A characterization of Approval Voting without the approval balloting assumption

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Keywords:
Informational basis, balloting procedure, Approval voting, Evaluative voting

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A characterization of Approval Voting without the approval balloting assumption*

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1 Introduction

Approval Voting is a voting procedure in which each individual voter submits a ballot that specifies whether she supports or not each of the available candidates for election, and the candidates who are supported by the largest number of voters are then elected. Since the seminal publications by Brams and Fishburn (Fishburn (1978), Brams and Fishburn (1978)), a burgeoning literature has been devoted to the analysis of the axiomatic properties of Approval Voting[1]. In the tradition of social choice theory, these theoretical studies share a common methodology: first, it is assumed that information concerning the voters' opinions comes in a specific, *dichotomous* format, namely approval ballots or, equivalently, dichotomous preferences[2]; then, it is asked how should such information be aggregated in order to make a decision on its basis. The consequent identification of a set of desirable normative principles that singles out Approval Voting among aggregators of ballots with a dichotomous structure can then be viewed either as a theoretical argument in support of Approval Voting, or as a defining exercise that can be useful in assessing the relative merits of such an aggregation procedure.

The purpose of this article is to provide an axiomatic characterization of Approval Voting without fixing the approval balloting procedure (or, crucially, any balloting procedure) from the start. Our motivation is twofold. First, we hope to enhance the comparability of Approval Voting with voting procedures that do not necessarily process the same ballot information. Interestingly, while there is a by now rich body of literature contrasting Approval Voting with other social choice methods that aggregate dichotomous ballot information, most notably the Plurality Rule[3], it is fair to say that most voting theorists would avoid taking a stand on whether Approval Voting is superior to the Borda Count or to Alternative Vote[4]. We do not believe that such state of affairs is purely fortuitous: Approval Voting is naturally comparable with rules that aggregate information of parsimonious nature,

[1] A comprehensive survey of axiomatic characterizations of approval voting (up to 2010) is to be found in Xu (2010); more recent contributions include Núñez and Valletta (2015), Maniquet and Mongin (2015) and Sato (2018).

[2] Approval ballots are subsets of the set of available candidates, interpreted to be the candidates of which a voter approves. Dichotomous preferences are weak orders over the set of candidates that admit at most two indifference classes, with the top indifference class being interpreted as containing the approved candidates.

[3] Examples include Ju (2010), Goodin and List (2006) and, more recently, Brandl and Peters (2019).

[4] On a more anecdotal note, during a symposium organized in 2010 in the north of France, a number of voting theorists voted on the question "What is the best voting rule for your town to use to elect the mayor?". All participants ranked Approval Voting above Plurality, while disagreement remained on the relative merits of the other 16 rules compared. Details can be found in Laslier (2012).

but how to *formally* compare it with preferential or evaluative voting is less clear[5]. Of course, one difficulty is that, mathematically, aggregation methods are fundamentally linked to the input they process. Yet another difficulty is that often, once applied to different informational environments, analogous normative principles can characterize competing voting methods, something that makes it hard "to argue in favor of one or the other. At least at a certain, intuitive level, arguments in favor of one can easily be turned into arguments in favor of the other"(Alós-Ferrer (2006), p. 622). For instance, Brams and Fishburn (1978) and Vorsatz (2008) show that Approval Voting coincides with the Condorcet rule and with (a natural adaptation of) the Borda count on the domain of dichotomous preferences (respectively). This brings us to our second, related, motivation. The axiomatic work that is conducted conditionally on a given informational environment is of considerable significance, insofar the type of information that is available, practical or even reasonable to collect is likely to depend on the particular voting context. However, if voting procedures are only analyzed on a given informational environment, questions concerning whether the choice of balloting procedures carries some *normative* relevance cannot be asked, so that, if indeed such normative relevance were to exist, it would go unwarranted.

