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Abstract

We analyze strictly competitive information design games between two designers and an agent. Before the agent takes a decision, designers disclose public information at multiple stages about persistent state parameters. We consider environments with arbitrary constraints on feasible information disclosure policies. Our main results characterize equilibrium payoffs and strategies for various timings of the game: simultaneous or alternating disclosures, with or without deadline. Without constraints on policies, information is disclosed in a single stage, but there may be no bound on the number stages used to disclose information when policies are constrained. As an application, we study competition in product demonstration and show that more information is revealed when there is a deadline. The format that provides the buyer with the most information is the sequential game with deadline in which the ex-ante strongest seller is the last mover.

Keywords: Bayesian persuasion; concavification; convexification; information design; Mertens-Zamir solution; product demonstration; splitting games; statistical experiments; stochastic games.

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1 Introduction

This paper analyzes the strategic interaction between two information designers who compete for influencing the action of an agent. The designers have opposite preferences and have access to different sources of information. A first key feature of our model is that designers are able to disclose information in multiple stages. Precisely, at each stage, designers disclose information publicly and choose new disclosure policies at the next stage. When designers are done releasing information, the agent chooses an action. The payoffs of the designers and of the agent depend on the realized state parameters and on the action taken. The second feature of our model is that we consider technological restrictions that designers may face when choosing information disclosure policies.

As in the literature on Bayesian persuasion initiated by Kamenica and Gentzkow (2011), the term "information designer" refers to a player who is uninformed about a state parameter, but is able to choose an information disclosure policy (a statistical experiment) about this parameter in order to modify agents' information. Contrary to cheap talk communication, there is no issue of credibility in information disclosure: the statistical experiment is publicly observable and verifiable. In addition, since an information designer is not privately informed about the state, the choice of the disclosure policy has no signaling effect. In practice, information designers can represent competing sellers who release information about a new product in order to influence buyers' expected valuations. For example, sellers can offer free samples or trial periods to social media influencers, they can control the access and content of online, press or magazine reviews, they can organize product testing and trade fairs, or they can make announcements about future product developments. Alternatively, information designers could be lobbyists who design the informativeness of some studies in order to influence a policy-maker. In these examples, each designer may want to release additional information as a function of the information revealed by the competing designer.

The aim of the paper is to study designers' equilibrium information disclosure policies and payoffs in such multi-stage information design problems. We consider various possible timings: designers may be able to disclose information simultaneously in each stage (the simultaneous-move game) or only a single designer may be able to disclose information in each stage (the sequential-move game). In addition, there may be a deadline, an a priori bound on the number of disclosure stages. If not, we will speak of "long" information design, even though information disclosure might end very quickly in equilibrium.

Contributions The main result of this paper is a characterization of the equilibrium payoffs of the information design games for the various timings, obtained under some assumptions on the environment. There are first standard continuity, convexity and compactness conditions: designers' utility functions are continuous with respect to the agent's beliefs, the correspondences of feasible disclosure policies are continuous (in some strong sense), with convex and compact values. Then, we make assumptions on the nature of feasible policies. We assume that non-informative policies are always feasible and importantly, that feasible policies are closed under iteration. That is, a distribution of beliefs that can be obtained by a two-step combination of experiments, can also be obtained by a single feasible experiment. Under this latter condition, multiple stages of information disclosure would become irrelevant if there was only one designer.[1]

Under this set of assumptions, we show that each version of the multi-stage information design game admits a Markovian equilibrium, i.e., such that designers' strategies depend only on the current beliefs of the agent and on calendar time. When there is no deadline, the equilibrium is furthermore stationary, i.e., depends only on current beliefs. In addition, if there is no constraint on the set of available information disclosure policies, there exists an equilibrium in which information is disclosed at the first stage only. That is, along the equilibrium path, at most one designer discloses some information at the first stage and information disclosure policies are uninformative thereafter. When information disclosure policies are constrained, we provide an example in which the number of disclosure stages is unbounded and information disclosure stops with probability one in finite time (see Section 3.3.2).

As in the usual case of one designer of Kamenica and Gentzkow (2011), the value of the game (equilibrium payoff of Designer 1) is derived from Designer 1's expected payoff $u(p, q)$ as a function of the agent's beliefs (p, q), from which equilibrium strategies can be backed out. First suppose that only Designer 1 is active (say because Designer 2 is constrained to be silent), then from Kamenica and Gentzkow (2011), Designer 1 would "concavify" the payoff function with respect to p, obtaining what we denote $\operatorname{cav} u(p, q)$. Likewise, if only Designer 2 was active, we would obtain the "convexification" with respect to q which we denote $\operatorname{vex} u(p, q)$.

Consider now the information design game in which designers move alternately until a deadline N. By backward induction, the value of the game is $\operatorname{vex} \operatorname{cav} u(p, q)$ if Designer 1 moves last, and $\operatorname{cav} \operatorname{vex} u(p, q)$ if Designer 2 moves last. In the first case, the optimal strategy of Designer 1 is to play non-revealing until the last stage and then to concavify optimally for the current beliefs (p, q).

In the information design game in which designers move simultaneously until a deadline N, then in equilibrium both remain silent up to stage $N - 1$. The value, which we call *the splitting game value*, is thus the same as for the one-shot simultaneous move game.

An important contribution of the paper is the equilibrium characterization of the games with no deadline. Surprisingly, the values and equilibrium strategies are the same for the games with simultaneous or alternating moves. We show that this value is the unique function $v(p, q)$, which we call *the Mertens-Zamir function*, satisfying the following system:[2]

$$v(p, q) = \operatorname{cav} \min(u, v)(p, q) = \operatorname{vex} \max(u, v)(p, q).$$

This system is key to the study of discounted zero-sum repeated games with incomplete information on both sides (Mertens and Zamir, 1971, 1977) and of zero-sum dynamic gambling games (Laraki and Renault, 2019). It allows to derive directly simple optimal strategies: when $u(p, q) \geq v(p, q)$, Designer 1 plays non-informatively, when $u(p, q) < v(p, q)$, Designer 1 concavifies $\min(u, v)$.

The value of every information design game is in between $\operatorname{cav} \operatorname{vex} u(p, q)$ and $\operatorname{vex} \operatorname{cav} u(p, q)$, so when these two quantities are equal, all multi-stage information design games have the same value. We provide examples in which $\operatorname{cav} \operatorname{vex} u$, $\operatorname{vex} \operatorname{cav} u$, the splitting game value and the Mertens-Zamir

[1] We show how this assumption can be relaxed in Section B.3 of the online Appendix.

[2] In Section B.2 we show how to approximate the Mertens-Zamir function with finite sets of admissible posterior beliefs and present an algorithm to compute it.

function are all different.

Application to product demonstration between two sellers We apply our results and methodology to a model of competitive product demonstration in which two sellers disclose information about their own product to a consumer. All players are initially uncertain about the buyer's valuations for the two products. The agent decides to buy from the seller for whom he has the highest expected valuation conditional on his information (see Boleslavsky and Cotton, 2015 for the analysis of the one-shot game). We show that, whatever be the prior expected valuations of the buyer, sellers' equilibrium strategies are always more informative with a deadline than without a deadline. Without deadline, only one designer discloses information, and the expected payoff of the agent is the same as without information disclosure. Additionally, when there is a deadline, the agent always prefers the sequential version of the game in which the seller with the highest ex-ante value discloses information last.

Related Literature The methodology and results of Bayesian persuasion (Kamenica and Gentzkow, 2011) and information design (e.g., Bergemann and Morris, 2016a,b, Mathevet, Perego, and Taneva, 2019, and Taneva, 2019) are deeply related to the seminal contributions in repeated games with incomplete information on one side (Aumann and Maschler, 1966, 1967, Aumann, Maschler, and Stearns, 1995) and to the literature on generalized principal-agent problems, correlated and communication equilibria (Aumann, 1974, Myerson, 1982, 1986, Forges, 1986, 1993). See, for example, the literature reviews in Kamenica (2018), Bergemann and Morris (2019) and Forges (2019). Bayesian persuasion models with constraints on information disclosure policies appear in Boleslavsky and Kim (2018) where the designer chooses from Bayes plausible distributions which satisfy some incentive constraints, and in Le Treust and Tomala (2019) where the designer is constrained to send noisy messages.

The strategic interaction between multiple information designers has been studied under the assumption of simultaneous and one-stage information disclosure by, among others, Gentzkow and Kamenica (2017), Albrecht (2017), Au and Kawai (2019b,a), Boleslavsky and Cotton (2015, 2018), and Koessler, Laclau, and Tomala (2019). Gentzkow and Kamenica (2017) consider the case in which each designer is always able to choose an information policy which is more informative than the other designer.[3] Albrecht (2017), Au and Kawai (2019b,a), and Boleslavsky and Cotton (2015, 2018) consider the case in which designers control independent pieces of information in applied examples. Koessler, Laclau, and Tomala (2019) provide existence results and properties of equilibria in games with multiple designers and multiple agents. They assume that designers disclose information simultaneously, and then agents take decisions simultaneously as well.

Multi-stage information design with a single designer has been studied in dynamic decision problems by, among others, Doval and Ely (2016), Ely (2017), Renault, Solan, and Vieille (2017) and Makris and Renou (2018). Since we assume that the state of nature is persistent and the decision problem is static (the agent takes a decision only once), multi-stage information design would be irrelevant in our model if there was only one designer. The dynamics of information design is interesting in our setting

[3]In our model, if designers can both reveal all the information about the payoff-relevant state (i.e., if their private states are perfectly correlated and all information disclosure policies are available), then the value of every multi-stage information design game coincides with the expected payoff under full information for the agent. See Section 5.3.

precisely because there are multiple designers.

Our methodology and results are closely related to the contributions in the literature on repeated games with incomplete information on *both* sides, splitting games and acyclic gambling games. The Mertens-Zamir function has been introduced by Mertens and Zamir (1971) (see also Mertens and Zamir, 1977, Sorin, 2002 and Mertens, Sorin, and Zamir, 2015) for the value function $u(p, q)$ of the one-shot incomplete information zero-sum game, which we replace by the indirect utility function of Designer 1. Mertens and Zamir (1971) have shown that the Mertens-Zamir function is the limit of the value of the infinitely repeated and discounted game when the discount factor tends to one.[4] It is also the value of zero-sum splitting games studied in Laraki (2001a,b) and Oliu-Barton (2017). Laraki and Renault (2019) have recently extended these results from splitting games to more general stochastic games, called acyclic gambling games. By considering the indirect utility function of the designers, given the beliefs and sequentially rational actions of the agent, our results are obtained by using the methodology of Laraki and Renault (2019). The main difference is that we consider terminal payoffs (when the agent takes his decision), while payoffs in splitting and acyclic gambling games are cumulated and discounted. We also consider and compare various possible timings of the game.

The timing of our multi-stage information design game is inspired by the timing of long cheap talk games (Forges, 1990, Aumann and Hart, 2003). In one of our examples, the equilibrium martingale of posteriors does not reach its limit within a bounded number of disclosure stages, similarly to the "four frogs" example in Aumann and Hart (1986) and Forges (1984, 1990).

The one-stage version of our application to competitive product demonstration between two sellers has been studied, among others, by Albrecht (2017), Au and Kawai (2019b) and Boleslavsky and Cotton (2015, 2018). Whitmeyer (2019) considers a multi-stage extension of this game with a finite horizon and discounted payoffs, and directly solve the game by backward induction. Like us, he shows that less information is revealed by the designers in the multi-stage game than in the static game.

Organization of the paper In Section 2 we present the model. The main results are in Section 3. The application to competition in product demonstration is developed in Section 4. Some generalizations and extensions are studied in Section 5: we provide sufficient conditions under which our results apply with discontinuous indirect utility functions (Section 5.1), we introduce stopping rules (Section 5.2), and we allow for correlated private states (Section 5.3). The technical proofs are relegated to the Appendix.

2 Model

Environment There are two *information designers* and a single *agent*. There is a finite set of *states* $K \times L$, which is endowed with a common prior probability distribution. At the start of the game, no player is informed about the state. Designer 1 is able to design public information about K, and Designer 2 is able to design public information about L. We assume for simplicity, that the prior probability distribution is the product of its marginal distributions: $p^0 \otimes q^0$, with $p^0 \in \Delta(K)$ and

[4] It is also the limit of the value of the undiscounted N-stage repeated game when $N \to \infty$. When the cav vex differs for the vex cav, the undiscounted infinitely repeated game has no value (see Aumann et al., 1995).

$q^0 \in \Delta(L)$, where $\Delta(X)$ denotes the set of Borel probability measures over a compact set X. The extension to correlated priors is studied in Section 5.3.

Designers produce and disclose publicly independent pieces of information, thus Designer 1 controls the public belief $p \in \Delta(K)$ in a Bayesian plausible way and similarly Designer 2 controls the public belief $q \in \Delta(L)$.

Information disclosure and admissible splittings An *information disclosure policy* for Designer 1 is a mapping $x : K \to \Delta(M)$, where M is a set of publicly observed messages. Such policies are also called (statistical or Blackwell) experiments. When the public belief over K is $p \in \Delta(K)$ and Designer 1 chooses the experiment x, this induces a probability distribution over posterior beliefs whose average is $p \in \Delta(K)$. Such a mean preserving spread is called a *splitting* of p. The set of splittings of $p \in \Delta(K)$ for Designer 1 is denoted by

$$\mathbb{S}(p) = \left\{ s \in \Delta(\Delta(K)) : \int_{\tilde{p} \in \Delta(K)} \tilde{p}\, ds(\tilde{p}) = p \right\}.$$

Similarly, the set of all splittings of $q \in \Delta(L)$ for Designer 2 is

$$\mathbb{T}(q) = \left\{ t \in \Delta(\Delta(L)) : \int_{\tilde{q} \in \Delta(L)} \tilde{q}\, dt(\tilde{q}) = q \right\}.$$

An important feature of this paper is that we consider technological constraints faced by information designers when choosing experiments. We model it by setting constraints on splittings of beliefs that designers are able to induce. Let $P \subseteq \Delta(K)$ and $Q \subseteq \Delta(L)$ be two compact sets, with $p^0 \in P$ and $q^0 \in Q$. We call $P \times Q$ the set of *admissible posteriors*. For every $(p, q) \in P \times Q$, we let $S(p) \subseteq \Delta(P) \cap \mathbb{S}(p)$ and $T(q) \subseteq \Delta(Q) \cap \mathbb{T}(p)$ be the set of *admissible splittings*, where the correspondences $S : P \rightrightarrows \Delta(P)$ and $T : Q \rightrightarrows \Delta(Q)$ have closed graph and non-empty convex, compact values. We assume that for every $(p, q) \in P \times Q$, $\delta_p \in S(p)$ and $\delta_q \in T(q)$ where δ_x denotes the Dirac measure at x, i.e. each designer is always able to choose a non-informative experiment.

In the unrestricted case, Designer 1 can choose any statistical experiment $x : K \to \Delta(M)$. The set of admissible posteriors and splittings are then the full sets: $P = \Delta(K)$ and $S(p) = \mathbb{S}(p)$. When there are constraints on information policies available to the designers, not all posteriors and all splittings are admissible. For example, if Designer 1 can only choose *deterministic* experiments, then $P = \{p \in \Delta(K) : \exists\, E \subseteq K, \text{ s.t. } p' = p^0(\cdot|E)\}$ is finite and $S(p) = \Delta(P) \cap \mathbb{S}(p)$. More generally, a designer may be constrained to choose experiments from an exogenous subset that thereby restricts the feasible posteriors and splittings.

Throughout the paper, we make two additional assumptions on admissible splittings. The first assumption states that iterating the admissible splittings does not enlarge the splitting possibilities. This is the case, for example, when all splittings are admissible. More generally, take p in P, $s \in S(p)$ and $f : P \to \Delta(P)$ a measurable selection of S, $f(p') \in S(p')$ for each $p' \in P$. Define a splitting $f * s$ where Designer 1 first draws a posterior p' from s, then a second posterior p'' from $f(p')$. Precisely, $f * s$ is the probability distribution on $\Delta(P)$ defined by $f * s(B) = \int f(B|p')ds(p')$ for each Borel set

$B \subseteq \Delta(P)$, where $f(B|p')$ denotes the probability of the Borel set B for the distribution $f(p')$. It easy to see that $f * s \in \mathbb{S}(p)$ (i.e. $\iint p'' df(p''|p')ds(p') = p$). Let us denote

$$S^2(p) = \{f * s : s \in S(p), f \text{ measurable selection of } S\},$$

the set of splittings of p obtained by iterating S twice.

Assumption 1. *$\forall\, (p, q) \in P \times Q$, $S^2(p) = S(p)$ and $T^2(q) = T(q)$.*

This assumption is satisfied when all splittings are admissible $S(p) = \Delta(P) \cap \mathbb{S}(p)$, $T(q) = \Delta(Q) \cap \mathbb{T}(q)$. As another example, consider feasible splittings given by constraints such as

$$S(p) = \left\{ s \in \Delta(P) \cap \mathbb{S}(p) : \int g_i(p')ds(p') \leq f_i(p), i = 1, \ldots, I \right\}$$

where f_i, g_i are measurable functions defined on P. Assumption 1 holds whenever each f_i is concave. A splitting problem with constraints which falls into this category is considered in Boleslavsky and Kim (2018).

The second assumption is a strong continuity condition for the correspondences S and T. Given a metric d inducing the Euclidean topology on P, we define for $s, s' \in \Delta(P)$ the associated Kantorovich-Rubinstein[5] distance between s and s' as $d_{KR}(s, s') = \sup \left\{ \left| \int f(p)ds(p) - \int f(p)ds'(p) \right|, f : P \to \mathbb{R}, 1\text{-Lipschitz for } d \right\}$.

