The contribution of intraday jumps to forecasting the density of returns

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Abstract

Recent contributions highlight the importance of intraday jumps in forecasting realized volatility at horizons up to one month. We extend the methodology developed in Maheu and McCurdy (2011) to exploit the information content of intraday data in forecasting the density of returns. Considering both intra-week periodicity and signed jumps, we estimate two variants of a bivariate model of returns and volatilities where the jump component is independently modeled. Our empirical results for four futures series (S&P 500, U.S. 10-year Treasury Note, USD/CAD exchange rate and WTI crude oil) highlight the importance of considering the continuous/jump decomposition of volatility for the purpose of density forecasting. Specifically, we show that models considering jumps apart from the continuous component consistently deliver better density forecasts for horizons up to one month and a half and, in two cases out of four, for horizons up to three months.

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1 Introduction

The extraction of the jump component in the dynamics of asset prices has witnessed a considerably growing body of literature. Of particular interest is the decomposition of realized volatility between its continuous and jump components. In a reference paper, Andersen *et al.* (2007a) provide convincing empirical evidence that disentangling jumps from the continuous component significantly improves realized volatility point forecast at horizons up to a trading month. The explanation for this result lies in the strong persistence in the continuous component and the absence of autocorrelation in the jump component.

Forecasting realized volatility has shown to be critical in empirical finance applications such as portfolio choice (Fleming et al. (2003), Liu (2009), Chou and Liu (2010)), risk management (Bali and Weinbaum (2007), Clements et al. (2008)) or derivatives pricing (Christoffersen et al. (2014), Corsi *et al.* (2013), Alitab *et al.* (2019)). Useful for these applications, volatility point forecasts, the often traditional focus, are better seen as the central points of ranges of uncertainty. Consequently, to provide a comprehensive description of the uncertainty associated with the point forecast, many professional forecasters and central banks now publish density forecasts, or more popularly fan charts. In contrast to interval forecasts, which give the probability that the outcome will fall within a stated interval, density forecasts provide a full description of the uncertainty associated with the forecast. [1] Recently, Hansen *et al.* (2011), Maheu and McCurdy (2011), Shephard and Sheppard (2010) and Liu and Maheu (2018) have suggested "complete" models of returns and volatility. In these models, returns and volatility are modeled simultaneously, thereby allowing density predictions for returns once assumptions are made about the conditional distribution of returns. In particular, Maheu and McCurdy (2011) propose a bivariate specification of returns and volatility and confirm, in the density forecasting context, the overall finding that intraday data enhance predictions.

Our paper is the first to merge these two strands of the recent literature on realized volatility and investigates whether the separation between the continuous and the jump components is of importance in predicting the density of returns. By using intraday data, it is possible to detect and extract jumps as the difference, when statistically significant, between realized volatility and a measure of realized volatility that is robust to jumps. This decomposition enables to embed jumps for forecasting purposes. We rely on the link between the conditional variance and the realized volatility provided in Andersen *et al.* (2003) to estimate five different nested models. The motivation behind considering jump-robust measures for realized volatility is that they simply have better predictive properties than non-jump-robust ones (see Shephard and Sheppard

[1] The academic literature has devoted increased emphasis to density forecasting. See among others, Corradi and Swanson (2006), Wallis (2007), Mitchell and Wallis (2011).

(2010)). Our empirical results for futures series of four different asset categories (financial index, sovereign bond, exchange rate and commodity) strongly argue in favor of separating and then modeling the two components when forecasting the density of returns up to three months. As such, we show that disentangling jumps from the continuous component do help in forecasting the density of returns.

In the general empirical framework developed in Maheu and McCurdy (2011), our approach builds on the parsimonious Heterogeneous Autoregressive (HAR) model developed by Corsi (2009) to capture the well-known long-memory dependence in volatility. The model has shown to be very successful in numerous applications dealing with the dynamics of the realized volatility (see Corsi *et al.* (2012)). As for the detection of jumps, we proceed with the test in Huang and Tauchen (2005) that we adapt to the median realized volatility (MedRV) proposed in Andersen et al. (2012) which is a jump-robust measure of realized volatility. Our choice for the MedRV measure is motivated by the empirical work in Theodossiou and Zikes (2011) and Dumitru and Urga (2012) who show that MedRV is, in most cases, a good alternative to the well-know bipower variation suggested earlier in Barndorff-Nielsen and Shephard (2004) which suffers from lower performances in finite samples. We extend the bivariate model in Maheu and McCurdy (2011) in allowing for a leverage effect whose usefulness is empirically demonstrated in our application. Finally, the comparisons between the different bivariate specifications for daily returns and realized volatilities are conducted using the predictive likelihood tests of Amisano and Giacomini (2007) which has the interesting property to focus on the whole distribution and provides, as such, a general measure of the density forecast soundness.

This paper makes essentially two contributions to the literature. First, we extend the framework of Maheu and McCurdy (2011) to show how jumps can be accounted for in their bivariate model. Critical to our approach is the adequacy of the conditional jump distribution used for forecasting purpose with the empirical distribution of jumps. Second, we investigate the information content of intraday jumps when it comes to forecasting the density of returns. We do so primarily through a thorough comparison of the performance of models based on jump-robust and non-jump-robust (naïve) measures of realized volatilities. Compared to the naïve measure of realized volatility, recognizing jumps provides a statistically significant improvement for horizons up to 30 days for the four futures series. We thus extend the seminal results in Andersen *et al.* (2007a) [2] in showing the importance of disentangling jumps from the continuous component to forecast not only the realized volatility but also the distribution of returns. Hence, our results potentially have practical implications for activities such as portfolio choice, construction of risk measures and derivatives pricing as these activities might benefit from improved return density

[2] See also Corsi *et al.* (2010), Duong and Swanson (2015), Patton and Sheppard (2015).

forecasts. As an illustration, we provide in the penultimate Section a forecasting exercise of the daily Value-at-Risk (VaR) for the S&P 500 futures and provide evidence that models including a specific jump component perform significantly better in this context.

The remainder of the paper is organized as follows. Section 2 summarizes the construction of the volatilities and jumps data sets. Section 3 details our choice of an adequate distribution for jumps. In Section 4, we present our modeling strategy as well as the methodology used to compare density forecasts. Section 5 discusses the empirical results, which are used in a short experiment to gauge the economic value or our findings through a VaR forecast exercise in Section 6. Finally, Section 7 provides concluding remarks.

2 Volatilities and jumps estimation

This section briefly describes the data as well as the methodology that has been used to detect jumps and estimate volatilities for the four futures series.

2.1 Data description

Our data consist of four futures markets from four different asset classes: stock index, Treasury bond, exchange rate and commodity. The motivation for using futures series lies in their higher informativeness in the sense of Hasbrouck (see Fleming *et al.* (2003)).[3] As is usual, we consider the continuous series of the front month contract using a rollover procedure which selects the largest volume each day to jump from one contract to the next.[4]. While all assets are all very liquid (and are therefore suitable for using realized estimators), we had to remove days where the trading activity has not been sufficient to compute these estimators. To this end, we filter our four time series with respect to three parameters: the length of the trading period in the day, the number of zero-returns and the number of transactions (Andersen *et al.* (2003)). We choose to work with open-to-close returns because overnight returns have shown to follow a very different dynamics (see Andersen et al. (2001a)). In addition, including overnight returns may alter our analysis when standardizing returns as we work with volatility computed with intraday transaction data. Some details about data are provided in Table 1.

[3]Fleming *et al.* (2001) in which futures price series are used note: “We use futures to avoid short sale constraints and microstructure effects, but our analysis generalizes to the underlying spot assets via standard no-arbitrage arguments." (p. 330) A model including a convenience yield allows to extend this argument to the case of commodities. In addition, using futures allows to avoid delivery considerations that are prone to create significant effects in cash markets.

[4]We do not build our continuous series using a fixed number of days prior to maturity, thus avoiding calendar effects.

— Table 1 about here —

2.2 Realized quantities

Let us assume that $p(t)$, the logarithm of the asset price, follows the general stochastic volatility jump diffusion model (see Andersen *et al.* (2007b)):

$$dp(t) = \mu(t)dt + \sigma(t)dW(t) + \kappa(t)dJ(t) \qquad with \qquad t \geq 0 \tag{1}$$

where the mean $\mu(.)$ (predictable drift) is assumed to be continuous and locally bounded, and the instantaneous (spot) volatility $\sigma(.)$ is strictly positive, bounded away from zero almost surely and càdlàg (right continuous with left limit). The mean, as well as the spot volatility, are both assumed to be independent from the driving standard Brownian motion $W(t)$.[5] The finite activity counting process, noted $J(t)$, is normalized such that $dJ(t) = 1$ when a jump occurs at time t, and $dJ(t) = 0$ otherwise. Finally, $\kappa(t)$ is the jump size at time t, which is assumed to be random. The process in equation (1), which belongs to the class of Brownian semimartingale processes with finite activity jumps, allows returns to exhibit leptokurticity and volatility clustering, which are both relevant empirical characteristics for financial time-series.

If we define $[p](t)$ as the quadratic variation of the process in equation (1), then:

$$[p](t) = \text{plim} \sum_{j=0}^{n-1} [p(\tau_{j+1}) - p(\tau_j)]^2 \tag{2}$$

where $0 = \tau_0 < \tau_1 < ... < \tau_n = t$ is any sequence of partitions, such that $\sup_j\{\tau_{j+1} - \tau_j\} \to 0$ for $n \to \infty$. The quadratic variation of process in equation (1) may then be expressed as:

$$[p](t) = \int_0^t \sigma^2(\tau)d\tau + \sum_{j=1}^{J(t)} \kappa^2(t_j) \tag{3}$$

where t_j are times when a jump occurs, implying $dJ(t_j) = 1$. Eq. (3) represents the continuous sample path along with the jump components of total return variation.

Because most of our analysis will focus on daily returns and volatilities, we consider the daily time interval as unity.[6] For each day t, we consider that $M + 1$ evenly spaced (calendar time

[5] The independence assumption is discussed in length in Barndorff-Nielsen and Shephard (2006), who explain that it rules out leverage and volatility feedback effects. The absence of leverage has been recently shown to be empirically relevant for S&P 500 index and futures returns (see Andersen *et al.* (2001a), Bollerslev *et al.* (2006), Bollerslev *et al.* (2009) and more recently Chorro *et al.* (2018), among others).

[6] To simplify the exposition, we rely on a slight abuse of notation when switching from continuous to discrete time and denote day with a t index from now on.

subsampling[7]) intraday observations are available for the logarithm of the asset price, noted $p_{t,j}$, thus allowing to compute M continuously compounded intraday log-returns each day as $r_{t,j} = p_{t,j} - p_{t,j-1}$ for $j = 1, ..., M$. The realized variance for day t is then given by the sum of squared intraday returns:

$$RV_{t,M} = \sum_{j=1}^{M} r_{t,j}^2 \tag{4}$$

Note that realized variance is the estimator of the total daily variance, and for this reason it remains dependent on the selected sampling frequency M. As frequency tends to infinity (if intraday observations are available as often as desired), then:

$$RV_{t,M} \rightarrow_p \int_{t-1}^{t} \sigma^2(\tau)d\tau + \sum_{j=J(t-1)}^{J(t)} \kappa^2(t_j) \tag{5}$$

Equation (5) illustrates that volatility, which is by nature a latent variable, is made "observable" by the theory of quadratic variation. Nevertheless, despite theory suggests that returns should be computed at the highest possible frequency, so that estimators converge asymptotically towards the true conditional volatility, it is well-known since Andersen and Bollerslev (1997, 1998) and Taylor and Xu (1997) that microstructure noise (due to price discreteness, bid-ask spread, non-synchronous trading, etc.)[8] may impact the realized volatility estimator at high frequency. To deal with this issue while making our results comparable with the rest of the literature, we follow the 5 minutes 'rule-of-thumb' in line with the findings in Liu *et al.* (2015). As our four series are highly liquid assets, this sampling interval is adequate to make our realized measures not to be impacted by the noise. Realized volatilities are plotted in Figures 1 to 4 along with price and return series.

— Figures 1 to 4 about here —

The next question addresses how to disentangle jumps from the continuous component. For this purpose, measures of diffusive volatility are necessary. Barndorff-Nielsen and Shephard (2004) tackle this issue by proposing the bipower variation (BPV) measure, which is computed as the scaled summation of the product of adjacent absolute returns. The BPV is a consistent estimator of integrated volatility, and allows to decompose the realized volatility into its diffusive and

[7]The case of alternative subsampling such as business (or transaction) time subsampling has attracted some attention in the literature (Ané and Geman (2000), Oomen (2006), Andersen *et al.* (2010)), as it allows to recover normality for standardized returns. Nevertheless, to our best knowledge, these ideas lack asymptotic theory and have not been extended yet to the analysis of jump-robust estimators.

