2 - \left(\mu_1+\mu_2\right)\lambda/\delta + 1/\delta^2,
\end{aligned}$$

for

$$\begin{aligned}
S &= \mu_1+\mu_2,\\
&= \delta^{-1}\iota\left[-1+\iota\gamma F'_K\delta\right] + \iota^2\gamma\frac{F''_{KK}}{F'_K}\frac{u'}{u''}\\
&= \delta^{-1}\iota\left[-1+\frac{\gamma}{1+\gamma\delta}\right] + \iota^2\gamma\frac{(1-\pi)\chi_c}{\sigma_F}\left(\frac{C^*}{K^*}\right),\\
P &= \mu_1\mu_2\\
&= -\left\{\frac{1}{\delta^2F'_K} + \left[F'_K + \frac{F''_{KK}}{F'_K}\frac{u'}{u''}\right]\delta^{-1}\iota^2\left[1+\delta\gamma^2\right]\right\} - 2/\delta\\
&= -\left\{\delta^{-1}\iota\left(1+\gamma\delta\right) + \left[\frac{1}{\delta\iota(1+\delta\gamma)} + \frac{(1-\pi)\chi_c}{\sigma_F}\left(\frac{C^*}{K^*}\right)\right]\delta^{-1}\iota^2\left[1+\delta\gamma^2\right]\right\} - 2/\delta.
\end{aligned}$$

Whence the occurrence of $P > 0$, $S^2 - 4P > 0$ and thus of $\mu_1, \mu_2 \in \mathbb{R}$. Q.E.D

D.2 Characterisation of the straight-line ${}_{1/\sigma_F}\Delta$ over the plane (P, D)

For future reference, first remark that the coordinates of A_δ and C_δ over the (S, P) plane list as:

$$\left(S_{A_\delta}, P_{A_\delta}\right) = (0, -4/\delta),$$
$$\left(S_{C_\delta}, P_{C_\delta}\right) = \left(0, -(1 + 1/\delta)^2\right).$$

A first remarkable property is then hat the straight-line is south-east orientated on the plane (S, P) and as $\sigma_F \in]0, +\infty[$ spans its interval, the coefficient S being indeed increasing as a function of σ_F while P is a decreasing one.

(i)-(ii) As a benchmark, firstly considering the occurrence of $\gamma = 1$, the slope of ${}_{1/\sigma^F}\Delta$ becomes: ${}_{1/\sigma^F}\Delta' = -(1 + 1/\delta)$, which is exactly the slope of $Q(-1) = 0$ or the locus B_1 in Appendix E. Remark then that $S_{1/\sigma_F=0,\gamma=1} = -1/(1+\delta)$ and $P_{1/\sigma_F=0,\gamma=1} = -\left[(1+1/\delta)^2\delta + 2\right]/\delta(1+\delta)$. Noticing further that, for S = 0, it is readily derived that $P = -2/\delta$, it is obtained that, for such a value, the straight-line ${}_{1/\sigma^F}\Delta$ is defined in the area Ψ_1. No positive values for P being admissible, the details of the statement follow.

Consider now the configuration with $\gamma \neq 1$. The expression of the slope is modified to ${}_{1/\sigma^F}\Delta' = -(1+\delta\gamma^2)/\delta\gamma$. Compare then this slope with the one of the locus B_1, *i.e.*, $-(1 + 1/\delta)$. It derives that:

$$-\left[\frac{1/\delta + \gamma^2}{\delta\gamma} - (1 + 1/\delta)\right] = -\frac{1}{\gamma}\left(\frac{1}{\delta} + \gamma^2 - \gamma - \frac{\gamma}{\delta}\right)$$
$$= -\frac{1}{\gamma}\left(\frac{1}{\delta} - \gamma\right)(1-\gamma).$$

Firstly considering the case $\gamma \in]0,1[$, it thus derives that the negative slope of the parameterized line ${}_{1/\sigma^F}\Delta$ is larger in absolute value than the one of the locus B_1 and thus of B_3 . For $\gamma > 1$, three configurations are conceivable that are respectively depicted by $\gamma \in]1, 1/\delta[$, $\gamma \in]1/\delta, \sqrt{4/\delta}[$ $\gamma \in]\sqrt{4/\delta}, +\infty[$. For $\gamma \in]1, 1/\delta[$, the slope of ${}_{1/\sigma^F}\Delta$ is lower in absolute value than the one of the locus B_1 . For $\gamma \in]1/\delta, \sqrt{4/\delta}[$, the slope of ${}_{1/\sigma^F}\Delta$ is lower in absolute value than the one of the locus B_1. Finally, and for $\gamma \in]\sqrt{4/\delta}, +\infty[$, the slope of ${}_{1/\sigma^F}\Delta$ is lower in absolute value than the one of the locus B_1.