Our result axiomatically singles out Approval Voting among the family of abstract voting procedures, i.e. mappings that take as input profiles of individual "signals" and return a set of winning candidates. These signals, that we call ballots, are abstract objects in a given set, interpreted as the admissible opinions that a voter is allowed to cast in a particular voting context. Specific typologies of ballots include approval ballots (cf. footnote 2), preferential ballots, i.e. weak or linear orders over the set of available candidates, evaluative ballots, i.e. list of numerical scores ("valuations") assigned to each candidate, or interesting combinations of the two (e.g. the preference-approval ballots introduced in Brams and Sanver (2009)). We adopt a variable-population framework that is comparable to the one proposed by Myerson (1995), though we do not require his anonymity condition on the voting procedure, that demands the result of an election to be fully described by a distribution that specifies how many of each kind of ballots have been submitted, nor the non-emptiness of the aggregation rule. Approval Voting is shown to be characterized by the classical *consistency* axiom[6], that roughly requires that when a candidate is selected by two different constituencies, she must also be selected by their union, together with two novel conditions. The first one is a requirement of expressive richness on the set of available ballots, that essentially states that

[5] By preferential and evaluative voting we generally mean voting procedures that aggregate preference rankings or cardinal evaluations, respectively.

[6] This terminology is due to Young (1974), Young (1975) and Saari (1990). Smith (1973) calls this condition "separability", while Myerson (1995), Pivato (2013) and Pivato (2014) call it "reinforcement".

any voting population can elect any number of candidates. The second condition, arguably the most restrictive, captures the idea that a voter cannot single-handedly control the electoral outcome in the sense that the sets of winners chosen by a voting population with or without the presence of an additional voter cannot be disjoint.

We view two features of our characterization as particularly important. First, the dichotomous nature of the informational basis of Approval Voting is a consequence of our conditions, rather than exogenously assumed. Secondly, as a corollary of our main result, if ballots are taken to be approval ballots, it is possible to characterize Approval Voting by replacing the condition of ballot richness with the well-known axiom of *faithfulness* (details will be provided in section 5), a result that, because of the novelty of the axiomatic conditions we impose, essentially departs from the existing axiomatizations of Approval Voting.

This article is organized as follows: next section introduces the framework and relevant notation. In section 3 we present and discuss the axiomatic conditions that turn out to characterize the Approval Voting procedure. The characterization theorem and its proof are provided in section 4. Concluding remarks are gathered in section 5, while the logical independence between the axioms is shown in the Appendix.

2 Framework

Throughout, we fix a finite **set of candidates** (or social alternatives) C and denote by $\mathcal{P}_*(C)$ the set of all non-empty subsets of C. Let $\mathcal{V}$ be an infinite set, interpreted to represent the universe of voters. An **electorate** is a finite subset of $\mathcal{V}$ and we denote by $\mathcal{E}$ the set of all electorates, with typical element V. Generic voters are denoted by i, j while a,b,c represent candidates. Let $\mathcal{X}$ be an infinite set representing the universe of ballots that voters may express in some voting situation. A **set of admissible ballots** is a subset X of $\mathcal{X}$ containing the actual ballots that a voter can possibly submit to the election chair in a given voting context. For every set of admissible ballots $X \subseteq \mathcal{X}$ and for every electorate $V \in \mathcal{E}$, a **voting profile**

$$B^V = (B^i)_{i \in V} \in X^V$$

is a collection of individual ballots. Let $\mathcal{B}_X$ be the set of all voting profiles that can be constructed from $\mathcal{V}$ and X. Given two disjoint electorates $V, W \in \mathcal{E}$ and two ballot profiles $B^V, B^W \in \mathcal{B}_X$, we denote by $B^{V \cup W} = (B^i)_{i \in V \cup W}$ the ballot profile obtained by merging B^V and B^W; similarly, given two electorates $V, W \in \mathcal{E}$ such that $W \subseteq V$, and a ballot profile $B^V \in \mathcal{B}_X$, we denote by $B^{V \setminus W} = (B^i)_{i \in V \setminus W}$ the ballot profile obtained by removing the ballots cast by voters in W from B^V. A **rule** f on $\mathcal{B}_X$ associates to any voting profile $B^V \in \mathcal{B}_X$ a (possibly empty) subset of the available candidates $f(B^V) \subseteq C$ (the "winners" of the election). We assume that a rule always satisfies $f(B^\emptyset) = C$[7]. Denote by $\mathcal{R}_X$ the set of rules on $\mathcal{B}_X$. A **voting procedure** is a couple (X, f) where X is a set of admissible ballots and f is a rule belonging to $\mathcal{R}_X$.