[8]See Hansen and Lunde (2006) for a thorough discussion of this issue and Andersen *et al.* (2011) for a theoretical and empirical analysis of the impact of microstructure noise on the forecast of realized volatility.

non-diffusive parts. As the sampling frequency increases, the presence of jumps should have no impact, because the return representing the jump is multiplied by a non-jump return which tends to zero asymptotically. This is true in case of rare jumps (one each day), when the probability of two consecutive jumps is negligible.

Nevertheless, the BPV has major drawbacks in empirical applications. First, in practice, the sampling frequency is not high enough to eliminate the influence of jumps. Indeed, a large (jump) intraday return is multiplied by an adjacent return that is not zero thereby resulting in an upward bias of the BPV. Second, the presence of zero-return that are multiplied twice (with the previous and the next intraday return) leads to a downward bias of the BPV. Some alternative jump-robust measures have been proposed in the literature to deal with the aforementioned issues. Among them, the MedRV (Andersen *et al.* (2012)) is very promising, as it has better properties in realistic settings and remains intuitive and easy to implement. The daily MedRV estimator reads as follows:

$$MedRV_{t,M} = \frac{\pi}{6 - 4\sqrt{3} + \pi} \left(\frac{M}{M-2} \right) \sum_{j=2}^{M-1} \text{median}(\mid r_{t,j-1} \mid, \mid r_{t,j} \mid, \mid r_{t,j+1} \mid)^2 \tag{6}$$

With the MedRV estimator, the impact of jumps completely vanishes except in the case of two consecutive jumps (which is extremely rare at the sampling frequencies used in empirical applications). In addition, the estimator is more robust to the occurrence of zero-returns except when many zero-returns are likely to be consecutive. We thus decide to consider the MedRV as a competitive alternative for our analysis.[9] To deal with the microstructure issue, we use staggered versions of MedRV as advocated in Huang and Tauchen (2005) and, as for the realized volatility estimator, we use the 5-minute sampling interval.[10]

2.3 Summary statistics

Table 2 displays descriptive statistics regarding our returns and volatility measures. From the analysis of the table, returns on the four available assets are showing features that are consistent with the stylized facts usually reported in the empirical financial literature: the distribution of their returns are asymmetric and leptokurtic. They show different levels of volatilities: the WTI's volatility is the largest, reaching 37%, when the 10-year bond futures' is 7.2%. All skewness are negative over the portion of history investigated here. Kurtosis are all higher than 3, while still reaching various levels. The distribution with the fatter tails is the bond futures one, while the one with the thinner tails is the USDCAD one. The table also provides information regarding

[9]Results using the BPV estimator are qualitatively similar and available from the authors upon request.

[10]In what follows, and for clarity of exposition, we now abstract from reporting the sampling frequency for realized measures.

the distribution of the two volatility measures that we will use in our empirical experiments: the RV and MedRV measures are variance estimates, and the table displays statistics for their square root, so that to focus on volatility-like measures. Several comments can be raised from this second part of the table. First, the average level of our volatility measures is lower than the standard deviation of returns, especially in the case of the MedRV measure: for the S&P500 index, the returns' standard deviation is 20.1% when the average square root of the RV is 13.1% and 12.4% in the case of MedRV. Similar cases can be raised from the other three assets. Second, the volatility skewness is positive, as it is distributed on the positive part of the real line. Third, the kurtosis is larger than 3, and its largest reading is obtained in the case of the S&P 500. These features are consistent for both the RV and the MedRV results.

— Table 2 about here —

2.4 Detecting jumps

Our methodology closely follows Boudt et al. (2011) which highlights the importance of consider intraweek periodicity for jump detection. Their methodology has been used in Lahaye et al. (2011). The estimation of intraweek periodicity is inspired by the early estimator of Taylor and Xu (1997). Boudt et al. (2011) modifies the estimator to make it more robust to the presence of jumps in the intraday data. The estimated intraweek periodicity is then used to test for jumps following the methodology developed in Lee and Mykland (2008) (see also Andersen et al. (2007b)). In particular, we consider the test statistic:

$$J_{t,i} \equiv \frac{|r_{t,i}|}{\widehat{\sigma_{t,i}}} \tag{7}$$

where $r_{t,i}$ is the i^{th} return in day t and $\sigma_{t,i}$ is the latent volatility at that time. $\sigma_{t,i}$ is estimated using an estimator which is robust to jumps.

While Boudt et al. (2011) follow Lee and Mykland (2008) in choosing the BPV estimator, we opt for the MedRV developed in Andersen et al. (2012) which has better finite-sample properties. The window on which the robust-to-jumps estimator has to be computed is a tricky question (see the discussion in Lahaye et al. (2011) Section 2.1). We follow Andersen et al. (2007b) and use the estimated volatility over the day t. Lee and Mykland (2008) suggests to use the distribution of the statistic's maximum to conclude on the presence of jumps. In particular, if:

$$J_{t,i} > G_{-1}(1-\alpha)S_n + C_n \tag{8}$$

with n, the total number of intraday returns in the full sample (number of days times number of intraday returns each day) and $G_{-1}(1-\alpha)$ the $1-\alpha$ quantile function of the standard Gumbel distribution,

$$C_n = \sqrt{2\log n} - \frac{\log \pi + \log(\log n)}{2\sqrt{2\log n}} \quad \text{and} \quad S_n = \frac{1}{\sqrt{2\log n}}.$$

The $1-\alpha$ quantile function of the standard Gumbel distribution is given by $-\log(-\log(1-\alpha))$.

We now introduce periodicity considerations in the original estimator of Lee and Mykland (2008) following Boudt et al. (2011). The main idea is to consider that the conditional volatility $\sigma_{t,i}$ is the product of a slowly-varying component $\delta_{t,i}$ and a deterministic circadian component $f_{t,i}$ whose aim is to model the intraweek periodicity. As in Taylor and Xu (1997), Boudt et al. (2011) assume that this deterministic component of the variance process sums to one:

$$\int_{t-1}^{t} f^2(s)ds = 1 \tag{9}$$

The modified version of the Lee and Mykland's test according to Boudt et al.' work is given as:

$$\text{Filt}J_{t,i} \equiv \frac{|r_{t,i}|}{\widehat{\delta_{t,i}}\widehat{f_{t,i}}} \tag{10}$$

where $\widehat{\delta_{t,i}}$ corresponds to the MedRV and $\widehat{f_{t,i}}$ is the estimated circadian component, both for day t.

The computation of the circadian component is as follows. First, we standardize intraday returns to allow for comparability across sample days. Intraday returns are standardized using MedRV for reasons presented above. Let the standardized i^{th} intraday return for day t be:

$$\bar{r}_{t,i} = \frac{r_{t,i}}{\sqrt{MedRV_t}} \tag{11}$$

Second, we use the nonparametric Weighted Standard Deviation (WSD) which has better efficiency under normality than classical estimators.[11] The WSD nonparametric periodicity estimator for the i^{th} intraday return in the sample days is given by:

$$\hat{f}_i^{\text{WSD}} = \frac{\text{WSD}_i}{\sqrt{\frac{1}{M}\sum_{j=1}^{M}\text{WSD}_j^2}} \tag{12}$$

with

$$\text{WSD}_j = \sqrt{1.081 \times \frac{\sum_{h=1}^{N_j} w_{h,j}\bar{r}_{h,j}^2}{\sum_{h=1}^{N_j} w_{h,j}}} \tag{13}$$

where M is the number of returns over day t and N_j the number of i^{th} intraday returns over a given period (in our case, one year). The weights are given by $w_{h,j} = w(\bar{r}_{h,j}/\hat{f}_j^{\text{ShortH}})$ with a

[11] In what follows, $\hat{f}_i^{\text{WSD}}$ will be used for $\widehat{f_{t,i}}$ in Eq. (10).

weighting function $w(z) = 1$ if $z^2 \leq 6.635$ and 0 otherwise. 6.635 represents the 99% quantile of the $\chi^2(1)$ distribution and:

$$\hat{f}_i^{\text{ShortH}} = \frac{\text{ShortH}_i}{\sqrt{\frac{1}{M}\sum_{j=1}^{M}\text{ShortH}_j^2}}.$$

The standardization in Eq. (12) ensures that the standardization condition in Eq. (9) is met.

The Shortest Half (ShortH) is defined as:

$$\text{ShortH}_j = 0.741 \times \min(\bar{r}_{(h_j),j} - \bar{r}_{(1),j}, \bar{r}_{(h_j+1),j} - \bar{r}_{(2),j}, \ldots, \bar{r}_{(n_j),j} - \bar{r}_{n_j-h_j+1,j}) \tag{14}$$

where $\bar{r}_{(1),j}, \ldots, \bar{r}_{(n_j),j}$ are the ordered standardized intraday returns for the j^{th} period of each day such that $\bar{r}_{(1),j} < \bar{r}_{(2),j} < \ldots < \bar{r}_{(n_j),j}$.

— Figure 5 about here —

We implement this methodology and allow for a different periodicity each day of the week and each year over the full sample. In other words, we consider a local window of one day as in Boudt et al. (2011) but our window is not local in the sense that we work with intraday data over diurnal trading sessions only while Boudt et al. (2011) work with 24-hour FX data. In Figure 5, we plot the average (over all years) periodicity factor for the four assets over the week.[12]. We observe a weekly profile that is very similar to the one in Boudt et al. (2011) and which exemplifies the variations in trading activity over the all week.
Jumps detected for the four series are plotted in Figures 1 to 4. The threshold for the jump test filtered statistic is chosen to be $\alpha = 0.01$ which is standard in the literature (cf. Dumitru and Urga (2012)). From the all Figures, it is clear that jumps tend to cluster in more volatile periods. Table 3 provides descriptive statistics for jumps detected using the Boudt et al.'s (2011) methodology. The number of days where a jump occurs is in line with the estimates in Andersen et al. (2007a) as it is in a range of 4 to 8%. It is clear that the FX rate is notably more jumpy than the other three time series.

— Table 3 about here —

3 Fitting the distribution of intraday jumps

In standard asset pricing models, jumps are usually characterized by a compound Poisson process such as in Merton (1976)'s model. The number of jumps follows a Poisson distribution and

[12] In all plots, the first value is the periodicity factor for the first 5-min return of the week. For instance, it corresponds to the 9:00-05 EST for the WTI. The last value is the periodicity factor for the last 5-min return of the week (i.e. 2:25-30 EST for the WTI).

the jump size is distributed as a continuous distribution such as the Gaussian or the double exponential distributions (see Kou and Wang (2004)). The type of jumps that are obtained with the intraday methodologies described earlier present features that turns their empirical distribution incompatible with that of continuous distributions. With those methodologies, jumps are infrequent and potentially large. Therefore, a distribution with thin tails should not be able to pass a goodness-of-fit test comparing it to a sample of jumps.

Furthermore, jumps are estimated by thresholding intraday returns. This means that it is very unlikely that the absolute value of the estimated jumps will reach a very low value. The empirical distribution of jumps will therefore take the shape of a curve with two peaks: one for the jumps with a negative value and one for the jumps with positive ones.

With these facts in mind, standard distributions – even the fat tailed ones – will not be able to pass an adequation test to such data-sets and another modeling direction has to be explored. Descriptive statistics for our jumps data-sets are presented in Table 3. We observe that jumps are exhibiting asymmetry and leptokurticity levels that are not compatible with a Gaussian distribution. Then, Figure 6 displays the empirical distribution of jumps estimated from a non parametric estimator. From those figures, the two peaks shape is obvious. In this section, we present our empirical approach to model the distribution of the jump sizes.

— Figure 6 about here —

For the sake of clarity in this section, we focus only on dates for which a jump has been detected and we denote by $\mathcal{N}$ such a set of dates. In the following, we suppose that the jumps $(J_t)_{t\in\mathcal{N}}$ are i.i.d random variables and we denote by $f_J : \mathbb{R} \to \mathbb{R}_+$ the corresponding density function. From Figure 6 we clearly see that f_J is not only bimodal but also very small in a vicinity $]a:b[$ of zero. In order to reproduce this empirical feature we propose the following natural candidate:

$$f_J(x) = \lambda \frac{f_1(x,\theta_1)1_{x<a}}{\int_{-\infty}^{a} f_1(x,\theta_1)dx} + (1-\lambda)\frac{f_2(x,\theta_2)1_{x>b}}{\int_{b}^{+\infty} f_2(x,\theta_2)dx} \tag{15}$$

where $\lambda \in]0,1[$ and where f_1 and f_2 are two arbitrary density functions depending on two vectorial parameters θ_1 and θ_2. If we denote by F_1 (resp. F_2) the distribution function associated to f_1 (resp. f_2) we can write:

$$f_J(x) = \lambda \frac{f_1(x,\theta_1)1_{x<a}}{F_1(a)} + (1-\lambda)\frac{f_2(x,\theta_2)1_{x>b}}{1-F_2(b)} \tag{16}$$

and f_J may be simply seen as a mixture of left-right truncated distributions. In order to offer the maximum flexibility in terms of goodness-of-fit given the jumps' asymmetric and leptokurtic distribution as well as their two-peak properties, the set of parameters θ_1 and θ_2 do not need to be the same.[13] In addition to the mixture and the truncation parameters, f_J also depends on θ_1 and θ_2 and we favor four different potential candidates for f_i, $i \in 1, 2$ selected for their parameter parsimony as well as their ability to fit the jumps' tail behaviors: the Gaussian distribution (mainly as a benchmark for leptokurticity), the scaled Student distribution[14], the Cauchy distribution[15] and the double exponential distributions.[16] Some of them have been used for modeling jump sizes – especially the first and the last one – when the other two have the ability to better fit tail behaviors. We discarded candidates coming from the Generalized hyperbolic distribution family: with 5 parameters, this distribution would require 12 parameters to be estimated when adding the threshold parameters a and b. We want to rely on models that stand a chance to pass goodness-of-fit tests while remaining as parsimonious as possible.

