Least Square Approximations and Linear Values of Cooperative Game

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Abstract Many important values for cooperative games are known to arise from least square optimization problems. The present investigation develops an optimization framework to explain and clarify this phenomenon in a general setting. The main result shows that every linear value results from some least square approximation problem and that, conversely, every least square approximation problem with linear constraints yields a linear value. This approach includes and extends previous results on so-called least square values and semivalues in the literature. In particular, it is demonstrated how known explicit formulas for solutions under additional assumptions easily follow from the general results presented here.

1 Introduction

Approximation of high-dimensional quantities or complicated functions by simpler functions with linear properties from low-dimensional spaces has countless applications in physics, economics, operations research *etc.* In these applications, the quality of the approximation is usually measured by the Gaussian principle of least squared error, which is also the guiding optimality criterion in the present investigation. Our study addresses a particular case of such an approximation context with many applications in different fields related to operations research, namely decision theory, game theory and the theory of pseudo-Boolean functions.

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Where N is a finite set with $n = |N|$ elements and collection 2^N of subsets, a set function $v : 2^N \to \mathbb{R}$ assigns to every subset of N a real number, and is by definition of exponential complexity (in n). Identifying subsets of N with their characteristic (incidence) vectors (and thus 2^N with $\{0,1\}^n$), a set function can be viewed as a so-called pseudo-Boolean function $f : \{0,1\}^n \to \mathbb{R}$ (*cf.* Hammer and Rudeanu [12]). Of particular interest are those set functions which vanish on the empty set, since they represent cooperative TU games with N being the set of players and the quantities $v(S)$ expressing the benefit created by the cooperation of the members of $S \subseteq N$ (see, *e.g.*, Peleg and Sudhölter [15]). Under the additional stipulation of monotonicity, *i.e.*, the property that $v(S) \leq v(T)$ holds whenever $S \subseteq T$, one arrives at so-called capacities, which are a fundamental tool in the analysis of decision making under uncertainty (*cf.* Schmeidler [18]) or relative to several criteria (Grabisch and Labreuche [8])[1].

Being of exponential complexity, a natural question is to try to approximate general set functions by simpler functions, the simplest being the additive set functions, which are completely determined by the values they take on the n singleton sets $\{i\}$ and are thus of linear complexity (in n). In the field of pseudo-Boolean functions, the question has been addressed by Hammer and Holzman [10] with respect to linear and quadratic approximations, while approximation of degree k was studied by Grabisch *et al.* [9]. In decision theory, linear approximation amounts to the approximation of a capacity μ by a probability measure P (an additive capacity satisfying the additional constraint that $P(N) = 1$).

In game theory, the approximation of a game v by an additive game (equivalently by a (payoff) vector in $\mathbb{R}^N$) is related to the concept of value or solution of a game: given v, find $x \in \mathbb{R}^N$ such that $\sum_{i \in N} x_i = v(N)$ and the x_i represent as faithfully as possible the contribution of the individual players i to the total benefit $v(N)$. A very natural approach for a value is to define it as the best least square approximation of v, under the constraint $\sum_{i \in N} x_i = v(N)$, the approximation being possibly weighted. Such values are called least square values. An early and important contribution to this cooperative solution concept is due to Charnes *et al.* [2], who gave the general solution for the weighted approximation with nonnegative weights, and exhibited the well-known Shapley value [19] as a least square value. Ruiz *et al.* [16], for example, generalized this approach and derived further values from least square approximation.

The aim of this paper is a general view on the set function approximation problem by placing it in the context of quadratic optimization and bringing well-known tools of convex analysis to bear on the problem. This approach generalizes existing results and points to interesting connections. Our formulation will remain general, although we will adopt most of the time the notation and ideas from cooperative game theory, due to the great interest in this field towards values and how to obtain them.

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[1] For a general treatment of set functions, games, capacities and their application in decision making, see [7].

Main result:

Linear values and least square values for cooperative games represent two sides of the same coin: *every* least square problem under linear constraints yields a linear value having the inessential game property (*cf.* Section 3) and *every* such linear value arises as a least square value.

The presentation is organized as follows. Section 2 describes the mathematical model and Section 3 reviews linear values in the present context. Section 4 concentrates on least square values, and establishes explicit formulas under mild conditions on the weights used in the approximation and generalizes the approach to the Shapley value and to an optimization problem given in Ruiz *et al.* [16]. We remark that, interestingly, the weights do not necessarily have to be all positive in our model. Finally, we show in Section 5 how Weber's [22] so-called probabilistic values arise naturally in the present context.