Assumption 2. *There exists metrics on P and Q, compatible with the Euclidean topologies on P and Q, such that the correspondences of admissible splittings $S : P \rightrightarrows \Delta(P)$ and $T : Q \rightrightarrows \Delta(Q)$ have closed graphs and are non-expansive, i.e.*

$$\forall\, p, p' \in P, \quad \forall\, s \in S(p), \exists\, s' \in S(p') \text{ s.t. } d_{KR}(s, s') \leq d(p, p'),$$
$$\forall\, q, q' \in Q, \quad \forall\, t \in T(q), \exists\, t' \in T(q') \text{ s.t. } d_{KR}(t, t') \leq d(q, q').$$

This assumption is satisfied in the unrestricted case, that is, when $P = \Delta(K)$, $Q = \Delta(L)$, $S(p) = \mathbb{S}(p)$ and $T(q) = \mathbb{T}(q)$, using $\|.\|_1$ on P and Q. More generally, it is satisfied when P and Q are product of simplices and all splittings are admissible, i.e., $S(p) = \Delta(P) \cap \mathbb{S}(p)$ and $T(q) = \Delta(Q) \cap \mathbb{T}(q)$ (see Theorem 1.17 in Laraki, 2004). It is also satisfied when states are binary ($|K| = |L| = 2$) and splittings are unrestricted, i.e., $S(p) = \Delta(P) \cap \mathbb{S}(p)$ and $T(q) = \Delta(Q) \cap \mathbb{T}(q)$ (using $d(p, p') = |p - p'|$, see Lemma 7 in the online Appendix B.1). It holds directly whenever P and Q are finite (using $d(p, p') = 1$ if $p \neq p'$). Finally, Assumption 2 can also be expressed through the concavifications of Lipschitz functions (see Lemma 8 in the online Appendix B.1). In the sequel, we fix metrics on P and Q satisfying Assumption 2.

Information design games We define now multi-stage games in which designers disclose information at stages $n = 1, 2, \ldots, N$ with $N \leq \infty$ and the agent takes a decision at the end of stage N. We

[5] This distance induces the weak* topology on $\Delta(P)$, which is also a compact metric space. It extends the distance on P: we have for p, p' in P, $d_{KR}(\delta_p, \delta'_p) = d(p, p')$.

consider both *simultaneous-move* games in which both designers disclose information in each stage, and *sequential-move* games in which Designer 1 is active at odd stages and Designer 2 at even stages.

The simultaneous-move information design game. The initial public belief is (p^0, q^0). At each stage $n = 1, 2, \ldots, N$ and each current belief (p^{n-1}, q^{n-1}), Designer 1 chooses an admissible splitting $s^n \in S(p^{n-1})$ and Designer 2 chooses an admissible splitting $t^n \in T(q^{n-1})$. The new posteriors p^n and q^n are drawn according to s^n and t^n. All players observe s^n, t^n, p^n and q^n, and the game proceeds to stage $n+1$. The agent chooses an action after stage N.

For $n \leq N$, a n-stage history is a sequence of splittings and posteriors

$$h^n = (p^0, q^0, s^1, t^1, \ldots, p^{n-1}, q^{n-1}, s^n, t^n, p^n, q^n).$$

Denote H^n the set of such histories. A strategy for Designer 1 is a sequence σ of measurable functions $(\sigma^n)_{n\leq N}$, where for every $n \leq N$, $\sigma^n : H^{n-1} \to \Delta(P)$ and for every history $h^{n-1} \in H^{n-1}$, $\sigma^n(h^{n-1}) \in S(p^{n-1})$ where p^{n-1} is the $(n-1)-$period posterior on K. Similarly, a strategy for Designer 2 is a sequence τ of measurable functions $(\tau^n)_{n\leq N}$, where for every $n \leq N$, $\tau^n : H^{n-1} \to \Delta(Q)$ and for every history $h^{n-1} \in H^{n-1}$, $\tau^n(h^{n-1}) \in T(q^{n-1})$ where q^{n-1} is $(n-1)-$period posterior on L.

The sequential-move information design game. The description of the game is the same except that Designer 1 moves only at odd stages and Designer 2, only at even stages. That is, player 1 is restricted to play a strategy σ such that for all histories and even stage n, $\sigma^n(p^0, q^0, s^1, t^1, \ldots, p^{n-1}, q^{n-1}) = \delta_{p_{n-1}}$, and player 2 is restricted to play a strategy τ such that for all histories and odd stage n, $\tau^n(p^0, q^0, s^1, t^1, \ldots, p^{n-1}, q^{n-1}) = \delta_{q_{n-1}}$.

We will distinguish *games with deadlines* ($N < \infty$) from *games without deadlines* ($N = \infty$). In Section 5.2, we also consider games with random termination in which at each stage the game between the designers continues to the next stage with probability $\delta \in (0, 1)$ or terminates with probability $(1 - \delta)$.

Agent's decision problem and players' payoffs The agent observes all splittings chosen by designers and all posteriors induced. After every play path h^N of the designers, the agent chooses an action z from a compact, convex subset Z of an euclidean space. The payoff of each player depends on the state $(k, l) \in K \times L$ and on the action $z \in Z$. The payoff of the agent is denoted by $\tilde{u}_A(z; k, l)$, the payoff of Designer 1 is denoted by $\tilde{u}(z; k, l)$. We assume that the game is strictly competitive between the designers, so the payoff of Designer 2 is $-\tilde{u}(z; k, l)$. For every state $(k, l) \in K \times L$, the utility functions $\tilde{u}(z; k, l), \tilde{u}_A(z; k, l)$ are continuous in z. When the agent has a finite set of actions A, we let $Z = \Delta(A)$ be the set of mixed actions and define payoffs linearly by taking expectation. We extend $\tilde{u}$ and $\tilde{u}_A$ as usual with: for z in Z, $p \in P$ and $q \in Q$,

$$\tilde{u}(z; p, q) = \sum_{(k,l)\in K\times L} p(k)q(l)\tilde{u}(z; k, l) \text{ and } \tilde{u}_A(z; p, q) = \sum_{(k,l)\in K\times L} p(k)q(l)\tilde{u}_A(z; k, l).$$

For every admissible posteriors $p \in P$ and $q \in Q$, we assume that the decision problem of the agent $\max_{z\in Z} \tilde{u}_A(z; p, q)$ has a unique solution $z(p, q)$, which is continuous in (p, q) by the Maximum Theorem. This assumption is satisfied, for example, when $\tilde{u}_A(z; k, l)$ is strictly concave in z. If $\max_{z\in Z} \tilde{u}_A(z; p, q)$

has multiple solutions, our results also apply as long as there is a continuous selection. Such a selection typically does not exist when the agent has a finite set of actions, Z is the set of mixed actions, and all posteriors are admissible (i.e., $P = \Delta(K)$ and $Q = \Delta(L)$). This is the case in the example presented in Section 4. Importantly, our results and methods can be extended to analyze this example (proofs have to be amended to account for the discontinuities), so the study of continuous games also gives insights for discontinuous models.

In all versions of the information design game, we assume that the agent's decision is sequentially rational, that is, the agent chooses $z(p, q)$ for all possible terminal posteriors $(p, q) \in P \times Q$. Thus, for each play path h^N of the designers, there is a uniquely defined decision of the agent. Let

$$u(p, q) = \sum_{(k,l)\in K\times L} p(k)q(l)\tilde{u}(z(p, q); k, l),$$

be Designer 1's expected payoff induced by the optimal action of the agent when public beliefs are given by $(p, q) \in P \times Q$. Notice that $u(p, q)$ is continuous in (p, q), since $\tilde{u}(z; k, l)$ is continuous in z and $z(p, q)$ is continuous in (p, q). We extend u and denote by $u(s, t)$ the expected value of u with respect to $s \in \Delta(P)$ and $t \in \Delta(Q)$,

$$u(s, t) = \iint u(p, q)ds(p)dt(q).$$

We use similar notation for the expected utility $u_A(p, q)$ of the agent.

Games between designers, equilibria and values Our main objects of interest are the two-player multi-stage games induced by the sequentially rational decision taken by the agent at the terminal stage. For $N \leq \infty$, we denote by $G_N(p^0, q^0)$ the game in which designers play simultaneously for N stages and by $G_N^{seq}(p^0, q^0)$ the game in which designers play sequentially for N stages and Designer 1 moves first.

Remark 1. The game $G_1^{seq}(p^0, q^0)$ is a one-designer game and corresponds to the model of Bayesian persuasion of Kamenica and Gentzkow (2011). The game $G_1(p^0, q^0)$ is a special class of the interactive information design games studied in Koessler et al. (2019), with a single agent and two designers with opposed preferences. The timing of the simultaneous-move game without deadline $G_\infty(p^0, q^0)$ is similar to a "long cheap talk" game, and the timing of the game $G_\infty^{seq}(p^0, q^0)$ is similar to a "long polite talk" game, as defined in Aumann and Hart (2003).

For any version of the multi-stage game, a pair of strategies (σ, τ) of the designers induces a distribution $\mathbb{P}_{\sigma,\tau}$ over play paths, which makes the random sequence of posteriors $(p^n, q^n)_{n\leq N}$ a martingale:

$$\forall\, n < N,\ \forall h^n = (p^0, q^0, ..., p^n, q^n),\ \ \mathbb{E}_{\sigma,\tau}[p^{n+1} \mid h^n] = p^n,\ \ \mathbb{E}_{\sigma,\tau}[q^{n+1} \mid h^n] = h^n.$$

When $N = \infty$, the martingale convergence theorem ensures that the random variable $\big(p^\infty, q^\infty\big) = \big(\lim_{n\to\infty} p^n, \lim_{n\to\infty} q^n\big)$ exists almost surely. Therefore, the terminal beliefs (p^N, q^N) are well defined for any $N \leq \infty$.

The expected payoff of Designer 1 in each version of the information design game is given by $U(\sigma, \tau) = \mathbb{E}_{\sigma,\tau} u(p^N, q^N)$ and the expected payoff of Designer 2 is $-U(\sigma, \tau)$. A pair of strategies (σ^*, τ^*)

is an ε-*equilibrium* of the information design game if

$$U(\sigma, \tau^*) - \varepsilon \leq U(\sigma^*, \tau^*) \leq U(\sigma^*, \tau) + \varepsilon,$$

for every σ and τ. It is an equilibrium if it is an ε-equilibrium for $\varepsilon = 0$. The information design game *has a value* V if the sup inf and inf sup payoffs coincide:

$$V = \sup_\sigma \inf_\tau U(\sigma, \tau) = \inf_\tau \sup_\sigma U(\sigma, \tau).$$

As is well known, the zero-sum information design game has a value if and only if it has ε-equilibria for every $\varepsilon > 0$, and (σ^*, τ^*) is an equilibrium if and only if $V = \min_\tau U(\sigma^*, \tau) = \max_\sigma U(\sigma, \tau^*)$. In this case σ^* and τ^* are called optimal strategies for Designers 1 and 2 respectively.

A strategy σ for Designer 1 is called *Markovian* if in each stage n, σ^n depends only on the posteriors (p^{n-1}, q^{n-1}) of the previous stage. If in addition σ^n depends only on (p^{n-1}, q^{n-1}) but not on n, then the strategy is called *stationary*. The same definitions apply to strategies of Designer 2. An equilibrium (σ^*, τ^*) is a *Markovian equilibrium* if σ^* and τ^* are Markovian. An equilibrium (σ^*, τ^*) is a *stationary equilibrium* if σ^* and τ^* are stationary.

3 Main results

In this section, we characterize equilibrium strategies and values for the multi-stage information design games presented in the previous section. We also provide some examples to illustrate the characterizations and the impact of the timing of the game on equilibrium strategies and payoffs.

3.1 Definitions and Preliminary Results

3.1.1 Concavity, convexity

It is well known from the theory of repeated games (Aumann and Maschler, 1966; Aumann et al., 1995) and of Bayesian persuasion (Kamenica and Gentzkow, 2011) that the concavification (or concave closure) of a function is a key concept, as it captures the optimal splitting for a single designer. In our setting, payoffs depend on two variables p, q and are zero-sum between designers. Thus ideally, Designer 1 would like to concavify with respect to p and Designer 2 would like to convexify with respect to q. Before studying the interplay between these notions, we need to define what concave/convex mean in our setting, since P and Q are not necessarily convex sets and splittings are restricted.

In the following Definitions 1 and 2, all functions defined on P (or Q, or $P \times Q$) are assumed to be measurable and bounded, and extended to elements of $\Delta(P)$ (or $\Delta(Q)$, or $\Delta(P) \times \Delta(Q)$) by (multi)-linearity.

Definition 1. *A function $w : P \to \mathbb{R}$ is S-*concave *if for all $p \in P$ and all $s \in S(p)$, $w(p) \geq w(s)$, where $w(s) = \int w(p')ds(p')$. A function $w : Q \to \mathbb{R}$ is T-*convex *if for all $q \in Q$ and all $t \in T(q)$, $w(q) \leq w(t)$. A function $w : P \times Q \to \mathbb{R}$ is S-concave if $w(\cdot, q)$ is S-concave for all q; and is T-convex if $w(p, \cdot)$ is T convex for all p.*

When P is a convex set and $S(p) = \Delta(P) \cap \mathbb{S}(p)$ for each p, w is concave in the usual sense: $w(p)$ is greater or equal to the expectation of w under any distribution with mean p. Thus, the definition is the generalization to all *admissible* distributions with mean p. For functions of two variables $w(p, q)$, S-concave means concave with respect to the p variable given the admissible splittings S.

Definition 2. *The* concavification *of $w : P \to \mathbb{R}$ is the smallest S-concave function pointwise greater than or equal to w. We denote it by*

$$\operatorname{cav} w = \inf\{g : g \geq w,\ g \text{ is } S\text{-concave}\}.$$

The convexification *of $w : Q \to \mathbb{R}$ is the largest T-convex function pointwise smaller than or equal to w. We denote it by*

$$\operatorname{vex} w = \sup\{g : g \leq w,\ g \text{ is } T\text{-convex}\}.$$

For a function $w : P \times Q \to \mathbb{R}$, $\operatorname{cav} w$ *will denote the concavification with respect to p of $w(\cdot, q)$ for q fixed and* $\operatorname{vex} w$ *will denote the convexification with respect to q of $w(p, \cdot)$ for p fixed.*

Lemma 1. *For a continuous function $w : P \times Q \to \mathbb{R}$, under Assumptions 1 and 2, the functions* $\operatorname{cav} w$ *and* $\operatorname{vex} w$ *are continuous on $P \times Q$ and for every $(p, q) \in P \times Q$:*

$$\operatorname{cav} w(p, q) = \max\{w(s, q) : s \in S(p)\} \text{ and } \operatorname{vex} w(p, q) = \min\{w(p, t) : t \in T(q)\}.$$

Most proofs are relegated to the Appendix.

3.1.2 Splitting value, cav vex, vex cav

Consider the one-shot zero sum game $G_1(p, q)$ where the set of strategies are $S(p)$, $T(q)$ for Designers 1 and 2 respectively and with payoff $u(s, t)$. Since the strategy sets are convex compact and the payoff $(s, t) \mapsto u(s, t)$ is continuous and bilinear, by the minmax theorem $G_1(p, q)$ has a value, denoted by $\mathsf{SV}(u)(p, q)$ and both players have optimal strategies:

$$\mathsf{SV}(u)(p, q) = \max_{s \in S(p)} \min_{t \in T(q)} u(s, t) = \min_{t \in T(q)} \max_{s \in S(p)} u(s, t).$$

We call $\mathsf{SV}(u)$ the *splitting value* function of u.

Lemma 2. *The functions* $\mathsf{SV}(u)$*,* $\operatorname{cav} \operatorname{vex} u$ *and* $\operatorname{vex} \operatorname{cav} u$ *are continuous, S-concave and T-convex. In addition, for every $(p, q) \in P \times Q$:*

$$\operatorname{vex} u(p, q) \leq \operatorname{cav} \operatorname{vex} u(p, q) \leq \mathsf{SV}(u)(p, q) \leq \operatorname{vex} \operatorname{cav} u(p, q) \leq \operatorname{cav} u(p, q).$$

Notice that all inequalities are equalities for Dirac measures $(p, q) = (\delta_k, \delta_l)$, since in this case $S(p) = \{\delta_p\}$ and $T(q) = \{\delta_q\}$.

3.1.3 Mertens-Zamir functions

The following definition is due to Mertens and Zamir (1971) (see also Laraki, 2001a,b, Oliu-Barton, 2017, Laraki and Renault, 2019 and the textbooks Sorin, 2002, Mertens et al., 2015.)

Definition 3. *Let $u : P \times Q \to \mathbb{R}$ be a continuous function. A* Mertens-Zamir (MZ) *function for u is a function $v : P \times Q \to \mathbb{R}$ such that for every $(p, q) \in P \times Q$:*

$$v(p, q) = \operatorname{cav} \min(u, v)(p, q) = \operatorname{vex} \max(u, v)(p, q).$$

Lemma 3. *If v is a MZ function for u, then v is S-concave and T-convex. In addition, for every $(p, q) \in P \times Q$:*

$$\operatorname{cav} \operatorname{vex} u(p, q) \leq v(p, q) \leq \operatorname{vex} \operatorname{cav} u(p, q).$$

As in Lemma 2, all inequalities are equalities for Dirac measures $(p, q) = (\delta_k, \delta_l)$.

Under our maintained assumptions on admissible splittings, there exists a unique continuous MZ function.

Proposition 1. *There exists a unique continuous MZ function for u, denoted $\mathsf{MZ}(u)$. Moreover, $\mathsf{MZ}(u)$ is the unique continuous function v which is S-concave, T-convex and such that for all $(p, q) \in P \times Q$:*

(P1) There exists $s \in S(p)$ such that $v(p, q) = v(s, q)$ and $v(p', q) \leq u(p', q)$, $\forall\, p' \in \mathsf{supp}(s)$, the support of s.

(P2) There exists $t \in T(q)$ such that $v(p, q) = v(p, t)$ and $v(p, q') \geq u(p, q')$, $\forall\, q' \in \mathsf{supp}(t)$.

This result is very useful since it allows to derive strategies from the Mertens-Zamir function. The proof is in the Appendix.

3.2 Values and equilibria of the multi-stage information design games

The main result of the paper is the following theorem.

