Mathematical Properties of Formulations of the Gas Transmission Problem

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Subject review

MATHEMATICAL PROPERTIES OF FORMULATIONS OF THE GAS TRANSMISSION PROBLEM

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Abstract: The paper presents the mathematical properties of several formulations for the gas transmission problem that account for the nonlinear flow pressure relations. The form of the nonlinear flow pressure relations is such that the model is in general nonconvex. However, we show here that under a restrictive condition (gas inlet or gas pressure fixed at every entry/outgoing node) the problem becomes convex. This result is obtained by use of the variational inequality theory. We also give a computational method to find a feasible solution to the problem and give a physical interpretation to this feasible solution.

Keywords: OR in natural resources: natural gas; variational inequalities theory: applied to prove convexity; convexity: sufficient conditions for

1 INTRODUCTION

The problem considered is a real world problem in the field of engineering applications. It is to determine the optimal transportation plan for a gas transmission company which must satisfy demands at different nodes at a minimal guaranteed pressure. The model consists of a linear objective function subject to a set of linear and nonlinear constraints. The linear constraints express flow conservation at each node of the network. The nonlinear equations give the relation between the flow along each arc and the pressure at its two ends.

This problem was first introduced by O'Neill et al. [12]. It was also considered by Wilson, Wallace and Furey [15], who also use integer variables to describe the state of the compressors. More recently, it was considered by André et al. [1] for the case of hydrogen. For a complete discussion of advantages and disadvantages of several mathematical models that address gas transport within the context of its technical and regulatory framework, see Koch et al [9]. For a review on the most relevant research works conducted to solve natural gas transportation problems via pipeline systems, see Rios-Mercado [14]. In the paper Rios-Mercado et al. [13] address the problem of minimizing the fuel consumption incurred by compressor stations in steady-state natural gas transmission networks. In the practical world, these types of instances are very large, in terms of the number of decision variables and the number of constraints, and very complex due to the presence of non-linearity and non-convexity in both the set of feasible solutions and the objective function. In this paper, the authors present a study of the properties of gas pipeline networks, and exploit them to develop a technique that can be used to reduce problem dimension significantly, making it more amenable to the solution.

In the present paper, we consider the same problem, show the convexity of the problem and present an auxiliary problem to find a feasible solution to the problem. Moreover, we give a physical interpretation to the auxiliary problem never published before.

Finally, Geißler et al. [7] present a solution algorithm for problems from steady-state gas transport optimization taking into account the nonlinear and nonconvex physics as well as discrete variables to control the active network devices. The proposed method is based on mixed-integer linear techniques using piecewise linear relaxations of the nonlinearities. For two recent theses on the subject, see A. Morsi [11] and J. Humpola [8] who also consider convex relaxation of the problems with only passive elements (pipes and valves). In [3], Borraz-Sanchez et al. consider a convex mixed-integer second-order cone relaxation for the gas expansion planning problem.

We present the problem formulation in section 2. Then, in section 3, attention is given to identifying circumstances that guarantee the problem to be convex. If gas inlet is fixed at every entry/outgoing node, the problem becomes convex. Then, in section 4, we show how to compute a feasible solution to the problem by solving a convex optimization problem where nonlinearities are only in the objective function. We also give a physical interpretation of this feasible solution. This gives a practical solution method to find an initial solution for the more complete problem. Section 5 explains the gain to proceed first through this auxiliary problem. Finally, Section 6 presents some conclusions.

2 PROBLEM FORMULATION

Consider the problem of a gas transmission company operating a network. If the company is an integrated one, it must decide the quantities of gas to buy from different producers in order to satisfy the demand spread over different nodes of the network at the minimal guaranteed pressure requested by the consumers. The problem can be considered at different levels of aggregation. The higher level is to look at the management of the gas purchases. Contracts often differ in their flexibility and each company only has limited storage capacities. The problem of planning the purchase and the storage activities is a multitemporal one. It is formulated as a single node problem, or using a simplified representation of the network. It is assumed here that this problem has been solved and that the gas transmission company considers the more disaggregated problem of optimizing the quantity taken from the different production contracts and from storage in order to meet the demand at some moment of time. The question is no longer focused on storage but on network operations.

More and more, the two functions (buying gas and transportation) are separated. For example, in many European countries, the former national gas company is separated in two or more companies: one or more for the distribution of gas and one for the operation of the network. If we consider the transportation company, the quantities of gas taken from contracts are fixed. The transportation company must decide on the transportation plan in order to satisfy several demands of the clients at a minimal transportation cost (which are essentially the compression costs).