Consider now the origin of the straight-line ${}_{1/\sigma^F}\Delta$. Its coordinates in the parameterized plane (S, P) are given by:

$$S_{1/\sigma_F=0} = \frac{1}{\delta}\left(-1 + \frac{\gamma}{1+\delta\gamma}\right),$$
$$P_{1/\sigma_F=0} = -\frac{1}{\delta}\left(1 + \gamma\delta + \frac{1}{\delta}\frac{1+\gamma^2}{(1+\delta)}\right).$$

Noticing that the first equation implies that $\gamma = -(S_{1/\sigma_F=0} + 1/\delta)/[\delta S_{1/\sigma_F=0} - (1/\delta)(1-\delta)]$, it is readily checked that this allows for restating $P_{1/\sigma_F=0}$ as a decreasing concave function of $S_{1/\sigma_F=0}$ that can be located on the plane and where it is noticed that its origin is associated with an infinitely negative slope for ${}_{1/\sigma^F}\Delta$ and when $\gamma = 0$. Q.E.D

E. Not to be Published: Eigenvalues Occurrences for Paired Roots Systems of Dimension 4 (Drugeon [17])

Lemma E.1 *Let* $\delta \in]0,1]$ *and consider the four unknowns* λ_1, $1/\delta\lambda_1$, λ_2, $1/\delta\lambda_2$ *defined by* $\mu_1 = \lambda_1 + 1/\delta\lambda_1$, $\mu_2 = \lambda_2 + 1/\delta\lambda_2$, *where* $\mu_1, \mu_2 \in \mathbb{R}$ *and satisfy* $\mu_1 + \mu_2 = S$ *and* $\mu_1\mu_2 = P$, $(S, P) \in \mathbb{R}^2$ *given. Consider then* *the following* (S, P) *boundaries graphed on Figure 1 for* $P < 0$:

$$B_1 : = \left\{(S, P) \in \mathbb{R} \times \mathbb{R}_- \mid P = -(1 + 1/\delta)^2 - (1 + 1/\delta)S\right\},$$
$$B_2 : = \left\{(S, P) \in \mathbb{R} \times \mathbb{R}_- \mid P = -(1 + 1/\delta)^2 + (1 + 1/\delta)S\right\},$$
$$B_3 : = \left\{(S, P) \in \mathbb{R} \times \mathbb{R}_- \mid P = -\sqrt{4/\delta}\left(S + \sqrt{4/\delta}\right)\right\},$$
$$B_4 : = \left\{(S, P) \in \mathbb{R} \times \mathbb{R}_- \mid P = \sqrt{4/\delta}\left(S - \sqrt{4/\delta}\right)\right\},$$

Consider then the areas $\Psi_1, \Psi_2, \Psi_3, \Psi_4, \Psi_5, \Psi_6$ *pictured on Figure 1 for* $P < 0$. *The 4-uple of unknowns* $\lambda_1, 1/\delta\lambda_1, \lambda_2, 1/\delta\lambda_2$ *is then to be described in the areas where* $P < 0$ *according to:*

(i) *within* Ψ_1*: four complex solutions with a modulus greater than one ;*

(ii) *within* Ψ_2*: two real solutions with absolute values greater than one and two complex solutions with a modulus greater than one ;*

(iii) *within* Ψ_3*: four real solutions with absolute values greater than one ;*

(iv) *within* Ψ_4*: three real solutions with a modulus greater than one and one real solution with a modulus lower than one ;*

(v) *within* Ψ_5*: two real solutions with absolute values greater than one and two real solutions with absolute values greater than one ;*

(vi) *within* Ψ_6*: two complex solutions a modulus greater than one, one real solution with an absolute value greater than one and one real solution with a modulus lower than one.*