Theorem 1. *The information design games $G_N(p^0, q^0)$, $G_N^{seq}(p^0, q^0)$ have Markovian equilibria for all $N \leq \infty$. The games $G_\infty(p^0, q^0)$ and $G_\infty^{seq}(p^0, q^0)$ have stationary equilibria. The values of these games are characterized as follows.*

1. *The games $G_\infty(p^0, q^0)$ and $G_\infty^{seq}(p^0, q^0)$ have the same value $V_\infty = V_\infty^{seq} = \mathsf{MZ}(u)(p^0, q^0)$.*
2. *The value of the game $G_N(p^0, q^0)$ is $V_N = \mathsf{SV}(u)(p^0, q^0)$ for every $N < \infty$.*
3. *The value of the game $G_1^{seq}(p^0, q^0)$ is $V_1^{seq} = \operatorname{cav} u(p^0, q^0)$.*
4. *The value of the game $G_N^{seq}(p^0, q^0)$ is $V_N^{seq} = \operatorname{vex} \operatorname{cav} u(p^0, q^0)$ for every $N > 1$, finite and odd.*
5. *The value of the game $G_N^{seq}(p^0, q^0)$ is $V_N^{seq} = \operatorname{cav} \operatorname{vex} u(p^0, q^0)$ for every N finite and even.*

An informal sketch of the proof is as follows. Consider the first point. From properties (P1) and (P2) we know that setting $v = \mathsf{MZ}(u)$, there exists $s \in S(p)$ such that $v(p,q) = v(s,q)$ and $v(p',q) \leq u(p',q)$, for all $p' \in \mathsf{supp}(s)$, and there exists $t \in T(q)$ such that $v(p,q) = v(p,t)$ and $v(p,q') \geq u(p,q')$, for all $q' \in \mathsf{supp}(t)$. These properties define naturally stationary strategies for both designers which turn out to form an equilibrium. Assume that $v(p,q)$ is the value of the continuation game starting at (p,q). From the point of view of Designer 1, (P1) means that it is possible to choose a splitting which preserves the expected continuation value. So suppose that $v(p,q) < u(p,q)$, then Designer 1 can play non-revealing without decreasing the equilibrium continuation payoff. Intuitively when $v(p,q) < u(p,q)$, Designer 1 would be content if the game stopped, he would get more than the value. If $v(p,q) > u(p,q)$, then Designer 1 does not want the game to stop and chooses a splitting s such that $v(p',q) \leq u(p',q)$ with probability one, in order to reach a point where the realized payoff u would potentially be greater than the value. The symmetry of (P1) and (P2) imply that $v(p,q)$ can be enforced by both players. Thus, this is the equilibrium payoff of this zero-sum game.

The second point says that when designers move simultaneously and there is a deadline, then in equilibrium they disclose no information until the last stage where they play the equilibrium of the one-shot game. No designer wants to deviate since revealing information can only be detrimental.

The remaining points deal with sequential games with deadlines. First, only the last two periods matter, before that, designers simply wait. Now if Designer 1 moves last, he will concavify with respect to p, thereby getting $\operatorname{cav} u$. By backward induction, Designer 2 convexifies this function and gets $\operatorname{vex}\operatorname{cav} u$. When Designer 2 moves last, we get $\operatorname{cav}\operatorname{vex} u$.

Proof of Theorem 1.

1. Let $v = \mathsf{MZ}(u)$, define a strategy σ of Designer 1 as follows. Given the posteriors $(p,q) \in P \times Q$ at stage n, player 1 chooses the non-revealing splitting δ_p if $u(p,q) \geq v(p,q)$, and if $u(p,q) < v(p,q)$, he chooses a splitting $s \in S(p)$ such that $v(p,q) = v(s,q)$ and $v(p',q) \leq u(p',q)$, $\forall\, p' \in \mathsf{supp}(s)$. From Proposition 1, this strategy is well defined. It has the property that for any strategy of Designer 2, $u(p^{n+1}, q^n) \geq v(p^{n+1}, q^n)$ almost surely. Passing to the limit gives $u(p^\infty, q^\infty) \geq v(p^\infty, q^\infty)$ almost surely and thus $\mathbb{E}u(p^\infty, q^\infty) \geq \mathbb{E}v(p^\infty, q^\infty)$. By T-convexity of v, $\mathbb{E}[v(p^{n+1}, q^{n+1}) \mid h^n] \geq \mathbb{E}[v(p^{n+1}, q^n) \mid h^n] = v(p^n, q^n)$ by construction of σ. It follows that $\mathbb{E}\, v(p^{n+1}, q^{n+1}) \geq \mathbb{E}v(p^n, q^n)$ and by induction $\mathbb{E}v(p^\infty, q^\infty) \geq \mathbb{E}\, v(p^n, q^n) \geq v(p^0, q^0)$. Thus, there is a strategy of Designer 1 such that for any strategy of Designer 2, $\mathbb{E}u(p^\infty, q^\infty) \geq v(p^0, q^0)$. By symmetry, this is the value of the game.

2. Define a strategy σ of Designer 1 as follows. At stages $n \leq N-1$, Designer 1 plays the non-revealing splitting $\delta_{p^{n-1}} \in S(p^{n-1})$ irrespective of the history. At stage $n = N$, Designer 1 chooses a splitting $s \in S(p^{N-1})$ that maximizes $\min_{t \in T(q^{N-1})} u(s,t)$. For any strategy of Designer 2, we have $p^{N-1} = p^0$ and the expected payoff is $\mathbb{E}\ \mathsf{SV}(p^0, q^{N-1}) \geq \mathsf{SV}(p^0, q^0)$, by T-convexity of $\mathsf{SV}(u)$. Hence, the payoff is at least $\mathsf{SV}(p^0, q^0)$ for any strategy of Designer 2. Similarly, Designer 2 can guarantee that the payoff is at most $\mathsf{SV}(p^0, q^0)$.

3. This follows directly from Lemma 1. When all posteriors and all splittings are allowed, this is the single-designer problem of Kamenica and Gentzkow (2011).

4. Define a strategy σ of Designer 1 as follows. At stages $n = 1, \ldots, N-2$, Designer 1 chooses the non-revealing splitting. At stage $n = N$, he chooses a splitting $s \in S(p^{N-1})$ that maximizes

$u(s, q^{N-1})$. For any strategy of Designer 2, the expected payoff is $\mathbb{E}\ \mathrm{cav}\, u(p^0, q^{N-1}) \geq \mathrm{vex}\,\mathrm{cav}\, u(p^0, q^0)$. Hence, Designer 1 has a strategy that guarantees a payoff of at least $\mathrm{vex}\,\mathrm{cav}\, u(p^0, q^0)$, irrespective of the strategy of Designer 2.

Define a strategy τ of Designer 2 as follows. At stages $n = 2, \ldots, N-3$, Designer 2 chooses the non-revealing splitting. At stages $n = N-1$ he chooses a splitting $t \in S(q^{N-2})$ that minimizes $\mathbb{E}_t\ \mathrm{cav}\, u(p^{N-2}, q)$. Whatever the strategy of Designer 1, the expected payoff is at most $\mathbb{E}\ \mathrm{vex}\,\mathrm{cav}\, u(p^{N-2}, q^0) \leq \mathrm{vex}\,\mathrm{cav}\, u(p^0, q^0)$, by S-concavity of $\mathrm{vex}\,\mathrm{cav}(p, q)$. Hence, Designer 2 has a strategy that guarantees that Designer 1's payoff is at most $\mathrm{vex}\,\mathrm{cav}\, u(p^0, q^0)$, for any strategy of player 1. Therefore, $\mathrm{vex}\,\mathrm{cav}\, u(p^0, q^0)$ is the value of the game $G_N^{seq}(p^0, q^0)$ for every $N > 1$, odd and finite.

5. The proof is the analogue of the proof of part 4.

In all versions of the game, we have constructed a Markovian equilibrium and a stationary equilibrium in the infinite horizon games. □

If $\mathrm{cav}\,\mathrm{vex}\, u(p^0, q^0) = \mathrm{vex}\,\mathrm{cav}\, u(p^0, q^0)$, then the values are the same for all versions of the game in which both designers are active (that is, all but $G_1^{seq}(p^0, q^0)$). In particular, if $u(p, q)$ is concave in p, then $V_\infty = V_1 = V_N = V_N^{seq} = \mathrm{vex}\, u(p^0, q^0)$ for every $N \geq 2$. If $u(p, q)$ is convex in q, then $V_\infty = V_1 = V_N = V_N^{seq} = \mathrm{cav}\, u(p^0, q^0)$ for every $N \geq 1$.

In games without deadlines $G_\infty(p^0, q^0)$, $G_\infty^{seq}(p^0, q^0)$, it follows from the above proof that under the equilibrium strategies we have $u(p^\infty, q^\infty) = v(p^\infty, q^\infty)$ almost surely. Thus, if the martingale stops, it must be at points where $u(p, q) = v(p, q)$. Further, if designers can always reach points where $u(p, q) = v(p, q)$, the martingale actually stops after the first period. This is the case under the conditions below.

Lemma 4. *Assume that P and Q are convex and that all splittings are admissible, $S(p) = \Delta(P) \cap \mathbb{S}(p)$, $T(q) = \Delta(Q) \cap \mathbb{T}(q)$. Then for any $(p, q) \in P \times Q$,*

- *if $u(p, q) \leq v(p, q)$, there exists $s \in S(p)$ such that $v(p', q) = u(p', q)$, $\forall\, p' \in \mathsf{supp}(s)$,*
- *if $u(p, q) \geq v(p, q)$, there exists $t \in T(q)$ such that $v(p, q') = u(p, q')$, $\forall\, q' \in \mathsf{supp}(t)$.*

This lemma is a direct extension of a result in Heuer (1992) and in Oliu-Barton (2017). For the sake of completeness, the proof is recalled in the Appendix. It follows that in games without deadline, at equilibrium, both designers play non-revealing and the martingale is constant if $u(p, q) = v(p, q)$, Designer 1 splits to a point p' such that $u(p', q) = v(p', q)$ and the martingale is constant thereafter if $u(p, q) < v(p, q)$, and Designer 2 splits to a point q' such that $u(p, q') = v(p, q')$ and the martingale is constant thereafter if $u(p, q) > v(p, q)$. Thus, when the sets of admissible posteriors are convex and all splittings are admissible, information disclosure lasts one period at most. In Section 3.3.2, we provide an example with finite sets of posteriors in which the number of disclosure periods is unbounded.

3.3 Examples

3.3.1 Illustrative example

As a simple illustration, consider binary states $K = L = \{0, 1\}$. Identify $p \in \Delta(K)$ with $p(1) \in [0, 1]$, and $q \in \Delta(L)$ with $q(1) \in [0, 1]$, and let $p^0 = q^0 = \frac{1}{2}$. Suppose that each designer has only two available disclosure policies: non-revealing or fully revealing, or equivalently that each designer can only use deterministic experiments. The possible posteriors are thus $P = Q = \left\{0, \frac{1}{2}, 1\right\}$ and all splittings (actually there are only two) are available on those sets. For each pair of feasible posteriors, the agent takes some (optimal) action which we abstract away from. We assume that the induced payoff for Designer 1 is given by

$$u = \begin{array}{|c|c|c|c|}\hline p=1 & 0 & 1 & 0 \\ \hline p=1/2 & 1 & 0 & 1 \\ \hline p=0 & 0 & 1 & 0 \\ \hline & q=0 & q=1/2 & q=1 \\ \hline \end{array}$$

Notice that Designer 1 would like to fully reveal his state when Designer 2 is silent at $q = 1/2$, and Designer 2 would like to reveal if Designer 1 has already revealed. u is neither S-concave nor T-convex. The concavification and convexification of u are given by:

$$\operatorname{cav} u = \begin{array}{|c|c|c|}\hline 0 & 1 & 0 \\ \hline 1 & 1 & 1 \\ \hline 0 & 1 & 0 \\ \hline \end{array} \qquad \operatorname{vex} u = \begin{array}{|c|c|c|}\hline 0 & 0 & 0 \\ \hline 1 & 0 & 1 \\ \hline 0 & 0 & 0 \\ \hline \end{array}$$

If designers can disclose information simultaneously at a single stage, then Designer 1 can guarantee an expected payoff of $\frac{1}{2}$ by revealing the state with probability $\frac{1}{2}$ and remaining silent with probability 1/2. Indeed, in this case, the posterior belief of the agent on K is $p = 0$ with probability $\frac{1}{4}$, $p = 1$ with probability $\frac{1}{4}$, and $p = \frac{1}{2}$ with probability $\frac{1}{2}$. Hence, the probability to get payoff 1 is equal to $\frac{1}{2}$ for any disclosure policy of Designer 2. Similarly, Designer 2 can guarantee that Designer 1's expected payoff is not higher than $\frac{1}{2}$ by revealing the state with probability $\frac{1}{2}$. This is the splitting value of the model, i.e., the equilibrium payoff of Designer 1 of the one-stage simultaneous move game.

Suppose now that designers play sequentially before a fixed deadline and that Designer 1 moves last before the agent takes the decision. In that case, Designer 1 can guarantee a payoff of 1 by simply waiting for the last stage playing the opposite of what Designer 2 did (disclose if Designer 2 has not disclosed before, do not disclose if Designer 2 has disclosed). This is the *vex cav* value. Similarly, if Designer 2 moves last, he can guarantee that Designer 1's payoff is 0 by waiting for the last stage and playing the same as Designer 1 (disclose if Designer 1 disclosed before, do not disclose otherwise). This is the *cav vex* value.

What is the equilibrium if there is no deadline? The game without deadline is not symmetric. Designer 2 can still apply the strategy above: when Designer 1 discloses, disclose right after, do not disclose otherwise. Clearly, the resulting payoff is 0, no matter what Designer 1 does. This is the Mertens-Zamir value of this example.

Summing up, we have:

$$\operatorname{vex}\operatorname{cav} u = \begin{array}{|c|c|c|}\hline 0 & 0 & 0 \\ \hline 1 & 1 & 1 \\ \hline 0 & 0 & 0 \\ \hline \end{array} \qquad \operatorname{cav}\operatorname{vex} u = \begin{array}{|c|c|c|}\hline 0 & 0 & 0 \\ \hline 1 & 0 & 1 \\ \hline 0 & 0 & 0 \\ \hline \end{array}$$

and

$$\mathsf{SV}(u) = \begin{array}{|c|c|c|}\hline 0 & 0 & 0 \\ \hline 1 & \frac{1}{2} & 1 \\ \hline 0 & 0 & 0 \\ \hline \end{array} \qquad \mathsf{MZ}(u) = \begin{array}{|c|c|c|}\hline 0 & 0 & 0 \\ \hline 1 & 0 & 1 \\ \hline 0 & 0 & 0 \\ \hline \end{array}$$

In the online Appendix B.4, we present an extension of this example where $P = \Delta(K)$, $Q = \Delta(L)$, all splittings are admissible, and $\mathsf{SV}(u) \neq MZ(u)$.

3.3.2 Unbounded disclosure periods

Let $|K| = |L| = 2$, $P = Q = \{0, 1/3, 2/3, 1\}$ and assume that all splittings are admissible on those sets. Consider the following utility function for Designer 1:

$$u = \begin{array}{|c|c|c|c|}\hline 0 & 0 & 1 & 1 \\ \hline 0 & 0 & 1 & 1 \\ \hline 1 & 1 & 0 & 0 \\ \hline 1 & 1 & 0 & 0 \\ \hline \end{array}$$

We have:

$$\operatorname{vex} u = \begin{array}{|c|c|c|c|}\hline 0 & 0 & \frac{1}{2} & 1 \\ \hline 0 & 0 & \frac{1}{2} & 1 \\ \hline 1 & \frac{1}{2} & 0 & 0 \\ \hline 1 & \frac{1}{2} & 0 & 0 \\ \hline \end{array} \qquad \operatorname{cav} u = \begin{array}{|c|c|c|c|}\hline 0 & 0 & 1 & 1 \\ \hline \frac{1}{2} & \frac{1}{2} & 1 & 1 \\ \hline 1 & 1 & \frac{1}{2} & \frac{1}{2} \\ \hline 1 & 1 & 0 & 0 \\ \hline \end{array}$$

$$\operatorname{cav} \operatorname{vex} u = \begin{array}{|c|c|c|c|}\hline 0 & 0 & \frac{1}{2} & 1 \\ \hline \frac{1}{2} & \frac{1}{4} & \frac{1}{2} & 1 \\ \hline 1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{2} \\ \hline 1 & \frac{1}{2} & 0 & 0 \\ \hline \end{array} \qquad \operatorname{vex} \operatorname{cav} u = \begin{array}{|c|c|c|c|}\hline 0 & 0 & \frac{1}{2} & 1 \\ \hline \frac{1}{2} & \frac{1}{2} & \frac{3}{4} & 1 \\ \hline 1 & \frac{3}{4} & \frac{1}{2} & \frac{1}{2} \\ \hline 1 & \frac{1}{2} & 0 & 0 \\ \hline \end{array}$$

Using $\operatorname{cav} \operatorname{vex} u \leq \mathsf{MZ}(u) \leq \operatorname{vex} \operatorname{cav} u$ and the symmetries of the example, the MZ value function can be written as

$$\mathsf{MZ}(u) = \begin{array}{|c|c|c|c|}\hline 0 & 0 & \frac{1}{2} & 1 \\ \hline \frac{1}{2} & x & y & 1 \\ \hline 1 & y & x & \frac{1}{2} \\ \hline 1 & \frac{1}{2} & 0 & 0 \\ \hline \end{array}$$

where $\frac{1}{4} \leq x \leq \frac{1}{2}$ and $\frac{1}{2} \leq y \leq \frac{3}{4}$. It is immediate to check that the solution of the Mertens-Zamir system $\mathsf{MZ}(u)(p,q) = \operatorname{cav} \min(u, \mathsf{MZ}(u))(p,q) = \operatorname{vex} \max(u, \mathsf{MZ}(u))(p,q)$ gives $x = \frac{1}{3}$ and $y = \frac{2}{3}$.[6] Notice that, as is the "four frogs" example of Forges (1984, 1990), the number of equilibrium disclosure periods in the infinite horizon information design game is unbounded, but disclosure stops with probability one in

[6] An algorithm to compute the solution more generally is provided in Section B.2.

finite time. It is also easy to check that the one-shot splitting value is

$$\mathsf{SV}(u) = \begin{array}{|c|c|c|c|} \hline 0 & 0 & \frac{1}{2} & 1 \\ \hline \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 1 \\ \hline 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \hline 1 & \frac{1}{2} & 0 & 0 \\ \hline \end{array}$$

For interior posteriors, both players split to the two closest posteriors with probability $\frac{1}{2}$. In this example, for every interior posteriors (p,q) we have $\operatorname{cav}\operatorname{vex} u(p,q) < \mathsf{MZ}(u)(p,q) \neq \mathsf{SV}(u)(p,q) < \operatorname{vex}\operatorname{cav} u(p,q)$.

4 Competition in product demonstration

In this section, we study in details the case in which the two designers are sellers of products of variable quality and the agent's decision is from which seller to buy the product. The agent prefers to buy from the seller with the highest expected quality. The static simultaneous-move game $G_1(p^0, q^0)$ is studied in Boleslavsky and Cotton (2015), who characterize the equilibrium and value of the game. The model is as follows.

The quality of the product of each seller can be either high (H) or low (L), denote $K = L = \{L, H\}$. Let $p^0 \in (0,1)$ be the prior probability that the state is H for Designer 1 and $q^0 \in (0,1)$ be the prior probability that the state is H for Designer 2. Denote by $p \in [0,1]$ and $q \in [0,1]$ the corresponding posteriors.