The parameters of each distribution are estimated over each jump dataset full sample by numerically maximizing the log-likelihood associated to each model. We then run a Kolmogorov-Smirnov test for each of the models and obtain the results presented in Table 4. "Single Gaussian" refers to a simple Gaussian distribution fitted to each time series. The single Gaussian distribution is rejected for the four datasets, as it is neither asymmetric nor bi-modal. The Gaussian bi-model distribution is rejected as well: it fails at capturing the tail behavior of the jumps' distribution. The Double exponential distribution is always rejected at a 5% risk level. Finally, the Cauchy and the scaled Student distributions are always accepted at a 5% risk level. However the acceptance probability are consistently higher for the Cauchy case: for this reason[17], we will rely on this distribution throughout the forthcoming empirical analysis. Figure 7 displays the empirical distribution of the estimated models vs. the empirical distribution of the estimated jumps, for a more graphical assessment of the Kolmogorv-Smirnov test's results.

— Figure 7 about here —

— Table 4 about here —

[13] Even, f_1 and f_2 could potentially not belong to the same family of distributions, while we do not explore this case in the present analysis.

[14] In this case $\theta_i = (\nu_i, \sigma_i) \in \mathbb{R}_+^* \times \mathbb{R}$, $f_i(x, \theta_i) = \frac{1}{\sigma_i \sqrt{\nu_i}\,\mathrm{B}(\frac{1}{2}, \frac{\nu_i}{2})} \left(1 + \frac{x^2}{\sigma_i^2 \nu_i}\right)^{-\frac{\nu_i+1}{2}}$ where B is the Beta function and $\forall x \in \mathbb{R}_+$ (the negative values are obtained by symmetry), $F_i(x) = 1 - \frac{\tilde{\mathrm{B}}\left(\frac{\sigma_i^2 \nu_i}{x^2+\sigma_i^2 \nu_i}, \frac{\nu_i}{2}, \frac{1}{2}\right)}{2\mathrm{B}\left(\frac{\nu_i}{2}, \frac{1}{2}\right)}$ where $\tilde{\mathrm{B}}$ is the incomplete Beta function.

[15] In this case, $\theta_i = (a_i, x_i^0) \in \mathbb{R}_+^* \times \mathbb{R}$, $f(x, \theta_i) = \frac{1}{\pi}\left[\frac{a_i}{(x-x_i^0)^2 + a_i^2}\right]$ and $F_i(x) = \frac{1}{\pi} \arctan\left(\frac{x - x_i^0}{a_i}\right) + \frac{1}{2}$.

[16] In this case $\theta_i = (\mu_i, b_i) \in \mathbb{R} \times \mathbb{R}_+^*$, $f(x, \theta_i) = \frac{1}{2b_i} \exp\left(-\frac{|x-\mu_i|}{b_i}\right)$ and $F_i(x) = \begin{cases} \frac{1}{2}\exp\left(\frac{x-\mu_i}{b_i}\right) & \text{if } x < \mu_i \\ 1 - \frac{1}{2}\exp\left(-\frac{x-\mu_i}{b_i}\right) & \text{if } x \geq \mu_i \end{cases}$.

[17] The fact that the scaled Student distribution both rely on parameters distributed over the real line and over $\mathbb{N}$ could also be problematic for the rolling estimation procedure that we intend on using.

4 The models

4.1 Linking conditional variance and realized volatility

An important idea in Maheu and McCurdy (2011) is to relate the conditional variance of daily returns σ_t^2 to the realized volatility estimator through a cross-equation restriction. Barndorff-Nielsen and Shephard (2002) and Andersen *et al.* (2003) show that under some empirically realistic assumptions, the conditional variance of daily returns should be equal to the conditional expectation of quadratic variation with respect to the past information.[18] Assuming that the realized volatility RV_t is an unbiased estimator of the quadratic variation, it follows that:

$$\sigma_t^2 = E_{t-1}(RV_t).$$

In other words, the *one-period-ahead* conditional expectation of the realized volatility should equal the "true" conditional volatility assuming the unbiasedness of the realized volatility estimator. Under the assumption of a log-normal distribution for the realized volatility[19], the conditional expectation may be simply written as:

$$\sigma_t^2 = E_{t-1}(RV_t) = \exp\left(E_{t-1}\log(RV_t) + \frac{1}{2}\text{Var}_{t-1}(\log(RV_t))\right). \tag{17}$$

Using the results presented in Section 2, we can extend this theory by writing the realized volatility RV as the sum of the median realized volatility MedRV and the jump component quadratic variation when the latter is significant:

$$\sigma_t^2 = E_{t-1}\left(MedRV_t + \sum_{j=J(t-1)}^{J(t)} \kappa^2(t_j)\right) \tag{18}$$

where

$$E_{t-1}(MedRV_t) = \exp\left(E_{t-1}\log(MedRV_t) + \frac{1}{2}\text{Var}_{t-1}(\log(MedRV_t))\right). \tag{19}$$

Hence, if we are interested in the impact of disentangling jumps from the rest of the volatility, using Eq. (18) constitutes an appropriate way to proceed.

We now turn to the specification of a predictive model for realized volatility measures, namely the HAR model. Note that the choice of this model is central for the role that conditional expectation in Eq. (17) and (18) will play in forecasting.

[18]These ideas were already developed in papers such as French et al. (1987) or Zhou (1996).

[19]Empirical evidence of this hypothesis can be found in early contribution such as Andersen *et al.* (2001a and b, 2003). Similar evidence for foreign exchange rates, futures markets, crude oil futures and the FTSE index may be found in Pong *et al.* (2004), Thomakos and Wang (2003), Wang *et al.* (2008) and Areal and Taylor (2002), respectively.

4.2 Heterogeneous autoregressive model of realized volatility

The HAR-RV model initially developed by Corsi (2009) has been used with success in a number of recent contributions (Andersen et al. (2007a), Corsi *et al.* (2008), Liu and Maheu (2009), Duong and Swanson (2015), Patton and Sheppard (2015) among many others). The economic intuition behind this model is that different groups of investors have different investment horizons, and consequently behave differently (see Müller *et al.* (1997) for the presentation of the HARCH original model relying on the Heterogeneous Hypothesis). The genuine HAR-RV model is formally a constrained AR(22) model using RV as the realized measures of variance but the HAR can naturally accommodate all realized measures (as MedRV) and transformations of these measures.[20] In particular, the log transformation has been found to perform very well and we choose to use it along with our different realized measures RM. The HAR model using daily, weekly and monthly[21] realized volatility components may be written as follows:

$$\log(RM_t) = \omega + \phi_1 \log(RM_{t-1}) + \phi_2 \log(RM_{t-5:t-1}) + \phi_3 \log(RM_{t-22:t-1}) + \eta u_t \tag{20}$$

where $\log(RM_{t-k:t-1}) \equiv \frac{1}{k}\sum_{j=1}^{k} \log(RM_{t-j})$.

In the following, the error term u_t is supposed to be Gaussian[22] to cope with Eqs. (17) and (19).

4.3 The bivariate model for daily returns and volatilities

In this section we present the bivariate model inspired by Maheu and McCurdy (2011). The main interest behind multivariate models consists in obtaining densities forecasts. More precisely our approach is made of three key ingredients: an independent jump component[23], a conditional variance (with or without jumps) that is linked to appropriate realized volatility measures and a conditionally Gaussian heterogeneous autoregressive model of realized volatility. Thus, in the model with at most one jump at time t the dynamics of the log-returns may be written as:

$$r_t = \mu + \sigma_t \epsilon_t + \mathbb{1}_t J_t \tag{21}$$

[20] Forsberg and Ghysels (2007) and Ghysels and Sohn (2009) note that other power transformations may be used to model the dynamics of the realized volatility. These studies show that for a number of stochastic volatility processes used in the financial literature the absolute value of the realized volatility is a better predictor of the future realized volatility, particularly for longer horizons. As we deal with quite short-term horizons here, we do not follow this approach in our empirical analysis.

[21] The optimal lag structure for the HAR model has been investigated in Craioveanu and Hillebrand (2010) who find that the genuine structure suggested in Corsi (2009) performs the best.

[22] As in Maheu and McCurdy (2011), conditional non-normality is introduced in the bivariate modeling via return equation and not in the variance one.

[23] According to the empirical descriptive statistics about jumps extracted using the detection methodology developed in Section 2, we favor in this section a model with at most one jump where the jump distribution is carefully estimated using the parametric family of distributions described in Section 3.

where σ_t is made observable through the relation $\sigma_t^2 = E_{t-1}(RM_t)$ with a well chosen realized measure RM fulfilling[24]

$$\begin{aligned} \log(RM_t) &= \omega + \phi_1 \log(RM_{t-1}) + \phi_2 \log(RM_{t-5:t-1}) + \phi_3 \log(RM_{t-22:t-1}) \\ &\quad + \gamma_1 \epsilon_{t-1} + \gamma_2 \mathbb{1}_{t-1} |J_{t-1}| + \eta u_t \end{aligned} \tag{22}$$

In the preceding equation, γ_1 is a parameter capturing a feedback effect from past returns to the subsequent increment in volatility. Finally, γ_2 is meant to materialize the feedback effect from past jumps to increases in volatility.[25] In Eqs.(21) and (22), we suppose that $((\epsilon_t, \mathbb{1}_t, J_t, u_t))_{t\in\{0,\dots,T\}}$ are independent random variables and we denote by $(\mathcal{F}_t)_{t\in\{0,\dots,T\}}$ the associated filtration.[26] Assuming that the u_t are i.i.d $N(0,1)$ we then have:

$$\sigma_t^2 = E_{t-1}(RM_t) = \exp\left(E_{t-1}\log(RM_t) + \frac{1}{2}\mathrm{Var}_{t-1}(\log(RM_t))\right). \tag{23}$$

Concerning the separated jump component, we suppose that $\mathbb{1}_{t-1}$ follows a binomial distribution, with a time independent parameter p representing the probability that a jump occurs at time t, and that the jumps J_t follows the mixture of left-right truncated Cauchy distributions described in Section 3.

When $p = 0$ (and thus RM=RV), this model nests the so-called HAR-RV model developed in Maheu and McCurdy (2011) and for our empirical investigations ϵ_t can either be Gaussian or a mixture of Gaussian distributions[27], the later being able to span a very large scope of couples of kurtosis and skewness as presented in Bertholon *et al.* (2006).[28] In this case, using the independence hypothesis, we obtain the conditional joint density of the pair (r_t, RV_t) and the bivariate model is estimated by conditional maximum-likelihood.

When $p \neq 0$ (and thus RM=MedRV), even if the conditional joint density of the pair (r_t, RV_t) is still available, we favor a two-steps estimation strategy that takes advantage of the jump detection presented in Section 2. In fact, the probability p is first estimated by the proportion

[24] As explained in Section 4.1, when the jump probability p is null we take RM=RV and RM=MedRV otherwise.

[25] On the impact of jumps in returns on volatility see for example Duffie et al. (2000), Eraker et al. (2003), Eraker (2004), Maheu et McCurdy (2004), Aït-Sahalia et al. (2015) and Carr and Wu (2011) among others. Guégan *et al.* (2013) show the empirical importance of jumps for option pricing.

[26] We can see easily that this filtration is also generated by the observations of the log-returns and of the realized measure RM until time t.

[27] The mixture of two Gaussian distributions has the density $f_{MN}(x) = \alpha n(x, \mu_1, \sigma_1) + (1-\alpha) n(x, \mu_2, \sigma_2)$ where $(\alpha, \mu_1, \mu_2, \sigma_1, \sigma_2) \in [0,1] \times \mathbb{R}^2 \times (\mathbb{R}^*)^2$ and where $n(., \mu_i, \sigma_i)$ is the density of a Gaussian random variable with expectation μ_i and standard deviation σ_i (see for example Kon (1984), Akgiray and Booth (1987), Tucker and Pond (1998) and Alexander and Lazar (2006)).