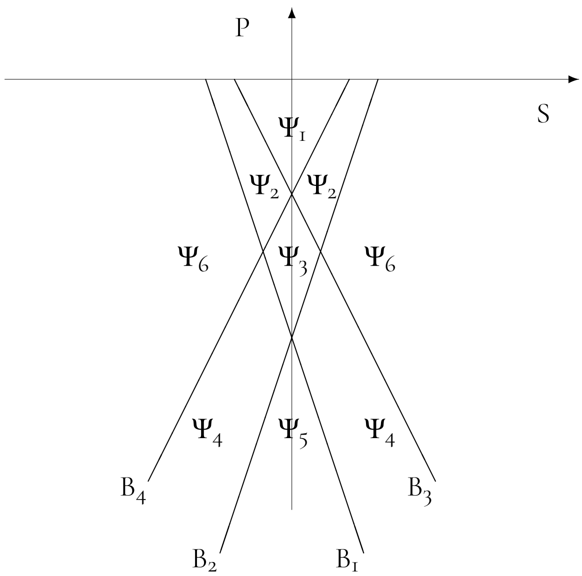

Figure 7: Stability and complexity for Paired Roots Systems

In order to establish the argument of the proof, first consider the formal definitions of the areas pictured through Figure 7 for $P < 0$:

$$\begin{aligned}
\Psi_1 :&= B_{3+} \cap B_{4-} \\
&= \Big\{(S,P) \in \mathbb{R}\times\mathbb{R} \;\Big|\; P > -(4/\delta)^{1/2}\big[S + (4/\delta)^{1/2}\big],\, P > (4/\delta)^{1/2}\big[S - (4/\delta)^{1/2}\big]\Big\}, \\
\Psi_2 :&= \Big(B_{1+} \cap B_{3-} \cap B_{4+}\Big) \cup \Big(B_{1+} \cap B_{3+} \cap B_{4-}\Big) \\
&= \Big\{(S,P) \in \mathbb{R}\times\mathbb{R} \;\Big|\; P > -(1+1/\delta)^2 - (1+1/\delta)S, \\
&\qquad P < -(4/\delta)^{1/2}\big[S + (4/\delta)^{1/2}\big],\, P > (4/\delta)^{1/2}\big[S - (4/\delta)^{1/2}\big] \text{ for } S < 0, \\
&\qquad P > -(1+1/\delta)^2 + (1+1/\delta)S, \\
&\qquad P > -(4/\delta)^{1/2}\big[S - (4/\delta)^{1/2}\big],\, P < (4/\delta)^{1/2}\big[S - (4/\delta)^{1/2}\big] \text{ for } S > 0\Big\}, \\
\Psi_3 :&= B_{1+} \cap B_{2+} \cap B_{3-} \cap B_{4-} \\
&= \Big\{(S,P) \in \mathbb{R}\times\mathbb{R}_- \;\Big|\; P > -(1+1/\delta)^2 - (1+1/\delta)S, \\
&\qquad P > -(1+1/\delta)^2 + (1+1/\delta)S,\, P < -(4/\delta)^{1/2}\big[S + (4/\delta)^{1/2}\big], \\
&\qquad\qquad P < (4/\delta)^{1/2}\big[S - (4/\delta)^{1/2}\big]\Big\}, \\
\Psi_4 :&= \Big(B_{1-} \cap B_{2+} \cap B_{4-}\Big) \cup \Big(B_{1+} \cap B_{2-} \cap B_{3+}\Big) \\
&= \Big\{(S,P) \in \mathbb{R}\times\mathbb{R}_- \;\Big|\; P < -(1+1/\delta)^2 - (1+1/\delta)S, \\
&\qquad P > -(1+1/\delta)^2 + (1+1/\delta)S,\, P < (4/\delta)^{1/2}\big[S - (4/\delta)^{1/2}\big] \text{ for } S < 0, \\
&\qquad P > -(1+1/\delta)^2 - (1+1/\delta)S, \\
&\qquad P < -(1+1/\delta)^2 + (1+1/\delta)S,\, P < -(4/\delta)^{1/2}\big[S + (4/\delta)^{1/2}\big] \text{ for } S > 0\Big\}, \\
\Psi_5 :&= B_{1-} \cap B_{2-} \\
&= \Big\{(S,P) \in \mathbb{R}\times\mathbb{R} \;\Big|\; P < -(1+1/\delta)^2 - (1+1/\delta)S, \\
&\qquad\qquad P < -(1+1/\delta)^2 + (1+1/\delta)S,\, P < -(4/\delta)^{1/2}\big[S + (4/\delta)^{1/2}\big]\Big\},
\end{aligned}$$

$$\begin{aligned}
\Psi_6 :&= \Big(B_{1-} \cap B_{3-} \cap B_{4+}\Big) \cup \Big(B_{2-} \cap B_{3+} \cap B_{4-}\Big) \\
&= \Big\{(S,P) \in \mathbb{R}\times\mathbb{R} \;\Big|\; P < -(1+1/\delta)^2 - (1+1/\delta)S, \\
&\qquad P < -(4/\delta)^{1/2}\big[S + (4/\delta)^{1/2}\big],\, P > (4/\delta)^{1/2}\big[S - (4/\delta)^{1/2}\big] \text{ for } S < 0, \\
&\qquad P < -(1+1/\delta)^2 + (1+1/\delta)S, \\
&\qquad P > -(4/\delta)^{1/2}\big[S + (4/\delta)^{1/2}\big],\, P < (4/\delta)^{1/2}\big[S - (4/\delta)^{1/2}\big] \text{ for } S > 0\Big\}.
\end{aligned}$$