The agent must choose either Designer 1 and Designer 2. His payoff is 1 if he chooses a designer whose state is H and 0 otherwise. Hence, he chooses Designer 1 if $p > q$, Designer 2 if $p < q$, and we assume that he randomizes uniformly if $p = q$. Designer 1's payoff is equal to 1 if he is chosen, and -1 otherwise. Given the posteriors (p,q), the expected payoff of Designer 1 is thus

$$u(p,q) = \mathbf{1}\{p > q\} - \mathbf{1}\{p < q\}.$$

Notice that $u(p,q)$ is discontinuous at $p = q$. The expected payoff of the agent is

$$u_A(p,q) = \max\{p,q\}.$$

This example can fit the following economic scenario. Designers are two competing firms (for example, Canon and Nikon) who are about to release a new product or system (for example, a new camera technology together with plans of lenses developments and compatibilities). The agent is a consumer (for example, a professional photographer) who plans to switch in the near future to one of these two new products. Ex-ante, the agent does not know his valuation for the products (it can be high (H) or low (L) for each product). Suppose he already owns a similar product from an older generation and is ready to take some time acquiring information before taking a decision. Firms can release information to consumers about their own product through product demonstration (e.g. press events, reviews, trade fairs, product testing or announcements of future compatible lenses). When choosing how to disclose information, the firm does not know the consumer's valuation for the product. Also, it does not know in advance the information feedbacks (for example, the ratings from reviews) its product will receive. Finally, firms cannot significantly adjust theirs prices but they are able to adjust their information policies to the signals generated by their competitor.[7] It follows that they have strictly opposed preferences, both want to attract the consumer.

Below, we characterize equilibrium strategies and values for the different timings of the information design games. We use those results to analyze informativeness of firm's strategies and the resulting welfare of the consumer. If we consider finite sets of admissible posteriors, the example fits the assumptions of Theorem 1. For instance, this represents situations in which the consumer reads reviews that apply some rating system (say from one to five stars, or some scores for relevant features of the product). To compare our results with the equilibria of the static game $G_1(p^0, q^0)$ studied in Boleslavsky and Cotton (2015), we do the equilibrium analysis when *all* splittings of *all* possible posteriors are admissible. Notice that when all splittings are admissible, designers' utility functions are discontinuous on the diagonal $p = q$.[8] To deal with this discontinuity, we extend some of the proofs of the previous section for this particular example.[9] In simultaneous-move games (with and without deadline), we show that equilibria exist. The sequential games with deadline admit ε-equilibria for all $\varepsilon > 0$ but not exact equilibria.

Before proceeding to the equilibrium analysis, let's first consider the fully revealing (FR) and non-revealing (NR) benchmarks. If information is fully revealed to the agent, then his payoff is 1 if the state is H for designer 1 or 2, and 0 if the state is L for both designers. The ex-ante expected payoff of the agent is therefore equal to

$$U_A^{FR} = 1 - (1 - p^0)(1 - q^0) = p^0 + q^0(1 - p^0).$$

This is an upper bound on the ex-ante expected payoff of the agent. A lower bound is obtained with no revelation of information. When no information is revealed, the agent's ex-ante expected payoff is

$$U_A^{NR} = u_A(p^0, q^0) = \max\{p^0, q^0\} < U_A^{FR}.$$

Simultaneous Product Demonstration with a Deadline ($G_N(p^0, q^0)$)

The equilibrium strategies and payoffs for the one-period simultaneous-move game have been characterized by Boleslavsky and Cotton (2015).[10] Assume without loss of generality that $p^0 \geq q^0$. The equilibrium strategies $s \in \mathbb{S}(p^0)$ and $t \in \mathbb{T}(q^0)$ are unique and given by:

If $p^0 \leq \frac{1}{2}$:

$$s = U[0, 2p^0] \quad \text{and} \quad t = \begin{cases} U[0, 2p^0] \text{ with prob. } \frac{q^0}{p^0}, \\ 0 \text{ with prob. } 1 - \frac{q^0}{p^0}. \end{cases}$$

[7]See, for example, Boleslavsky, Cotton, and Gurnani (2016) for more details on product demonstration and on the flexibility of prices in a similar scenario with a single information designer.

[8]The designers' utility functions are discontinuous at $p = q$, for any tie-breaking rule adopted by the agent.

[9]An alternative approach would be to assume a continuous utility by letting the agent tremble and choose Designer 1 with a probability which is a continuous function of $p - q$, increasing from 0 to 1 (e.g. a logit rule).

[10]The solution is similar to solutions of lotto and electoral competition games (see e.g. Bell and Cover, 1980, Sahuguet and Persico, 2006 and Hart, 2008).

If $p^0 > \frac{1}{2}$:

$$s = \begin{cases} U[0, 2(1-p^0)] \text{ with prob. } \frac{1}{p^0} - 1 \\ 1 \text{ with prob. } 2 - \frac{1}{p^0} \end{cases} \quad \text{and} \quad t = \begin{cases} 0 \text{ with prob. } 1 - \frac{q^0}{p^0}, \\ U[0, 2(1-p^0)] \text{ with prob. } \frac{q^0}{p^0}(\frac{1}{p^0} - 1), \\ 1 \text{ with prob. } \frac{q^0}{p^0}(2 - \frac{1}{p^0}). \end{cases}$$

Hence, the one-shot splitting value exists and is given by:

$$\mathsf{SV}(u)(p^0, q^0) = 1 - \frac{q^0}{p^0}.$$

The proof of Theorem 1, point 2, extends directly to this game: For $N < \infty$, the value of the game $G_N(p^0, q^0)$ is the splitting value $\mathsf{SV}(u)(p^0, q^0)$; the equilibrium strategies of the designers are non-revealing at the first $N-1$ stages, and they use the splittings above at the last stage.

When $p^0 \le \frac{1}{2}$, the ex-ante expected utility of the agent is

$$U_A^{SV} = \Big(1 - \frac{q^0}{p^0}\Big)p^0 + \frac{q^0}{p^0}\Big(\frac{4}{3}p^0\Big) = p^0 + \frac{1}{3}q^0.$$

When $p^0 > \frac{1}{2}$, the ex-ante expected utility of the agent is

$$U_A^{SV} = \Big(\frac{1}{p^0} - 1\Big)\Big[\Big(1 - \frac{q^0}{p^0}\Big)(1-p^0) + \frac{q^0}{p^0}\Big(\frac{1}{p^0} - 1\Big)\frac{4}{3}(1-p^0) + \frac{q^0}{p^0}\Big(2 - \frac{1}{p^0}\Big)\Big] + \Big(2 - \frac{1}{p^0}\Big).$$

Product Demonstration with no Deadline ($G_\infty(p^0, q^0)$ and $G_\infty^{seq}(p^0, q^0)$)

For every p and q, let

$$v(p, q) = \frac{p - q}{\max(p, q)}.$$

It is easy to check that for every p and q we have $v(p, q) = \operatorname{cav}\min(u, v)(p, q) = \operatorname{vex}\max(u, v)(p, q)$ (see Figure 1). Hence, despite the discontinuity of u, there exists a solution to the Mertens-Zamir system. We have:

Lemma 5. *The value of the games $G_\infty(p^0, q^0)$ and $G_\infty^{seq}(p^0, q^0)$ is $v(p^0, q^0) = \frac{p^0 - q^0}{\max(p^0, q^0)}$.*

The equilibrium strategies are the following. The (stationary) equilibrium strategy σ of Designer 1 is such that given posteriors (p, q) at stage n, he plays the non revealing splitting δ_q if $p \ge q$, and the splitting $s = \frac{p}{q}\delta_q + (1 - \frac{p}{q})\delta_0$ if $p < q$. The equilibrium strategy τ of Designer 2 is such that given posteriors (p, q) at stage n, he plays the non revealing splitting δ_q if $p \le q$, and the splitting $s = \frac{q}{p}\delta_p + (1 - \frac{q}{p})\delta_0$ if $p > q$. Observe that for fixed q, we have $v(p, q) = \frac{p-q}{\max(p,q)} > u(p, q)$ when $p < q$ and $v(\cdot, q)$ is linear on $[0, q)$. When $p \ge q$, $v(p, q) \le u(p, q)$. Therefore, σ is as in the proof of Theorem 1 and satisfies $v(s, q) = v(p, q) \le u(s, q)$.

Notice that with an initial prior $p^0 \ge q^0$, along the equilibrium path, Designer 1 uses a non-revealing splitting and Designer 2 splits the prior q^0 to $q = p^0$ with probability $\frac{q^0}{p^0}$ and to $q = 0$ with probability $1 - \frac{q^0}{p^0}$. Even though the values of the games $G_1(p^0, q^0)$ and $G_\infty(p^0, q^0)$ are the same for the designers,

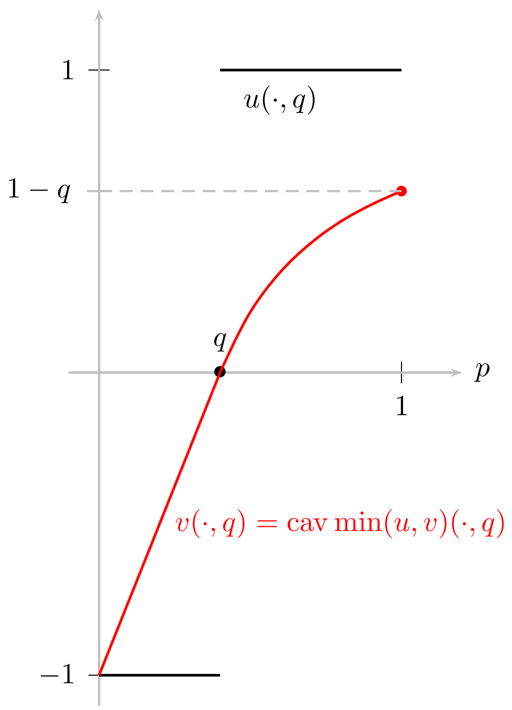

Figure 1: The MZ function $v(\cdot, q)$ for a given q.

the induced equilibrium outcomes are very different. In particular, players' strategies are always more informative in the simultaneous-move game with deadline than in the games without deadline. The equilibrium payoffs of the agent in $G_\infty(p^0, q^0)$ and $G_\infty^{seq}(p^0, q^0)$ are actually equal to p^0, as in the non-revealing case:

$$\text{For } p^0 \geq q^0, \;\; U_A^{MZ} = U_A^{NR} = p^0.$$

The proof that this is an equilibrium is in Section 5.1 where we give a more general result, Proposition 2, dealing with discontinuous utilities.

Sequential Product Demonstration with a Deadline ($G_N^{seq}(p^0, q^0)$, $N < \infty$)

Now, we consider the sequential-move games with deadlines and calculate cav vex u and vex cav u. The concavification of $u(p, q)$ with respect to p and its convexification with respect to q are given by

$$\text{cav}\, u(p,q) = \begin{cases} 1 & \text{if } p > q \\ -1 + \frac{2p}{q} & \text{if } p \leq q \text{ and } 0 < q < 1 \\ 0 & \text{if } p = q = 0 \\ -1 + p & \text{if } q = 1 \end{cases} \qquad \text{vex}\, u(p,q) = \begin{cases} -1 & \text{if } q > p \\ 1 - \frac{2q}{p} & \text{if } q \leq p \text{ and } 0 < p < 1 \\ 0 & \text{if } p = q = 0 \\ 1 - q & \text{if } p = 1. \end{cases}$$

Notice that cav u is discontinuous at $p = q = 0$ and $q = 1$, and vex u is discontinuous at $p = q = 0$ and $p = 1$. When only Designer 1 can disclose information at (p^0, q^0), the equilibrium is non-revealing.

When only Designer 2 can disclose information, he splits q^0 to p^0 (more precisely, to $p^0+\varepsilon$, with $\varepsilon \to 0$) with probability $\frac{q^0}{p^0}$ and to 0 with probability $1-\frac{q^0}{p^0}$. In both cases, the agent's expected payoff is the non-revealing payoff:

$$U_A^{\text{cav}} = U_A^{\text{vex}} = U_A^{NR} = p^0.$$

To compute cav vex u, fix q and consider the optimal splitting of p for Designer 1 when his utility is given by vex $u(p,q)$. There are three cases (see Figure 2).

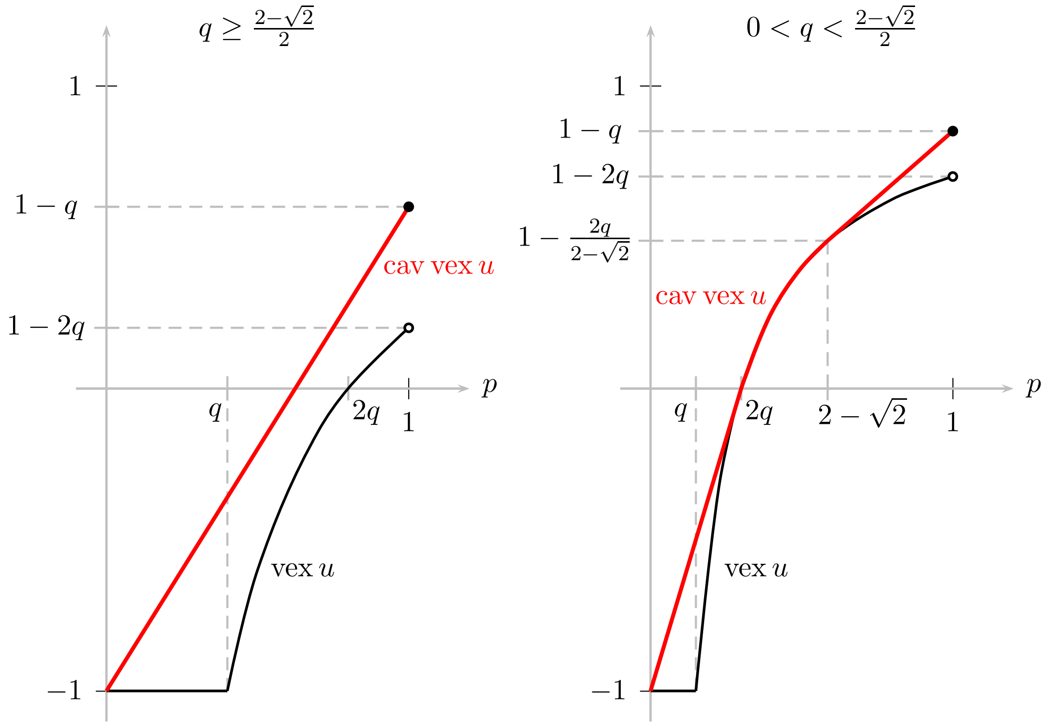

Figure 2: Concavification of vex $u(p,q)$ when $q \geq \frac{2-\sqrt{2}}{2}$ (left figure) and $0 < q < \frac{2-\sqrt{2}}{2}$ (right figure).

(i) If $q \geq \frac{2-\sqrt{2}}{2}$, then the optimal splitting of Designer 1 is full revelation: he splits p to the posterior 1 with probability p, and 0 with probability $1-p$.

(ii) If $0 < q < \frac{2-\sqrt{2}}{2}$, then the optimal splitting of Designer 1 depends on p. If $p < 2q$ he splits p to the posterior $2q$ with probability $\frac{p}{2q}$, and 0 with probability $1-\frac{p}{2q}$. If $2q \leq p \leq 2-\sqrt{2}$ he plays a non-revealing splitting. If $p > 2-\sqrt{2}$ he splits p to the posterior $2-\sqrt{2}$ with probability $\frac{1-p}{\sqrt{2}-1}$, and 1 with the complementary probability.

(iii) If $q=0$, then the optimal splitting of Designer 1 is non-revealing.

Hence:

$$\text{cav vex}\, u(p,q) = \begin{cases} 0 & \text{if } p=q=0 \\ -1+p(2-q) & \text{if } 2-\sqrt{2} \leq 2q \\ 1-\frac{2q}{p} & \text{if } 2q \leq p \leq 2-\sqrt{2} \text{ and } p>0 \\ -1+\frac{p}{2q} & \text{if } p \leq 2q < 2-\sqrt{2} \\ 1-\frac{q(4-p-2\sqrt{2})}{3-2\sqrt{2}} & \text{if } 2q \leq 2-\sqrt{2} \leq p. \end{cases}$$

Notice that $\operatorname{cav}\operatorname{vex} u$ is discontinuous at $p = q = 0$. Also $\operatorname{cav}\operatorname{vex} u(p,q) < \mathsf{SV}(u)(p,q)$ for every $p, q \in (0,1)$. Similarly, we get

$$\operatorname{vex}\operatorname{cav} u(p,q) = \begin{cases} 0 & \text{if } p = q = 0 \\ 1 - q(2-p) & \text{if } 2-\sqrt{2} \leq 2p \\ -1 + \frac{2p}{q} & \text{if } 2p \leq q \leq 2-\sqrt{2} \text{ and } q > 0 \\ 1 - \frac{q}{2p} & \text{if } q \leq 2p < 2-\sqrt{2} \\ -1 + \frac{p(4-q-2\sqrt{2})}{3-2\sqrt{2}} & \text{if } 2p \leq 2-\sqrt{2} \leq q. \end{cases}$$

These computations directly allow to describe the ε-equilibrium strategies of $G_N^{seq}(p^0, q^0)$ when $\varepsilon \to 0$. We describe below designers' behavior along the equilibrium path. In all cases, designers play non revealing up to stage $N-2$.

Assume that $N \geq 2$ is even (that is, Designer 2 plays last) and $(p^0, q^0) \in (0,1)^2$.

(i) $q^0 > \frac{2-\sqrt{2}}{2}$. At stage $N-1$, Designer 1 fully discloses information. Then at stage N, Designer 2 fully discloses information if $p^{N-1} = 1$, and plays non-revealing if $p^{N-1} = 0$. The agent always choose the best designer, so his expected payoff is $U_A^{\text{cav vex}} = U_A^{FR}$.

(ii) $q^0 < \frac{2-\sqrt{2}}{2}$.

(a) $p^0 < 2q^0$. At stage $N-1$, Designer 1 splits p^0 to $2q^0$ and 0. Then at stage N, Designer 2 splits q^0 to $2q^0$ $(+\varepsilon)$ and 0 if $p^{N-1} = 2q^0$, and plays non-revealing ($q^N = q^0$) if $p^{N-1} = 0$. In the former case, the expected payoff of the agent is $2q^0$, and in the latter case it is q^0. Hence, the expected utility of the agent is $U_A^{\text{cav vex}} = \frac{p^0}{2} + q^0 \in (U_A^{NR}, U_A^{FR})$. Notice that when $p^0 < \frac{1}{2}$ we have $U_A^{\text{cav vex}} > U_A^{SV}$ iff $p^0 < \frac{4q^0}{3}$. When $p^0 > \frac{1}{2}$ we have $U_A^{\text{cav vex}} > U_A^{SV}$.