2 Least square approximations and linear operators

For integers $k, m \geq 1$, we denote by $\mathbb{R}^k$ the vector space of all k-dimensional (column) vectors and by $\mathbb{R}^{m\times k}$ the vector space of all $(m \times k)$-matrices $M = [m_{ij}]$ with coefficients m_{ij}. Generally, M^T denotes the transpose of a matrix (or coefficient vector) M.

Let $Q = [q_{ij}]$ be an arbitrary positive definite symmetric $(k \times k)$-matrix with coefficients q_{ij} and recall that Q defines an inner product *via*

$$\langle x|y\rangle_Q = x^T Q y = \sum_{j=1}^{k}\sum_{i=1}^{k} q_{ij} x_i y_j$$

with the associated Q-*norm* $\|x\|_Q = \sqrt{\langle x|x\rangle_Q}$ on $\mathbb{R}^k$. Note that the choice $Q = I$ of the identity matrix I yields the usual Euclidian norm $\|x\| = \|x\|_Q$.

Fix now a matrix $A \in \mathbb{R}^{m\times k}$, a linear map $b : \mathbb{R}^k \to \mathbb{R}^m$ as well as a linear map $c : \mathbb{R}^k \to \mathbb{R}^k$. For any $v \in \mathbb{R}^k$, denote by $\hat{v} = \hat{v}(A, b, c)$ the optimal solution of the quadratic minimization problem

$$\min_{Ax=b(v)} \|c(v) - x\|_Q^2. \tag{1}$$

So, if the system $Ax = b(v)$ of linear equations has at least one solution, $\hat{v}$ is the (uniquely determined) best approximation of $c(v)$ in the solution space of $Ax = b(v)$ relative to the norm $\|\cdot\|_Q$.

The following observation is the key for our analysis of linear values. We give its short and concise proof in terms of the well-known Karush-Kuhn-Tucker (KKT) optimality conditions[2], from which we will derive explicit formulas for linear values in Section 4.

Lemma 1 *Assume that $Ax = b(v)$ has a solution for every $v \in \mathbb{R}^k$ and that the map $c : \mathbb{R}^k \to \mathbb{R}^k$ is linear. Then $v \mapsto \hat{v}$ is a well-defined linear operator.*

Proof. Problem (1) is equivalent to the quadratic optimization problem

$$\min_{Ax=b(v)} \frac{1}{2} x^T Q x - c'(v)^T x, \tag{2}$$

with $c' = Qc$. Given that Q is positive definite, it is well-known that x is the unique optimal solution for problem (2) if and only if there is a vector y such that the associated KKT-conditions

$$\begin{array}{lll} Qx + A^T y & = & c'(v) \\ Ax & = & b(v) \end{array} \tag{3}$$

are satisfied. Since b and c' are linear functions in v, one immediately deduces from (3) that also the optimal solutions of (1) are linear functions in v.

⋄

As a final remark, it is easy to see that every linear operator arises from some parametrized quadratic minimization problem of type (1). The game theoretically important observation is the opposite observation: in the context of cooperative games (see next sections), quadratic approximation problems typically give rise to linear operators on the space of cooperative games.

3 Values of cooperative games

Let N be a set of *players* of finite cardinality $n = |N|$ and let $\mathcal{N}$ be the collection of non-empty subsets $S \subseteq N$. A *cooperative TU game* is a function $v : \mathcal{N} \to \mathbb{R}$ (which is usually thought to be extended to all subsets of N *via* $v(\emptyset) = 0$). So the set $\mathcal{G} = \mathbb{R}^{\mathcal{N}}$ of all cooperative TU games on N is a vector space and isomorphic to $\mathbb{R}^k$ with $k = |\mathcal{N}| = 2^n - 1$.

