Takeuti's proof theory in the context of the Kyoto School

Ryota Akiyoshi, Andrew Arana

Takeuti's proof theory in the context of the Kyoto School

Ryota Akiyoshi and Andrew Arana*

L'infini et l'éternité n'existent que dans
le cœur, dans la pensée.

Shuzo Kuki, *Propos sur le temps*

Abstract

Gaisi Takeuti (1926–2017) is one of the most distinguished logicians in proof-theory after Hilbert and Gentzen. He extensively extended Hilbert's program in the sense that he formulated Gentzen's sequent calculus, conjectured that cut-elimination holds for it (Takeuti's conjecture), and obtained several stunning results in the 1950–60s towards the solution of his conjecture. Though he has been known chiefly as a great mathematician, he wrote many papers in English and Japanese where he expressed his philosophical thoughts. In particular, he used several keywords such as "active intuition" and "self-reflection" from Nishida's philosophy.

In this paper, we aim to describe a general outline of our project to investigate Takeuti's philosophy of mathematics. In particular, after reviewing Takeuti's proof-theoretic results briefly, we describe some key elements in Takeuti's texts. By explaining these texts, we point out the connection between Takeuti's proof theory and Nishida's philosophy and explain the future goals of our project.

1. Introduction

In this article we will introduce our project of showing that Takeuti's finitist program had its own character and goals, related to but distinct from the classical finitist program of Hilbert and Bernays. Takeuti's program, rooted in his readings of Hilbert and Bernays but also in his background in the philosophy of the Kyoto school of Japanese philosophy, envisions finitism as not only realizing our human limitations as provers but also our relation to what Takeuti called "infinite mind", modern mathematics with its full embrace of infinitary set theory. Finitism realizes the connection between our finite minds and this infinite mind. In embracing this connection between the finite and the infinite, Takeuti's programme also permits the clarification of the mathematical means available to the finitist, going beyond primitive recursive arithmetic. Thus Takeuti's finitism permits his classical Gentzenian proof of the consistency of arithmetic, and indeed points forward toward finitist proofs of consistency of higher-order theories as well.

2. Proof-theoretic background

Proof theory is the study of mathematical proofs as mathematical objects themselves, by means of ordinary mathematical methods. This branch of mathematical logic was created by David Hilbert with the goal of proving metamathematical results, like the independence of certain axioms from other axioms, and the consistency of axiom systems. Hilbert set his sights on proving the consistency of elementary arithmetic, because of its foundational importance for other parts of mathematics, especially analysis. For such a proof to have foundational force, the resources used by the proof should be more secure than those whose consistency is under investigation. Hilbert identified a part of mathematical reasoning that he judged to be ideally secure that he called *finitary* mathematics. He would describe his finitism as follows:

> Our treatment of the basics of number theory and algebra was meant to demonstrate how to apply and implement direct contentual inference that takes place in thought experiments on intuitively conceived objects and is free of axiomatic assumptions. Let us call this kind of inference "*finitist*" inference for short, and likewise the methodological attitude underlying this kind of inference as the "finitist" attitude or the "finitist" standpoint. [...] With each use of the word "finitist", we convey the idea that the relevant consideration, assertion, or definition is confined to objects that are conceivable in principle, and processes that can be effectively executed in principle, and thus it remains within the scope of a concrete treatment. (Hilbert & Bernays, 1968, p. 32; translated in Hilbert & Bernays, 2011, p. 32)

Thus finitist reasoning employs only concrete objects and concrete operations on those objects, as opposed to abstract objects, complete infinite totalities, and abstract operations on these. These concrete objects and operations are given directly in intuition. A typical example of this kind would be the syntactic object "||", a stroke designating the natural number

2. A concrete operation on such an object is the concatenation operation. Elementary operations on natural numbers such as addition, subtraction, and multiplication are finitistically acceptable.

Hilbert emphasized that the concreteness of finitary reasoning included the surveyability of its objects in intuition, as a condition of its epistemic security (Hilbert, 1926, p. 171). A universal quantification over an infinite domain, say the natural numbers, is restricted to $\forall x A(x)$ where $A(x)$ is decidable. Moreover, the meaning of an existential statement $\exists x A(x)$ with a decidable $A(x)$ is that there exists a witness (a concrete natural number) satisfying $A(x)$.