[28] The importance of modeling higher moments of returns is emphasized in Jondeau and Rockinger (2003).

of jumpy days in our datasets and the parameters of the jump density f_J are obtained from the classical maximum-likelihood for independent observations. In a second step, remarking that the conditional density of the log-returns is given by[29]:

$$f_{r_t}(z \mid \mathcal{F}_{t-1}) = p \int_{-\infty}^{+\infty} f_\epsilon(\frac{x-\mu}{\sigma_t}) f_J(z-x) \frac{dx}{\sigma_t} + \frac{(1-p)}{\sigma_t} f_\epsilon(\frac{z-\mu}{\sigma_t}) \tag{24}$$

where f_ϵ is the density of the innovations in the return equation, we obtain the remaining parameters using the explicit conditional joint density of the pair $(r_t, MedRV_t)$ and conditional maximum-likelihood.

Across our empirical experiences, we focus on the five following models:

- RV-Gaussian: A HAR-RV model estimated on a Gaussian distribution for which $\epsilon_t \sim N(0,1)$, $p=0$ and $\gamma_2 = 0$
- RV-MN: A HAR-RV model estimated on a mixture-of-normals distribution for which $\epsilon_t \sim MN(\alpha, \mu_1, \sigma_1, \mu_2, \sigma_2)$, $p=0$ and $\gamma_2 = 0$
- MedRV-MN: A HAR-MedRV model estimated on a mixture-of-normals distribution for which $\epsilon_t \sim MN(\alpha, \mu_1, \sigma_1, \mu_2, \sigma_2)$, $p=0$ and $\gamma_2 = 0$
- MedRV – Jumps1: A HAR-MedRV model along with jumps but no feedback for which $\epsilon_t \sim N(0,1)$, $p \neq 0$ and $\gamma_2 = 0$
- MedRV – Jumps2: A HAR-MedRV model along with jumps and feedback effect for which $\epsilon_t \sim N(0,1)$, $p \neq 0$ and $\gamma_2 \neq 0$.

4.4 Test methodology

To evaluate the relative accuracy of competing forecasts, we rely on the test statistics developed by Diebold and Mariano (1995) in the context of the comparison of density forecasts (Amisano and Giacomini (2007)). In our presentation we follow Maheu and McCurdy (2011), focusing on the ability of the approach to test multi-period forecasts on the same out of sample log-returns.

For $\mathcal{M} \in \{\mathcal{A}, \mathcal{B}\}$ we consider two competing models and we denote by $\theta^{\mathcal{M}}$ the corresponding parameters that drive the log-returns and the intraday volatility measure dynamics. Starting

[29] In the estimation process, the density $f_{r_t}(z \mid \mathcal{F}_{t-1})$ is approximated running the *integrate* command of the stats package of the software R that is particularly well suited for numerical integration on unbounded domains.

from a sample $(r_1, ...r_T)$ of size[30] T we want to test forecast horizons $1 \leq k \leq k_{max} = 60$ through rolling-window forecasting schemes of size τ.[31] Thus, for $k \geq 1$, the average predictive likelihood for the forecast horizon k is given by:

$$D_{\mathcal{M},k} = \frac{1}{T - \tau - k_{\max} + 1} \sum_{t=\tau+k_{\max}-k}^{T-k} \log f_{\mathcal{M},k}(r_{t+k}, \widehat{\theta_{t+k}^{\mathcal{M}}}|\mathcal{F}_t), \tag{25}$$

where $\widehat{\theta_{t+k}^{\mathcal{M}}}$ is the maximum likelihood estimator of the model $\mathcal{M}$ obtained from the sample $(r_{t-\tau+1}, ..., r_t)$ of length τ and where

$$f_{\mathcal{M},k}(r_{t+k}, \widehat{\theta_{t+k}^{\mathcal{M}}}|\mathcal{F}_t)$$

is the conditional density of the log-returns at time $t + k$, given $\mathcal{F}_t$ and $\widehat{\theta_{t+k}^{\mathcal{M}}}$, evaluated at the realized log-return r_{t+k}. Thus, for each forecast horizon, we are left with $T - \tau - k_{max} + 1$ predictions. The particular form of (25) allow us to obtain a term structure of average predictive likelihoods, $(D_{\mathcal{M},1}, ..., D_{\mathcal{M},k_{max}})$, to compare the performance of alternative models $\mathcal{M}$ over an identical set of out-of-sample data points $(r_{\tau+k_{\max}}, ..., r_T)$.

For practical use, the terms $f_{\mathcal{M},k}(r_{t+k}, \widehat{\theta_{t+k}^{\mathcal{M}}}|\mathcal{F}_t)$ in (25) have to be computed but unfortunately, in our setting, these conditional densities do not have an explicit form except for $k = 1$ and have, in general, to be evaluated generating independent realizations of r_{t+k} given $\mathcal{F}_t$ and using classical density kernel estimators. Nevertheless, remarking that

$$f_{\mathcal{M},k}(r_{t+k}, \widehat{\theta_{t+k}^{\mathcal{M}}}|\mathcal{F}_t) = \int_{\mathbb{R}_+} f_{\mathcal{M},k}(r_{t+k}, \widehat{\theta_{t+k}^{\mathcal{M}}}|\sigma_{t+k}^2) p(\sigma_{t+k}|\mathcal{F}_t) d\sigma_{t+k}^2 \tag{26}$$

we have

$$f_{\mathcal{M},k}(r_{t+k}, \widehat{\theta_{t+k}^{\mathcal{M}}}|\mathcal{F}_t) \approx {}^{32} \frac{1}{N} \sum_{i=1}^{N} f_{\mathcal{M},k}(r_{t+k}, \widehat{\theta_{t+k}^{\mathcal{M}}}|(\sigma_{t+k}^2)^i) \tag{27}$$

where $f_{\mathcal{M},k}(y_{t+k}, \widehat{\theta_{t+k}^{\mathcal{M}}}|\sigma_{t+k}^2)$ is now perfectly known and where $(\sigma_{t+k}^2)^i$ are independent realizations of σ_{t+k}^2 generated from the preceding dynamics with parameters $\widehat{\theta_{t+k}^{\mathcal{M}}}$ and starting values σ_t and r_t as explained in Maheu and McCurdy (2011).

According to Amisano and Giacomini (2007), under the null hypothesis of equal performance, the statistic based on predictive likelihoods of horizon k for models $\mathcal{A}$ and $\mathcal{B}$,

$$t_{\mathcal{A},\mathcal{B}}^k = \frac{(D_{\mathcal{A},k} - D_{\mathcal{B},k})\sqrt{T - \tau - k_{max} + 1}}{\hat{\sigma}_{\mathcal{A},\mathcal{B},k}} \tag{28}$$

[30] For the datasets considered in this paper and described in Table 1 we have $T = 7,662$ for the S&P 500, $T = 5,713$ for the 10-year Treasury Note, $T = 6,869$ for the USCAD, and $T = 6,279$ for the WTI.

[31] In the empirical part, we estimate the models on a rolling window of $\tau = 1200$ daily observations.

[32] In practice, numerical results will be obtained using $N = 10000$ Monte Carlo simulations.

is asymptotically standard normal, where $\hat{\sigma}_{\mathcal{A},\mathcal{B},k}$ is a properly selected estimator for the variance of:

$$\log f_{\mathcal{A},k}(r_{t+k}, \widehat{\theta^{\mathcal{A}}_{t+k}}|\mathcal{F}_t) - \log f_{\mathcal{B},k}(r_{t+k}, \widehat{\theta^{\mathcal{B}}_{t+k}}|\mathcal{F}_t).$$

Here, as proposed in Amisano and Giacomini (2007), we use a Newey-West estimator[33], with a lag chosen to be the integer part of $hmax * 0.15$, that takes into account heteroskedasticity and autocorrelation due tothe overlapping nature of our exercise. One of the main interest of this approach comes from the fact that the two models can indifferently be nested or not and can be estimated using very different techniques as soon as they are based on a finite estimation window.[34]

5 Empirical results

5.1 Full-sample analysis

Tables 5 to 8 present the estimation results for the four assets. The tables display the average values for each parameter across our rolling estimations, as well as the 5 and 95% quantiles. For most of the parameters, both quantiles are of the same sign, pointing into the direction of statistically significant parameters.

Parameter values are in line with the literature and in particular with Maheu and MacCurdy (2011). Across our estimations, log realized risk measures exhibit a persistent behaviour, with most ϕ_i parameters positive while their sum remains below 1. The η parameter is also estimated to have a value close to what Maheu and MacCurdy (2011) found in their numerical experiments. Also, γ_1 is consistently estimated to be negative in all data sets. It is found to be different from zero for the S&P 500 and the WTI, highlighting that negative returns contribute positively to the subsequent volatility. In the case of Treasury bonds and CAD, this parameter is not found to be statistically significant when the conditional distribution of returns is chosen to be asymmetric. This conclusion both holds for the mixture of Gaussian distribution and the discontinuous jump distribution introduced in Section 3. As such, the γ_1 parameter measuring the leverage effect – that is an asymmetric feedback from past returns into subsequent volatilities – is found to be weaker in line with the findings of Chorro *et al.* (2018).

— Tables 5 to 8 about here —

[33]This estimator may be computed using the command *NeweyWest* of the package *sandwich* of the R software.

[34]In our empirical part, some nested models such as those including or not a feedback effect will be compared.

Turning to the jump-to-volatility feedback parameter γ_2, it is estimated to be positive across all four time series of returns, potentially indicating that jumps have a tendency to be the trigger episodes of periods of heightened volatility. However, our results also suggest that this parameter is not significant, but in the case of the WTI. As such, in spite of the positivity of the estimated parameters, this parameter does not seem to a have statistically material impact on the modeling of the returns' distribution.

Finally, turning to the parameters associated to the jump part of the estimated models, our estimates depict jump distributions for which negative jumps happen more often than positive ones. This conclusion stems from the estimated values of the λ parameter. It ranges from 0.52 (CAD) to 0.595 (WTI), implying the negative jumps are more frequent than positive ones. This ties well with the estimates presented in Table 3, in which the most negative average for jumps is found for the WTI and the least negative for the CAD. Also consistent with Table 3, the parameter for the Bernouilli distribution is estimated to be higher for the CAD (7.8%) and for the 10-year Treasury Note (4.9%) than for the other two: jumps are more frequent for these two series of returns. Turning to the Cauchy distribution parameters, the left and right tail distributions are driven by estimated parameters that display a comparable set of absolute values. One notable exception is the WTI for which the left tail Cauchy distribution has a location parameter x_1^0 the absolute value of which is higher than x_2^0, hinting that the the shape of both distributions can be different. The threshold parameters a and b are found to be statistically significant and of opposite signs, justifying the interest of our approach. An interesting exception is found with the 10-year Treasury Note for which a is not statistically different from 0. In this case, jumps are either located around zero or above zero, a shape that the methodology proposed here can deal with. Finally, the parameters estimates related to the jump part of the models are overall hinting towards jumps having an asymmetric distribution, in a similar spirit to the MN model estimates but using realized jumps instead of returns. For all MN-based models, α is found to be below 50% illustrating that negative extreme values are estimated to happen more often than positive ones. This result is consistent across our models and data sets.

Tables 9 to 12 and Figure 8 display the results of the Amisano and Giacomini (2007) comparison of density forecasts test. From these tables and plots, several conclusions arise. First, for forecast horizons ranging from 1 and 30 days, the models using the MedRV volatility measure as well as jumps based on the discontinuous distributions introduced earlier statistically dominate the rest of the models across the four datasets. This is the main conclusion of our empirical work. In other words, for forecasting horizon of up to a month and a half, disentangling jumps from volatility improves our ability to forecast the distribution of the four very diverse types of returns on financial assets that are considered in this article.

— Tables 9 to 12 about here —

— Figure 8 about here —

Second, the jump-to-volatility feedback component does not improve the forecast ability of the models for all assets but in the cases of a forecast horizon of 60 days for CAD and the WTI, for which a statistically significant parameter had been found: the jump-to-volatility feedback component only seems to matter in the case of oil prices for forecast horizons of 20 and 60 days. In the case of oil, for distant horizon density forecasts, the Gaussian RV model still outperforms the rest of its competitors. Finally, in the case of the S&P 500, for a 60-day forecast horizon the Gaussian RV model dominates the rest of tested models. Still, for such a forecast horizon, the jump-based models remain equivalent to the MedRV-MN model, a sign that the jump estimation methodologies yield accurate jump estimates.