We say that (X, f) is an **Approval Voting Procedure** if for all $i \in \mathcal{V}$, there exists a surjection $\varphi_i : X \to \mathcal{P}_*(C)$ such that for all $V \in \mathcal{E}$ and for all $B^V \in \mathcal{B}_X$,

$$f(B^V) = \underset{a \in C}{\operatorname{argmax}} \, |\{i \in V : a \in \varphi_i(B^i)\}|. \tag{1}$$

Remark that, mathematically, the above definition is an extension of the standard definition of Approval Voting, which is obtained by identifying X with $\mathcal{P}_*(C)$ and letting $\varphi_i = Id$, for every $i \in \mathcal{V}$. Such extension includes more generally all voting procedures that map the information contained in the signals cast by voters into approval ballots, and then aggregate such profiles of approval ballots so as to select the most approved candidate(s). This reflects the more general intuition that, in the abstract framework that we adopt, voting procedures embody two defining features: their informational basis and their aggregative rationale. The informational basis of Approval Voting (as defined in 1) is captured by the mappings $\{\varphi_i\}_{i \in \mathcal{V}}$. They outline the fact that the information available in any signal $B^i \in X$ that voter $i \in \mathcal{V}$ may cast is *translated* (or "interpreted") into an approval ballot $\varphi_i(B^i) \subseteq C$, and that *only* such approval ballot information matters for determining the winner of an election, in the sense that for every $V, W \in \mathcal{E}$ and $B^V, B^W \in \mathcal{B}_X$,

$$\{\varphi_i(B^i)\}_{i \in V} = \{\varphi_j(B^j)\}_{j \in W} \implies f(B^V) = f(B^W). \tag{2}$$

Finally, remark that, given an Approval Voting procedure (X, f) and a collection of bijections $\{\mu_i : \mathcal{P}_*(C) \to \mathcal{P}_*(C)\}_{i \in \mathcal{V}}$, the voting procedure (X, g) defined by:

$$g(B^V) = \underset{a \in C}{\operatorname{argmax}} \, |\{i \in V : a \in \mu_i(\varphi_i(B^i))\}| \tag{3}$$

is also an Approval Voting procedure. Hence, while several voting procedures can be Approval Voting, each of them is "expressively" equivalent, in the sense that, for every voter $i \in \mathcal{V}$, what ultimately matters is the procedural interpretation of a signal - that is, how candidates are sorted into "approved" and "non-approved" via the translation mapping φ_i - rather than the signal itself.

[7] This property is sometimes called the axiom of *general abstention* (see e.g. Houy (2007)).

3 Axiomatic properties

In this section, we present the axiomatic conditions that will be shown to characterize the Approval Voting procedure. As mentioned in the introduction, our characterization makes use of the well-known social-theoretic axiom of *consistency*[8], which demands that, when merging two constituencies, all candidates - and only them - who are selected by both constituencies, if existent, be selected.

Consistency. Let (X, f) be a voting procedure. For any $B^V, B^{V'} \in \mathcal{B}_X$ such that $V \cap V' = \emptyset$ and $f(B^V) \cap f(B^{V'}) \neq \emptyset$,

$$f(B^{V \cup V'}) = f(B^V) \cap f(B^{V'}).$$

Next axiom is a richness condition on the ballot space, requiring any voting population to be able to elect any subset of the candidates.

Ballot richness. Let (X, f) be a voting procedure. For every $V \in \mathcal{E}$ and $A \in \mathcal{P}_*(C)$, there exists $B^V \in \mathcal{B}_X$ such that $f(B^V) = A$.

Notice that a necessary condition for the existence of an aggregation rule f on X such that (X, f) satisfies Ballot richness is that there exists a surjection from X to $\mathcal{P}_*(C)$, or, equivalently, that the cardinality of X is larger than the one of $\mathcal{P}_*(C)$. Moreover, when restricting attention to single-voter electorates, Ballot richness requires every voter to be able to vote for any number of the alternatives, reflecting the "one candidate-one vote" philosophy that is typical of Approval Voting, but also of the class of scoring rules or of most preferential voting rules in which voters are allowed to express indifference among candidates. Finally, the last condition demands that in no circumstance adding a voter to the electorate excludes all previously winning candidates.