Hilbert also stressed that finitary reasoning is "the fundamental mode of thought that I hold to be necessary for mathematics and for all scientific thought, understanding, and communication, and without which mental activity is not possible at all" (Hilbert, 1931, p. 486; translated in Mancosu, 1998, p. 267). Thus a finitary proof of the consistency of PA should be accepted by all thinking agents.

Hilbert thus sought a finitistically acceptable proof of the consistency of arithmetic. Gödel's incompleteness theorems posed a problem for this goal, because they suggested that proving the consistency of arithmetic would require resources stronger, provability-wise, than arithmetic. Such resources would seem not to be finitary, since finitism seems to be only a fragment of arithmetic. Gerhard Gentzen, a member of Hilbert's school at Göttingen, responded to this difficulty by giving a proof of the consistency of arithmetic that pinpointed precisely what stronger resources were sufficient while being plausibly finitary (Gentzen, 1936, 1938). This proof can be sketched as follows, letting "PA" designate the usual collection of first-order axioms of elementary arithmetic. Gentzen first defined a system of ordinals and an ordering of these ordinals of type ϵ_0. He then showed how to assign these ordinals to proofs in PA, according to the rules of inference they use, including mathematical induction. He then gave a procedure for reducing these proofs so that each proof of inconsistency is reduced to another proof of inconsistency with a smaller ordinal. If there is a proof of inconsistency, this procedure generates an infinitely decreasing sequence of such ordinals. By the well-ordering of the ordering of type ϵ_0, such a sequence is impossible. Thus there is no proof of inconsistency in PA.

In order to meet the challenge to Hilbert's consistency program posed by Gödel's theorems, Gentzen tried to formulate his proof so that it would be finitistically acceptable. He maintained that his formulations of infinite ordinals and transfinite induction on these ordinals were finitistically acceptable, on account of their concreteness. (We shall return to this matter shortly.) Thus while his proof was not purely arithmetic, it was still, he argued, finitistically acceptable, and hence provided a justification of the consistency of arithmetic, even if not quite in the way that Hilbert had originally envisioned.

On the face of it, it seems implausible that transfinite induction to an infinite ordinal should be finitistically acceptable. Gentzen maintained, however, that the particular transfinite induction that he had used was indeed finitist. His argument was that we can visualize the steps of this transfinite induction, even if only "remotely" on account of its complexity, and this visualizability licences its finitariness (Gentzen, 1936; Szabo, 1969, p. 196). Here Gentzen adopted the surveyability condition of Hilbert's finitist standpoint.

It is here that Takeuti's unique contributions begin, so we shall turn next to those.

3. Takeuti's program

In this section, we briefly explain Takeuti's proof-theoretic works extending Gentzen's consistency result for PA into higher-order case in 1950's. For this purpose, he formulated a higher-order sequent calculus called *GLC* and conjectured that the cut-elimination theorem holds for it in 1953 (Takeuti, 1953). Also, it is proved that the consistency of analysis (the theory of real numbers) follows from his conjecture.

To explain Takeuti's partial solutions to the conjecture, we focus on the second-order subsystem of it called GLC^1. The language of GLC^1 is obtained by adding a predicate variable X and second-order quantifiers $\forall X, \exists X$ to first-order ones. Hence, it has additional rules for second-order quantifiers $\forall X, \exists X$. For example, the rule of $\exists X$ says that $\exists X A(X)$ can be deduced from $A(T)$ where T is a set term. So, this rule says that if the property A holds for the particular set T, then there is a set X satisfying A as well. Here is the main difference from the first-order case deducing $\exists x A(x)$ from $A(t)$; T might contain arbitrary number of logical symbols, hence $A(T)$ could be much more complicated than $A(X)$ while $A(t)$ is simpler than $\exists x A(x)$ since $\exists x$ does not occur in the former. Also, the right rule of $\forall X$

says that $\forall X A(X)$ is deducible from $A(X)$. We remark that mathematical induction is not included in GLC^1 since it is a "logical" calculus (1) .

Using an analogy with the key case of Gentzen's cut-elimination for the first-order logic (Gentzen, 1935), the crucial case is the following:

$$\dfrac{\dfrac{\Gamma, A(Y)}{\Gamma, \forall X A} \bigwedge_{\forall X A} \quad \dfrac{\Gamma, \neg A(T)}{\Gamma, \exists X \neg A} \bigvee_{\exists X A}}{\Gamma}$$

If we follow Gentzen's method of cut-elimination method for the first-order case, then we will insert a new cut over $A(T)$. Since T might be very complicated, we cannot say the result obtained by this method is "simpler" than before.