For the sake for comparison, Table 13 presents the results in the case of the S&P500 futures on a sample period similar to Maheu and McCurdy (2011). This table makes it easier to understand the interest of the modeling approach presented here. From this table, two main conclusions can be reached. First, consistently with Maheu and McCurdy (2011), the RV-G ("HAR" in their article) model is found to deliver a strong forecasting performance. Over this sub-sample, the RV-G model is found to be comparable to the jump model with feedback and even dominates the model without feedback for a forecasting horizon of 60 trading days. This confirms the strength of their original model. Second, for shorter horizons for which jumps really matter, the jump models are outperforming, delivering a significantly superior forecasting performance: our article therefore extends their original findings, casting light on when – in terms of forecasting horizon – jumps measured from intra-day datasets can help improve returns density forecasts.

— Table 13 about here —

5.2 How do models with jumps perform in crisis period?

One question naturally arising from the use of a jump focused model is to gauge whether it does better during volatile periods or not. Table 14 presents the outcome of density forecasts based only on periods with a higher volatility. For the sake of space saving, we only present the results in the case of the S&P500 futures for which periods of increased volatility are probably more regularly studied in the literature[35]. Volatility periods are defined here as days for which the realized volatility is higher than its 80th percentile[36]. During such periods, the jump based models

[35]For the rest of the assets, qualitatively similar results have been obtained, that are available upon request but not presented here to save space.

[36]Results leading to similar conclusions have been obtained when using the 70th and 90th percentile instead.

are found to show an even stronger out-performance than usual (when comparing to the initial Table 9), as their test statistics are found to be largely increased. Across the three forecasting horizons retained here, the jump model are strongly outperforming their peers. This results is in line with the observation that jumps tend to cluster in more volatile periods (see Figures 1 to 4).

— Table 14 about here —

6 Some economic value of modeling jumps

Instead of only relying on a statistical criterion based on different likelihood ratio tests, we conclude this section by illustrating the economic importance of having jumps in the forecast questioning their impact on the computation of the so-called Value-at-Risk (VaR). Classically, the daily conditional VaR is defined by

$$P(r_t \leq -VaR_{t|t-1} \mid \mathcal{F}_{t-1}) = \alpha$$

(α will be 1% for the applications).
For the bivariate models described in Equation (21) the one period ahead VaR may be obtained easily once the parameters are properly estimated:

- When $p = 0$, we have
$$VaR_{t|t-1} = -\sigma_t \hat{F}_{\epsilon_t}(\alpha) - \mu$$
where $\hat{F}_{\epsilon_t}(\alpha)$ is the α quantile of ϵ_t[37].

- When $p \neq 0$, we have
$$VaR_{t|t-1} = -\hat{F}_{r_t}(\alpha)$$
where $\hat{F}_{r_t}(\alpha)$ is the α quantile related to the conditional density (24) that was already computed to estimate the model[38].

To compare the empirical performances of models in modeling the daily Value-at-Risk we use the quantile loss function test of Angelidis et al. (2004) that is an adapted version of the Diebold and Mariano (1995) methodology for our purpose: As an out of sample period we take $[0, T]$ for the year 2008. First we compute $q(\alpha)$ the α quantile of the daily log-returns on $[0, T]$. For $t \in [0, T-1]$, we estimate the model using the previous 1200 points, we compute $VaR_{t+1|t}$ and the loss function Ψ_t of the test at time t: $\Psi_t = (Y_{t+1} - VaR_{t+1|t})^2$ if $Y_{t+1} \leq VaR_{t+1|t}$ or $(q(\alpha) - VaR_{t+1|t})^2$ otherwise.

[37] Computed running the *qnorm* and *qmixnorm* commands of the bda package of the software R.
[38] To compute the quantile related to (24) one can use for example the command *inverseCDF* of the HDInterval package of the software R.

For two models A and B we define $e_t = \Psi_t^A - \Psi_t^B$ and under the hypothesis of same forecasting performances on $[0, T]$ we have

$$\frac{\sum_{t=0}^{T-1} e_t}{\sqrt{T}\hat{\sigma}_T} \longrightarrow \mathcal{N}(0,1)$$

where $\hat{\sigma}_T$ is the Newey-West estimator of the standard deviation of the $e_t's$. The results for the test of Angelidis et al. (2004) are reported in Table (15) for the four time series studied in this paper. The models including jumps in the dynamics always outperform their competitors at a 5% level. This illustrates a potential economic value to the modeling approach presented in this paper: jumps matter to compute risk metrics such as the Value-at-Risk.

— Table 15 about here —

7 Concluding remarks

Based on a set of nested competing models, this article aims at testing whether disentangling jumps from volatility using intraday data helps forecast the density of returns. Using four different data sets – equity index, bonds, commodities and currencies – we find consistent evidence that for short to medium term horizons, the density forecasts based on such an approach improves over its competitors. This result is obtained by relying on a jump modeling that takes into account the specificities of the intraday-based jump estimates. Whilst numerous authors have considered the informational content of continuous *vs.* jump components for volatility forecasting, none have thought to address the particular question of density forecasting. Detection of such information for different classes of assets would indicate the ability of new econometric models to anticipate the evolution of density returns in a fundamentally different way compared to more traditional forecasting models.

One of the main conclusion of this article is that jumps are important mainly for shorter dated forecasts (up to a month and a half). This result is consistent with the rest of the literature, see for example Corsi et al. (2012) Table 1.2 and Audrino and Hu (2016) Table 4 as well as Asai and McAleer (2017) in the case of co-jumps. The economic rationale to that phenomenon stems from the fact that jumps should be reflective of a "price discovery" moment – that is a permanent shock to the asset's price – and therefore should not lead to a mean-reversion type of pattern as it is the case for volatility. The case of volatility is very different: a volatility shock leads to an increase in volatility which usually reverts back to its long run level eventually. Jumps are therefore meant to help forecast shorter dated densities while the mean reversion feature of volatility is probably more important for longer dated ones.

A natural potential extension to the present paper would be to investigate further the economic value of the forecast improvement evidenced in our results. This could be done in the context of option pricing, portfolio choice and the computation of alternative risk measures. In particular, extending the results in Christoffersen et al. (2014), Corsi *et al.* (2013) and Alitab *et al.* (2019), while not trivial, is a promising venue for future research. In a very different vein, it might also be of interest to consider the informational content of various additional economic variables such as in Christiansen et al. (2012) or Paye (2012).

Figures

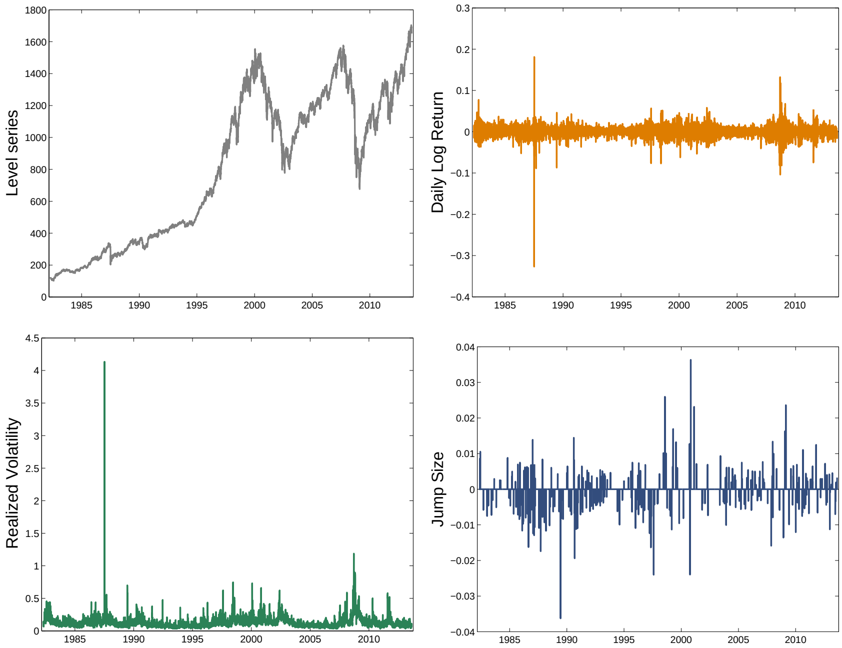

Figure 1: S&P 500 index futures (04/21/1982 – 08/16/2013).

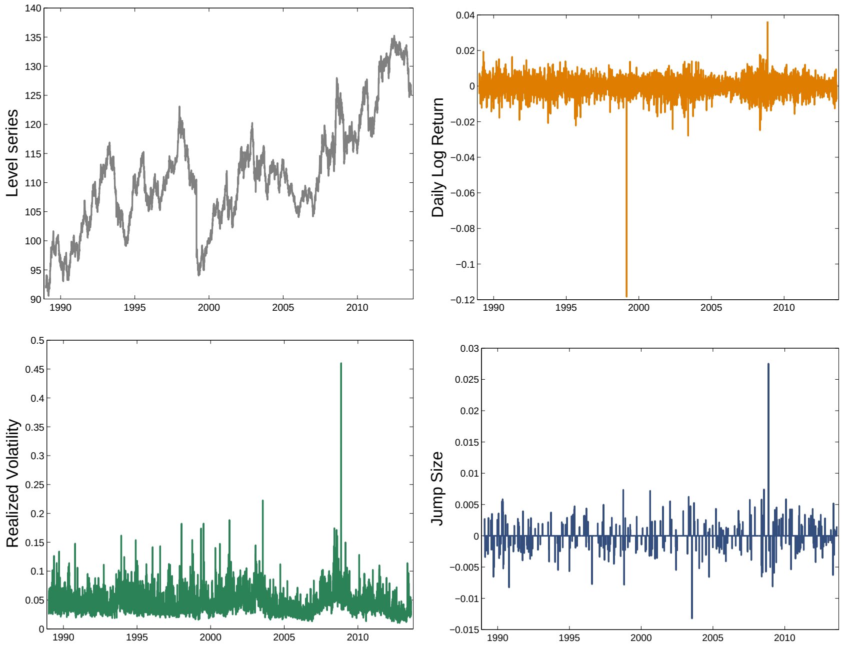

Figure 2: 10-year Treasury Note futures (01/03/1989 – 08/16/2013).

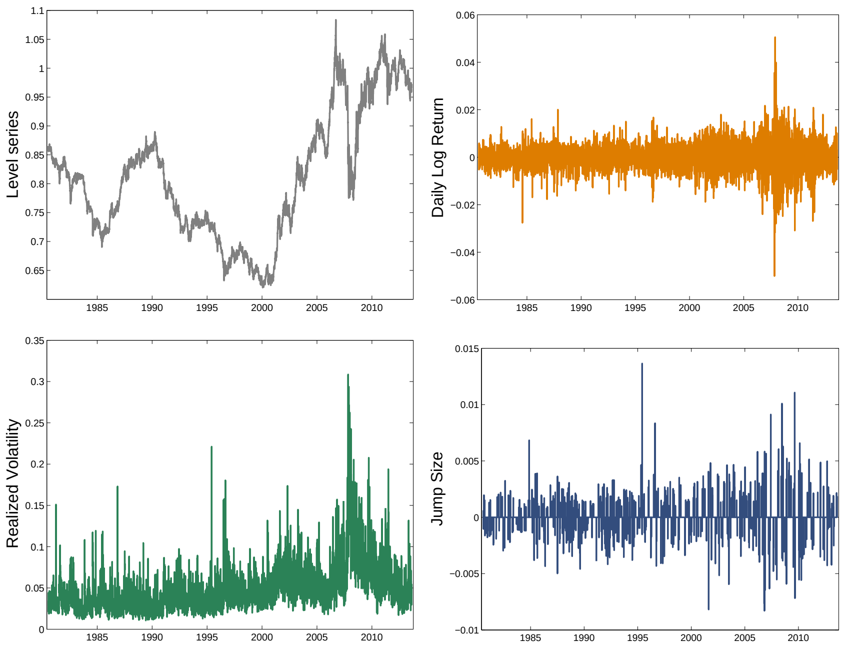

Figure 3: USD/CAD exchange rate futures (07/21/1980 – 08/16/2013).