The *additive (cooperative) games* correspond to those members $x \in \mathbb{R}^{\mathcal{N}}$ that satisfy the homogeneous system of linear equations

$$x(S) - \sum_{i \in S} x(\{i\}) = 0 \quad (S \in \mathcal{N}) \tag{4}$$

and one may be interested in the approximation of a game $v \in \mathcal{G}$ by an additive game with certain properties. More general approximations might be of interest. For example, the linear constraints

[2] see Faigle *et al.* [5] or any other textbook on mathematical optimization

$$\sum_{i \in N} x(\{i\}) = v(N) \tag{5}$$

$$\sum_{S \in \mathcal{N}'} x(S) = \sum_{S \in \mathcal{N}'} v(S) \tag{6}$$

for some subcollection $\mathcal{N}' \subseteq \mathcal{N}$ would stipulate an approximation of v by a game that induces an efficient value (the first equality) and, furthermore, preserves the total sum of the $v(S)$ for some specific subsets (second equality). Observe that all equalities (4) to (6) are of the form $Ax = b(v)$ with b linear in v. Hence, least square approximation problems of a game v (or of its image by a linear map c) by a game satisfying some of the above equalities (*e.g.,* an additive game) fall under the case covered by (1). Then Lemma 1 applies, from which it follows that such approximations are linear in v.

In cooperative game theory, a function $\Phi : \mathcal{G} \to \mathbb{R}^N$ is called a *value* for $\mathcal{G}$. It is well known that there is an isomorphism between additive games v and vectors x in $\mathbb{R}^N$, letting $v(\{i\}) = x_i$, $i \in N$. Consequently, values can be seen as additive games, and Lemma 1 shows that least square approximations of games by additive games yield values which are linear on $\mathcal{G}$. Moreover, supposing c to be the identity map, uniqueness of the solution of (1) implies that the induced linear value $\hat{v}$ has the *inessential game property*:

Φ has the inessential game property if v additive implies $\Phi_i(v) = v(\{i\}), \forall i \in N$,

that is, $\Phi(v) \equiv v$ in the sense of the above defined isomorphism.

This discussion leads us naturally to the concept of least square values, developed in the next section.

4 Least square values

A *least square value* is a value obtained as the optimal solution of a least square problem of the following type:

$$\min_{x \in \mathbb{R}^N} \sum_{S \in \mathcal{N}} \alpha_S (v(S) - x(S))^2 \quad \text{s.t.} \quad \sum_{i \in N} x_i = v(N), \tag{7}$$

where we set $x(S) = \sum_{i \in S} x_i$. So (7) asks for the best (α-weighted) least square approximation of a game v by an additive game x under the additional efficiency constraint $x(N) = v(N)$. We recognize here problem (1) with c being the identity operator, Q a diagonal matrix, and the constraints $Ax = b(v)$ being given by (4) and (5).

This problem has a long history. Hammer and Holzman ([10])[3] studied both the above version and the unconstrained version with equal weights ($\alpha_S = 1\ \forall S$), and

[3] later published in [11]

proved that the optimal solutions of the unconstrained version yield the Banzhaf value [1] (see also Section 5 below). More general versions of the unconstrained problem were solved by Grabisch *et al.* [9] with the approximation being relative to the space of k-additive games (*i.e.*, games whose Möbius transform vanishes for subsets of size greater than k)[4].

In 1988, Charnes *et al.* [2] gave a solution for the case with the coefficients α_S being *uniform* (*i.e.*, $\alpha_S = \alpha_T$ whenever $|S| = |T|$) and strictly positive. As a particular case, the Shapley value was shown to result from the coefficient choice

$$\alpha_S = \alpha_s = \binom{n-2}{s-1} = \frac{(n-2)!}{(s-1)!(n-1-s)!} \qquad (s = |S|). \tag{8}$$

In 1998, Ruiz *et al.* [16] analyzed problem (7) in a more general approach for uniform and nonnegative weights. They also noted several properties of least square values (like linearity) and even characterized them. Kultti and Salonen [13] generalized the results of Ruiz *et al.* by considering a distance minimization problem for an arbitrary norm under additivity and efficiency, which amounts to considering our problem (1) with c being the identity and constraints (4) and (5). One of their major results is that Φ is a linear and efficient value having the inessential game property if and only if it is the unique solution of a certain distance minimization problem. Later, Tanino [21] exploited the results of Kultti and Salonen by proposing to perform optimization using the Möbius representation of games (*i.e.*, to work in the basis of unanimity games). Thanks to this, Tanino exhibited the norm yielding any *sharing values* (*i.e.*, produced by a sharing of the dividends (Möbius transform)), also known as *selectope elements* (see Derks, Haller and Peters [3]).

In this section, we will first present a general framework for dealing with such situations and then illustrate it with the example of regular weight approximations and probabilistic values.