Takeuti's crucial idea to resolve this difficulty is to analyze infinitary structures (which are expressible in ordinals) behind this finitary proof. Hence, his idea is nothing else but a considerable extension of Gentzen's consistency proof for PA. This new system of ordinals discovered by Takeuti is called "ordinal diagrams" (expressing his reduction steps for a subsystem of analysis) (Takeuti, 1957). For the termination proof of his reduction steps, he assigned ordinal diagrams to proof-figures and showed in 1958 (Takeuti, 1958b) that if a proof-figure d is reduced into another one $r(d)$, then the corresponding ordinals are decreasing. The point here is that the definition of his reduction steps and the assignment of ordinals must be done simultaneously in order to complete the termination proof. He successively published partial solutions to this conjecture (Takeuti, 1955a,b, 1956c,d, 1958a), and in particular he proved the consistency of an impredicative subsystem of analysis corresponding to $(\Pi^1_1 - CA)_0$ in a modern terminology. To conclude his consistency proof, it is necessary to prove the key lemma saying that his ordinal diagrams are well-founded. Indeed, Takeuti had tried to prove the well-foundedness of ordinal diagrams several times. His first argument is given in his seminal paper in 1957 (Takeuti, 1957) where ordinal diagrams were introduced. However, since this proof is not very satisfying according to his philosophical standpoint, other proofs are presented several times (2) .

4. Takeuti's philosophical perspective

Reflecting on Takeuti's foundational projects as explained in the previous section, the following two thematic aims emerge: 1) to extend his finitist standpoint and to prove the consistency of a strong subsystem of his second-order or higher-order logic; and 2) to reflect on what is achieved by such results.

To investigate this, we focus on the paper「数学について」(About mathematics) published in 1972 (Takeuti, 1972). A key term in this paper is "infinite mind", which Takeuti explains was suggested by Gödel in Takeuti (1985). First, let us quote the following passage.

> Usually, it is essential for modern mathematics that it supposes an infinite mind and conjectures what it does. By infinite mind, I mean a mind which can investigate infinitely many things by checking one by one. For example, the law of excluded middle holds for an infinite mind because it can check whether $A(x)$ or $\neg A(x)$ one by one. Similarly, it can see $\{x|A(x)\}$ since it can check whether A(x) or not one by one. By contrast, a human mind is clearly a finite mind. (Takeuti, 1972, p. 172; translated by the authors)

An infinite mind, then, can solve the foundational problems of the formalist project: among other capacities, it can confirm the consistency of formal theories. Takeuti writes a few years later that it "must be able to operate on infinitely many objects as freely as we operate on finitely many objects" using concrete methods such as mathematical induction (Takeuti, 1985, p. 255). He views modern mathematics, encompassing set theory, as an investigation by finite human minds of the capacities of an infinite mind.

One might think that an infinite mind is an analogy to God as conceived in the Western world, but this is not quite right since Takeuti thinks that an infinite mind is essentially connected with finite minds, as we will discuss in the next section.

Takeuti explains that testing this "conjecture" about an infinite mind leads to three problems for the foundations of mathematics.

1. To formulate the function of an infinite mind.
2. To justify our intuition of the world of an infinite mind using only our finite mind.
3. To formulate the function of a finite mind.

Concerning (1), Takeuti means finding new axioms of set theory, because set theory expresses the functions of an infinite mind. Concerning (2), Takeuti means his proof-theoretic projects, including consistency proofs. Let us quote his passage:

> 2) to justify the world of our infinite mind in the world of a finite mind. We want to confirm the rationality of modern mathematics since the world of our mind is finite and the world of modern mathematics is just conjecture about our infinite mind. For this, we need to develop mathematics of the world of a finite mind sufficiently. (Takeuti, 1972, p. 173; translated by the authors)

He writes that we need to confirm the "rationality" of modern mathematics (such as set theory or analysis) from the perspective of our finite minds, and this requires a finitist treatment. The word "rationality" is translated from 「合理性」, which means that the world of an infinite mind comports with the world of a finite mind. This can be investigated, among other ways, by checking that the former is not contradictory with the latter. The expressions "world of our infinite mind" and "world of a finite mind" also call for further discussion, which we postpone to the next section.