The network of a gas transmission company consists of several supply points where the gas is injected into the system, several demand points where gas flows out the system, and other intermediate nodes where the gas is simply rerouted. Pipelines are represented by arcs linking the nodes. Some of them can include compressors. We do not include the compressors in the present formulation.

The following mathematical notation is used. The network is defined as the pair (N, A) where $N = \{1, 2, ..., n\}$ is the set of nodes and $A \subseteq N \times N$ is the set of arcs connecting these nodes. A network example is represented in Fig. 1.

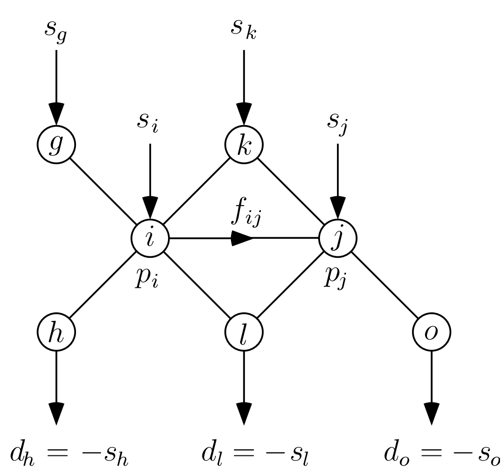

Figure 1 Network example

To each node i of the network a number p_i is associated, which represents the gas pressure at this node. We distinguish three types of nodes: the set of demand nodes, denoted N_d, the set of supply nodes, denoted N_s, and the set of connection nodes, denoted N_c. The gas supply at node i is denoted s_i. The gas demand at node i is denoted d_i. A gas flow f_{ij} is associated to each arc (i,j) from i to j. The arcs here correspond to pipelines.

The **constraints of the model** are as follows. The *flow conservation equation* at node i simply establishes the gas balance at this node. Mathematically, the gas balance at a node i is written as follows:

$$\forall i \in N, \quad \sum_{j|(i,j)\in A} f_{ij} + d_i = \sum_{j|(j,i)\in A} f_{ji} + s_i \qquad (1)$$

At a supply node i, the gas inflow s_i must remain within the lower and upper bounds specified in the contract. A gas contract specifies a nominal daily quantity to be taken by the transmission company from the producer. Still, depending on the flexibility of the contract, the transmission company has the possibility of taking a quantity ranging from a certain fraction lower than one (e.g. 80 % ⋯ 90 %) to a certain fraction higher than one (e.g. 110 % ⋯ 115 %) of the nominal contracted quantity. Mathematically:

$$\forall i \in N_s, \ \underline{s}_i \le s_i \le \overline{s}_i \qquad (2)$$

It is clear that if we consider the problem of the pure transmission company, these quantities are fixed, in other words:

$$\forall i \in N_s, \ s_i = \overline{s}_i$$

The gas must be provided at the demand nodes at a minimal pressure $\underline{p}_i$ which is requested by the consumers. At the supply nodes, the pressure is bounded above by the maximum pressure $\overline{p}_i$ that the producer can provide. In general, these two pressure bounds can be summarized in the following form:

$$\forall i \in N, \ \underline{p}_i \le p_i \le \overline{p}_i \qquad (3)$$

Now, consider the constraints on the arcs. The relation between the flow f_{ij} in the arc (i, j) and the pressure p_i and p_j is of the following form (see O'Neill et al. [12]):

$$\forall (i,j) \in A, \ \text{sign}\,(f_{ij}) f_{ij}^2 = C_{ij}^2 (p_i^2 - p_j^2) \qquad (4)$$

where C_{ij} is a constant which mainly depends on the length and on the diameter of the pipe. Thus, for each pipeline where the gas can move in both directions, we consider that the f_{ij} are unrestricted in sign. If $f_{ij} < 0$, the flow $-f_{ij}$ goes from node j to node i. This form also answers to the question: what about the case of $p_i = p_j$? In this case, the flow f_{ij} is equal to zero.

The **objective function** of the integrated transmission company is to minimize the total cost of the supplies. We can write:

$$\min z = \sum_{j \in N_S} c_j s_j$$

where c_j is the purchase price of the gas delivered at node j. In the case of a pure transmission company, the objective is to minimize the compressor costs (if the pressure goes under the minimal pressure requested by the client, one can increase the pressure using compressors).