The properties of the real—in a rather weak sense since both μ_1 and μ_2 are real, though none of λ_1, λ_2,

λ_3 and λ_4 may be real— shall be determined by the careful examination of three coefficients, namely

$$Q(1 + 1/\delta) = \left[(1 + 1/\delta) - \mu_1\right]\left[(1 + 1/\delta) - \mu_2\right]$$
$$= (1 + 1/\delta)^2 - (1 + 1/\delta)\left(\mu_1 + \mu_2\right) + \mu_1\mu_2 \gtreqless 0$$
$$Q[-(1 + 1/\delta)] = \left[-(1 + 1/\delta) - \mu_1\right]\left[-(1 + 1/\delta) - \mu_2\right]$$
$$= (1 + 1/\delta)^2 + (1 + 1/\delta)\left(\mu_1 + \mu_2\right) + \mu_1\mu_2 \gtreqless 0$$
$$\mu_1\mu_2 \gtreqless (1 + 1/\delta)^2.$$

Recall indeed that any of the μ_i, $i = 1, 2$, is defined as $\mu_i = \lambda_i + 1/\delta\lambda_i$, for μ_i a real coefficient. This restates as a second-order equation in λ_i that is parameterised by μ_i, namely a pair of second-order polynomials:

$$Q_{\mu_i}\left(\lambda_i\right) = \left(\lambda_i\right)^2 - \mu_i\lambda_i + 1/\delta,$$

whose critical values are given by:

$$Q_{\mu_i}(+1) = 1 - \mu_i + 1/\delta,$$
$$Q_{\mu_i}(-1) = 1 + \mu_i + 1/\delta,$$
$$Q_{\mu_i}(0) = 1/\delta$$

whose roots are real if and only if $\Delta_{\mu_i} = \left(\mu_i\right)^2 - 4/\delta \gtreqless 0$ and where it is remarked that

$$Q(1 + 1/\delta) = Q_{\mu_1}(+1)Q_{\mu_2}(+1)/\lambda_1\lambda_2,$$
$$Q[-(1 + 1/\delta)] = Q_{\mu_1}(-1)Q_{\mu_2}(-1)/\lambda_1\lambda_2,$$

a/ The complex case

Facing then with the areas for which μ_1 and μ_2 are real, *i.e.*, $\mathscr{P} = \mu_1\mu_2 < S^2/4 = \left(\mu_1 + \mu_2\right)^2/4$ and recalling that μ_i relates to λ_i according to $\mu_i = \lambda_i + 1/\delta\lambda_i$, it derives that both μ_i and λ_i are real when $\Delta_{\mu_i} = \left(\mu_i\right)^2 - 4/\delta > 0$, *i.e.*, $\mu_i < -(4/\delta)^{1/2}$ or $\mu_i > (4/\delta)^{1/2}$. In opposition to this, μ_i is real whereas λ_i is complex when $\Delta_{\mu_i} = \left(\mu_i\right)^2 - 4/\delta < 0$, *i.e.*, $-(4/\delta)^{1/2} < \mu_i < (4/\delta)^{1/2}$.

(i) Consider then the area Ψ_1 with $0 < P < 4/\delta$ and $0 < S < 2(\delta/4)^{1/2}$, that gives $0 < \mu_1 < (\delta/4)^{1/2}$ and $0 < \mu_2 < (\delta/4)^{1/2}$, that in turn imply $\Delta_{\mu_1} < 0$ and $\Delta_{\mu_2} < 0$, $\lambda_1, \lambda_2, \lambda_3$ and λ_4 being then complex. More precisely, note that

$$\lambda_1 = \alpha + i\beta \quad \Longrightarrow \quad \lambda_3 = \frac{\alpha - i\beta}{\delta\left(\alpha^2 + \beta^2\right)} = \frac{1}{\delta(\alpha + i\beta)} = \frac{1}{\delta\lambda_1}$$
$$\lambda_2 = \alpha - i\beta \quad \Longrightarrow \quad \lambda_4 = \frac{\alpha + i\beta}{\delta\left(\alpha^2 + \beta^2\right)} = \frac{1}{\delta(\alpha - i\beta)} = \frac{1}{\delta\lambda_2},$$