(b) $p^0 \in (2q^0, 2-\sqrt{2})$. At stage $N-1$, Designer 1 does not disclose information (hence, $p^{N-1} = p^0$). Then at stage N, Designer 2 splits q^0 to p^0 $(+\varepsilon)$ and 0. The agent's expected payoff is then equal to $U_A^{\text{cav vex}} = p^0 = U_A^{NR}$.

(c) $p^0 > 2-\sqrt{2}$. At stage $N-1$, Designer 1 splits p^0 to 1 with probability $\frac{p^0-(2-\sqrt{2})}{\sqrt{2}-1}$ and $2-\sqrt{2}$ with probability $\frac{1-p^0}{\sqrt{2}-1}$. Then at stage N, Designer 2 plays fully-revealing ($q^N = 0$ or $q^N = 1$) if $p^{N-1} = 1$, and splits q^0 to $2-\sqrt{2}$ $(+\varepsilon)$ and 0 if $p^{N-1} = 2-\sqrt{2}$. In the former case, the expected payoff of the agent is 1, and in the latter case it is $2-\sqrt{2}$. The agent's expected payoff is then equal to $U_A^{\text{cav vex}} = p^0 = U_A^{NR}$.

By symmetry, the equilibrium strategies are similar when $N \geq 3$ is odd (that is, when Designer 1 plays last). The expected equilibrium payoff of the agent in the N-stage sequential game as a function of the priors is summarized below for N even ($U_A^{\text{cav vex}}$) and $N \geq 3$ odd ($U_A^{\text{vex cav}}$):

$$U_A^{\text{cav vex}} = \begin{cases} U_A^{FR} & \text{if } q^0 > \frac{2-\sqrt{2}}{2} \\ U_A^{NR} & \text{if } q^0 < \frac{2-\sqrt{2}}{2} \text{ and } p^0 > 2q^0 \\ \frac{p^0}{2} + q^0 \in (U_A^{NR}, U_A^{SV}) & \text{if } q^0 < \frac{2-\sqrt{2}}{2} \text{ and } \frac{4}{3}q^0 < p^0 < 2q^0 \\ \frac{p^0}{2} + q^0 \in (U_A^{SV}, U_A^{FR}) & \text{if } q^0 < \frac{2-\sqrt{2}}{2} \text{ and } p^0 < \frac{4}{3}q^0 \end{cases}$$

$$U_A^{\text{vex cav}} = \begin{cases} U_A^{FR} & \text{if } p^0 > \frac{2-\sqrt{2}}{2} \\ U_A^{NR} & \text{if } p^0 < \frac{2-\sqrt{2}}{2} \text{ and } q^0 > 2p^0 \\ \frac{q^0}{2} + p^0 \in (U_A^{NR}, U_A^{SV}) & \text{if } p^0 < \frac{2-\sqrt{2}}{2} \text{ and } \frac{4}{3}p^0 < q^0 < 2p^0 \\ \frac{q^0}{2} + p^0 \in (U_A^{SV}, U_A^{FR}) & \text{if } p^0 < \frac{2-\sqrt{2}}{2} \text{ and } q^0 < \frac{4}{3}p^0 \end{cases}$$

When p^0 and q^0 are greater than $\frac{2-\sqrt{2}}{2}$, the equilibrium expected payoff of the agent in $G_N^{seq}(p^0, q^0)$ is maximum for all $N \geq 2$. The agent gets the fully-revealing payoff U_A^{FR} which is higher than his payoff U_A^{SV} in the simultaneous-move games with deadline, which is itself higher than his expected payoff in information design games with no deadline. Every sequential game also gives a higher expected payoff to the agent than the simultaneous game when p^0 and q^0 are less than $\frac{2-\sqrt{2}}{2}$ but close to each other (when $\frac{3}{4}q^0 < p^0 < \frac{4}{3}q^0$). Otherwise, when p^0 and q^0 are lower than $\frac{2-\sqrt{2}}{2}$, the agent prefers the sequential game in which Designer 1 plays last to the simultaneous game if $p^0 > \frac{3}{4}q^0$, and prefers the sequential game in which Designer 2 plays last to the simultaneous game if $p^0 > \frac{4}{3}q^0$.

We summarize those comparisons as follows:

1. *The buyer prefers that sellers have a deadline.* Indeed, designers' equilibrium strategies are more informative with a deadline than without a deadline. With no deadline, whether information disclosure is simultaneous or sequential, only one designer discloses information, and the expected payoff of the agent is the same as without information disclosure.

2. *The buyer prefers that the seller with the highest ex-ante expected valuation disclose last.* Indeed, when there is a deadline, the agent is better off in the sequential game than in the simultaneous game whenever the designer with the highest ex-ante value has a last mover advantage.

5 Extensions and Further Results

5.1 Discontinuous utilities

In the example studied in Section 4, the utility function of the designers is a discontinuous function of the posteriors. Nevertheless, we are able to characterize the value of the game and to find equilibrium strategies, using the intuition from Theorem 1. The next result gives sufficient conditions under which the existence of an MZ function implies that it is the value of the game.

Proposition 2. *Assume that there exists a function $v : P \times Q \to \mathbb{R}$ which is S-concave, T-convex and satisfies (P1) and (P2) for all $(p, q) \in P \times Q$. Suppose further that v is u.s.c. with respect to q, l.s.c. with respect to p, that the sets*

$$\{(p, q) \in P \times Q : v(p, q) \leq u(p, q)\} \text{ and } \{(p, q) \in P \times Q : v(p, q) \geq u(p, q)\},$$

are closed, that v is l.s.c. on the closure of $\{(p, q) \in P \times Q : v(p, q) < u(p, q)\}$, and u.s.c on the closure of $\{(p, q) \in P \times Q : v(p, q) > u(p, q)\}$. Then $v(p^0, q^0)$ is the value of the games $G_\infty(p^0, q^0)$ and $G_\infty^{seq}(p^0, q^0)$ and the equilibrium is as constructed in the proof of Theorem 1.

The proof is in the Appendix. There, we show that the stationary strategies given by (P1) and (P2) form an equilibrium of the game, under the above weakening of continuity. These conditions are met in the example of Section 4.

5.2 Stopping rules

In the games $G_\infty(p^0, q^0)$ and $G_\infty^{seq}(p^0, q^0)$ there is no deadline, so potentially the number of times information is disclosed could be unbounded. We study now two new variations of the game in which the game stops either exogenously or endogenously.

In the first variation of the game, we assume that the game stops whenever both designers choose a non-revealing splitting at some stage. For instance, in a debate or a committee discussion, people are asked if they have additional evidence at each period, and if not the discussion stops. We have the following proposition.

Proposition 3. *The simultaneous information design game without deadline in which the game stops whenever both designers choose a non-revealing splitting at some stage, has a stationary equilibrium. The value of this game is* $\mathsf{MZ}(u)(p^0, q^0)$.

The same holds for the sequential game without deadline which stops whenever both designers play non-revealing consecutively. The proof is identical to that of the first point of Theorem 1.

In the second variation of the game denoted by $G_\delta(p^0, q^0)$, the game terminates after each stage with exogenous probability $1 - \delta \in (0, 1)$ and the agent takes an action. With probability δ, the game continues to the next stage. It is easy to see that this game is equivalent to a discounted game in which a short-lived agent takes a decision at each period and designers maximize/minimize the discounted average payoff. This game is a discounted stochastic game, called a splitting game in Laraki (2001a) and gambling game in Laraki and Renault (2019).

Proposition 4 (Laraki and Renault, 2019)**.** *The game* $G_\delta(p^0, q^0)$ *has a value* V_δ, *and* $\lim_{\delta\to 1} V_\delta = \mathsf{MZ}(u)(p^0, q^0)$.

Thus, our games $G_\infty(p^0, q^0)$ and $G_\infty^{seq}(p^0, q^0)$ can be approximated by discounted games with high discount factor.

5.3 Correlated information

Theorem 1 also extends to correlated priors. A first approach consists in analyzing a modified game with independent priors. Precisely, given any prior $\mu^0 \in \Delta(K \times L)$ and utility functions $\tilde{u}_i(z; k, l)$, i being Designer 1, Designer 2 or the agent, we define a modified game with stochastically independent states by letting $\hat{\mu}^0(k, l) = (\sum_{\tilde{l}\in L} \mu^0(k, \tilde{l})) \times (\sum_{\tilde{k}\in K} \mu^0(\tilde{k}, l))$ and for every i:

$$\hat{\tilde{u}}_i(z; k, l) = \frac{\mu^0(k, l)}{\hat{\mu}^0(k, l)} \tilde{u}_i(z; k, l),$$

for every (k,l) in the support of μ^0.[11] Notice that this modification preserves the continuity of players' utility functions, as well as the zero-sum property between the two designers. The equilibria of this modified multi-stage information design game (without correlation) coincides with the equilibria of the original multi-stage information design game (with correlation). A second approach consists in defining concavification and convexification for correlated distributions and to extend the notions of cav, vex and MZ. The reader is referred to Ponssard and Sorin (1980), Sorin (1984), Mertens et al. (2015) and Oliu-Barton (2017).

Consider now the particular case in which designers disclose information about a common payoff-relevant state, i.e., their private states are perfectly correlated, as in Gentzkow and Kamenica, 2017. Denote $\Omega = K = L$ the common set of states and assume that all information disclosure policies are available. In all versions of the information design game, full revelation is an equilibrium. Indeed, when a designer fully discloses the state, the other one is indifferent and may fully disclose as well. Thus, the value of each version of the game is the one obtained by full revelation. It is easy to verify that this value coincides with the cav vex and vex cav values, and thus also with the SV and MZ values. Indeed, all those functions have to be both concave and convex with respect to the prior. Therefore they must be linear and equal to $\sum_\omega \mu(\omega)u(z(\omega),\omega)$, where $z(\omega)$ is the optimal action of the agent when he knows that the state is ω.

A Appendix: Proofs

Proof of Lemma 1 We first prove the formula $\operatorname{cav} w(p,q) = \max\{w(s,q) : s \in S(p)\}$. Fix q and let $h(p) := \max\{w(s,q) : s \in S(p)\}$. Consider a function $g : P \to \mathbb{R}$, S-concave such that $g(\cdot) \geq w(\cdot, q)$. This pointwise inequality implies that for any $s \in S(p)$, $g(s) \geq w(s,q)$ and since g is S-concave,

$$g(p) \geq g(s) \geq w(s,q).$$

Since this holds for any $s \in S(p)$, this implies $g(p) \geq h(p)$, and since this holds for any $g(\cdot) \geq w(\cdot, q)$, we get $\operatorname{cav} w(p,q) \geq h(p)$.

To get the converse inequality, since $h(p) \geq w(p,q)$, it is enough to prove that h is S-concave. So take $s \in S(p)$ and consider $h(s) = \int h(p')ds(p')$. Define a measurable selection of S, $f : P \to \Delta(P)$ such that $f(p') \in \arg\max\{u(s',p') : s' \in S(p')\}$. We have for each $p' \in P$, $h(p') = w(f(p'),q) = \int w(p'',q)df(p''|p')$. We get then,

$$h(s) = \iint w(p'',q)df(p''|p')ds(p') = \int w(\tilde{p},q)d(f * s)(\tilde{p}) = w(f * s, q) \leq h(p)$$

since $f * s \in S(p)$ from Assumption 1.

We thus have $\operatorname{cav} w(p,q) = \max\{w(s,q) : s \in S(p)\}$ for all p and q. Notice that by Assumption 2, the correspondence $(p,q) \mapsto S(p)$ is both upper- and lower-hemi continuous. Applying the Maximum Theorem twice, this implies that $\operatorname{cav} w$ is continuous on $P \times Q$. □

[11] Such transformations can be found in Aumann et al. (1995, Chap. 2, section 4.2, pages 100–104) and in Myerson (1985).

Proof of Lemma 2

1. *Continuity.* From Lemma 1, $\operatorname{vex} u, \operatorname{cav} u, \operatorname{cav}\operatorname{vex} u, \operatorname{vex}\operatorname{cav} u$ are continuous. By assumption 2, the correspondence $(p,q) \mapsto S(p) \times T(q)$ is both upper- and lower-hemi continuous, hence by the Maximum Theorem $\mathsf{SV}(u)$ is continuous.

2. *The function* $\operatorname{cav}\operatorname{vex} u$ *is* T*-convex.* Denote $F(p,q) = \operatorname{vex} u(p,q)$ and take $s \in S(p)$ such that

$$\operatorname{cav}\operatorname{vex} u(p,q) = F(s,q) = \int F(p',q)ds(p').$$

Since F is T-convex, $F(p',q) \leq F(p',t)$ for each $t \in T(q)$. Thus,

$$\operatorname{cav}\operatorname{vex} u(p,q) = \int F(p',q)ds(p') \leq \int F(p',t)ds(p') \leq \int \operatorname{cav}\operatorname{vex} u(p',t)ds(p') \leq \operatorname{cav}\operatorname{vex} u(p,t),$$

where the last inequality is from $\operatorname{cav}\operatorname{vex} u$ being S-concave. By symmetry, $\operatorname{vex}\operatorname{cav} u$ is S-concave.

3. *The function* $\mathsf{SV}(u)$ *is* S*-concave.* Fix (p,q) in $P \times Q$ and $s \in S(p)$, we have to show that $\mathsf{SV}(u)(p,q) \geq \mathsf{SV}(u)(s,q)$. Let $f : P \to \Delta(P)$ be a measurable selection of S such that for each p' in P, $f(p') \in S(p')$ is an optimal strategy of Designer 1 in the game $G_1(p',q)$. We have for each p' in P, $\forall\, t \in T(q),\ u(f(p'),t) \geq \mathsf{SV}(u)(p',q)$. By taking expectation with respect to s, this gives

$$\forall\, t \in T(q),\ \int u(f(p'),t)ds(p') \geq \int \mathsf{SV}(u)(p',q)ds(p') = \mathsf{SV}(u)(s,q).$$

We get $\forall\, t \in T(q),\ u(f * s, t) \geq \mathsf{SV}(u)(s,q)$. Since $f * s \in S(p)$, we obtain that the value $\mathsf{SV}(u)(p,q)$ of $G_1(p,q)$ is at least $\mathsf{SV}(u)(s,q)$. Hence $\mathsf{SV}(u)$ is S-concave.

4. *For each* $(p,q) \in P \times Q$, $\operatorname{cav}\operatorname{vex} u(p,q) \leq \mathsf{SV}(u)(p,q)$. It suffices to show that there exists $s \in S(p)$ such that for all $t \in T(q)$, $u(s,t) \geq \operatorname{cav}\operatorname{vex} u(p,q)$. Choose s such that $\operatorname{cav}\operatorname{vex} u(p,q) = \operatorname{vex} u(s,q)$. Then,

$$\forall\, t \in T(q),\ u(s,t) \geq \operatorname{vex} u(s,q) = \operatorname{cav}\operatorname{vex} u(p,q).$$

The other inequalities are either trivial or deduced by symmetry. □

Proof of Lemma 3 If $v = \operatorname{cav}\min(u,v)$ then it is S-concave, and if $v = \operatorname{vex}\max(u,v)$, then it is T-convex. To see the inequality $\operatorname{cav}\operatorname{vex} u(p,q) \leq \mathsf{MZ}(u)(p,q)$, observe that $\max(u, \mathsf{MZ}(u)) \geq u$, thus

$$\mathsf{MZ}(u) = \operatorname{vex}\max(u, \mathsf{MZ}(u)) \geq \operatorname{vex} u.$$

Since $\mathsf{MZ}(u)$ is S-concave, this implies $\mathsf{MZ}(u) \geq \operatorname{cav}\operatorname{vex} u$. The other inequality is obtained by symmetry. □

Proof of Proposition 1 Existence and uniqueness of MZ functions is due to Mertens and Zamir (1971, 1977) for the unrestricted case $P = \Delta(K), Q = \Delta(L), S(p) = \mathbb{S}(p), T(q) = \mathbb{T}(q)$ and to Laraki and Renault (2019), Theorem 3.3., Propositions 7.7 and 7.9, under Assumptions 1 and 2. To prove Proposition 1, we first show the following lemma, which gives a useful property of concavification and optimal splittings: at an optimal splitting s such that $\operatorname{cav} w(p) = w(s)$, it must be that $\operatorname{cav} w(p') = w(p')$

on the support of s. This implies that

$$\operatorname{cav} w(s) = \int \operatorname{cav} w(p')ds(p') = \int w(p')ds(p') = \operatorname{cav} w(p),$$

so $\operatorname{cav} w$ is "linear" on the support of s.

Lemma 6. *Let $w : P \to \mathbb{R}$ be a continuous function. For each $p \in P$ and each $s \in S(p)$ such that* $\operatorname{cav} w(p) = w(s) = \max\{w(s') : s' \in S(p)\}$, *we have*

$$s(\{p' \in P : w(p') = \operatorname{cav} w(p')\}) = 1.$$

Proof. Suppose to the contrary that $\operatorname{cav} w(p) = w(s)$ and $s(\{p' \in P : \operatorname{cav} w(p') > w(p')\}) > 0$. Then there exists $\varepsilon, \alpha > 0$ such that $s(\{p' \in P : \operatorname{cav} w(p') \geq w(p') + \varepsilon\}) = \alpha > 0$. Let us denote $B = \{p' \in P : \operatorname{cav} w(p') \geq w(p') + \varepsilon\}$ and define a measurable selection f of S such that $\operatorname{cav} w(p') = w(f(p'))$ for all $p' \in B$ and $f(p') = \delta_{p'}$ for all $p' \notin B$, and consider the splitting $f * s$. We have

$$w(f * s) = \int_B \operatorname{cav} w(p')ds(p') + \int_{P\setminus B} w(p')ds(p') \geq \varepsilon\alpha + \int w(p')ds(p').$$

Since $f * s \in S(p)$, this contradicts $w(s) = \max\{w(s') : s' \in S(p)\}$. ■

To complete the proof of the proposition, we first show that if $v = \mathsf{MZ}(u)$, then v is S-concave, T-convex and (P1), (P2) hold. $\mathsf{MZ}(u)$ is S-concave and T-convex, and by symmetry, it is enough to prove (P1). Suppose that $v = \operatorname{cav} \min(u, v)$ and consider some $(p, q) \in P \times Q$. From Lemma 6, there exists $s \in S(p)$ such that $v(p, q) = \min(u, v)(s, q)$ and $v(p', q) = \min(u, v)(p', q)$, $\forall\, p' \in \mathsf{supp}(s)$. It follows that $v(p', q) \leq u(p', q)$, $\forall\, p' \in \mathsf{supp}(s)$ and $v(p, q) = v(s, q)$, so (P1) holds.