Substituting new variables π_i defined as the square of the pressure variables p_i^2 and defining $\pi_i = p_i^2$, the problem of the integrated company can be summarized as follows:

$$\min z = \sum_{j \in N_S} c_j s_j$$
$$\text{s.t.} \begin{cases} \sum_{j|(i,j)\in A} f_{ij} - \sum_{j|(j,i)\in A} f_{ji} - s_i = 0 & \forall i \in N \\ \underline{s}_i \le s_i \le \overline{s}_i & \forall i \in N \\ \underline{\pi}_i \le \pi_i \le \overline{\pi}_i & \forall i \in N \\ \text{sign}(f_{ij}) f_{ij}^2 - C_{ij}^2(\pi_i - \pi_j) = 0 & \forall (i,j) \in A \end{cases} \quad (5)$$

3 CONVEXITY OF THE SOLUTION SET

It is well known that it is easier to compute a global solution for convex problems than for nonconvex ones. It is thus relevant to identify under which circumstances problem (5) is convex. The objective function being linear, we need only identify when the constraints define a convex set.

Proposition 1. If π_i or s_i is fixed $\forall i \in N_d \cup N_s$, then the feasible set for the gas transmission problem is convex.

Proof: See Koch [9, page 128].

Remark also that the assumption of Proposition 1 is satisfied in the case of a pure transmission company for which, as previously said:

$$\forall i \in N_s,\ s_i = \overline{s}_i .$$

4 COMPUTATION OF A FEASIBLE SOLUTION

Let us now consider the problem of the computation of a feasible solution. Following the same lines as Maugis [10], we shall show that a feasible solution can be found by solving the following mathematical programming problem:

$$\min h = \sum_{(i,j)\in A} \frac{|f_{ij}| f_{ij}^2}{3C_{ij}^2}$$
$$\text{s.t.} \quad \sum_{j|(j,i)\in A} f_{ji} - \sum_{j|(i,j)\in A} f_{ij} + s_i = 0 \quad \forall i \in N \quad (6)$$

We shall also give a physical interpretation to the objective function of this mathematical problem.

Proposition 2. The optimal solution of problem (6) is a feasible solution to the gas transmission problem (5).

Proof: Noting by π_i, the Lagrange multiplier associated to constraint of node i for (6), Maugis [10] has proved that the nonlinear flow-pressure equations of problem (5) are the Kuhn Tucker optimality conditions for the convex problem (6).

Since the objective function is strictly convex, there is only a single optimal solution in f. Note that uniqueness is guaranteed only for the flow variables and not for the pressure variables.

This mathematical programming problem can be given an interesting physical interpretation in the case of a pure transmission company (i.e. when $s_i = \overline{s}_i$, $\forall i$). Extending the work of Maugis for distribution network, we have the following proposition.

Proposition 3. The objective function of problem (6) corresponds up to a multiplicative constant to the mechanical energy dissipated per unit of time in the pipes (the mechanical energy being defined as the energy necessary for compressing f_{ij} from pressure p_j to pressure p_i).

Proof. At node i, the power W_i given by a volumetric outflow of Q_i units of gas per second at pressure p_i can be calculated in the following manner, where the total energy released by the gas when changing pressure from the reference pressure p^0 to pressure p_i is:

$$W_i = \int_{p^0}^{p_i} Q(p)\mathrm{d}p .$$

By using the perfect gas state relation ($p^0 Q^0 = pQ$), we can write:

$$W_i = \int_{p^0}^{p_i} p^0 Q^0 \frac{\mathrm{d}p}{p} = p^0 Q^0 \log\left(\frac{p_i}{p^0}\right).$$

Denote the volumetric flow going though arc (i, j) under standard conditions by Q_{ij}^0 and the pressures at the two ends of the arc by p_i and p_j. The power lost in arc (i, j) can be calculated by:

$$W_{ij} = W_i - W_j = Q_{ij}^0 p^0 \log\left(\frac{p_i}{p_j}\right) = Q_{ij}^0 \frac{p^0}{2} \log\left(\frac{p_i^2}{p_j^2}\right).$$

Introducing the mean of square of pressure p_M defined by

$$p_M^2 = \frac{p_i^2 + p_j^2}{2}$$

the power discharge W_{ij} can be expressed through the head discharge variable $H_{ij} = p_i^2 - p_j^2$ as:

$$W_{ij} = Q_{ij}^0 \frac{p^0}{2} \log \left(\frac{p_M^2 + \frac{H_{ij}}{2}}{p_M^2 - \frac{H_{ij}}{2}} \right) =$$
$$= Q_{ij}^0 \frac{p^0}{2} \left[\log \left(1 + \frac{H_{ij}}{2p_M^2} \right) - \log \left(1 - \frac{H_{ij}}{2p_M^2} \right) \right]$$