No single-voter overrides. Let (X, f) be a voting procedure. For every $V \in \mathcal{E}$, $i \in \mathcal{V} \setminus V$ and $B^V, B^{\{i\}} \in \mathcal{B}_X$ such that either $f(B^V)$ or $f(B^{V \cup \{i\}})$ is nonempty,

$$f(B^V) \cap f(B^{V \cup \{i\}}) \neq \emptyset.$$

Intuitively, No single-voter overrides is a democratic principle requiring the addition of a single voter not to overrule the decision of the majority. It is comparable to the (logically independent, but intuitively much weaker) property of *no minority overrides* that was used by Pivato (2014) to characterize the range voting procedure. The latter essentially requires the existence of at least one voting

[8] *Consistency* has been used in most of the existing axiomatic characterizations of Approval Voting or, more generally, scoring rules with variable electorate (see Xu (2010) for the case of Approval Voting and Myerson (1995) and Pivato (2013) for axiomatic derivations of scoring rules).

profile in which an additional voter cannot single-handedly change the electoral outcome[9].

4 Result

We are now ready to state our characterization result.

Theorem 1 A voting procedure satisfies CONSISTENCY, BALLOT RICHNESS and NO SINGLE-VOTER OVERRIDES if and only if it is an Approval Voting procedure.

Proof.

Necessity of the axioms is easy to prove, so we only establish sufficiency.

Let (X, f) be a voting procedure satisfying BALLOT RICHNESS, CONSISTENCY and NO SINGLE-VOTER OVERRIDES. For every $a \in C$, $V \in \mathcal{E}$ and $B^V \in \mathcal{B}_X$, let

$$I_a(B^V) := \{i \in V \,:\, a \in f(B^{\{i\}})\},$$

and

$$A(B^V) = \underset{a \in C}{\operatorname{argmax}}\, |I_a(B^V)|.$$

Moreover, for every $i \in \mathcal{V}$, let the mapping $\varphi_i : X \to \mathcal{P}_*(C)$ be defined by: for every $x \in X$, $\varphi_i(x) = f(B^{\{i\}})$, where $B^i = x$. By BALLOT RICHNESS, φ_i is surjective for every $i \in \mathcal{V}$. Hence, to establish the theorem, it suffices to show that for every $V \in \mathcal{E}$ and $B^V \in \mathcal{B}_X$,

$$f(B^V) = A(B^V). \tag{P}$$

We do so by induction. Let $P(n)$ be the statement *"property (P) holds for every electorate of size n"*.

First notice that whenever $V \in \mathcal{E}$ is such that $|V| = 1$, property (P) holds, i.e. $P(1)$ holds. Next, fix a natural number $n \geq 1$ and suppose that $P(n)$ holds. We show that this implies that $P(n+1)$ also holds. The proof is divided into several steps.

Step 1: For every $B^V \in \mathcal{B}_X$, $f(B^V) \neq \emptyset$.

[9] Formally, *no minority overrides* demands that there exists a finite set $V \subsetneq \mathcal{V}$, a voter $i \in \mathcal{V} \setminus V$ and $B^V \in \mathcal{B}_X$ such that for every $B^{\{i\}} \in \mathcal{B}_X$, $f(B^V) = f(B^{V \cup \{i\}})$.

No single-voter overrides implies that for every $B^{\{i\}} \in \mathcal{B}_X$,

$$f(B^{\{i\}}) \neq \emptyset, \tag{4}$$

(since by definition $f(B^{\emptyset}) = C$) and that moreover for every $B^{\{i\}}, B^{\{j\}} \in \mathcal{B}_X$,

$$f(B^{\{i\}}) \cap f(B^{\{i,j\}}) \neq \emptyset. \tag{5}$$

Equations 4 and 5 together imply that $f(B^{\{i,j\}})$ is nonempty. Proceeding by induction, it is straightforward to observe that for every $B^V \in \mathcal{B}_X$, $f(B^V)$ is nonempty.

Step 2: For all electorate V of size $n+1$, if there exists $a \in f(B^V)$ such that $a \notin A(B^V)$ then $A(B^V) \subseteq f(B^V)$.