Second, we show that if v is S-concave, T-convex and (P1), (P2) hold, then $v = \mathsf{MZ}(u)$. Suppose that v is S-concave and T-convex and let $w(p, q) = \min(u, v)(p, q)$. Since $w \leq v$, $\operatorname{cav} w \leq \operatorname{cav} v$. From (P1), there exists $s \in S(p)$ such that $v(p, q) = v(s, q)$ and $w(p', q) = v(p', q) \leq u(p', q)$ for all p' in the support of s. So $v(p, q) = v(s, q) = w(s, q) \leq \operatorname{cav} w(s, q)$. Thus $v = \operatorname{cav} \min(u, v)$ and by symmetry, $v = \operatorname{vex} \max(u, v)$. □

Proof of Lemma 4 This property has been proved by Heuer (1992) in the case $P = \Delta(K)$ and $Q = \Delta(L)$, see also Oliu-Barton (2017). For the convenience of the reader, we present the argument which stems from the following observation. Consider a continuous function $w : P \to \mathbb{R}$. For each p such that $w(p) < \operatorname{cav} w(p)$, there exists a splitting with finite support $s = \sum_m \lambda_m \delta_{p_m}$ such that $\operatorname{cav} w(p) = \sum_m \lambda_m w(p_m)$ and $w(p') < \operatorname{cav} w(p')$ for each p' in the relative interior of the convex hull of $\{p_m : m\}$. This is because $(p, \operatorname{cav} w(p))$ lies on a face of the convex hull hypograph of the function w when $w(p) < \operatorname{cav} w(p)$. One just has to obtain this point as a "minimal" convex combination of extreme points of the hypograph, minimal in the sense that the convex hull of its extreme points is minimal for inclusion, among the splittings s such that $\operatorname{cav} w(p) = w(s)$.

Fix q and apply this logic to $v(p,q) = \text{cav}\min(u,v)(p,q)$. We get that

$$v(p,q) = \sum_m \lambda_m \min(u,v)(p_m,q),$$

with $v(p_m,q) \le u(p_m,q)$ for all m and $\min(u,v)(p',q) < v(p',q)$ for all p' in the relative interior of the convex hull of $\{p_m : m\}$. By continuity, this implies that $v(p_m,q) = u(p_m,q)$ for all m. □

Proof of Proposition 2 Consider the stationary strategy σ of Designer 1 which plays non-revealing if $v(p,q) \le u(p,q)$, and s given by (P1) otherwise. Consider any strategy τ of Designer 2. The definition of σ gives that for each n

$$\mathbb{E}[v(p^{n+1},q^n)|p^n,q^n] = v(p^n,q^n).$$

Since v is T-convex,

$$\mathbb{E}[v(p^{n+1},q^{n+1})|p^{n+1},q^n] \ge v(p^{n+1},q^n).$$

Taking expectation, for each n, $\mathbb{E}[v(p^{n+1},q^{n+1})] \ge \mathbb{E}[v(p^{n+1},q^n)] \ge \mathbb{E}[v(p^n,q^n)] \ge v(p_0,q_0)$ by induction. Denote $X = \{(p,q) \in P \times Q : v(p,q) \le u(p,q)\}$, from the construction, $(p^{n+1},q^n) \in X$ almost surely for each n. Since by assumption X is closed, $(p^\infty,q^\infty) \in X$ a.s., that is $u(p^\infty,q^\infty) \ge v(p^\infty,q^\infty)$ a.s.

Claim 1. $\limsup_n v(p^n,q^n) \le v(p^\infty,q^\infty)$ *a.s.*

Proof. Fix a realized play path and consider a converging subsequence of $(v(p^n,q^n))_n$ denoted $(v(p^n_*,q^n_*))_n$. We show that $\lim_n v(p^n_*,q^n_*) \le v(p^\infty,q^\infty)$. There are 2 cases.

1) Suppose there exists n_0 such that for all $n \ge n_0$, $(p^n_*,q^n_*) \in X$. Then for each $n \ge n_0$, $p^{n+1}_* = p^n_* = p^\infty$ and $\lim_n v(p^n_*,q^n_*) = \lim_n v(p^\infty,q^n_*)$. Since v is u.s.c. in q, $\lim_n v(p^\infty,q^n_*) \le v(p^\infty,q^\infty)$.

2) Otherwise, there exists a sub-sequence (p^n_{**},q^n_{**}) of (p^n_*,q^n_*) with values in $\{(p,q) : v(p,q) > u(p,q)\}$. Since v is u.s.c. on the closure this set, we obtain: $v(p^\infty,q^\infty) \ge \lim_n v(p^n_{**},q^n_{**}) = \lim_n v(p^n_*,q^n_*)$. ■

It follows that

$$v(p_0,q_0) \le \limsup_n \mathbb{E}(v(p_n,q_n)) \le \mathbb{E}(\limsup_n v(p_n,q_n)) \le \mathbb{E}(v(p_\infty,q_\infty)) \le \mathbb{E}(u(p_\infty,q_\infty)),$$

which concludes the proof. □

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B Online Appendix: Supplementary Materials

B.1 Supplementary results

Lemma 7. *Let $K = \{0,1\}$, P be a compact subset of $\Delta(K)$ indentified with $[0,1]$ and $S(p) = \Delta(P) \cap \mathbb{S}(p)$. Using $d(p,p') = |p - p'|$, the correspondence S is non-expansive.*

Proof. Let $m = \min P$ and $M = \max P$. Take p, p' in P with $p' < p$ and $s \in S(p)$. Define

$$s' = \frac{p' - m}{p - m} s + \frac{p - p'}{p - m} \delta_m \in S(p').$$

For any 1-Lipschitz function f we have

$$f(s') - f(s) = \frac{p - p'}{p - m} \int (f(m) - f(x)) ds(x) \leq \frac{p - p'}{p - m} \int (x - m) ds(x) = p - p'.$$

As this holds for f and $-f$, we get $|f(s') - f(s)| \leq |p - p'|$ as required. For $p' > p$, repeat the above argument replacing m with M. ■

The next lemma is a characterization of non-expansive correspondences through concavification.

Lemma 8. *The correspondence S is non-expansive if and only if for any 1-Lipschitz $f : P \to \mathbb{R}$, $\operatorname{cav} f$ is also 1-Lipschitz.*

Proof. Suppose that S is non-expansive, fix p, p' and $s \in S(p)$ such that $\operatorname{cav} f(p) = f(s)$. We have,

$$\operatorname{cav} f(p) - \operatorname{cav} f(p') = f(s) - \operatorname{cav} f(p') \leq f(s) - f(s') \leq d_{KR}(s, s') \leq d(p, p'),$$

for the s' in $S(p')$ given by non-expansiveness. Thus, exchanging p and p', we get that $\operatorname{cav} f$ is 1-Lipschitz.

Conversely, fix p, p', $s \in S(p)$. We want to show that

$$\exists\, s' \in S(p') \text{ such that } \forall\, f \in \mathcal{L}_1, |f(s') - f(s)| \leq d(p, p'),$$

where $\mathcal{L}_1$ denotes the set of 1-Lipschitz function. For any $f \in \mathcal{L}_1$, we know that $\operatorname{cav} f \in \mathcal{L}_1$ thus $\operatorname{cav} f(p) \leq \operatorname{cav} f(p') + d(p, p')$. Also, there exists $s' \in S(p')$ such that $\operatorname{cav} f(p') = f(s')$. Thus,

$$\forall\, f \in \mathcal{L}_1, \exists\, s' \in S(p'), \text{ s.t. } f(s) \leq f(s') + d(p, p').$$

Since $\mathcal{L}_1$ is convex and closed and $S(p')$ is convex and compact, the minmax Theorem implies that

$$\exists\, s' \in S(p'), \forall\, f \in \mathcal{L}_1, \text{ s.t. } f(s) \leq f(s') + d(p, p').$$

As this is true for f and $-f$, we get $|f(s) - f(s')| \leq d(p, p')$ as desired. ■

B.2 Computing the Mertens-Zamir value

The Mertens-Zamir function is defined as a fixed point $v = \operatorname{cav}\min(u, v) = \operatorname{vex}\max(u, v)$ in a set of functions, and its computation is difficult in general. We present some algorithmic results for computing it. The first result shows an iterative procedure converging to the MZ value.

Proposition 5. *Let $u : P \times Q \to \mathbb{R}$ be a continuous function. Define inductively two sequences of functions $\{\underline{u}_k\}_{k=1}^{\infty}$, $\{\overline{u}_k\}_{k=1}^{\infty}$ as follows:*

$$\underline{u}_0 = \overline{u}_0 = u,\ \underline{u}_{k+1} = \operatorname{cav}\operatorname{vex}\max(u, \underline{u}_k),\ \overline{u}_{k+1} = \operatorname{vex}\operatorname{cav}\min(u, \overline{u}_k).$$

Then $\{\underline{u}_k\}_{k=1}^{\infty}$ is monotonically increasing, $\{\overline{u}_k\}_{k=1}^{\infty}$ is monotonically decreasing, and both sequences converge uniformly to $\mathsf{MZ}(u)$.

Proof of Proposition 5. These properties are proved by Mertens and Zamir (1977) when P and Q are convex and compact and all splittings are admissible $S(p) = P \cap \mathbb{S}(p)$, $T(q) = Q \cap \mathbb{T}(q)$. A minor adaptation of the proof in Mertens and Zamir (1977) shows that they also hold in our more general context. □

In particular, Proposition 5 gives an algorithm for calculating the MZ value when P and Q are finite and all splittings are admissible. Each step of the algorithm is a series of simple cav and vex operations. The algorithm can be made more explicit for binary states $K = L = \{0, 1\}$. We identify $\Delta(K)$ and $\Delta(L)$ with the interval $[0, 1]$, and we identify $p \in \Delta(K)$ with $p(1) \in [0, 1]$ and $q \in \Delta(L)$ with $q(1) \in [0, 1]$. Let $P = \{p_0, p_1, \dots, p_{M-1}, p_M\} \subset [0, 1]$ with $p_i < p_{i+1}$ for $i = 0, \dots, M-1$. Similarly $Q = \{q_0, q_1, \dots, q_{N-1}, q_N\} \subset [0, 1]$ with $q_j < q_{j+1}$ for $j = 0, \dots, N-1$. An algorithm for computing $\operatorname{cav} u$ is as follows. Start from the function u. For each $j = 0, \dots, N$,

- for each $i = 1, \dots, M-1$, if
$$u(p_i, q_j) < \frac{p_{i+1} - p_i}{p_{i+1} - p_{i-1}} u(p_{i-1}, q_j) + \frac{p_i - p_{i-1}}{p_{i+1} - p_{i-1}} u(p_{i+1}, q_j),$$
then replace $u(p_i, q_j)$ by $\frac{p_{i+1}-p_i}{p_{i+1}-p_{i-1}} u(p_{i-1}, q_j) + \frac{p_i-p_{i-1}}{p_{i+1}-p_{i-1}} u(p_{i+1}, q_j)$;
- repeat until $u(p_i, q_j) \geq \frac{p_{i+1}-p_i}{p_{i+1}-p_{i-1}} u(p_{i-1}, q_j) + \frac{p_i-p_{i-1}}{p_{i+1}-p_{i-1}} u(p_{i+1}, q_j)$ for all $i = 1, \dots, M-1$.

We now present another algorithm to compute the MZ value when states are binary ($K = L = \{0, 1\}$), P and Q are finite, and all splittings are admissible. To simplify the exposition, we assume that P and Q are uniform grids, i.e., there exist positive integers M and N such that $P = \{\frac{i}{M}, i \in I\}$ and $Q = \{\frac{j}{N}, j \in J\}$, with $I = \{0, 1, \dots, M\}$ and $J = \{0, 1, \dots, N\}$ (the approach can be easily generalized to irregular grids). The payoff function u can here be simply represented by a matrix in $\mathbb{R}^{I \times J}$, that we denote with some abuse of notation by $u = (u_{i,j})$, where $u_{i,j} = u(p_i, p_j)$ for every $(i, j) \in I \times J$. Given a matrix w in $\mathbb{R}^{I \times J}$, we say that:

$$\begin{aligned} w \text{ is column-concave if}: &\quad \forall j \in J, \forall i \in \{1, \dots, M-1\},\ w_{i,j} \geq \tfrac{1}{2}(w_{i-1,j} + w_{i+1,j}), \\ w \text{ is row-convex if}: &\quad \forall i \in I, \forall j \in \{1, \dots, N-1\}, w_{i,j} \leq \tfrac{1}{2}(w_{i,j-1} + w_{i,j+1}). \end{aligned}$$

We have the following characterization of the MZ value.

Lemma 9. *Given a matrix u in $\mathbb{R}^{I\times J}$, there exists a unique matrix v in $\mathbb{R}^{I\times J}$ which is column-concave and row-convex, and such that for all $(i,j) \in I \times J$:*

if $v_{i,j} > u_{i,j}$, then $0 < i < M$ and $v_{i,j} = \frac{1}{2}(v_{i-1,j} + v_{i+1,j})$, and
if $v_{i,j} < u_{i,j}$, then $0 < j < N$ and $v_{i,j} = \frac{1}{2}(v_{i,j-1} + v_{i,j+1})$.

Moreover $v_{i,j} = \mathsf{MZ}(u)(i/M, j/N)$ for all i and j, where $\mathsf{MZ}(u)$ is the value of the information design games with payoff $u : P \times Q \to \mathbb{R}$ and no deadline. Abusing notations, we write $v = \mathsf{MZ}(u)$.

Proof. We have to show that the matrix v corresponding to $\mathsf{MZ}(u)$ is the unique matrix satisfying the conditions of the lemma. We first show that v satisfies these conditions. By Proposition 1, v is column-concave and row-convex, and for all $(p,q) \in P \times Q$:

(P1) There exists $s \in S(p)$ such that $v(p,q) = v(s,q)$ and $v(p',q) \le u(p',q)$, $\forall\, p' \in \mathsf{supp}(s)$,

(P2) There exists $t \in T(q)$ such that $v(p,q) = v(p,t)$ and $v(p,q') \ge u(p,q')$, $\forall\, q' \in \mathsf{supp}(t)$.

Consider (i,j) in $I \times J$ such that $v_{i,j} > u_{i,j}$. The splitting s obtained from (P1) at $(p,q) = (i/M, j/N)$ must be informative and we can find i' and i'' in I such that $i' \le i-1 < i+1 \le i''$ and i'/M, i''/M belong to the support of s. Since $v(p,q) = v(s,q)$ and v is column-concave, $v(\cdot, w)$ has to be affine on $[i'/M, i''/M]$ and in particular on $[(i-1)/M, (i+1)/M]$. So $v_{i,j} = \frac{1}{2}(v_{i-1,j} + v_{i+1,j})$. The case where $v_{i,j} < u_{i,j}$ is treated symmetrically.

Now, we prove uniqueness. Assume that two matrices v and w satisfy the conditions of Lemma 9. Define $\alpha = \max_{(i,j)\in I\times J}\{v_{i,j} - w_{i,j}\}$, $Z = \operatorname{argmax}_{(i,j)\in I\times J}\{v_{i,j} - w_{i,j}\}$ and let (i_0, j_0) be an element of Z minimizing the sum of coordinates $i + j$.

Suppose that $v_{i_0,j_0} > u_{i_0,j_0}$. Then $v_{i_0,j_0} = \frac{1}{2}(v_{i_0-1,j_0} + v_{i_0+1,j_0})$ and since w is column-concave, $w_{i_0,j_0} \ge \frac{1}{2}(w_{i_0-1,j_0} + w_{i_0+1,j_0})$. We get $\alpha = v_{i_0,j_0} - w_{i_0,j_0} \le \frac{1}{2}((v_{i_0-1,j_0} - w_{i_0-1,j_0}) + (v_{i_0+1,j_0} - w_{i_0+1,j_0})) \le \frac{1}{2}((v_{i_0-1,j_0} - w_{i_0-1,j_0}) + \alpha)$, hence $(i_0 - 1, j_0) \in Z$ which contradicts the minimality of (i_0, j_0). So $v_{i_0,j_0} \le u_{i_0,j_0}$.

Suppose now that $w_{i_0,j_0} < u_{i_0,j_0}$. Then $w_{i_0,j_0} = \frac{1}{2}(w_{i_0,j_0-1} + w_{i_0,j_0+1})$ and since v is row-convex, $v_{i_0,j_0} \le \frac{1}{2}(v_{i_0,j_0-1} + v_{i_0,j_0+1})$. We obtain $\alpha = v_{i_0,j_0} - w_{i_0,j_0} \le \frac{1}{2}((v_{i_0,j_0-1} - w_{i_0,j_0-1}) + (v_{i_0,j_0-1} - w_{i_0,j_0+1}))$, so $(i_0, j_0 - 1) \in Z$ which contradicts the minimality of (i_0, j_0). So $w_{i_0,j_0} \ge u_{i_0,j_0}$.

We deduce $\alpha = v_{i_0,j_0} - w_{i_0,j_0} \le u_{i_0,j_0} - u_{i_0,j_0} = 0$, hence $v \le w$. By symmetry, $v = w$. ∎

Equipped with Lemma 9, the algorithm for computing $\mathsf{MZ}(u)$ is as follows.

- First concavify u_{i0} and u_{iN} w.r.t. i, convexify u_{0j} and u_{Mj} w.r.t. j.
- Take two subsets A_+, A_- of $I \times J$ and postulate that for all i, j, $v_{ij} > u_{ij}$ if and only if $(i,j) \in A_+$ and $v_{ij} < u_{ij}$ if and only if $(i,j) \in A_-$. Write and solve the linear system:

$$\begin{aligned} v_{ij} &= \frac{1}{2}v_{i-1,j} + \frac{1}{2}v_{i+1,j} \text{ for } (i,j) \in A_+, \\ v_{ij} &= \frac{1}{2}v_{i,j-1} + \frac{1}{2}v_{i,j+1} \text{ for } (i,j) \in A_-, \\ v_{ij} &= u_{ij} \text{ for } (i,j) \notin A_- \cup A_+. \end{aligned}$$

- If the solution of the system does not satisfy $v_{ij} > u_{ij}$ for all $(i,j) \in A_+$ and $v_{ij} < u_{ij}$ for all $(i,j) \in A_-$, or does not satisfy S-concavity and T-convexity, try another pair of subsets.