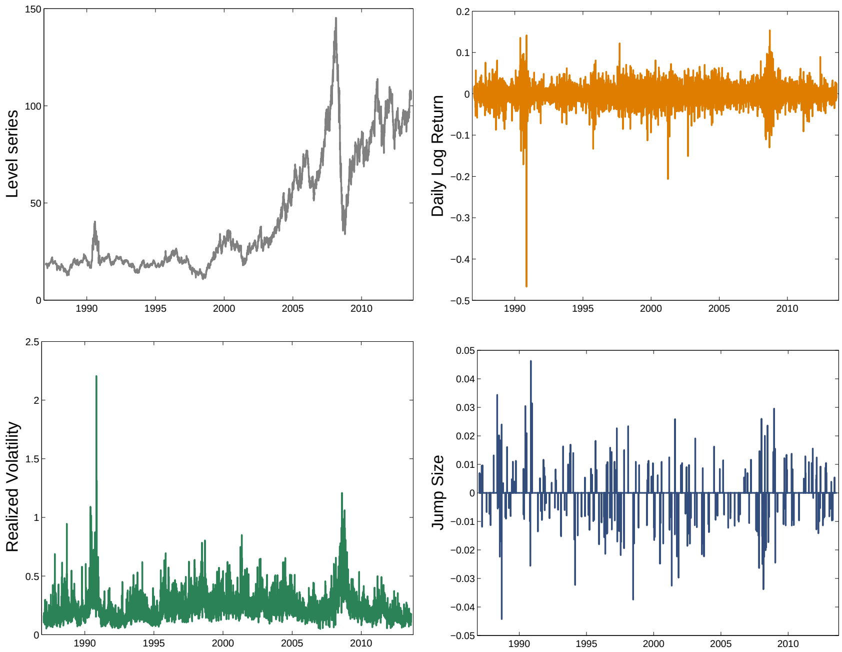

Figure 4: WTI crude oil futures (01/02/1987 – 08/16/2013).

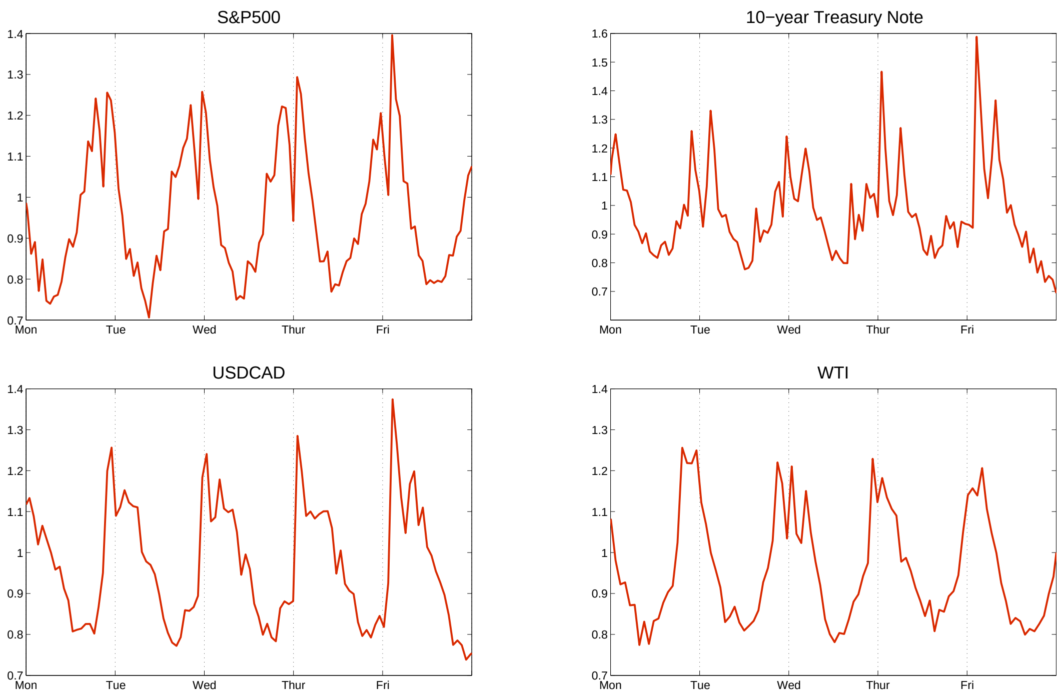

Figure 5: WSD periodicity for S&P 500, 10-year Treasury Note, USDCAD and WTI.

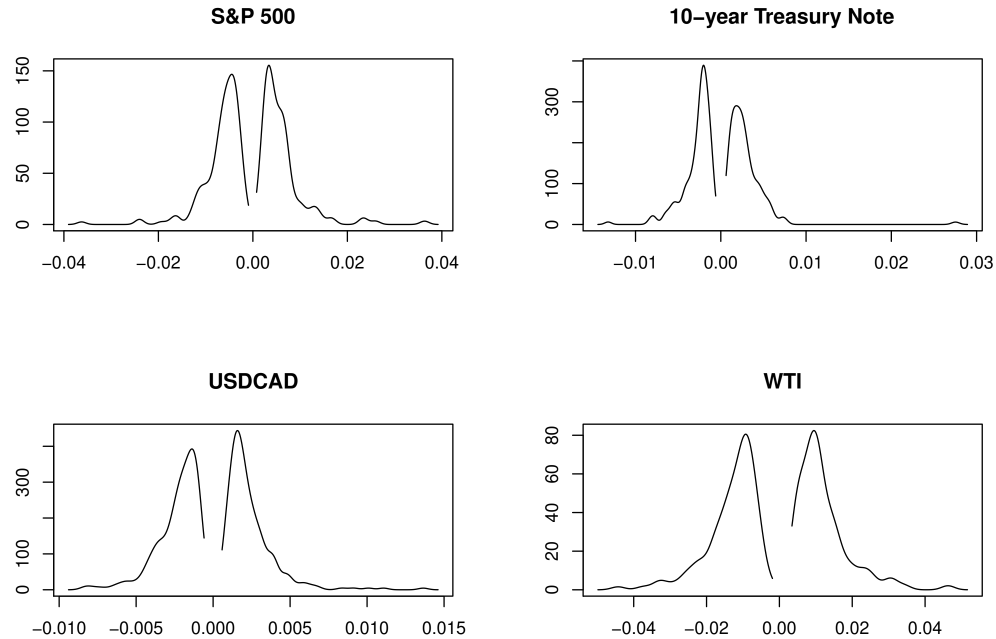

Figure 6: Empirical distribution of jumps for S&P 500, 10-year Treasury Note, USDCAD and WTI. The empirical distribution has been estimated using a non-parametric kernel.

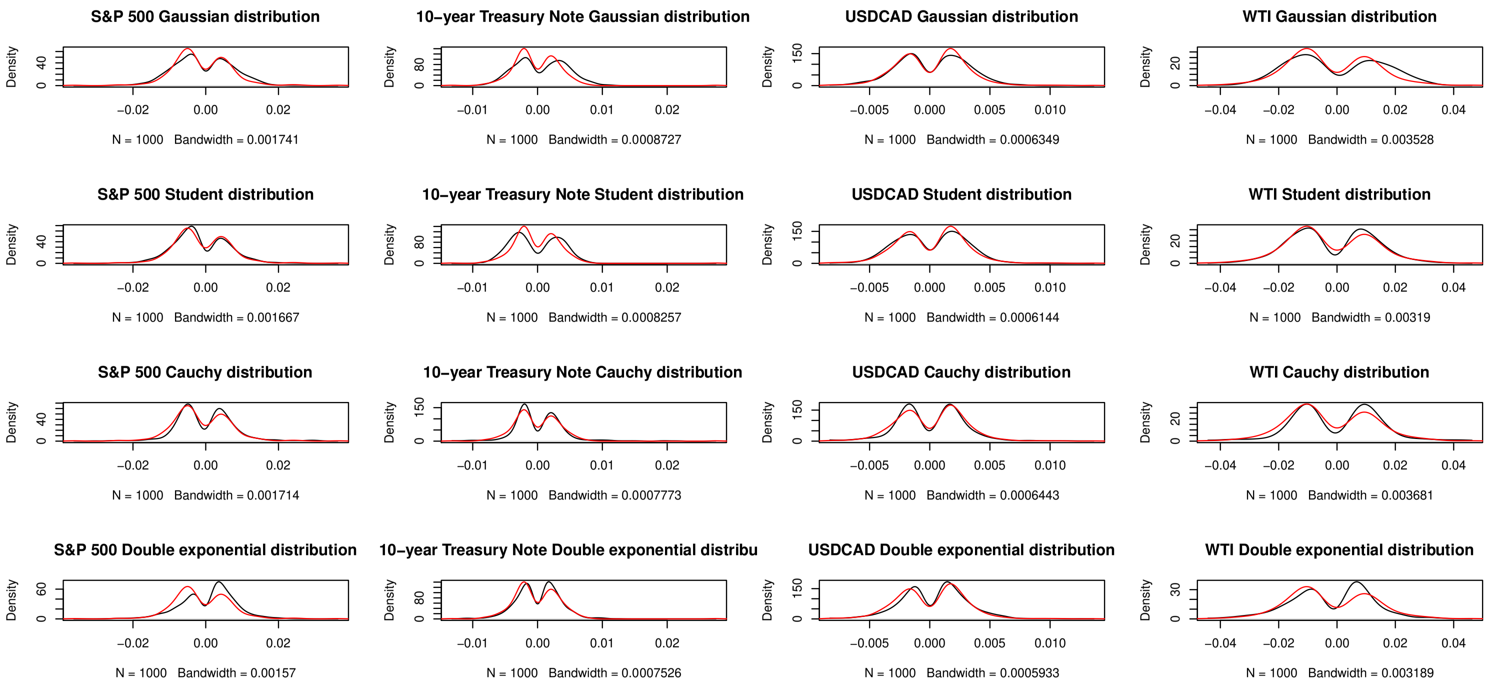

Figure 7: Empirical distribution (light line) of jumps for S&P 500, 10-year Treasury Note, USDCAD and WTI vs. estimated distributions (dark line).

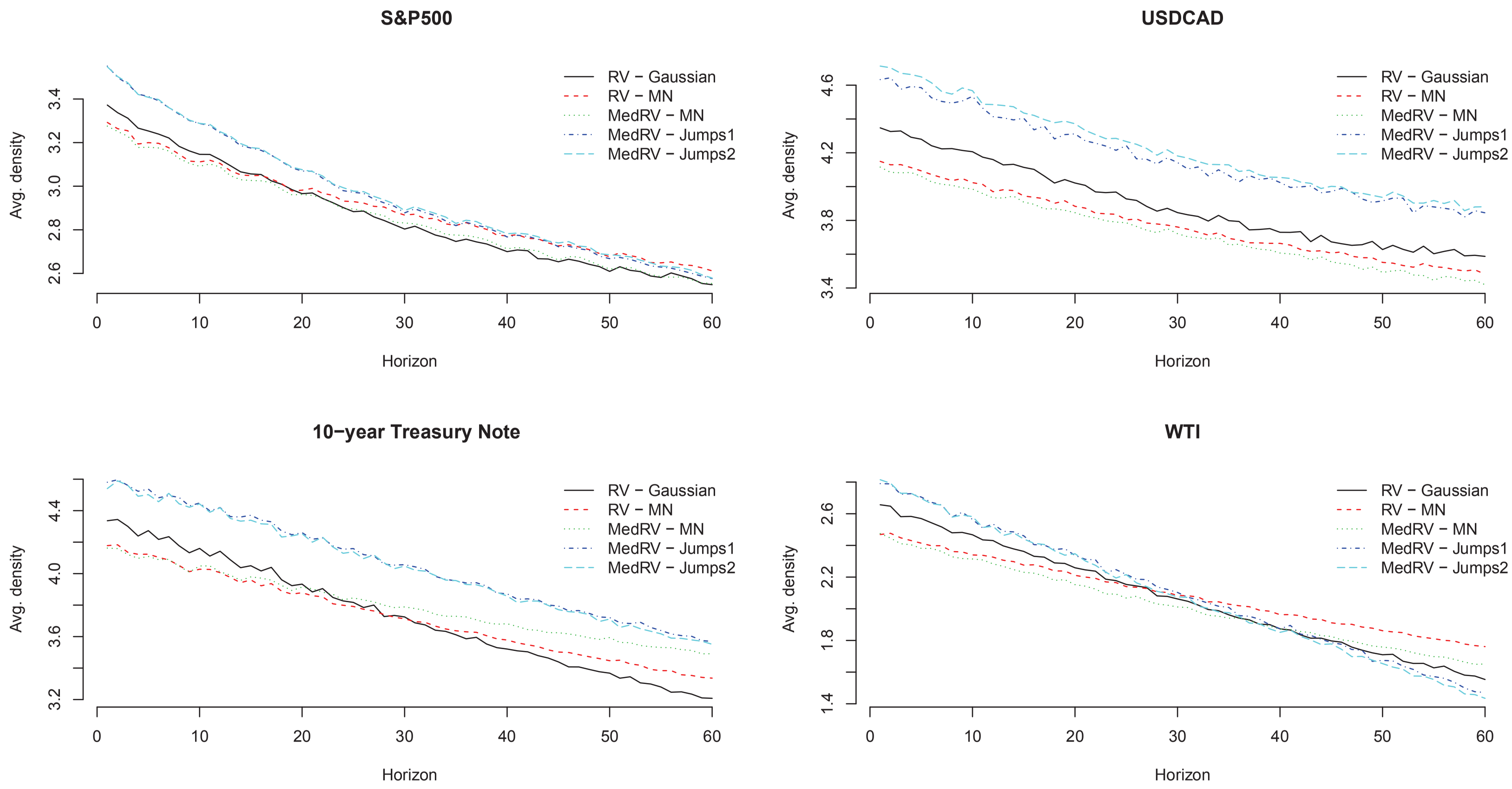

Figure 8: Average log density per forecasting horizon for each model across the four datasets.