Let $a \in f(B^V)$ be such that $a \notin A(B^V)$ and let b be an element of $A(B^V)$. We want to show that b belongs to $f(B^V)$. We begin by showing that $\exists i_0 \in V$ such that $b \notin f(B^{\{i_0\}})$. To see why, suppose by contradiction that $b \in \bigcap_{i \in V} f(B^{\{i\}})$. Then by Consistency

$$\bigcap_{i \in V} f(B^{\{i\}}) = f(B^V). \tag{6}$$

On the other hand, $|I_b(B^V)| = |V| > |I_a(B^V)|$ because a does not belong to $A(B^V)$ by assumption. It follows that

$$a \notin \bigcap_{i \in V} f(B^{\{i\}}) = f(B^V), \tag{7}$$

a contradiction with our hypothesis. Hence, we deduce that $\exists i_0 \in V$ such that $b \notin f(B^{\{i_0\}})$.

Next, since $b \in A(B^V)$ and $b \notin f(B^{\{i_0\}})$, we have that

$$b \in A(B^{V \setminus \{i_0\}}). \tag{8}$$

Moreover, since $a \notin A(B^V)$,

$$|I_a(B^{V \setminus \{i_0\}})| \leq |I_a(B^V)| < |I_b(B^V)| = |I_b(B^{V \setminus \{i_0\}})|, \tag{9}$$

from which we deduce that

$$a \notin A(B^{V \setminus \{i_0\}}). \tag{10}$$

Using the induction hypothesis, we have that

$$f(B^{V \setminus \{i_0\}}) = A(B^{V \setminus \{i_0\}}). \tag{11}$$

Combining formulae 8 and 10 with formula 11, we obtain that

$$b \in f(B^{V\setminus\{i_0\}}) \text{ and } a \notin f(B^{V\setminus\{i_0\}}). \tag{12}$$

Next, let $k \in \mathcal{V} \setminus V$ be a voter and fix $B^{\{k\}} \in \mathcal{B}_X$ such that $f(B^{\{k\}}) = \{a, b\}$ (this exists by Ballot richness).

Formula 12 and Consistency imply that

$$f(B^{(V\setminus\{i_0\})\cup\{k\}}) = f(B^{V\setminus\{i_0\}}) \cap f(B^{\{k\}}) = \{b\}. \tag{13}$$

Meanwhile formula 13 and No single-voter overrides imply that

$$b \in f(B^{V\cup\{k\}}). \tag{14}$$

On the other hand, $a \in f(B^V) \cap f(B^{\{k\}})$, so by Consistency again,

$$f(B^{V\cup\{k\}}) = f(B^V) \cap f(B^{\{k\}}). \tag{15}$$

From equations 14 and 15, we deduce that $b \in f(B^V)$, as desired.

Step 3: For no electorate V of size $n + 1$, $f(B^V) \subsetneq A(B^V)$.

By contradiction, suppose that there is some $V \in \mathcal{E}$ such that $|V| = n + 1$ and $f(B^V) \subsetneq A(B^V)$. Then, $A(B^V)$ contains at least two elements because by **step 1**, $f(B^V) \neq \emptyset$. Let a be an element of $A(B^V)$ such that $a \notin f(B^V)$. Two cases arise:

- **Case 1: $\forall c \in f(B^V)$, $I_c(B^V) \subseteq I_a(B^V)$.** Let $i_1 \in V$ be such that $f(B^{\{i_1\}}) \cap f(B^V) \neq \emptyset$ (such a voter exists since by assumption $f(B^V) \subsetneq A(B^V)$). We now show that $f(B^V) \subseteq f(B^{\{i_1\}})$. To do so, let $c \in f(B^{\{i_1\}}) \cap f(B^V)$ and suppose by contradiction that there exists $d \in f(B^V)$ such that $d \notin f(B^{\{i_1\}})$. Then, $i_1 \notin I_d(B^V)$, so

$$|I_a(B^V)| = |I_a(B^V) \cap I_d(B^V)| + |I_a(B^V) \cap (V \setminus I_d(B^V))| > |I_a(B^V) \cap I_d(B^V)|, \tag{16}$$

where the inequality follows from the fact that $i_1 \in V \setminus I_d(B^V)$. Since $I_d(B^V) \subseteq I_a(B^V)$, we have that

$$|I_a(B^V) \cap I_d(B^V)| = |I_d(B^V)|. \tag{17}$$

Combining equations 16 and 17 we obtain $|I_a(B^V)| > |I_d(B^V)|$. However, by hypothesis, $d \in f(B^V)$ implies $d \in A(B^V)$. It follows that $|I_a(B^V)| = |I_d(B^V)|$, a contradiction. We deduce that

$$f(B^V) \subseteq f(B^{\{i_1\}}). \tag{18}$$

From formula 18 we immediately deduce that

$$f(B^{V\setminus\{i_1\}}) \cap f(B^V) \subseteq f(B^{V\setminus\{i_1\}}) \cap f(B^{\{i_1\}}). \tag{19}$$