- Otherwise, $v = \mathsf{MZ}(u)$.

The linear system above is associated to a diagonally dominant matrix, hence it has a unique solution. Since there is a unique $v = \mathsf{MZ}(u)$, there is a unique pair A_+, A_- on which the algorithm stops eventually.

Let us illustrate this algorithm with the matrix

$$u = \begin{array}{|c|c|c|c|}\hline 6 & 4 & 2 & 1 \\ \hline 4 & 2 & 3 & 3 \\ \hline 1 & 4 & 2 & 5 \\ \hline 0 & 3 & 4 & 7 \\ \hline \end{array}$$

The first step is the concavification of the first and fourth column, and the convexification of the first and fourth rows. We obtain

$$u' = \begin{array}{|c|c|c|c|}\hline 6 & 4 & 2 & 1 \\ \hline 4 & 2 & 3 & 3 \\ \hline 2 & 4 & 2 & 5 \\ \hline 0 & 2 & 4 & 7 \\ \hline \end{array}$$

It is easy to see that $v = \mathsf{MZ}(u) = \mathsf{MZ}(u')$, that v coincide with u' on the the first and fourth column and on the first and fourth rows. So v can be written:

$$v = \begin{array}{|c|c|c|c|}\hline 6 & 4 & 2 & 1 \\ \hline 4 & a & b & 3 \\ \hline 2 & c & d & 5 \\ \hline 0 & 2 & 4 & 7 \\ \hline \end{array}$$

We have to compare a, b, c, d to the corresponding entries in u. Let us compute lower and upper bounds for v:

$$\operatorname{cav} u' = \begin{array}{|c|c|c|c|}\hline 6 & 4 & 2 & 1 \\ \hline 4 & 4 & 3 & 3 \\ \hline 2 & 4 & 3 & 5 \\ \hline 0 & 2 & 4 & 7 \\ \hline \end{array} \qquad \operatorname{vex} u' = \begin{array}{|c|c|c|c|}\hline 6 & 4 & 2 & 1 \\ \hline 4 & 2 & \frac{5}{2} & 3 \\ \hline 2 & 2 & 2 & 5 \\ \hline 0 & 2 & 4 & 7 \\ \hline \end{array}$$

$$\operatorname{vex}\ \operatorname{cav} u' = \begin{array}{|c|c|c|c|}\hline 6 & 4 & 2 & 1 \\ \hline 4 & \frac{7}{2} & 3 & 3 \\ \hline 2 & \frac{5}{2} & 3 & 5 \\ \hline 0 & 2 & 4 & 7 \\ \hline \end{array} \qquad \operatorname{cav}\ \operatorname{vex} u' = \begin{array}{|c|c|c|c|}\hline 6 & 4 & 2 & 1 \\ \hline 4 & \frac{10}{3} & \frac{8}{3} & 3 \\ \hline 2 & \frac{8}{3} & \frac{10}{3} & 5 \\ \hline 0 & 2 & 4 & 7 \\ \hline \end{array}$$

We have $\operatorname{cav}\operatorname{vex} u' \leq v \leq \operatorname{vex}\operatorname{cav} u'$, so $a > 2$, $d > 2$, $b \leq 3$ and $c < 4$. The entries corresponding to a and b belong to the region where $(v > u)$ and the entry corresponding to c belong to the region $(v < u)$. From Lemma 9, we obtain: $a = (4+c)/2, c = (2+d)/2, d = (b+4)/2$. There are now 2 possible cases:

$b < 3$ and $b = 3$.

If $b < 3$, we have $b = (a+3)/2$, and the system has a unique solution given by $a = 51/15$, $b = 48/15$, $c = 42/15$ and $d = 54/15$. We find a contradiction since $48/15 > 3$.

So necessarily $b = 3$. The system has a unique solution given by $a = 27/8$, $b = 3$, $c = 11/4$ and $d = 7/2$. Finally,

$$v = \begin{array}{|c|c|c|c|} \hline 6 & 4 & 2 & 1 \\ \hline 4 & \frac{27}{8} & \frac{24}{8} & 3 \\ \hline 2 & \frac{22}{8} & \frac{28}{8} & 5 \\ \hline 0 & 2 & 4 & 7 \\ \hline \end{array}$$

At equilibrium here, information disclosure stops after maximum 3 stages.

The next result shows that computing the MZ function for finite sets allows to approximate the MZ function for models with binary states $K = L = \{0, 1\}$ and no restriction on admissible posteriors and admissible splittings: $P = Q \cong [0, 1]$, $S(p) = \mathbb{S}(p)$, $T(q) = \mathbb{T}(q)$.

Proposition 6. *Let $u : [0,1] \times [0,1] \to \mathbb{R}$ be a continuous function. For each $k \geq 1$, denote $P_k = Q_k = \left\{0, \frac{1}{k}, \frac{2}{k}, \dots, \frac{k-1}{k}, 1\right\}$, let $u_k : P_k \times Q_k \to \mathbb{R}$ be the restriction of u on $P_k \times Q_k$ and $v_k : P_k \times Q_k \to \mathbb{R}$ be the MZ function of u_k. Let also $w_k : [0,1] \times [0,1] \to \mathbb{R}$ be the extension of v_k by piecewise bi-linearity. That is on each square, $[\frac{i}{k}, \frac{i+1}{k}] \times [\frac{j}{k}, \frac{j+1}{k}]$, $w_k(p,q)$ is bi-linear and coincides with v_k on the corners of the square.*

Then $(w_k)_k$ uniformly converges to $v = \mathsf{MZ}(u)$. Moreover, if u is C-Lipschitz for $d((p,q),(p',q')) = |p - p'| + |q - q'|$, then for each $k \geq 1$,

$$\|v - w_k\|_\infty \leq \frac{4C}{k}.$$

As a consequence, an approximation of $\mathsf{MZ}(u)$ is found by taking a fine discretization of $[0, 1]$ and computing the MZ value of the information design game in which designers are constrained by the grid.

Proof. We use the distance $d((p,q),(p',q')) = |p - p'| + |q - q'|$ on the square $[0,1]^2$. Since the function $u : P \times Q \to \mathbb{R}$ is continuous on compact sets, it is also uniformly continuous and therefore there exists a function $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{x \to 0} \omega(x) = 0$ called a modulus of continuity such that

$$\forall\, p, p' \in P, q, q' \in Q,\ |u(p,q) - u(p',q')| \leq \omega(d((p,q),(p',q')).$$

Moreover, ω can be chosen non-decreasing and concave. Laraki and Renault (2019), Lemma 9.1. prove that if $v = \mathsf{MZ}(u)$, then v also has ω as modulus of continuity, i.e.

$$\forall\, p, p' \in P, q, q' \in Q,\ |v(p,q) - v(p',q')| \leq \omega(d((p,q),(p',q'))).$$

In particular if u is C-Lipschitz, $\omega(x) = Cx$ and v is also C-Lipschitz. With the notations of Proposition 6, for each $k \geq 1$, u_k also has ω as modulus of continuity, this is also true for v_k from Laraki and Renault (2019) and it is easy to see that this holds for w_k as well.

Fix $k \geq 1$ and consider the information design game $G_\infty(p^0, q^0)$ in which the payoff function is u, $P = P_k$, $Q = \Delta(L)$ and no further restrictions on admissible splittings. Denote $A_k = B_k = \{0, 1, \dots, k\}$

and define a strategy σ of Designer 1 as follows. For $(p^n, q^n) \in P_k \times \Delta(L)$ the posteriors of stage n, let i^n in A_k and j^n in B_k be such that $p^n = \frac{i^n}{k}$ and $q^n \in [\frac{j^n}{k}, \frac{j^n+1}{k})$.

a) If $w_k(\frac{i^n}{k}, \frac{j^n}{k}) > u(\frac{i^n}{k}, \frac{j^n}{k})$ and $w_k(\frac{i^n}{k}, \frac{j^n+1}{k}) > u(\frac{i^n}{k}, \frac{j^n+1}{k})$ then $\sigma(p^n, q^n)$ splits at stage n uniformly between $p^n - \frac{1}{k}$ and $p^n + \frac{1}{k}$.

b) Otherwise $\sigma(p^n, q^n)$ is non-revealing.

Fix any strategy τ of player 2 and consider the induced random sequence of posteriors $(p^n, q^n)_n$. Suppose that case a) holds at stage n. Since $w_k(\frac{i^n}{k}, \frac{j^n}{k}) > u(\frac{i^n}{k}, \frac{j^n}{k})$ and $w_k = \mathsf{MZ}(u_k)$ on the grid, we have

$$w_k\Big(\frac{i^n}{k}, \frac{j^n}{k}\Big) = \frac{1}{2} w_k\Big(\frac{i^n - 1}{k}, \frac{j^n}{k}\Big) + \frac{1}{2} w_k\Big(\frac{i^n + 1}{k}, \frac{j^n}{k}\Big).$$

Similarly $w_k(\frac{i^n}{k}, \frac{j^n+1}{k}) = \frac{1}{2} w_k(\frac{i^n-1}{k}, \frac{j^n+1}{k}) + \frac{1}{2} w_k(\frac{i^n+1}{k}, \frac{j^n+1}{k})$, so from bi-linearity,

$$w_k\Big(\frac{i^n}{k}, q^n\Big) = \frac{1}{2} w_k\Big(\frac{i^n - 1}{k}, q^n\Big) + \frac{1}{2} w_k\Big(\frac{i^n + 1}{k}, q^n\Big),$$

and we obtain

$$\mathbb{E}[w_k(p^{n+1}, q^n) | p^n, q^n] = w_k(p^n, q^n).$$

This equality obviously holds in case b), so it holds almost surely for every n.

Next, for any given p in P_k, the mapping $q \mapsto w_k(p, q)$ is convex. To see this, notice that the T-convexity of w_k implies that for $0 < j < k$,

$$w_k\Big(p, \frac{j}{k}\Big) \le \frac{1}{2}\Big(w_k(p, \frac{j-1}{k}) + w_k(p, \frac{j+1}{k})\Big).$$

Thus $q \mapsto w_k(p, q)$ is continuous, piecewise linear with a slope which is non-decreasing, hence it is convex. As a consequence

$$\mathbb{E}(w_k(p^{n+1}, q^{n+1})) \ge \mathbb{E}(w_k(p^{n+1}, q^n)) = \mathbb{E}(w_k(p^n, q^n)) \ge w_k(p^0, q^0),$$

and taking limit gives

$$\mathbb{E}(w_k(p^\infty, q^\infty)) \ge w_k(p^0, q^0).$$

Now since P_k is finite, for almost all realized sequence $(p^n, q^n)_n$, there exists n_0 such that $p^n = p^\infty$ for all $n \ge n_0$. So for $n \ge n_0$, we have $w_k(p^\infty, \frac{j^n}{k}) \le u(p^\infty, \frac{j^n}{k})$ or $w_k(p^\infty, \frac{j^n+1}{k}) \le u(p^\infty, \frac{j^n+1}{k})$. Denoting ω the modulus of continuity of u, we have $u(p^\infty, q^n) \ge u(p^\infty, \frac{j^n}{k}) - \omega(\frac{1}{k})$ and $u(p^\infty, q^n) \ge u(p^\infty, \frac{j^n+1}{k}) - \omega(\frac{1}{k})$. Also $w_k(p^\infty, q^n) \le w_k(p^\infty, \frac{j^n}{k}) + \omega(\frac{1}{k})$ and $w_k(p^\infty, q^n) \le w_k(p^\infty, \frac{j^n+1}{k}) + \omega(\frac{1}{k})$. We obtain for all $n \ge n_0$, $w_k(p^\infty, q^n) \le u(p^\infty, q^n) + 2\omega(\frac{1}{k})$. Taking limit we get that almost surely,

$$w_k(p^\infty, q^\infty) \le u(p^\infty, q^\infty) + 2\omega\Big(\frac{1}{k}\Big).$$

It follows that

$$\mathbb{E}(u(p^\infty, q^\infty)) \ge w_k(p^0, q^0) - 2\omega\Big(\frac{1}{k}\Big).$$

Therefore by playing σ, Designer 1 guarantees the payoff $w_k(p^0,q^0)-2\omega(\frac{1}{k})$ in $G_\infty(p^0,q^0)$. Hence we have

$$\forall\,(p^0,q^0)\in P_k\times\Delta(L),\ \ v(p^0,q^0)\geq w_k(p^0,q^0)-2\omega\Big(\frac{1}{k}\Big).$$

Consider now (p^0,q^0) in $\Delta(K)\times\Delta(L)$. For p'^0 in P_k such that $d(p^0,p'^0)\leq\frac{1}{k}$ we have:

$$v(p^0,q^0)\geq v(p'^0,q^0)-\omega\Big(\frac{1}{k}\Big)\geq w_k(p'^0,q^0)-3\omega\Big(\frac{1}{k}\Big)\geq w_k(p^0,q^0)-4\omega\Big(\frac{1}{k}\Big).$$

By exchanging the roles of player 1 and 2, we get $v(p^0,q^0)\leq w_k(p^0,q^0)+4\omega(\frac{1}{k})$ and finally

$$\|v-w_k\|_\infty\leq 4\omega\Big(\frac{1}{k}\Big).$$

■

Minor adaptations of this proof yield the following result for irregular grids.

Proposition 7. *Let $u:[0,1]\times[0,1]\to\mathbb{R}$ be a continuous function with modulus of continuity ω for the distance $d((p,q),(p',q'))=|p-p'|+|q-q'|$. Let $P=\{p_0,p_1,\ldots,p_{M-1},p_M\}$, $Q=\{q_0,q_1,\ldots,q_{N-1},q_N\}$ and denote*

$$m(P,Q)=\max\big\{\max\{|p_{i+1}-p_i|:i=0,\ldots,M-1\};\ \max\{|q_{j+1}-q_j|:j=0,\ldots,N-1\}\big\}.$$

Let u_{PQ} be the restriction of u on $P\times Q$, v_{PQ} be the MZ function of u_{PQ} and w_{PQ} be the piecewise bi-linear extension of v_{PQ}. We have,

$$\|MZ(u)-w_{PQ}\|_\infty\leq 4\omega\Big(m(P,Q)\Big).$$

B.3 General splittings on finite sets of admissible posteriors

In this section, we show that when P and Q are finite, very few assumptions are needed to prove that the games $G_\infty(p^0,q^0)$ and $G_\infty^{seq}(p^0,q^0)$ have values. That is, we take P and Q finite, suppose that for all (p,q), $\delta_p\in S(p)$ and $\delta_q\in T(q)$, but make no further assumption on the correspondences S,T.

For all (p,q) in $P\times Q$, define:

$$v_-(p,q)=\sup\big\{v(s,q):\exists\,\alpha\in[0,1),\exists\,s\in\Delta(P\backslash\{p\})\text{ s.t. }\alpha\delta_p+(1-\alpha)s\in S(p)\big\},$$

$$v_+(p,q)=\inf\big\{v(p,t):\exists\,\alpha\in[0,1),\exists\,t\in\Delta(Q\backslash\{q\})\text{ s.t. }\alpha\delta_q+(1-\alpha)t\in T(q)\big\}.$$

For instance, one may think of the example where $P=\{0,\frac{1}{k},\frac{2}{k},....,1\}$, and Designer 1 at posterior $p\in\{\frac{1}{k},...,\frac{k-1}{k}\}$ can either stay at p or split between the two closest neighbours $p-\frac{1}{k}$ and $p+\frac{1}{k}$. Then $v_-(p,q)=\frac{1}{2}(v(p-\frac{1}{k},q)+v(p+\frac{1}{k},q))$.

Proposition 8. *The games $G_\infty(p^0,q^0)$ and $G_\infty^{seq}(p^0,q^0)$ admit the same value which is the unique function $v:P\times Q\to\mathbb{R}$ which is S-concave, T-convex, and such that for all (p,q) in $P\times Q$:*

(P1') if $v(p,q)>u(p,q)$, then $v(p,q)=v_-(p,q)$;

(P2') if $v(p,q) < u(p,q)$, then $v(p,q) = v_+(p,q)$.

Moreover, the value is obtained in pure strategies: designers have pure ε-optimal strategies for all $\varepsilon > 0$.

This result relaxes the assumptions of compactness and convexity of the correspondences S, T. Importantly, it relaxes Assumption 1 which states that admissible splittings are closed under iteration. In the proof we consider correspondences S^∞, T^∞ which are the "closures" of S, T defined by taking convex hull, topological closure and closure under repetition. The first part of the proof (Lemma 10) shows that the MZ function associated to S^∞, T^∞ satisfies *(P1')* and *(P2')*. The second part (Lemma 11) shows that each Designer can guarantee the value up to any $\varepsilon > 0$. Since admissible splittings are not closed under iteration, it may require several stages to achieve a desired splitting. As a result, ε-optimal strategies are not Markovian.

Lemma 10. *There exists $v : P \times Q \to \mathbb{R}$ which is S-concave, T-convex and such that for all (p,q) in $P \times Q$:*

(P1') if $v(p,q) > u(p,q)$, then $v(p,q) = v_-(p,q)$;

(P2') if $v(p,q) < u(p,q)$, then $v(p,q) = v_+(p,q)$.

Proof. Define the correspondence $S : \Delta(P) \rightrightarrows \Delta(P)$ as follows:

$$\forall s \in \Delta(P),\ S(s) = \left\{ \sum_{p \in P} s(p) l(p) : \forall p \in P,\ l(p) \in \overline{\mathrm{co}}\, S(p) \right\},$$

where $s(p) \in [0,1]$ denotes the probability of p under the splitting s and $\overline{\mathrm{co}}\, S(p)$ is the closure of the convex hull of $S(p)$. Define then the iterates of S as follows: $S^0(s) = \{s\}$ and for all $n \geq 1$, $S^{n+1}(s) = \{s'' \in S^n(s') : s' \in S(s)\}$. Finally let $S^\infty(s)$ be the closure of $\cup_{n=0}^\infty S^n(s)$. Since P and Q are finite, S and T are non-expansive. From Laraki and Renault (2019), Theorem 3.1, there exists a unique function $v : P \times Q \to \mathbb{R}$ which is S-concave, T-convex and such that for every (p,q) in $P \times Q$:

(P1") If $v(p,q) > u(p,q)$, there exists s in $S^\infty(\delta_p)$ such that $v(p,q) = v(s,q) \leq u(s,q)$;

(P2") If $v(p,q) < u(p,q)$, there exists t in $T^\infty(\delta_q)$ such that $v(p,q) = v(p,t) \geq u(p,t)$.