Tables

Futures contract	Exchange	Sample	Trading Hours	Nb of days
S&P 500	CME	04/21/1982–08/16/2013	8:30 a.m.–3:15 p.m. (CT)	7,662
10-year Treasury Note	CME-CBOT	01/03/1989–08/16/2013	7:20 a.m.–2:00 p.m. (CT)	5,713
USDCAD	CME	07/21/1980–08/16/2013	7:20 a.m.–2:00 p.m. (CT)	6,869
WTI	CME-NYMEX	01/02/1987–08/16/2013	9:00 a.m.–2:30 p.m. (CT)	6,279

Table 1: Futures data.

		S&P 500	10-year Treasury Note	USDCAD	WTI
Returns statistics	Average return	0.087	0.013	0.004	0.072
	Standard deviation	0.201	0.072	0.079	0.376
	Skewness	-1.984	-3.376	-0.150	-1.493
	Kurtosis	72.849	87.579	10.057	31.039
RV statistics	Average volatility - RV	0.131	0.045	0.048	0.236
	Volatility of volatility - RV	0.095	0.021	0.027	0.119
	Skewness of volatility - RV	13.355	2.964	2.208	2.668
	Kurtosis of volatility - RV	458.792	36.097	11.747	22.075
MedRV statistics	Average volatility - MedRV	0.124	0.043	0.045	0.221
	Volatility of volatility - MedRV	0.091	0.019	0.026	0.114
	Skewness volatility - MedRV	11.195	1.626	2.423	2.915
	Kurtosis of volatility - MedRV	330.545	8.080	14.124	29.043

Table 2: Descriptive statistics

	Ann. Average	Ann. Std. Dev.	Skewness	Kurtosis	% of jumpy days
S&P 500	-0.296	0.125	0.308	5.825	3.942
10-year Treasury Note	-0.036	0.057	1.310	13.433	5.43
USDCAD	0.040	0.044	0.263	4.131	7.57
WTI	-0.562	0.228	0.190	2.833	3.934

Table 3: Descriptive statistics of the jump dataset

	Single Gaussian	Gaussian	Scaled Student	Cauchy	Double exponential
S&P 500	0%	9%	41%	96%	4%
10-year Treasury Note	0%	2%	14%	82%	0%
USDCAD	0%	0%	31%	67%	0%
WTI	0%	16%	80%	84%	1%

Table 4: Test results for the Kolmogorov-Smirnov goodness-of-fit test.

Note: The table reports the p-values associated to the null hypothesis of an adequation of the proposed distribution to the time series.

					RV - Gaussian										
	ω	ϕ_1	ϕ_2	ϕ_3	γ_1	η									
Average value	-1.501	0.204	0.377	0.268	-0.11	0.598									
95% quantile	-0.446	0.344	0.551	0.479	-0.072	0.682									
5% quantile	-2.985	0.099	0.152	0.152	-0.145	0.513									
					RV - MN										
	ω	ϕ_1	ϕ_2	ϕ_3	γ_1	η	α	μ_1	σ_1	μ_2	σ_2				
Average value	-1.58	0.21	0.377	0.262	-0.103	0.600	0.325	-0.308	0.888	0.37	1.318				
95% quantile	-0.532	0.334	0.549	0.448	-0.069	0.685	0.50	0.277	1.19	0.677	1.636				
5% quantile	-2.985	0.113	0.188	0.163	-0.131	0.514	0.023	-1.058	0.472	0.07	1.125				
					MedRV - MN										
	ω	ϕ_1	ϕ_2	ϕ_3	γ_1	η	α	μ_1	σ_1	μ_2	σ_2				
Average value	-1.623	0.221	0.356	0.271	-0.096	0.631	0.3	0.147	1.105	0.276	1.399				
95% quantile	-0.59	0.323	0.495	0.431	-0.056	0.703	0.5	1.641	2.138	0.641	1.68				
5% quantile	-2.985	0.138	0.155	0.166	-0.134	0.559	0.026	-0.557	0.568	-0.051	1.123				
					MedRV - Jumps - No feedback										
	ω	ϕ_1	ϕ_2	ϕ_3	γ_1	η	a	b	x_1^0	x_2^0	a_1	a_2	λ	p	
Average value	-1.433	0.217	0.365	0.275	-0.1	0.631	-0.001	0.002	-0.005	0.005	0.002	0.003	0.575	0.039	
95% quantile	-0.469	0.371	0.504	0.436	-0.053	0.704	-0.001	0.003	-0.004	0.007	0.003	0.005	0.632	0.055	
5% quantile	-2.709	0.109	0.147	0.162	-0.136	0.559	-0.002	0.001	-0.007	0.003	0.001	0.001	0.519	0.024	
					MedRV - Jumps - Feedback										
	ω	ϕ_1	ϕ_2	ϕ_3	γ_1	γ_2	η	a	b	x_1^0	x_2^0	a_1	a_2	λ	p
Average value	-1.458	0.216	0.365	0.274	-0.099	16.898	0.632	-0.001	0.002	-0.005	0.005	0.002	0.003	0.575	0.039
95% quantile	-0.469	0.37	0.505	0.438	-0.05	58.218	0.704	-0.001	0.003	-0.004	0.007	0.003	0.005	0.632	0.055
5% quantile	-2.758	0.108	0.149	0.158	-0.136	-4.807	0.559	-0.002	0.001	-0.007	0.003	0.001	0.001	0.519	0.024

Table 5: S&P 500 parameters estimates for the five HAR-based models.

Note: The estimates presented in this table are obtained by maximizing the log-likelihood associated with the models presented in Section 4 and using the S&P 500 index futures data. The estimation has been performed over a 1260 days rolling period. "Average" presents the average value obtained across the estimation. The table also displays the associated 5 and 95% quantiles.

RV - Gaussian															
	ω	ϕ_1	ϕ_2	ϕ_3	γ_1	η									
Average value	-2.446	0.073	0.148	0.567	-0.008	0.607									
95% quantile	-0.827	0.229	0.308	0.8	0.009	0.642									
5% quantile	-4.878	0	0	0.328	-0.031	0.572									
RV - MN															
	ω	ϕ_1	ϕ_2	ϕ_3	γ_1	η	α	μ_1	σ_1	μ_2	σ_2				
Average value	-2.369	0.113	0.181	0.515	-0.002	0.623	0.028	-3.654	7.397	0.092	1.337				
95% quantile	-0.718	0.243	0.301	0.787	0.013	0.679	0.081	-0.148	31.008	0.186	1.457				
5% quantile	-4.528	0.048	0	0.327	-0.019	0.567	0	-15.703	0.99	0.048	1.24				
MedRV - MN															
	ω	ϕ_1	ϕ_2	ϕ_3	γ_1	η	α	μ_1	σ_1	μ_2	σ_2				
Average value	-2.326	0.123	0.23	0.463	-0.003	0.61	0.057	-3.61	8.578	0.088	1.403				
95% quantile	-0.635	0.202	0.316	0.728	0.011	0.663	0.125	0.321	26.833	0.184	1.566				
5% quantile	-4.309	0.072	0.141	0.306	-0.019	0.558	0.002	-14.217	0.788	0.025	1.267				
MedRV - Jumps - No feedback															
	ω	ϕ_1	ϕ_2	ϕ_3	γ_1	η	a	b	x_1^0	x_2^0	a_1	a_2	λ	p	
Average value	-2.162	0.146	0.131	0.534	-0.01	0.595	-0.001	0.001	-0.001	0.002	0.002	0.002	0.563	0.049	
95% quantile	-0.58	0.384	0.277	0.833	0.01	0.636	0	0.001	0.001	0.002	0.003	0.002	0.623	0.066	
5% quantile	-4.8	0.035	0	0.338	-0.03	0.555	-0.001	0.001	-0.002	0	0.001	0.001	0.448	0.04	
MedRV - Jumps - Feedback															
	ω	ϕ_1	ϕ_2	ϕ_3	γ_1	γ_2	η	a	b	x_1^0	x_2^0	a_1	a_2	λ	p
Average value	-2.312	0.142	0.131	0.526	-0.011	4.071	0.591	-0.001	0.001	-0.001	0.002	0.002	0.002	0.563	0.049
95% quantile	-0.639	0.377	0.277	0.834	0.009	37.485	0.623	0	0.001	0.001	0.002	0.003	0.002	0.623	0.066
5% quantile	-5.1	0.032	0	0.334	-0.031	-22.575	0.558	-0.001	0.001	-0.002	0	0.001	0.001	0.448	0.04

Table 6: 10-year Treasury Note parameters estimates for the five HAR-based models.

Note: The estimates presented in this table are obtained by maximizing the log-likelihood associated with the models presented in Section 4 and using the 10-year Treasury Note futures data. The estimation has been performed over a 1260 days rolling period. "Average" presents the average value obtained across the estimation. The table also displays the associated 5 and 95% quantiles.

RV - Gaussian															
	ω	ϕ_1	ϕ_2	ϕ_3	γ_1	η									
Average value	-2.765	0.118	0.251	0.399	-0.045	0.637									
95% quantile	-0.442	0.308	0.4	0.691	0	0.711									
5% quantile	-5.441	0	0.103	0.036	-0.075	0.564									
RV - MN															
	ω	ϕ_1	ϕ_2	ϕ_3	γ_1	η	α	μ_1	σ_1	μ_2	σ_2				
Average value	-2.328	0.137	0.259	0.422	-0.034	0.637	0.158	0.097	2.622	0.022	1.395				
95% quantile	-0.659	0.289	0.353	0.668	-0.003	0.705	0.456	0.384	3.647	0.168	1.691				
5% quantile	-4.107	0.001	0.164	0.115	-0.06	0.569	0.055	-0.29	1.137	-0.119	1.239				
MedRV - MN															
	ω	ϕ_1	ϕ_2	ϕ_3	γ_1	η	α	μ_1	σ_1	μ_2	σ_2				
Average value	-2.413	0.128	0.251	0.436	-0.033	0.644	0.213	0.09	2.742	0.016	1.436				
95% quantile	-0.766	0.255	0.356	0.724	-0.002	0.696	0.366	0.414	3.689	0.166	1.629				
5% quantile	-4.356	0.043	0.095	0.145	-0.058	0.592	0.063	-0.162	2.123	-0.118	1.267				
MedRV - Jumps - No feedback															
	ω	ϕ_1	ϕ_2	ϕ_3	γ_1	η	a	b	x_1^0	x_2^0	a_1	a_2	λ	p	
Average value	-2.858	0.101	0.234	0.427	-0.044	0.644	-0.001	0.001	-0.001	0.001	0.002	0.001	0.52	0.078	
95% quantile	-0.369	0.296	0.416	0.791	0.003	0.695	-0.001	0.001	0.002	0.003	0.003	0.003	0.575	0.112	
5% quantile	-5.933	0.001	0.048	0.043	-0.079	0.593	-0.001	0	-0.003	-0.004	0.001	0	0.411	0.059	
MedRV - Jumps - Feedback															
	ω	ϕ_1	ϕ_2	ϕ_3	γ_1	γ_2	η	a	b	x_1^0	x_2^0	a_1	a_2	λ	p
Average value	-2.892	0.103	0.237	0.418	-0.044	12.599	0.65	-0.001	0.001	-0.001	0.001	0.002	0.001	0.52	0.078
95% quantile	-0.618	0.298	0.429	0.791	0.002	58.71	0.7	-0.001	0.001	0.002	0.003	0.003	0.003	0.575	0.112
5% quantile	-5.925	0.001	0.057	0.049	-0.08	-31.095	0.599	-0.001	0	-0.003	-0.004	0.001	0	0.411	0.059

Table 7: USDCAD parameters estimates for the five HAR-based models.

Note: The estimates presented in this table are obtained by maximizing the log-likelihood associated with the models presented in Section 4 and using the USDCAD futures data. The estimation has been performed over a 1260 days rolling period. "Average" presents the average value obtained across the estimation. The table also displays the associated 5 and 95% quantiles.