Meanwhile, No single-voter overrides implies that

$$f(B^{V\setminus\{i_1\}}) \cap f(B^V) \neq \emptyset. \tag{20}$$

Therefore, there exists $b \in f(B^{V\setminus\{i_1\}}) \cap f(B^V)$. By formula 18, $b \in f(B^{\{i_1\}})$ so $i_1 \in I_b(B^V) \subseteq I_a(B^V)$, where the last inclusion is due to our working hypothesis. Hence,

$$a \in f(B^{\{i_1\}}). \tag{21}$$

Next, by the induction hypothesis, $f(B^{V\setminus\{i_1\}}) = A(B^{V\setminus\{i_1\}})$, so

$$b \in A(B^{V\setminus\{i_1\}}). \tag{22}$$

This, together with the fact that $b \in f(B^{\{i_1\}})$, implies that $b \in A(B^V)$. But by assumption, $a \in A(B^V)$, so

$$\{a, b\} \subseteq f(B^{\{i_1\}}) \cap A(B^V). \tag{23}$$

Combining the latter inclusion with formula 22, we obtain that $a \in A(B^{V\setminus\{i_1\}})$. Using again the induction hypothesis, $a \in f(B^{V\setminus\{i_1\}})$. Finally, Consistency and formula 21 imply that $a \in f(B^V) = f(B^{V\setminus\{i_1\}}) \cap f(B^{\{i_1\}})$, a contradiction.

- **Case 2: $\exists c \in f(B^V)$, $\exists i_c \in V$ such that $c \in f(B^{\{i_c\}})$ and $a \notin f(B^{\{i_c\}})$.** Using the induction hypothesis, we have

$$f(B^{V\setminus\{i_c\}}) = A(B^{V\setminus\{i_c\}}). \tag{24}$$

Since $a \in A(B^V)$, $c \in f(B^{\{i_c\}})$ and $a \notin f(B^{\{i_c\}})$, formula 24 implies that

$$a \in f(B^{V\setminus\{i_c\}}) \text{ and } c \notin f(B^{V\setminus\{i_c\}}). \tag{25}$$

Let $\ell \in \mathcal{V} \setminus V$ be a voter and fix $B^{\{\ell\}} \in \mathcal{B}_X$ such that $f(B^\ell) = \{a, c\}$ (such a ballot exists by Ballot richness). Then, Consistency implies that

$$f(B^{(V\setminus\{i_c\})\cup\{\ell\}}) = f(B^{V\setminus\{i_c\}}) \cap f(B^\ell) = \{a\}, \tag{26}$$

while at the same time

$$f(B^{V\cup\{\ell\}}) = f(B^V) \cap f(B^\ell) = \{c\}. \tag{27}$$

Together, equations 26 and 27 imply that

$$f(B^{(V\setminus\{i_c\})\cup\{\ell\}}) \cap f(B^{V\cup\{\ell\}}) = \emptyset, \tag{28}$$

in contradiction with No single-voter overrides.

Step 4: For every electorate $\boldsymbol{V}$ of size $\boldsymbol{n+1}$, $\boldsymbol{A(B^V) \subseteq f(B^V)}$.

Let $V \in \mathcal{E}$ be such that $|V| = n+1$ and $B^V \in \mathcal{B}_X$. If $f(B^V) \not\subseteq A(B^V)$, then by **step 2**, $A(B^V) \subseteq f(B^V)$. On the other hand, if $f(B^V) \subseteq A(B^V)$, then by **step 3** necessarily $f(B^V) = A(B^V)$, so again $A(B^V) \subseteq f(B^V)$, as desired.

Step 5: For all electorate $\boldsymbol{V}$ of size $\boldsymbol{n+1}$, $\boldsymbol{f(B^V) \subseteq A(B^V)}$.