In noting that this inquiry concerns the rationality of modern infinitary mathematics from a finite perspective, we note that Takeuti does not require a "justification" of the former, which would require investigations from the most reliable or "absolute" position, as the finitist consistency program was understood by Hilbert. Rather, Takeuti sought to understand the relative relationship between an infinite mind and a finite mind. Hence Takeuti adopted a broader evaluative position of infinitary mathematics than Hilbert did, and accordingly used the broader term "rationality" 「合理性」. As part of this broader evaluative position, Takeuti considered several other foundational standpoints in the paper, such as an ω-mind and a constructible mind. This point is related with (3). Takeuti states that we need to develop his finitistic standpoint by formulating the function of a finite mind for carrying out the program of the consistency proofs. Therefore, for Takeuti, the investigations of set theory and proof theory are related to one another, and both developments should be interconnected in an essential way.

The significance of the following passage is now clear:

> From these considerations, it turns out that a common problem, to formulate the

> function of a mind, whether finite or infinite, is an important task in foundations of mathematics. [...] (Takeuti, 1972, p. 173; translated by the authors)

We turn next to Takeuti's understanding of formalism. Even before obtaining partial but great achievements in the 1950s for his fundamental conjecture, he discussed the power of using symbols in his paper 「形式主義の立場から II」 (From the viewpoint of formalism II) (Takeuti, 1956b). Near the end, Takeuti discusses a power of the use of symbols in mathematics.

> Just by writing any real number a, though we do not know or recognize very much about each real number, this notation has the ability to make us have an illusion as if we know about them very well. (Takeuti, 1956b, p. 298; translated by the authors)

He then applies this idea to the notation $\{x|A(x)\}$ in set theory as well.

> By writing $\{x|A(x)\}$, this notation has the ability [for the mathematician, RA and AA] to make us have an illusion as if we can look at the totality of something which we cannot see and grasp. (Takeuti, 1956b, p. 299; translated by the authors)

The point here is that we treat a set as if we can grasp it by writing the notation $\{x|A(x)\}$, although we cannot check it one by one. Here, we remark that his passage reminds us of the ability to grasp mathematical entities. As we will see below, this is not just an analogy, but expresses his philosophical thinking about mathematics.

After these, Takeuti writes that he has explained several things which were left in his earlier paper 「形式主義の立場から I」 (From the viewpoint of formalism I) (Takeuti, 1956a). He then asks some further questions in the following passage. With Hilbert's conception of formalism in our mind, this passage is somewhat surprising.

> Thought-objectification in mathematics via symbol has arrived at proofs in mathematics in a formalism via propositions (sets) and logical concepts. What is the significance of it? Or, will it have a great influence on mathematics similar to the case of set? For answering these questions, it seems that formalism is too immature. This new gateway of the new virgin field is too difficult to open. Is formalism too much a premature baby? For me, the consistency problems seem a touchstone for formalism and the foundation of formalism rather than the foundation of mathemat-

> ics. Still I believe in a bright future for formalism, though one laden with difficulties. (Takeuti, 1956b, p. 299; translated by the authors)

This passage shows that Takeuti's conception of formalism is quite different from Hilbert's, for the following reason. Hilbert saw his finite standpoint as delineating a mode of thought common to all thinking agents, through the intuition of signs. He seems to have taken intuition to be sufficiently clearly understood to serve as a foundational notion of his finitism. This is a contestable position. For instance, Gödel noted in a letter to Bernays that "Hilbert's finitism (through the requirement of being "intuitive" [Anschaulichkeit]) has a quite unnatural boundary" (Gödel, 2003, p. 273). Takeuti seems to have stood with Gödel here, taking his general program of searching for consistency proofs to be, in part, an inquiry into the nature of formalism itself.

5. A surprising connection: Takeuti's philosophy and Nishida's philosophy in Kyoto school

In this section, we introduce the surprising connection of Takeuti's thinking with the Japanese philosophical movement centered at Kyoto University called the "Kyoto School" 「京都学派」. As we explain below, he uses some keywords due to Kitaro Nishida （西田幾多郎） for explaining his philosophical ideas about the foundations of mathematics.

To understand this point, we focus on another passage in the paper in 1956.

> In the above example, I have frequently used the word "illusion", but indeed I hope the readers to replace it by "intuition". In this sense, I redcall the intuition for grasping a function or a set in general, the active volitional intuition.
>
> Indeed, when we try to grasp a function or a set, this active volition works. And, what would be thought by this active volition, the object of this active volition, or the volition itself would be called a function or a set. (Takeuti, 1956b, p. 299)

This passage occurs just after the previous one and it looks very abrupt. Indeed, this kind of abruptness is a feature of Takeuti's writings about philosophical ideas. Here, we would like to understand his philosophical background in the context of Japanese philosophy. We can then better understand his peculiar conception of formalism.