We now prove that this function v satisfies (P1'). Consider (p,q) such that $v_-(p,q) < v(p,q)$ and take $\varepsilon > 0$ such that $v_-(p,q) \leq v(p,q) - \varepsilon$. From the defintion of v_-, for any s in $S(p)$, there exists $s' \in \Delta(P)$ such that $s = s(p)\delta_p + (1 - s(p))s'$, $p \notin \mathsf{supp}(s')$ and $v(s',q) \leq v(p,q) - \varepsilon$. It follows that

$$v(s,q) = s(p)v(p,q) + (1 - s(p))v(s',q) \leq v(p,q) - \varepsilon(1 - s(p)).$$

The inequality $v(s,q) \leq v(p,q) - \varepsilon(1 - s(p))$ then directly extends to any s in $\overline{\mathrm{co}}\, S(p) = S(\delta_p)$. Also, for each s in $S(\delta_p)$, there exists $s' \in \Delta(P)$ such that $s = s(p)\delta_p + (1 - s(p))s'$, $p \notin \mathsf{supp}(s')$ and $v(s',q) \leq v(p,q) - \varepsilon$.

Assume now by induction that for some $n \geq 2$, $v(s,q) \leq v(p,q) - \varepsilon(1 - s(p))$ holds for each $s \in S^{n-1}(\delta_p)$ and consider a given s_n in $S^n(\delta_p)$. There exists s_1 in $S(\delta_p)$ such that $s_n \in S^{n-1}(s_1)$ and

s_1 can be written $s_1 = s_1(p)\delta_p + (1 - s_1(p))s'_1$, with $p \notin \mathsf{supp}(s'_1)$ and $v(s'_1, q) \leq v(p, q) - \varepsilon$. We can write $s_n = s_1(p)s_{n-1} + (1 - s_1(p))s'_n$, with $s_{n-1} \in S^{n-1}(\delta_p)$ and $s'_n \in S^{n-1}(s'_1)$. From the induction hypothesis, $v(s_{n-1}, q) \leq v(p, q) - \varepsilon + \varepsilon s_{n-1}(p)$. S-concavity implies $v(\varsigma', q) \leq v(\varsigma, q)$ for any ς, ς' with $\varsigma' \in S(\varsigma)$, so $v(s'_n, q) \leq v(s'_1, q) \leq v(p, q) - \varepsilon$. This gives

$$v(s_n, q) \leq v(p, q) - \varepsilon + \varepsilon s_1(p)s_{n-1}(p) \leq v(p, q) - \varepsilon + \varepsilon s_n(p).$$

By induction we get:

$$\forall s \in S^\infty(\delta_p),\ v(s, q) \leq v(p, q) - \varepsilon(1 - s(p)).$$

Finally, assume that $v(p, q) > u(p, q)$. By (P1"), there exists s in $S^\infty(\delta_p)$ such that $v(p, q) = v(s, q) \leq u(s, q)$. We deduce that $s \neq \delta_p$ and $v(p, q) = v(s, q) \leq v(p, q) - \varepsilon(1 - s(p))$, a contradiction. Hence $v(p, q) \leq u(p, q)$ and we have proved that (P1') holds. By symmetry (P2') also holds. ■

Lemma 11. *$G_\infty(p^0, q^0)$ has a value and it is given by $v(p^0, q^0)$.*

Proof. We fix $\varepsilon > 0$ and define a strategy $\sigma = (\sigma^n)_{n\geq 1}$ for Designer 1 by a main phase and transitions phases. At each stage n, the strategy depends on the history of beliefs $(p^0, q^0, \ldots, p^{n-1}, q^{n-1})$ and on the current phase. The play is initially in the main phase.

- At any stage $n \geq 1$ in the main phase:
  - if $u(p^{n-1}, q^{n-1}) \geq v(p^{n-1}, q^{n-1})$, play $\sigma_n = \delta_{p^{n-1}}$ and remain in the main phase at stage $n + 1$.
  - If $u(p^{n-1}, q^{n-1}) < v(p^{n-1}, q^{n-1})$, from (P1'), there exists $s_n \in S(p^{n-1})$, $s'_n \in \Delta(P\backslash\{p^{n-1}\})$ and $\alpha_n \in [0, 1)$ such that $s_n = \alpha_n\delta_{p^{n-1}} + (1 - \alpha_n)s'_n$ and $v(s'_n, q^{n-1}) \geq v(p^{n-1}, q^{n-1}) - \frac{\varepsilon}{2^n}$.

  The strategy enters a transition phase where it plays s_n at each stage $m > n$ as long as $p^{n-1} = p^n = \cdots = p^{m-1}$. At the first stage $m > n$ where $p^{m-1} \neq p^{n-1}$, σ returns to the main phase at stage m.

Consider any strategy τ of player 2, let $(p^n, q^n)_{n\geq 0}$ be the induced martingale and (p^∞, q^∞) be its almost sure limit. Since P and Q are finite, there exists almost surely a stage n_0 such that for each $n \geq n_0$, $(p^n, q^n) = (p^\infty, q^\infty)$. Also, since a transition phase starting at (p^n, q^n) ends up almost surely at some posterior $p' \neq p_n$, it must be the case that

$$u(p^\infty, q^\infty) \geq v(p^\infty, q^\infty),\ \text{almost surely.}$$

Now we define sequences of stopping times $(l_i)_{i\geq 1}, (m_i)_{i\geq 1}$ with values in $\{1, 2, \ldots, \infty\}$ as follows:

- Let l_1 be the first stage $n \geq 1$ where Designer 1 is in the main phase and enters the first transition phase, let $m_1 \geq l_1$ be the last stage of the first transition phase. For $i \geq 2$, let $l_i > m_{i-1}$ be the first stage of the i-th transition phase and $m_i \geq l_i$ be the last stage of the i-th transition phase.

Fix $i \geq 1$. We have $u(p^{l_i-1}, q^{l_i-1}) < v(p^{l_i-1}, q^{l_i-1})$ and at all stages $n = l_i, \ldots, m_i$, Designer 1 plays $s_{l_i} = \alpha_{l_i}\delta_{p^{l_i-1}} + (1 - \alpha_{l_i})s'_{l_i}$, with $s'_{l_i} \in \Delta(P\backslash\{p^{l_i-1}\})$, $\alpha_{l_i} \in [0, 1)$ and $v(s'_{l_i}, q^{l_i-1}) \geq v(p^{l_i-1}, q^{l_i-1}) - \frac{\varepsilon}{2^{l_i}}$. At stages $n = m_i + 1, \ldots, l_{i+1}$, the play is in the main phase and $p^{l_{i+1}-1} = p^{m_i}$. We have:

$$\mathbb{E}[v(p^{l_{i+1}-1}, q^{l_{i+1}-1}) \mid h^{l_i}] = \mathbb{E}[v(p^{m_i}, q^{l_{i+1}-1}) \mid h^{l_i}] \geq \mathbb{E}[v(p^{m_i}, q^{l_i-1}) \mid h^{l_i}] \geq v(p^{l_i-1}, q^{l_i-1}) - \frac{\varepsilon}{2^{l_i}},$$

where the first inequality uses that v is T-convex. As a consequence,

$$\mathbb{E}(v(p^{l_{i+1}-1}, q^{l_{i+1}-1})) \geq v(p^0, q^0) - \varepsilon \sum_{j=1}^{i} \frac{1}{2^{l_j}},$$

and thus $\mathbb{E}(v(p^\infty, q^\infty)) \geq v(p^0, q^0) - \varepsilon$. We get $\mathbb{E}(u(p^\infty, q^\infty)) \geq \mathbb{E}(v(p^\infty, q^\infty)) \geq v(p_1, q_1) - \varepsilon$, thus Designer 1 guarantees the payoff $v(p_0, q_0)$ in $G_\infty(p^0, q^0)$ up to any $\varepsilon > 0$. By symmetry, this is also true for Designer 2 and $v(p^0, q^0)$ is the value of the game. The proof is identical for $G_\infty^{seq}(p^0, q^0)$. ■

As an illustration, consider the following example which has no 0-optimal strategy. Let $K = \{0, 1\}$, $P = \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$, $S(p) = \{\delta_p\}$ for $p \in \{0, \frac{1}{4}, \frac{3}{4}, 1\}$, and

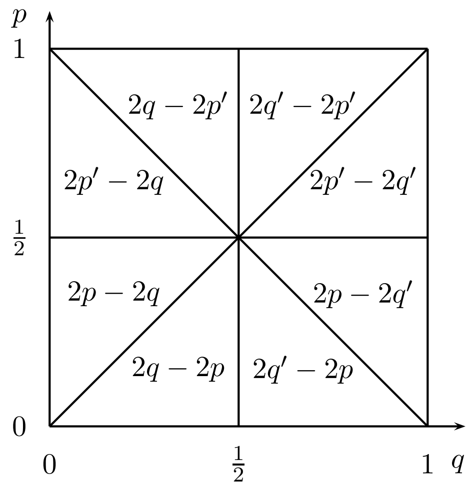
$$S\Big(\frac{1}{2}\Big) = \Big\{(1 - 2\varepsilon - 2\varepsilon^2)\delta_{\frac{1}{2}} + \varepsilon\delta_{\frac{1}{4}} + \varepsilon\delta_{\frac{3}{4}} + \varepsilon^2\delta_0 + \varepsilon^2\delta_1 : \varepsilon \in \Big[0, \frac{1}{4}\Big]\Big\}.$$

That is, at all posteriors but $\frac{1}{2}$, Designer 1 cannot split the belief. At $\frac{1}{2}$, Designer 1 can split but the posterior is likely to remain $\frac{1}{2}$, and is much more likely to be $\frac{1}{4}$ or $\frac{3}{4}$ than 0 or 1. Suppose that for each q in Q, $u(0, q) = u(1/2, q) = u(1, q) = 0$, and $u(1/4, q) = u(3/4, q) = 1$. Then the value v satisfies $v(0, q) = v(1, q) = 0$, $v(\frac{1}{4}, q) = v(\frac{1}{2}, q) = v(\frac{3}{4}, q) = 1$. v is S-concave and we have $v(\frac{1}{2}, q) = v_-(\frac{1}{2}, q)$. The point is that the splitting $\frac{1}{2}\delta_{\frac{1}{4}} + \frac{1}{2}\delta_{\frac{3}{4}}$ is not feasible at $\frac{1}{2}$, but it can be approximately achieved with many stages by choosing a very small ε.

B.4 An extension of the illustrative example of Section 3.3.1

We extend the example of Section 3.3.1 to the case in which all splittings are admissible for the designers. To do so, we define designers' payoffs for all possible posteriors, not only for posteriors is $\{0, \frac{1}{2}, 1\}$. Consider the following payoff function, where $p' = 1 - p$ and $q' = 1 - q$:

$u(p, q) =$



This function reduces to the previous discrete one when only the posteriors in $\{0, \frac{1}{2}, 1\}$ are admissible. It is equivalent to example VI.7.4 in Mertens et al. (2015, page 375). Mertens et al. (2015) have shown that the MZ value function is given by Figure 3. The blue lines represent the set of values (p, q) for which $\mathsf{MZ}(u)(p, q) = u(p, q)$ (the equations of these curves in different areas are given in blue); in the red rectangle, we have $\mathsf{MZ}(u)(p,q) = \frac{1}{4}$; black arrows represent increasing linear functions (corresponding to cases in which optimal strategies are such that only player 1 is revealing information); and black lines represent constant values (corresponding to cases in which optimal strategies are such that only player 2 is revealing information). One can easily check that this function satisfies conditions (P1) and (P2) of Proposition 1. The value functions of the sequential games with deadline are given in Figures 4 and 5.

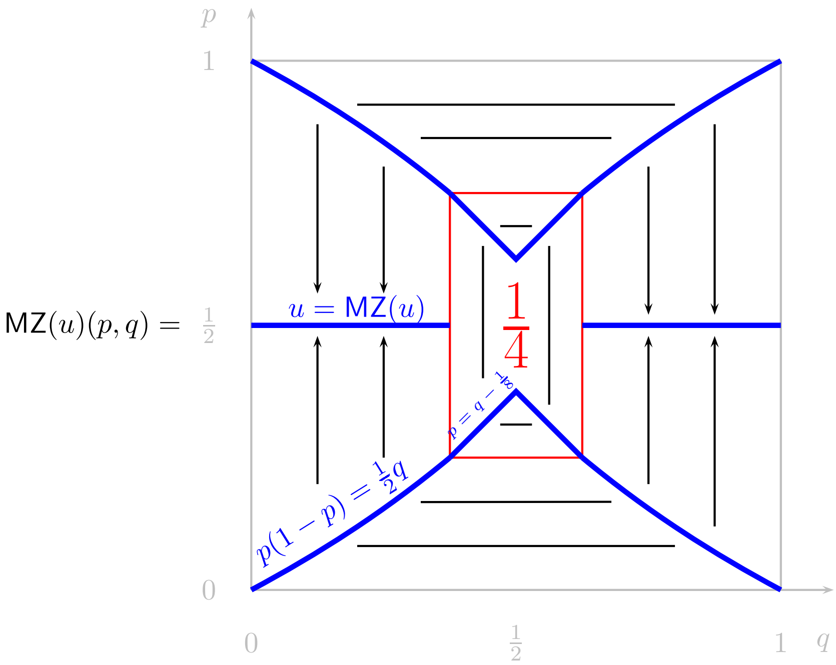


Figure 3: MZ value function in the extended example of Section 3.3.1.

The MZ value of this continuous game at $(\frac{1}{2}, \frac{1}{2})$ is $\mathsf{MZ}(u)(\frac{1}{2}, \frac{1}{2}) = \frac{1}{4}$, while it is equal to 0 when the set of admissible posteriors is restricted to $\left\{0, \frac{1}{2}, 1\right\}$. The optimal strategy of the 1-period simultaneous-move game with admissible posteriors $\left\{0, \frac{1}{2}, 1\right\}$ remains optimal in the extended game when the prior is $(\frac{1}{2}, \frac{1}{2})$. Indeed, an optimal strategy of Designer 1 (resp. Designer 2) is to play non-revealing with probability $\frac{1}{2}$ and to fully reveal information with probability $\frac{1}{2}$. Hence, the one-shot splitting value of u at $(\frac{1}{2}, \frac{1}{2})$ is $\mathsf{SV}(u)(\frac{1}{2}, \frac{1}{2}) = \frac{1}{2}$. We can also check that $\operatorname{cav}\operatorname{vex} u(\frac{1}{2}, \frac{1}{2}) = 0$ and $\operatorname{vex}\operatorname{cav} u(\frac{1}{2}, \frac{1}{2}) = \frac{1}{2}$. Summing up, in this example we have:

$$\operatorname{cav}\operatorname{vex} u\left(\frac{1}{2}, \frac{1}{2}\right) = 0 < \mathsf{MZ}(u)\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{4} < \operatorname{vex}\operatorname{cav} u\left(\frac{1}{2}, \frac{1}{2}\right) = \mathsf{SV}(u)\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{2}.$$

This example shows that, even in the standard case where $P = \Delta(K)$, $Q = \Delta(L)$ and all splittings are admissible, one may have $\mathsf{SV}(u) \neq \mathsf{MZ}(u)$. To our knowledge, no such example can be found in the literature on splitting games.

$\mathrm{vex}\,\mathrm{cav}\,u(p,q) =$

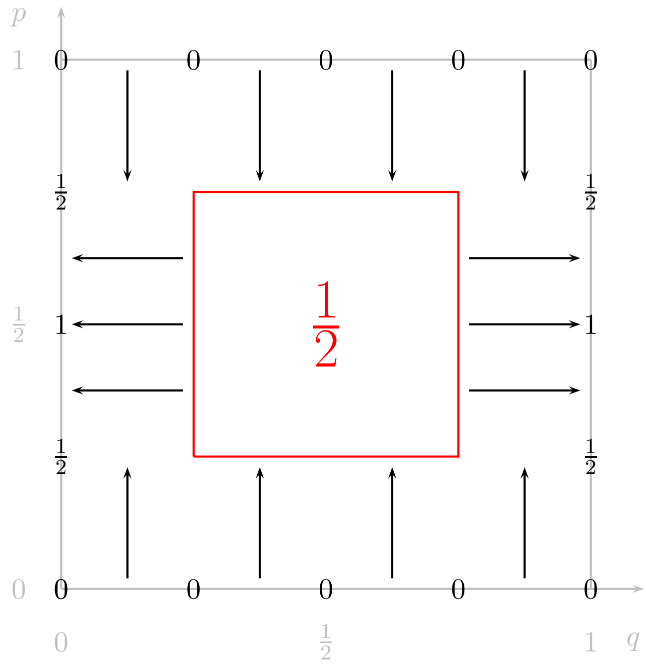


Figure 4: Value function of the sequential game with a deadline for $N = 3, 5, 7, \dots$.

$\mathrm{cav}\,\mathrm{vex}\,u(p,q) =$

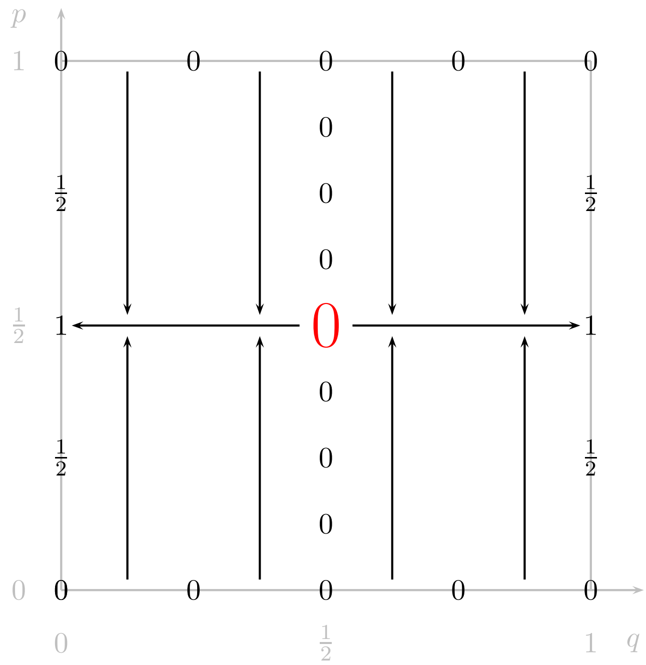


Figure 5: Value function of the sequential game with a deadline for $N = 2, 4, 6, \dots$.