Suppose by contradiction that there is a candidate a such that $a \in f(B^V)$, $a \notin A(B^V)$ and let b be another candidate such that $b \in A(B^V)$ (this exists because $A(\cdot)$ is always non-empty). By **step 4** $b \in f(B^V)$ because V is an electorate of size $n+1$. Since $a \notin A(B^V)$ and $b \in A(B^V)$, there exists a voter $i_2 \in V$ such that $b \in f(B^{\{i_2\}})$ and $a \notin f(B^{\{i_2\}})$. Let $m \in \mathcal{V} \setminus V$ be a voter and fix $B^{\{m\}} \in \mathcal{B}_X$ such that $f(B^{\{m\}}) = \{a, b\}$ (again, this is possible by Ballot richness). On the one hand, it is easy to check that

$$b \in A(B^{(V\setminus\{i_2\})\cup\{m\}}). \tag{29}$$

Since $(V \setminus \{i_2\}) \cup \{m\}$ is an electorate of size $n+1$, formula 29 and **step 4** imply that

$$b \in f(B^{(V\setminus\{i_2\})\cup\{m\}}). \tag{30}$$

Since $b \in f(B^{\{i_2\}})$, by 30 we obtain

$$f(B^{(V\setminus\{i_2\})\cup\{m\}}) \cap f(B^{\{i_2\}}) \neq \emptyset \tag{31}$$

and by Consistency this intersection is equal to $f(B^{V\cup\{m\}})$. It follows that

$$a \notin f(B^{V\cup\{m\}}), \tag{32}$$

because $a \notin f(B^{\{i_2\}})$. On the other hand, since $a \in f(B^V)$ and $a \in f(B^{\{m\}})$, Consistency implies that $a \in f(B^{V\cup\{m\}})$, contradicting equation 32. We conclude that for all electorate V of size $n+1$, $f(B^V) \subseteq A(B^V)$.

Step 6: conclusion. Combining **steps 4** and **5**, we conclude that for all electorate V of size $n+1$, $f(B^V) = A(B^V)$. Since both the base case and the inductive step have been performed, by mathematical induction the statement $P(n)$ holds for all $n \geq 1$, completing the proof of the theorem.

5 Discussion and concluding remarks

Most of the axiomatic work in social choice theory presupposes a given informational environment, and there are sound reasons to do so. In this article, we offer a complementary perspective that explicitly addresses questions about the normative relevance of the informational basis of voting procedures. We provide an axiomatic characterization of Approval Voting without the approval balloting assumption. That is, both the informational basis of Approval Voting and its aggregative rationale are simultaneously derived from our axiomatic conditions, which are therefore shown to effectively impose the reduction of whatever information available in the ballots to dichotomous information. This does not in principle rule out the possibility of collecting richer information from the voters, nor of effectively using such richer information when available. In principle, it is possible to collect richer information without exploiting it for the purpose of determining the winner of the election. This feature in not unusual in real-world elections. For instance, voters in India are allowed to cast "None Of The Above" ballots which, although counted, do not impact the result of the election process. Alternatively, richer information could be collected and reduced to dichotomous information, say, by asking voters to assign a numerical grade between 0 and 10 to candidates, and then considering the candidate(s) whose grade is above 6 to be approved. This highlights the consequentialist perspective of our analysis: once one accepts the axioms of theorem 1, any two ballots that induce the same set of approved candidates will have equivalent instrumental value. Yet, they may well be substantially different w.r.t. other types of considerations that can be invoked for the practical purpose of designing a voting procedure. For instance, experimental evidence suggests that the labeling of ballots has a psychological meaning to voters that can affect the output of a voting procedure, and that voters value the possibility of expressing non-dichotomous opinions even though the additional information is not exploited when determining the winner of the election (see for instance Baujard et al. (2018)). Similarly, legal or political considerations may also point at the direction of enhancing the expressive freedom of voters, e.g. when explicitly introducing an "abstention" ballot allowing voters to exercise their right not to vote, while maintaining their right to secrecy.