Note that since H_{ij} is small with respect to p_M^2, we obtain the following as a first order approximation

$$W_{ij} \approx Q_{ij}^0 p^0 \frac{H_{ij}}{p_M^2}.$$

The power discharge through the whole network is thus (we suppose that p_M is similar for each arc (i, j) and can be factored out in the following sum):

$$W = \sum_{(i,j)\in A} W_{ij} \approx \frac{p^0}{p_M^2} \sum_{(i,j)\in A} Q_{ij}^0 (p_i^2 - p_j^2) =$$
$$= \frac{p^0}{p_M^2} \sum_{(i,j)\in A} \frac{\left(Q_{ij}^0\right)^3}{C_{ij}^2} \operatorname{sign}(Q_{ij})$$

which corresponds, up to a multiplicative constant, to the first term of the objective of problem (6). We can thus conclude that the function h corresponds to the mechanical energy dissipated per unit time in the network due to the flow of gas in the pipes up to a multiplicative constant.

This proposition was suggested to the author by Mr. Zarea of Gaz de France.

5 UTILITY OF PROPOSITION

As said in the introduction, the motivation for our work is algorithmic. We are looking for the solution of the nonlinear nonconvex problem (5). Looking for a sufficient condition for the problem (5) to be convex, we found the following condition: fix the gas net inlet at each node. We have also emphasized that this condition is satisfied for a pure transmission company.

We have also showed that a feasible point to the non-convex problem (5) can be found as the solution of a strictly convex problem (6). The use of the solution of this auxiliary problem as a starting point for the general non-convex problem has two main advantages:

- For a non-convex optimization problem, it is well known that starting far from the optimal solution can give a local optimal solution far from the global optimum. We hope to reduce this risk with our starting point. In fact, we have proved in proposition 3 that the initial solution computed by the auxiliary problem as a meaningful interpretation: the minimization of the energy used to push the gas through the pipes.
- Using this point as a starting point can reduce computational times for the general problem.

The gain in processing time achieved by resorting to the first problem was studied and earlier proved by De Wolf and Smeers [5] for several representations of the Belgian gas transmission network, the gain of efficiency increase with the size of the problem. For small examples, the reduction of the computational time achieved in the problem (5) is completely lost due to the time spent in solving the problem (6). In contrast, for larger problems the cost of processing first the problem (6) is largely compensated by the savings achieved in the complete problem. Specifically, the global time necessary to successively solve the two problems is about half the time needed for solving the problem (5) directly from scratch. For the greater sizes of problems considered now (see for example Geißler [7]), the utilization of this auxiliary problem is totally justified.

More recently, De Wolf and Bakhouya [6] use the same auxiliary problem to find an initial solution for a larger network, namely the French high pressure natural gas network. They also show the important gain in computer time to solve first this auxiliary problem. Note that the physical interpretation given by proposition 3 explains why this starting point is a very good initial solution for the complete problem. In fact, in proposition 3, we have proved that the objective function corresponds to the minimization of the mechanical energy dissipated per unit of time in the pipes (the mechanical energy being defined as the energy necessary for compressing f_{ij} from pressure p_j to pressure pi). And the objective of a pure transmission company (the new case considered by De Wolf and Bakhouya [6]) is precisely the minimization of the used of compressors to push the gas through the pipes.

6 CONCLUSIONS

Several aspects of the gas transmission problem have been considered in this paper. The gas transmission problem is convex if the gas net inlet is fixed at all supplies and demand nodes. We have shown how to compute a feasible solution to the problem by solving a strictly convex problem with nonlinearity only in the objective function.

Based on these results, an auxiliary problem can be defined for producing an initial solution to the general problem. This auxiliary problem has a natural physical interpretation namely the minimization of the mechanical power dissipated in the pipes. This also corresponds to the objective for a pure transmission company. The use of this starting point can reduce the computing time as pointed by De Wolf and Smeers [5].

Even for more recent and realistic situation where the gas selling company and the network operators are separated (See De Wolf and Bakhouya [6]), this formulation gives a good starting point for the general nonconvex problem.

The tool there appears also to be useful in investment problem. This was used with success by De Wolf and Smeers [4] for the two stages problem of optimal dimensioning and operating of a gas transmission problem. More recently, for the case where the gas selling company and the network operators are separated, it also allows to solve this difficult non convex integer problem (see Andre et al [1, 2] for the case of hydrogen).

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