One point to be noted here is that he uses the word "intuition"「直観」 or "volition" 「意志」for explaining the power of symbols in mathematics. At first sight, this word suggests Hilbert's use of the word "intuition" in characterizing his finitist standpoint. However, if we read the last sentence carefully, it turns out that the intuition or volition itself is a mathematical object, e.g., a function. This is one feature of Japanese philosophy which is articulated by Nishida. In particular, the word "active intuition"「行為的直観」is a keyword in Nishida's philosophy [(3)] . In Nishida's view, active intuition comprises the fundamental mode of experience of the world that does not suppose a separation between object and subject [(4)] . In the context of mathematics, this can be understood as making no separation between the act by a subject of constructing an object such as a set, and the object so constructed itself. Thus both this act and the object of this act can be identified as an active intuition, as Takeuti says in the passage in question. Thus Takeuti's remark makes more sense in the context of Nishida's thought.

Furthermore, we can make further sense of Takeuti's broad evaluative position of the relationship between infinitary mathematics and finitary mathematics, discussed in the previous section as "rationality" 「合理性」, in which set theory and proof theory are interconnected to one another in an essential way. While set theory is often thought of as an investigation of the pure *object* that is a set, and proof theory as an investigation of the demonstrative powers of a *subject*, in neglecting the separation between object and subject Nishida's standpoint provides a way to interconnect these two investigations. Thus we can make better sense of Takeuti's understanding of formalism when seeing it within the framework of Nishida's work.

We next make a historical remark about Takeuti's use of "active volitional intuition" 「行為的意志的直観」[(5)] . As we have seen, this goes back to the quotation from "From the viewpoint of formalism II" in 1956 (Takeuti, 1956b). Takeuti attributed his use of the word "active volitional intuition" to Joichi Suetsuna（末綱恕一）(1898–1970), who was a mathematician with a strong interest in the foundations of mathematics and Buddhist philosophy (Kegon「華厳」) [(6)] . Suetsuna had close contacts with Nishida and with the writer Daisetsu T. Suzuki（鈴木大拙）(1870–1966) about Zen in Japan [(7)] . He already mentions Nishida's keywords like "active intuition" in the book 『論理と数理』 (*Logic and Mathematics*) published in 1947 (Suetsuna, 1947). It is much beyond the scope of this paper to describe Nishida's philosophy or the tradition of Japanese philosophy based on Zen, but we indicate the connection of Takeuti's thinking with Nishida's philosophy below. In particular, we will discuss two notions that we saw in Takeuti's writings in the last section, that of the "world" of finite and infinite minds, and that of "self-reflection", both of which play critical roles in Takeuti's views of the foundations of mathematics.

Firstly, we discuss the notion of "world" 「世界」. Takeuti's word "the world of a finite mind" or "the world of an infinite mind" might be a bit strange if "world" means a universe and there is a strict separation between the world and a (finite or infinite) mind. In particular, this contrast between "world" and "mind" would be drastic if we consider that the world as the universe might contain infinitary many things including all mathematical objects while our minds are finite in the sense that our ability to remember things is limited.

However, we may resolve this difficulty by reading Takeuti's passage in the context of Nishida's philosophy, though we only sketch our resolution for reasons of space. We rely on a recent analysis of Nishida's theory of "self" 「自己」 due to Deguchi (2017). He introduces an important distinction between a "narrow" self and a "broad" self in Nishida's philosophy.

To explain this distinction briefly, let us note that according to Nishida, the world is the totality of all the entities, as well as the creator of those entities (including itself). According to Deguchi, a narrow self is created by the world and is "expressed" (「表現される」) or "reflected" (「映される」) by the world. On the other hand, a broad self is a narrow self as a producer in the world furnished with its materials, and thus a narrow self having some interaction with other entities in the world.

To make sense of Takeuti's talk of worlds of minds, we interpret Takeuti's notion of "mind" as a narrow self in Nishida's philosophy. In this reading, narrow selves and the world are related by means of reflection. Moreover, Takeuti's reference to several minds indicates that there might be several narrow selves in the world. On this reading, Takeuti's investigations in the foundations of mathematics are regarded as attempts to investigate the relationship between narrow selves. For example, we interpret Gentzen's consistency proof of *PA* as a work establishing a relation between the narrow self having exactly the (infinitary) mathematical power of *PA* and another narrow mind exactly having the power of a Hilbertian finite mind augmented with the capacity to carry out transfinite induction up to ϵ_0. In future work we will interpret Takeuti's own partial but remarkable solutions to his fundamental conjecture by means of this reading. It would also be interesting to interpret Takeuti's results in set theory, and to make sense of Takeuti's view that proof theory and set theory are internally connected, by means of this Nishidean interpretation.