But then remark that, in order for this to be associated with real values for $\mu_i = \lambda_i + 1/\delta\lambda_i$, $i = 1, 2$, one is to get

$$\Im\left(\mu_1\right) = \Im\left(\alpha\left[1 + \frac{1}{\delta\left(\alpha^2 + \beta^2\right)}\right] + i\beta\left[1 - \frac{1}{\delta\left(\alpha^2 + \beta^2\right)}\right]\right) = 0,$$

that can only hold for $1 - 1/\delta\left(\alpha^2 + \beta^2\right) = 0$, that in turn gives $\alpha^2 + \beta^2 = 1/\delta > 1$. Completing the same line of argument for μ_2, λ_2 and λ_4, it derives that the area is associated with four complex conjugate eigenvalues with more greater than one modulus.

(ii) Consider then the area Ψ_2 restricted to $0 < P < 4/\delta$ and $0 < S < 2(\delta/4)^{1/2}$. The holding of $P - (\delta/4)^{1/2}\left[S - (\delta/4)^{1/2}\right] = \left(\mu_1 - (\delta/4)^{1/2}\right)\left(\mu_2 - (\delta/4)^{1/2}\right) < 0$ further indicates that, *e.g.*, $\mu_1 < (\delta/4)^{1/2}$ and $\mu_2 > (\delta/4)^{1/2}$. The same argument as above indicates that λ_1 and $\lambda_3 = 1/\delta\lambda_1 = \bar{\lambda}_1$ are such that $|\lambda_1| > 1$ and $|\lambda_3| > 1$. Noticing that this implies $Q_{\mu_1}(+1) > 0$ and $Q_{\mu_1}(-1) > 0$, the uniform holding of $Q(1+1/\delta) > 0$ and $Q[-(1+1/\delta)] > 0$ over the area Ψ_2 then implies the one of $Q_{\mu_2}(+1) > 0$ and $Q_{\mu_2}(-1) > 0$. Recalling that $Q_{\mu_2}(0) = 1/\delta > 1$, it is obtained that the two remaining real roots λ_2 and λ_4 are such that $|\lambda_2| > 1$ and $|\lambda_4| > 1$.

(iii) Finally consider then the area Ψ_6 restricted to $P > 0$ and $S > 0$. The holding of $P - (\delta/4)^{1/2}\left[S - (\delta/4)^{1/2}\right] = \left(\mu_1 - (\delta/4)^{1/2}\right)\left(\mu_2 - (\delta/4)^{1/2}\right) < 0$ further indicates that, *e.g.*, $\mu_1 < (\delta/4)^{1/2}$ and $\mu_2 > (\delta/4)^{1/2}$. The same argument as above indicates that λ_1 and $\lambda_3 = 1/\delta\lambda_1 = \bar{\lambda}_1$ are such that $|\lambda_1| > 1$ and $|\lambda_3| > 1$. Noticing that this implies $Q_{\mu_1}(+1) > 0$ and $Q_{\mu_1}(-1) > 0$, the uniform holding of $Q(1+1/\delta) < 0$ and $Q[-(1+1/\delta)] > 0$ over the area Ψ_6 then implies the one of $Q_{\mu_2}(+1) < 0$ and $Q_{\mu_2}(-1) > 0$, that in its turn reformulates as $\mu_2 > 1 + 1/\delta$. Recalling again that $Q_{\mu_2}(0) = 1/\delta > 1$, it is obtained that the two remaining real roots λ_2 and λ_4 are, *e.g.*, such that $|\lambda_2| < 1$ and $|\lambda_4| > 1$.

b/ The real case

The consideration of the various conceivable configurations on the coefficients $Q_{\mu_i}(-1)$, $Q_{\mu_i}(1)$, $Q_{\mu_{i'}}(-1)$ and $Q_{\mu_{i'}}(1)$ then deliver a complete characterisation in the real case. Three distinct areas are to be considered in their turn. More explicitly, Ψ_3 for which $Q[-(1+1/\delta)] > 0$, $Q(1+1/\delta) > 0$ and $-(1+1/\delta)^2 < \mu_i\mu_{i'} < (1+1/\delta)^2$, Ψ_5 for which $Q[-(1+1/\delta)] < 0$, $Q(1+1/\delta) < 0$ and finally Ψ_4 for which $Q[-(1+1/\delta)]Q(1+1/\delta) < 0$.