Another, related, feature of our characterization is that the translation mappings may depend on the identity of the voter, so that the same ballot could in principle receive different interpretations depending, say, on the age of the voter who casts it. Therefore, the Approval Voting procedure (as defined in 1) does not necessarily satisfy the usual definition of anonymity, i.e. it is possible that for two electorates $V, W \in \mathcal{E}$ and ballot profiles $B^V, B^W \in \mathcal{B}_X$, $\{B^i\}_{i\in V} = \{B^j\}_{j\in W}$ and yet $f(B^V) \neq f(B^W)$. Nevertheless, property 2 shows that the Approval Voting

procedure satisfies anonymity w.r.t. ballot interpretations rather than ballots, in line with our consequentialist approach to ballot information. A characterization of Approval Voting procedures in which the interpretation of ballots is independent of the voters' identity can be obtained by adding to the axiom set used in theorem 1 the requirement that for every $i, j \in \mathcal{V}$, if $B^i = B^j$, then $f(B^{\{i\}}) = f(B^{\{j\}})$.
Finally, as a corollary of our result, *if approval ballots are assumed*, Approval Voting is seen to be characterized by the widely used axiom of *faithfulness*, requiring the aggregation rule to return the set of candidates approved by a voter whenever she is the only voter, together with *consistency* and *no single-voter overrides*. This is because, under the approval balloting restriction, *faithfulness* and *consistency* imply *ballot richness*, that can therefore be dropped. A novel feature of such result is that it does not hinge on the common requirements of *anonymity* and *non-emptiness*, assumptions that appear in most of the existing axiomatizations of Approval Voting.
While we only studied a specific voting procedure, it seems meaningful to extend this type of analysis to other voting methods, at the very least because the absence of agreement as per what type of information is to be asked to voters, or used in social choice procedures, seems to call for a formal treatment thereof. For instance, this could help elucidate the consequences of replacing the dichotomous informational basis of Approval Voting with a trichotomous one - allowing voters to express approval, disapproval and neutrality towards candidates - as required by an alternative to Approval Voting, the dis&approval voting procedure, that has received some attention in recent years (see Hillinger (2005), Alcantud and Laruelle (2014) or Gonzalez et al. (2019)).

6 Appendix: independence of the axioms

We now show that the axiomatic conditions of theorem 4 are logically independent by identifying examples of voting procedures satisfying all but one given condition. To make clear that their independence is not due to the fact that the domain of voting procedures (as opposed to the one of approval ballot aggregators) is rather large, we provide examples of voting procedures that only make use of dichotomous ballot information (essentially, they aggregate approval ballots). In what follows, for any weak order R over C, let $t(R)$ denote its top indifference class; that is, $t(R) = \{a \in C : aRb \text{ for all } b \in C\}$.

Example 1. Let (X, f_1) (the "plurality voting procedure") be defined by: for every voter $i \in \mathcal{V}$, there exists a surjection $\varphi_i^{f_1} : X \to \{\{a\}_{a \in C}\}$ such that for every finite set of voters $V \subseteq \mathcal{V}$ and $B^V \in \mathcal{B}_X$,

$$f(B^V) = argmax_{a \in C} |\{i \in V : \{a\} = \varphi_i^{f_1}(B^i)\}|. \tag{33}$$

The Plurality voting procedure satisfies all axioms of theorem 4 but Ballot richness.

Example 2. Let (X, f_2) be defined by: for every voter $i \in \mathcal{V}$, there exists a surjection $\varphi_i^{f_2} : X \rightarrow \mathcal{P}_*(C)$ such that for every finite set of voters $V \subseteq \mathcal{V}$ and $B^V \in \mathcal{B}_X$,

$$f_2(B^V) = \begin{cases} A, & \text{if} \quad A = \varphi_i^{f_2}((B^i)) \text{ for all } i \in \mathcal{V} \\ C, & \text{otherwise.} \end{cases} \tag{34}$$

Then (X, f_2) satisfies all our axioms but Consistency.

Example 3. Let (X, f_3) be defined by: for every voter $i \in \mathcal{V}$, there exists a surjection $\varphi_i^{f_3} : X \rightarrow \mathcal{P}_*(C)$ such that for every finite set of voters $V \subseteq \mathcal{V}$ and $B^V \in \mathcal{B}_X$,

$$f_3(B^V) = \bigcap_{i \in \mathcal{V}} \varphi_i^{f_3}(B^i) \tag{35}$$

Then (X, f_3) satisfying all our axioms but No single-voter Overrides.

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