RV - Gaussian

	ω	ϕ_1	ϕ_2	ϕ_3	γ_1	η
Average value	-1.212	0.098	0.305	0.451	-0.043	0.619
95% quantile	0.112	0.222	0.504	0.588	-0.013	0.677
5% quantile	-3.003	0.003	0.137	0.304	-0.074	0.619

RV - MN

	ω	ϕ_1	ϕ_2	ϕ_3	γ_1	η	α	μ_1	σ_1	μ_2	σ_2
Average value	-1.331	0.126	0.324	0.406	-0.035	0.613	0.07	-1.045	2.3	0.087	1.334
95% quantile	-0.515	0.308	0.489	0.546	-0.013	0.655	0.173	0.982	3.582	0.148	1.487
5% quantile	-2.478	0.017	0.178	0.284	-0.066	0.613	0.006	-4.463	0.535	0.024	1.221

MedRV - MN

	ω	ϕ_1	ϕ_2	ϕ_3	γ_1	η	α	μ_1	σ_1	μ_2	σ_2
Average value	-1.399	0.116	0.305	0.433	-0.031	0.65	0.094	-0.14	2.74	0.077	1.413
95% quantile	-0.548	0.289	0.445	0.572	-0.015	0.698	0.201	1.085	3.595	0.162	1.529
5% quantile	-2.587	0.007	0.16	0.311	-0.057	0.65	0.012	-3.508	2.16	0.016	1.277

MedRV - Jumps - No feedback

	ω	ϕ_1	ϕ_2	ϕ_3	γ_1	η	a	b	x_1^0	x_2^0	a_1	a_2	λ	p
Average value	-1.2	0.081	0.291	0.483	-0.041	0.652	-0.005	0.004	-0.009	0.008	0.005	0.005	0.595	0.038
95% quantile	0.403	0.19	0.468	0.661	-0.017	0.713	-0.001	0.008	0	0.012	0.011	0.011	0.692	0.047
5% quantile	-3.297	0.003	0.102	0.293	-0.065	0.652	-0.008	0.002	-0.014	-0.003	0.003	0.001	0.508	0.029

MedRV - Jumps - Feedback

	ω	ϕ_1	ϕ_2	ϕ_3	γ_1	γ_2	η	a	b	x_1^0	x_2^0	a_1	a_2	λ	p
Average value	-1.235	0.081	0.29	0.481	-0.04	6.904	0.653	-0.005	0.004	-0.009	0.008	0.005	0.005	0.595	0.038
95% quantile	0.346	0.184	0.464	0.66	-0.016	16.958	0.716	-0.001	0.008	0	0.012	0.011	0.011	0.692	0.047
5% quantile	-3.305	0.003	0.105	0.293	-0.065	0.345	0.653	-0.008	0.002	-0.014	-0.003	0.003	0.001	0.508	0.029

Table 8: WTI parameters estimates for the five HAR-based models.

Note: The estimates presented in this table are obtained by maximizing the log-likelihood associated with the models presented in Section 4 and using the WTI futures data. The estimation has been performed over a 1260 days rolling period. "Average" presents the average value obtained across the estimation. The table also displays the associated 5 and 95% quantiles.

Horizon = 5 days					
	RV-G	RV-MN	MedRV-MN	MedRV-Jump 1	MedRV-Jump 2
RV-G		-2.3	-2.37	46.14	49.29
RV-MN			-2.09	14.55	14.7
MedRV-MN				12.22	12.29
MedRV-Jump 1					-0.32
MedRV-Jump 2					
Horizon = 20 days					
	RV-G	RV-MN	MedRV-MN	MedRV-Jump 1	MedRV-Jump 2
RV-G		0.22	-1.4	25.56	24.64
RV-MN			-1.85	23.17	25.66
MedRV-MN				17.21	18.06
MedRV-Jump 1					0.98
MedRV-Jump 2					
Horizon = 60 days					
	RV-G	RV-MN	MedRV-MN	MedRV-Jump 1	MedRV-Jump 2
RV-G		13.74	8.38	4.43	3.31
RV-MN			-6.12	-4.65	-4.16
MedRV-MN				-1.18	-1.06
MedRV-Jump 1					-0.25
MedRV-Jump 2					

Table 9: S&P 500 forecasting test results

Note: This table presents the Amisano and Giacomini (2007) forecasting density test results for various forecasting horizons, ranging from 5 to 60 days. The table reads as follow: for a forecasting horizon of 5 days and when comparing the density forecasts obtained with a MedRV-Jump 2 model vs. a MedRV-MN model, the statistics is equal to 12.29. The MedRV-Jump 2 model is therefore found to dominate the MedRV-MN model given the sample used here.

Horizon = 5 days					
	RV-G	RV-MN	MedRV-MN	MedRV-Jump 1	MedRV-Jump 2
RV-G		-5.55	-4.1	25.28	24
RV-MN			6.58	15.29	13.68
MedRV-MN				14.84	13.86
MedRV-Jump 1					-1.4
MedRV-Jump 2					
Horizon = 20 days					
	RV-G	RV-MN	MedRV-MN	MedRV-Jump 1	MedRV-Jump 2
RV-G		-9.35	-0.12	35.14	33.01
RV-MN			11.21	31.95	27.12
MedRV-MN				35.02	28.4
MedRV-Jump 1					0.66
MedRV-Jump 2					
Horizon = 60 days					
	RV-G	RV-MN	MedRV-MN	MedRV-Jump 1	MedRV-Jump 2
RV-G		5.77	12.95	23.57	27.46
RV-MN			8.33	17.03	20.98
MedRV-MN				16.34	16.28
MedRV-Jump 1					1.05
MedRV-Jump 2					

Table 10: 10-year Treasury Note forecasting test results

Note: This table presents the Amisano and Giacomini (2007) forecasting density test results for various forecasting horizons, ranging from 5 to 60 days. The table reads as follow: for a forecasting horizon of 5 days and when comparing the density forecasts obtained with a MedRV-Jump 2 model vs. a MedRV-MN model, the statistics is equal to 13.86. The MedRV-Jump 2 model is therefore found to dominate the MedRV-MN model given the sample used here.

Horizon = 5 days					
	RV-G	RV-MN	MedRV-MN	MedRV-Jump 1	MedRV-Jump 2
RV-G		-5.99	-5.68	46.52	48.72
RV-MN			-1.14	28.03	27.96
MedRV-MN				26.3	26.88
MedRV-Jump 1					-1.68
MedRV-Jump 2					
Horizon = 20 days					
	RV-G	RV-MN	MedRV-MN	MedRV-Jump 1	MedRV-Jump 2
RV-G		-7.52	-9.15	31.08	34.15
RV-MN			-4.35	32.22	44.97
MedRV-MN				33.57	48.53
MedRV-Jump 1					-1.76
MedRV-Jump 2					
Horizon = 60 days					
	RV-G	RV-MN	MedRV-MN	MedRV-Jump 1	MedRV-Jump 2
RV-G		-3.57	-5.5	13.18	9.8
RV-MN			-2.65	15.34	13.92
MedRV-MN				16.88	14.98
MedRV-Jump 1					-2.61
MedRV-Jump 2					

Table 11: USDCAD forecasting test results

Note: This table presents the Amisano and Giacomini (2007) forecasting density test results for various forecasting horizons, ranging from 5 to 60 days. The table reads as follow: for a forecasting horizon of 5 days and when comparing the density forecasts obtained with a MedRV-Jump 2 model vs. a MedRV-MN model, the statistics is equal to 26.88. The MedRV-Jump 2 model is therefore found to dominate the MedRV-MN model given the sample used here.

Horizon = 5 days					
	RV-G	RV-MN	MedRV-MN	MedRV-Jump 1	MedRV-Jump 2
RV-G		-4.8	-5.13	33.29	29.32
RV-MN			-7.7	11.54	9.92
MedRV-MN				10.39	9.11
MedRV-Jump 1					-1.04
MedRV-Jump 2					
Horizon = 20 days					
	RV-G	RV-MN	MedRV-MN	MedRV-Jump 1	MedRV-Jump 2
RV-G		-1.64	-5.25	10.49	7.9
RV-MN			-17.31	7	5.34
MedRV-MN				9.46	8.73
MedRV-Jump 1					-2.87
MedRV-Jump 2					
Horizon = 60 days					
	RV-G	RV-MN	MedRV-MN	MedRV-Jump 1	MedRV-Jump 2
RV-G		15.11	12.6	-7.33	-9.14
RV-MN			-21.32	-12.17	-12.57
MedRV-MN				-11.04	-11.53
MedRV-Jump 1					-2.91
MedRV-Jump 2					

Table 12: WTI forecasting test results

Note: This table presents the Amisano and Giacomini (2007) forecasting density test results for various forecasting horizons, ranging from 5 to 60 days. The table reads as follow: for a forecasting horizon of 5 days and when comparing the density forecasts obtained with a MedRV-Jump 2 model vs. a MedRV-MN model, the statistics is equal to 9.11. The MedRV-Jump 2 model is therefore found to dominate the MedRV-MN model given the sample used here.

Horizon = 5 days					
	RV-G	RV-MN	MedRV-MN	MedRV-Jump 1	MedRV-Jump 2
RV-G		-2.51	-6.84	19.71	19.42
RV-MN			-5.11	42.28	42.79
MedRV-MN				12.68	12.42
MedRV-Jump 1					-0.18
MedRV-Jump 2					
Horizon = 20 days					
	RV-G	RV-MN	MedRV-MN	MedRV-Jump 1	MedRV-Jump 2
RV-G		-5.19	-4.88	12.03	14.01
RV-MN			-1.78	41.47	44.13
MedRV-MN				13.91	14.89
MedRV-Jump 1					3.87
MedRV-Jump 2					
Horizon = 60 days					
	RV-G	RV-MN	MedRV-MN	MedRV-Jump 1	MedRV-Jump 2
RV-G		-24.08	8.16	-2.29	0.58
RV-MN			19.03	45.23	44.38
MedRV-MN				-5.82	-4.45
MedRV-Jump 1					7.43
MedRV-Jump 2					

Table 13: S&P 500 forecasting test results for the period January 2, 1996 to August 29, 2007 considered in Maheu and McCurdy (2011)

Note: This table presents the Amisano and Giacomini (2007) forecasting density test results for various forecasting horizons, ranging from 5 to 60 days. The table reads as follow: for a forecasting horizon of 5 days and when comparing the density forecasts obtained with a MedRV-Jump 2 model vs. a MedRV-MN model, the statistics is equal to 12.42. The MedRV-Jump 2 model is therefore found to dominate the MedRV-MN model given the sample used here.

Horizon = 5 days					
	RV-G	RV-MN	MedRV-MN	MedRV-Jump 1	MedRV-Jump 2
RV-G		-25.09	-35.67	125.92	143.32
RV-MN			4.34	114.42	112.63
MedRV-MN				158.15	140.08
MedRV-Jump 1					-2.14
MedRV-Jump 2					
Horizon = 20 days					
	RV-G	RV-MN	MedRV-MN	MedRV-Jump 1	MedRV-Jump 2
RV-G		-10.90	-12.88	132.36	126.03
RV-MN			4.09	77.56	101.78
MedRV-MN				99.44	138.66
MedRV-Jump 1					-1.99
MedRV-Jump 2					
Horizon = 60 days					
	RV-G	RV-MN	MedRV-MN	MedRV-Jump 1	MedRV-Jump 2
RV-G		-17.74	-20.69	125.74	140,40
RV-MN			5.42	127.78	103.32
MedRV-MN				160.44	127.45
MedRV-Jump 1					-2.02
MedRV-Jump 2					

Table 14: S&P 500 forecasting test results for crisis periods only corresponding to realized volatility greater than the 80th percentile.

Note: This table presents the Amisano and Giacomini (2007) forecasting density test results for various forecasting horizons, ranging from 5 to 60 days. The table reads as follow: for a forecasting horizon of 5 days and when comparing the density forecasts obtained with a MedRV-Jump 2 model vs. a MedRV-MN model, the statistics is equal to 140.08. The MedRV-Jump 2 model is therefore found to dominate the MedRV-MN model given the sample used here.

	RV-G	RV-MN	MedRV-MN	MedRV-Jump 1	MedRV-Jump 2
S&P 500					
RV-G		-1.71	-1.70	3.74	3.74
RV-MN			-2.33	2.44	2.44
MedRV-MN				2.45	2.45
MedRV-Jump 1					1.97
MedRV-Jump 2					
TY					
RV-G		5.16	5.09	5.19	5.19
RV-MN			-7.24	6.62	6.57
MedRV-MN				8.84	8.78
MedRV-Jump 1					3.04
MedRV-Jump 2					
USDCAD					
RV-G		5.77	5.76	5.81	5.81
RV-MN			1.37	7.14	7.12
MedRV-MN				6.71	6.70
MedRV-Jump 1					-1.78
MedRV-Jump 2					
WTI					
RV-G		6.51	6.50	6.57	6.57
RV-MN			-1.17	4.45	4.46
MedRV-MN				4.22	4.20
MedRV-Jump 1					-1.17
MedRV-Jump 2					

Table 15: Testing the VaR forecasting performances

Note: This table presents the values of the Angelidis et al. (2004) test statistics. The table reads as follow: for the S&P 500 and when comparing the VaR forecasts obtained with a MedRV-Jump 2 model vs. a MedRV-MN model, the statistics is equal to 2.45. The MedRV-Jump 2 model is therefore found to dominate the MedRV-MN model given the sample used here.

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