4.1 Weighted approximation

For the sake of generality, consider a general linear subspace $\mathcal{F} \subseteq \mathbb{R}^{\mathcal{N}}$ of dimension $k = \dim \mathcal{F}$, relative to which the approximation will be made.

Let $W = [w_{ST}] \in \mathbb{R}^{\mathcal{N}\times\mathcal{N}}$ be a given symmetric matrix of weights w_{ST}. Let $c : \mathbb{R}^{\mathcal{N}} \to \mathbb{R}^{\mathcal{N}}$ be a linear function and consider, for any game v, the optimization problem

$$\min_{u\in\mathcal{F}} \; (v-u)^T W(v-u) + c^T(v-u) \quad \text{with } c = c(v), \tag{9}$$

which is equivalent with

$$\min_{u\in\mathcal{F}} \; u^T W u - \tilde{c}^T u, \tag{10}$$

[4] see also Ding [4], and Marichal and Mathonet [14]

where $\tilde{c} \in \mathbb{R}^{\mathcal{N}}$ has the components $\tilde{c}_S = c_S + 2\sum_T w_{ST} v_T$. A further simplification is possible by choosing a basis $B = \{b_1, \dots, b_k\}$ for $\mathcal{F}$. With the identification

$$x = (x_1, \dots, x_k) \in \mathbb{R}^k \quad \longleftrightarrow \quad u = \sum_{i=1}^{k} x_i b_i \in \mathcal{F},$$

problem (10) becomes

$$\min_{x \in \mathbb{R}^k} \; \sum_{i=1}^{k} \sum_{j=1}^{k} q_{ij} x_i x_j - \sum_{i=1}^{k} \overline{c}_i x_i \tag{11}$$

with the coefficients

$$q_{ij} = \sum_S \sum_T w_{ST} b_i(S) b_j(T) \quad \text{and} \quad \overline{c}_i = \sum_S \tilde{c}_S b_i(S).$$

Note that $\overline{c} : \mathbb{R}^{\mathcal{N}} \to \mathbb{R}^k$ is a linear function in v.

Let $A \in \mathbb{R}^{m \times k}$ be a constraint matrix and $b : \mathbb{R}^{\mathcal{N}} \to \mathbb{R}^m$ a linear function such that $Ax = b(v)$ has a solution for every $v \in \mathbb{R}^{\mathcal{N}}$. If $Q = [q_{ij}] \in \mathbb{R}^{k \times k}$ is positive definite, the problem

$$\min_{x \in \mathbb{R}^k} \; \sum_{i=1}^{k} \sum_{j=1}^{k} q_{ij} x_i x_j - \sum_{i=1}^{k} \overline{c}_i x_i \quad \text{s.t.} \quad Ax = b \tag{12}$$

has the form (2), and therefore by Lemma 1 has a unique optimal solution x^* which is linear in v.

In the model (7), for example, $\mathcal{F}$ is the space C of all additive games and has dimension n. The matrix W is diagonal with the diagonal elements $w_{SS} = \alpha_S$. If $\alpha_S > 0$ holds for all S, then W is positive definite and the linearity of the implied value $v \mapsto \hat{v}$ follows directly from Lemma 1.

Otherwise, let us choose for B the basis of unanimity games $\zeta_i, i \in N$, for C (like in Tanino [21]), where

$$\zeta_i(S) = \begin{cases} 1 \text{ if } i \in S, \\ 0 \text{ if } i \notin S. \end{cases}$$

The associated matrix $Q = [q_{ij}]$ in model (7) has the coefficients

$$q_{ij} = \sum_{S \in \mathcal{N}} \alpha_S \zeta_i(S) \zeta_j(S) = \sum_{S \ni \{i,j\}} \alpha_S. \tag{13}$$

For establishing a linear value, it suffices that Q be positive definite, which is possible even when some of the α_S are negative (see Examples 4.1 and 4.2 below).