One might wonder whether the notion of broad self is necessary in order to interpret Takeuti's thoughts in the context of Nishida's philosophy, since the notion of narrow self is seemingly sufficient for interpreting Takeuti's notion of mind. We reply as follows. Firstly, Takeuti says explicitly that minds can operate or create mathematical objects such as numbers, functions, or sets. In the context of Nishida's philosophy, such mathematical objects would not belong to a narrow self, but to the world. Hence, the notion of broad self is also necessary, or at least natural, for interpreting Takeuti's writings.

Secondly, we discuss the notion of "self-reflection" 「自己反省」 in Takeuti's writings. We begin by noting its usage in the following passage:

> Now, for example, consider a finite mind. If the function of a finite mind is completely finite, then our mathematics remains in finite. However, our finite mind is the so-called potential infinity. In other words, we can indicate infinitary objects like $0, 1, 2, 3, \ldots$ [...] Why? This is because our mind has the ability of self-reflection, that is, the ability to observe what we are doing and to know what we are doing. (Takeuti, 1972, p. 173; translated by the authors)

To understand this passage, let us consider the number 1 as a stroke |. Since adding one stroke is clearly possible, we can consider 2 as ||. Iterating this process, we consider any finite number occurring in the sequence $0, 1, 2, \ldots$. Now, following Takeuti, we may reflect on what we are doing. It is not so easy to characterize the limit of the ability of self-reflection, but Takeuti clearly considers the possibility that we can observe the process of "adding a stroke beginning with the empty sequence". Therefore, we can consider the set of natural numbers as potentially infinite.

Interestingly, Takeuti claims the same thing for an infinite mind as well.

> This holds true for an infinite mind as well. When an infinite mind creates sets

> from the empty set, though I explain the construction of ordinals very briefly in my lecture or explanation, if we think this over, it is clear that the self-reflection of an infinite mind plays the role here. In particular, the function of self-reflection is most important for a mind which can create something internally when a set is not confined to a closed set and extends further. How to formulate this function of the self-reflection of a mind? (Takeuti, 1972, p. 173; translated by the authors)

It is not very clear what kind of picture for an infinite mind is considered by Takeuti in this passage, but as another quotation below shows, Takeuti thought this kind of approach would be useful for "a fundamental problem of set theory", that is for carrying out a program to obtain a new axiom of set theory.

> In the case of an infinite mind, this problem appears implicitly and explicitly when I consider the paper about formalization principle or a fundamental problem of set-theory. (Takeuti, 1972, p. 173; translated by the authors)

Hence, for Takeuti, it is crucially important to investigate the function of "self-reflection" in relation to active volitional intuition. If we recall the title of Nishida's important book 『自覚における直観と反省』(*Intuition and Reflection in Self-Consciousness*) (Nishida, 1917), then it would be promising to consider the close similarity between Takeuti's mathematical philosophy and Nishida's one. Hence, we claim that Takeuti's mathematical results should be understood in the context of Japanese philosophy, and we will investigate this point in future works. In particular, his detailed argument concerning the well-ordering up to ϵ_0 would be the first clue for us.

6. Conclusion

We have seen that Takeuti's proof-theoretic works form a program in the foundations of mathematics that is related to but distinct from Hilbert's finitist, formalist program. By contrast with Hilbert, Takeuti saw formalism as a view still being developed, and what precisely distinguishes finitary mathematics from infinitary mathematics as an open problem. We have indicated how Takeuti's views may be illuminated by understanding his intellectual context within the Kyoto School of Japanese thought, in particular with the thought of

Nishida. Much remains to be done in this project, but we hope to have indicated how philosophically fruitful a reconciliation of Takeuti's formal works with his Japanese intellectual context can be, not only for our understanding of Hilbert's thought but more generally for the foundations of mathematics.

Notes

∗ We would like thank Mark van Atten for helpful discussions on these subjects. Special thanks to Hidenori Kurokawa for his very valuable comments on an early version of this paper. The first author is partially supported by Core-to-Core Program (A. Advanced Research Networks) and KAKENHI 19K00022.