(i) Then consider the area Ψ_3. The holding of $Q(1+1/\delta) > 0$ implies $Q_{\mu_i}(1)Q_{\mu_{i'}}(1) > 0$, that can only be reconciled with $-(1+1/\delta)^2 < \mu_i\mu_{i'} < (1+1/\delta)^2$ for $\mu_i < 1+1/\delta, \mu_{i'} < 1+1/\delta$ and thus for $Q_{\mu_i}(1) > 0$, $Q_{\mu_{i'}}(1) > 0$. The holding of $Q[-(1+1/\delta)] > 0$ implies $Q_{\mu_i}(-1)Q_{\mu_{i'}}(-1) > 0$, that can only be reconciled with $-(1+1/\delta)^2 < \mu_i\mu_{i'} < (1+1/\delta)^2$ for $\mu_i > -(1+1/\delta), \mu_{i'} > -(1+1/\delta)$ and thus for $Q_{\mu_i}(-1) > 0$, $Q_{\mu_{i'}}(-1) > 0$. The simultaneous occurrences of $Q_{\mu_i}(1)Q_{\mu_i}(-1) > 0$ and $Q_{\mu_{i'}}(1)Q_{\mu_{i'}}(-1) > 0$ eventually implies that, *e.g.*, $|\lambda_1| > 1$, $|\lambda_3| = |1/\delta\lambda_1| > 1$, $|\lambda_2| > 1$, $|\lambda_4| = |1/\delta\lambda_2| > 1$.

(ii) Then consider the area Ψ_5. The holdings of $Q(1+1/\delta)0 <$ and $Q[-(1+1/\delta)] < 0$ respectively imply the ones of $Q_{\mu_i}(1)Q_{\mu_{i'}}(1) < 0$ and $Q_{\mu_i}(-1)Q_{\mu_{i'}}(-1) < 0$. Consider, *e.g.*, a configuration with $1 + 1/\delta - \mu_i > 0$ and $1 + 1/\delta - \mu_{i'} < 0$, or $Q_{\mu_i}(1) > 0, Q_{\mu_{i'}}(1) < 0$. Though both $Q_{\mu_i}(-1) > 0, Q_{\mu_{i'}}(-1) < 0$ and $Q_{\mu_i}(-1) < 0, Q_{\mu_{i'}}(-1) > 0$ remain *a priori* conceivable, it is readily noticed that this latter case implies $\mu_{i'} < -(1+1/\delta)$, a contradiction with the retained configuration $\mu_{i'} + (1+\delta) > 0$ retained for the holding of $Q(1+1/\delta) > 0$, hence $Q_{\mu_i}(-1) > 0, Q_{\mu_{i'}}(-1) < 0$. The simultaneous occurrences of $Q_{\mu_i}(1)Q_{\mu_i}(-1) < 0$ and $Q_{\mu_{i'}}(1)Q_{\mu_{i'}}(-1) < 0$ eventually imply that, *e.g.*, $|\lambda_1| > 1$, $|\lambda_3| = |1/\delta\lambda_1| < 1$, $|\lambda_2| > 1$.

(iii) Finally consider the area Ψ_4 and let, *e.g.*, $Q(1+1/\delta) < 0$ and $Q[-(1+1/\delta)] > 0$, that respectively imply $Q_{\mu_i}(1)Q_{\mu_{i'}}(1) < 0$ and $Q_{\mu_i}(-1)Q_{\mu_{i'}}(-1) > 0$. Let then, *e.g.*, $Q_{\mu_i}(1) < 0, Q_{\mu_{i'}}(1) > 0$. The required holding of $Q_{\mu_i}(-1) > 0, Q_{\mu_{i'}}(-1) > 0$ then translates as $Q_{\mu_i}(1)Q_{\mu_i}(-1) > 0$ and $Q_{\mu_{i'}}(1)Q_{\mu_{i'}}(-1) > 0$ eventually implies that, *e.g.*, $|\lambda_1| > 1$, $|\lambda_3| = |1/\delta\lambda_1| < 1$, $|\lambda_2| > 1$, $|\lambda_4| = |1/\delta\lambda_2| > 1$. The remaining configurations associated with $Q(1+1/\delta)Q[-(1+1/\delta)] < 0$ follow similar lines of arguments.

Q.E.D