4.2 Regular weights

While Lemma 1 guarantees the existence of linear values resulting from approximation, explicit formulas can be given under additional assumptions on the weights. Restricting ourselves to objectives of type

$$\sum_{S\in\mathcal{N}} \alpha_S (v_S - u_S)^2 + \sum_{S\in\mathcal{N}} c_S u_S,$$

we propose a simple framework that nevertheless includes all the cases treated in the literature so far. We say that the weights α_S are *regular* if the resulting matrix Q has just two types of coefficients q_{ij}, *i.e.*, if there are real numbers p, q such that

$$q_{ij} = \begin{cases} q \text{ if } i = j \\ p \text{ if } i \neq j. \end{cases}$$

Example Assume that the weights α_S are uniform and set $\alpha(|S|) = \alpha_S$. Then formula (13) yields

$$q_{ij} = \sum_{s=2}^{n} \binom{n-2}{s-2} \alpha(s) \quad \text{and} \quad q_{ii} = \sum_{s=1}^{n} \binom{n-1}{s-1} \alpha(s)$$

holds for all $i \neq j$. So $Q = [q_{ij}]$ is regular. □

Lemma 2 *Let $Q = [q_{ii}] \in \mathbb{R}^{k\times k}$ be regular with $q = q_{ii}$ and $p = q_{ij}$ for $i \neq j$. Then Q is positive definite if and only if $q > p \geq 0$.*

Proof. For any $x \in \mathbb{R}^k$, we have after some algebra

$$x^T Q x = (q-p) \sum_{i=1}^{k} x_i^2 + p\overline{x}^2$$

where $\overline{x} = \sum_{i=1}^{n} x_i$, which makes the claim of the Lemma obvious.

◇

Note that our model allows for possibly negative uniform coefficients, as shown in the following example.

Example Let $n = 3$. We get $p = \alpha_2 + \alpha_3$ and $q = \alpha_1 + 2\alpha_2 + \alpha_3$. Letting $\alpha > 0$, the following vectors $(\alpha_1, \alpha_2, \alpha_3)$ lead to a positive definite matrix Q:

$$(0, \alpha, 0), \quad (\alpha, 0, \alpha), \quad (0, \alpha, -\alpha), \text{ etc.}$$

For the remainder of this section, let $Q \in \mathbb{R}^{N\times N}$ be a regular matrix with parameters $q > p \geq 0$, $c \in \mathbb{R}^N$ a vector and $g \in \mathbb{R}$ a scalar. Setting $\mathbf{1}^T = (1, 1, \ldots, 1)$, the optimization problem

$$\min_{x \in \mathbb{R}^N} \; x^T Q x - c^T x \quad \text{s.t.} \quad \mathbf{1}^T x = x(N) = g \tag{14}$$

has a unique optimal solution $x^* \in \mathbb{R}^N$. Moreover, there is a unique scalar $z^* \in \mathbb{R}$ such that (x^*, z^*) is the unique solution of the associated KKT-system

$$\begin{array}{rcl} Qx \; - z\mathbf{1} &=& c/2 \\ \mathbf{1}^T x \quad\quad &=& g. \end{array} \tag{15}$$

Verifying this KKT-system, the proof of the following explicit solution formulas is straightforward.

Theorem 1 *If Q is regular, the solution (x^*, z^*) of the KKT-system (15) is:*

$$\begin{aligned} z^* &= (2(q + (n-1)p)g - C)/n \quad (\textit{with } C = c\mathbf{1}^T = \textstyle\sum_{i \in N} c_i) \\ x_i^* &= (c_i + z^* - 2pg)/(2q - 2p) \quad (i \in N). \end{aligned}$$

If Q is furthermore positive definite, then x^* is an optimal solution for (14).

$\diamond$

In the case of uniform weights $\alpha(s)$, the formulas in Theorem 1 yield the formulas derived by Charnes *et al.* [2] for problem (7). To demonstrate the scope of Theorem 1, let us look at the extremal problem[5] studied by Ruiz *et al.* [17]

$$\min_{x \in \mathbb{R}^N} \sum_{S \subseteq N} m_S d(x, S)^2 \text{ s.t. } x(N) = v(N), \tag{16}$$

where $m_S > 0$ and

$$d(x, S) = \frac{v(S) - x(S)}{|S|} - \frac{v(N \setminus S) - x(N \setminus S)}{n - |S|}.$$

Letting $v^*(S) = v(N) - v(N \setminus S)$ and

$$\overline{v}(S) = \frac{(n - |S|)v(S) + |S|v^*(S)}{n}$$

(and thus $n\overline{v}(N) = v(N)$), we find that problem (16) becomes

$$\min_{x \in \mathbb{R}^N} \sum_{S \subseteq N} \alpha_S (\overline{v}(S) - x(S))^2 \text{ s.t. } x(N) = n\overline{v}(N).$$

with $\alpha_S = n^2 m_S (|S|^2 (n - |S|)^2)^{-1}$. Because $v \mapsto \overline{v}$ and $v \mapsto g(v) = n\overline{v}(N)$ are linear mappings, the optimal solutions of (16) yield an efficient linear value for any choice of parameters m_S such that the associated matrix Q is positive definite.