(1) We note that Frege-Dedekind's definition of the set of natural numbers is representable in this language: $\mathbb{N}(t) :\equiv \forall X(X(0) \wedge \forall x(X(x) \to X(x+1)) \to X(t))$. Moreover, higher-order constructive ordinals are also representable in it.

(2) For example, a detailed explanation of his standpoint and argument to justify ordinals up to ϵ_0 are presented in Takeuti (1975, 1987). For Takeuti, ϵ_0 should be a simple and good example or starting point to explain his philosophical standpoint. It is very clear from the texts that he intends to extend this kind of approach to bigger ordinals, in particular to his ordinal diagrams.

(3) Takeuti already uses this word in a chapter of a book 『自然数論』(*The Theory of Natural Numbers*) (Kawada & Takeuti, 1951, p. 139). The last chapter 「基礎論の立場から」 (From a viewpoint of foundations of mathematics) of this book is written down by Takeuti. Also, another keyword 「自覚」(self-consciousness) in Nishida's philosophy is used there.

(4) For this feature of active intuition, see the preface of the new typesetting in 1936 of 『善の研究』(*An Inquiry into the Good*) (Nishida, 1911). In particular, he writes "The world of acting intuition–the world of poiesis–is none other than the world of pure experience." (Nishida, 1911, p. xxxiii)

(5) We thank Hidenori Kurokawa, who suggested to us to translate 「意志的」 into "volitional".

(6) Indeed, Takeuti writes about his terminology in the following way.

> Here, the word "active volitional intuition" is a coined word under Suetsuna's influence (Suetsuna scolded me that "volitional" is not suitable). Suetsuna claims that the whole of the natural numbers or real numbers is grasped by active intuition. His active intuition would be a part of my active volitional intuition, but it seems to hold in his case that one [element in his active intuition] is constructive, and another [element in his active intuition] can see [the whole of natural numbers or real numbers] as a fact [as a totality] not depending on volition but on

movement.

The friendship between Takeuti and Suetsuna is observed in the fact that he dedicated the paper "On the recursive functions of ordinal numbers" in 1960 to Suetsuna (Takeuti, 1960).

(7) We note also that Kanazawa is the hometown of Takeuti, Nishida, and Suzuki. We refer to Takahashi (2011) for a discussion of the friendship between Suetsuna, Suzuki, and Nishida.

References

Deguchi, Y. (2017). 'Late Nishida and Dialetheism,' in *Daigokai Nicchu Tetsugaku Forum "Shisaku To Taiwa Ni Yoru Nicchu Kankei No Shinka: Kyokai Wo Kakyo Suru Tetsugaku No Yakuwari" Yokoshu (Proceedings of the 5th Philosophy Forum of Japan and China "Deepening Japan-China Relation by Thinking and Dialogue: Role of Philosophy Bridging over the Boundaries")*, 359–385: Institute of Philosophy of Chinese Academy of Social Sciences, The Philosophical Association of Japan, Institute of Humanities, Human and Social Sciences, Ritsumeikan University.

Gentzen, G. (1935). 'Untersuchungen über das logische Schließen,' *Mathematische Zeitschrift*, *39*, 176–210, 405–431. English translation in Szabo (1969).

——— (1936). 'Die Widerspruchsfreiheit der reinen Zahlentheorie,' *Mathematische Annalen*, *112*, 494–565. English translation in Szabo (1969).

——— (1938). 'Neue Fassung des Widerspruchsfreiheitsbeweises für die reine Zahlentheorie,' *Forschungen zur Logik und zur Grundlegung der exakten Wissenschaften, Neue Folge*, *4*, 19–44. English translation in Szabo (1969).

Gödel, K. (2003). 'Letter of Gödel to Bernays, 25 July 1969,' in Feferman, S. et al. eds. *Kurt Gödel Collected Works, Vol. IV: Correspondence A–G*, Oxford: Oxford University Press, 270–273.

van Heijenoort, J. ed. (1967). *From Frege to Gödel: A Source Book in Mathematical Logic, 1879–1931*, Cambridge, Mass.: Harvard University Press.

Hilbert, D. & Bernays, P. (1968). *Grundlagen der Mathematik. I*, Berlin: Springer-Verlag.

——— (2011). *Foundations of Mathematics I*, *v.1. pt.1*, London: College Publications. Edited by Wirth, Claus-Peter, Joerg Siekmann, Michael Gabbay, and Don Gabbay. Commented translation by Claus-Peter Wirth.