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[5] see also Sun *et al.* [20] for similar problems

If furthermore the weights m_S (and hence the α_S) are uniform, Q is regular and the optimal solution can be explicitly computed from the formulas of Theorem 1.

5 Probabilistic values

Weber [22] introduced the idea of a *probabilistic* value arising as the expected marginal contribution of players relative to a probability distribution on the coalitions. For example, a *semivalue* is a probabilistic value relative to probabilities that are equal on coalitions of equal cardinality.

For our purposes, it suffices to think of the *marginal contribution* of an element $i \in N$ as a linear functional $\partial_i : \mathcal{G} \times \mathcal{N} \to \mathbb{R}$, where $\partial_i^v(S)$ is interpreted as the marginal contribution of $i \in N$ to the coalition $S \subseteq N$ relative to the characteristic function v.

Probabilistic values can be studied quite naturally in the context of weighted approximations. Indeed, let p be an arbitrary probability distribution on $\mathcal{N}$. Then the expected marginal contribution of $i \in N$ relative to the game v is

$$E(\partial_i^v) = \sum_{S \subseteq N} \partial_i^v(S) p_S.$$

Let $\mu_i \in \mathbb{R}$ be an estimate value for the marginal contribution of $i \in N$. Then the expected observed deviation from μ_i is

$$\sigma(\mu_i) = \sqrt{\sum_{S \in \mathcal{N}} p_S(\partial_i^v(S) - \mu_i)^2}.$$

A well-known fact in statistics says that the deviation function $\mu_i \mapsto \sigma(\mu_i)$ has the unique minimizer $\mu = E(\partial_i^v)$, which can also be immediately deduced from the KKT conditions for the least square problem

$$\min_{\mu \in \mathbb{R}} \sum_{S \in \mathcal{N}} p_S(\partial_i^v(S) - \mu)^2.$$

The above problem is a special case of (1), with $c(v) = \partial_i^v$, Q the diagonal matrix with diagonal terms p_S, while the approximation subspace is just $\mathbb{R}$ and does not depend on v.

The values of Shapley and Banzhaf.

Shapley's [19] model assumes that player i contributes to a coalition S only if $i \in S$ holds and that, in this case, i's marginal contribution is evaluated as

$$\partial_i^v(S) = v(S) - v(S \setminus i).$$

So only coalitions in $\mathcal{N}_i = \{S \subseteq N \mid i \in S\}$ need to be considered. In order to speak about the "average marginal contribution", the model furthermore assumes:

(i) The cardinalities $|X|$ of the coalitions $X \in \mathcal{N}_i$ are distributed uniformly.
(ii) The coalitions $X \in \mathcal{N}_i$ of the same cardinality $|X| = s$ are distributed uniformly.

Under these probabilistic assumptions, the coalition $S \in \mathcal{N}_i$ of cardinality $|S| = s$ occurs with probability

$$p_S = \frac{1}{n} \cdot \frac{1}{\binom{n-1}{s-1}} = \frac{(s-1)!(n-s)!}{n!}, \tag{17}$$

which exhibits the Shapley value as a probabilistic (and hence approximation) value:

$$\sum_{S \in \mathcal{N}_i} p_S[v(S) - v(S \setminus i)] = \sum_{S \in \mathcal{N}} p_S[v(S) - v(S \setminus i)] = \Phi_i^{\mathrm{Sh}}(v).$$

REMARK. Among the probabilistic values, the Shapley value can also be characterized as the one with the largest entropy (Faigle and Voss [6]).

In contrast to the Shapley model, the assumption that all coalitions in $\mathcal{N}_i$ are equally likely assigns to any coalition $S \in \mathcal{N}_i$ the probability

$$\overline{p}_S = \frac{1}{2^{n-1}} \tag{18}$$

with the Banzhaf value [1] as the associated probabilistic value:

$$\sum_{S \in \mathcal{N}_i} \overline{p}_S[v(S) - v(S \setminus i)] = \sum_{S \in \mathcal{N}} \overline{p}_S[v(S) - v(S \setminus i)] = B_i^v.$$

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