Hilbert, D. (1926). 'Über das Unendliche,' *Mathematische Annalen*, *95*, 161–190. English translation by Stefan Bauer-Mengelberg in Heijenoort (1967).

——— (1931). 'Die Grundlagen der Elementaren Zahlentheorie,' *Mathematische Annalen*, *104*, 485–94. English translation by William Ewald in Mancosu (1998).

Kawada, Y. & Takeuti, G. (1951). *Shizensuron (The Theory of Natural Numbers)*: Kawade Shobo, 2nd edition.

Mancosu, P. ed. (1998). *From Brouwer to Hilbert: The Debate on the Foundations of Mathematics in the 1920s*, Oxford: Oxford University Press.

Nishida, K. (1911). *Zen No Kenkyu (An Inquiry into the Good)*: Kodo Kan. translated into English under the title *An Inquiry into the Good* by Masao Abe and Christopher Ives, Yale University Press, 1992.

——— (1917). *Jikaku Ni Okeru Chokkan To Hansei (Intuition and Reflection in Self-Consciousness)*: Iwanami Shoten. translated into English under the title *Intuition and Reflection in Self-Consciousness* by Valdo H. Viglielmo, Toshinori Takeuchi, and Joseph S. O'Leary, SUNY Series in Philosophy, SUNY Press, 1987.

Suetsuna, J. (1947). *Ronri to Suuri (Logic and Mathematics)*, Kyoyo Bunko: Koubundo.

Szabo, M. E. ed. (1969). *The Collected Papers of Gerhard Gentzen*, *55* of Studies in Logic and the Foundations of Mathematics, Amsterdam: North-Holland.

Takahashi, S. (2011). 'Kegonkyo ni miryousareta suugakusya Suetsuna Joichi (Mathematician Suetsuna Joichi fascinated by Kegonkyo),' *Gendai Mikkyo*, *22*, 135–152.

Takeuti, G. (1953). 'On a generalized logic calculus,' *Japanese Journal of Mathematics*, *23*, 39–96.

——— (1955a). 'On the fundamental conjecture of GLC I,' *Journal of the Mathematical Society of Japan*, *7*, *3*, 249–275.

——— (1955b). 'On the fundamental conjecture of GLC II,' *Journal of the Mathematical Society of Japan*, *7*, *4*, 394–408.

——— (1956a). 'Keishikisyugi no tachiba kara I (From the point of view of formalism I),' *Kagakukisoron Kenkyu*, *2*, *2*, 265–269.

——— (1956b). 'Keishikisyugi no tachiba kara II (From the point of view of formalism II),' *Kagakukisoron Kenkyu (Annals of the Japan Association for Philosophy of Science)*, *2*, *3*, 295–299.

——— (1956c). 'On the fundamental conjecture of GLC III,' *Journal of the Mathematical Society of Japan*, *8*, *1*, 54–64.

——— (1956d). 'On the fundamental conjecture of GLC IV,' *Journal of the Mathematical Society of Japan*, *8*, *2*, 145–155.

——— (1957). 'Ordinal diagrams,' *Journal of the Mathematical Society of Japan*, *9*, *4*, 386–394.

——— (1958a). 'On the fundamental conjecture of GLC V,' *Journal of the Mathematical Society of Japan*, *10*, *2*, 121–134.

——— (1958b). 'On the theory of ordinal numbers, II,' *Journal of the Mathematical Society of Japan*, *10*, *1*, 106–120.

——— (1960). 'On the recursive functions of ordinal numbers,' *Journal of the Mathematical Society of Japan*, *12*, *2*, 119–128.

——— (1972). 'Suugaku ni tsuite (About mathematics),' *Kagakukisoron Kenkyu (Annals of the Japan Association for Philosophy of Science)*, *4*, 170–174.

——— (1975). *Proof Theory*, *81* of Studies in Logic and the Foundations of Mathematics, Amsterdam: North-Holland, 1st edition.

——— (1985). 'Proof theory and set theory,' *Synthese*, *62*, *2*, 255–263.

——— (1987). *Proof Theory*, *81* of Studies in Logic and the Foundations of Mathematics, Amsterdam: North-Holland, 2nd edition. Reissued by Dover Publications, 2013.

〔Waseda Institute for Advanced Study and Keio University, Tokyo
Université Paris 1 Panthéon-Sorbonne, and IHPST, Paris〕