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Abstract

Models in Multicriteria Decision Analysis (MCDA) can be analyzed by means of an importance index and an interaction index for every group of criteria. We consider first discrete models in MCDA, without further restriction, which amounts to considering multichoice games, that is, cooperative games with several levels of participation. We propose and axiomatize two interaction indices for multichoice games: the signed interaction index and the absolute interaction index. In a second part, we consider the continuous case, supposing that the continuous model is obtained from a discrete one by means of the Choquet integral. We show that, as in the case of classical games, the interaction index defined for continuous aggregation functions coincides with the (signed) interaction index, up to a normalizing coefficient.

Keywords: multicriteria decision analysis, interaction, multichoice game, Choquet integral

1 Introduction

An important issue in MultiCriteria Decision Analysis (MCDA) is to be able to analyze and explain a numerical model, obtained by elicitation of preferences of the decision maker. A classical way to do this is to assess the importance of each criterion (see a general approach to define an importance index in (Ridaoui et al., 2017a)). This description of the model may appear to be sufficient in the case of simple models, which are additive in essence (e.g., additive utility models), as it is well known that they imply mutual preferential independence of criteria (Keeney and Raiffa, 1976). However, in case of more complex models, the preferential independence among criteria does not hold any more, and *interaction* appears among criteria, so that a description of the model by the sole importance indices is not sufficient any more. For example, for models where aggregation of preference is done through a Choquet integral w.r.t. a capacity, an interaction index is defined for any group of criteria (Grabisch and Labreuche, 2010), which is a generalization of the interaction index for pairs of criteria proposed by Murofushi and Soneda (1993). Roughly speaking, a positive interaction index induces a conjunctive behavior (like the minimum operator), while a negative interaction index induces a disjunctive behavior (maximum).

The aim of the paper is to propose an axiomatic foundation of an interaction index for a MCDA model with no special restriction (and in particular, mutual preferential independence is not supposed to hold). In a first step, the attributes are supposed to be defined on a finite universe. Then, such a model is equivalent to what is called a multichoice game in game theory (Hsiao and Raghavan, 1993), that is, a game on a set of players N, where each player can play at a level of participation represented by an integer between 0 and k. Up to our knowledge, there is no definition of an interaction index for multichoice games. Nevertheless, there exists a general form of interaction index for games on lattices (Grabisch and Labreuche, 2007), and multichoice games with k levels can be considered as games on the lattice $(k+1)^N$. This interaction index is defined, however, for any element of the lattice $x \in (k+1)^N$, i.e., any profile of participation of the players. This does not make sense for our purpose, since we are looking for an interaction index defined for *groups* of players/criteria. It is the contribution of this paper to provide two definitions for an interaction index, and to give a characterization of them. The first one is a natural generalization of the interaction index for classical games, which we call *signed interaction index* as it can take positive or negative values. The second one is an *absolute interaction index*, because it cumulates only amounts of interaction without considering its signs. The latter permits to avoid cancellation of local interaction effects, yielding counterintuitive results.

The paper is organized as follows. Section 2 introduces the necessary material and notation. Section 3 summarizes previous works on the interaction index (the case of classical games and the case of games on lattices). Our work on the importance index for multichoice games is summarized in Section 4, since some of the axioms are necessary for our approach. Section 5 and 6 give the main results of the paper, which are the definition and characterization of two interaction indices for multichoice games, and consequently for general discrete MCDA models. In Section 7, we address the continuous case, supposing that the model is obtained from a discrete one via the Choquet integral.

2 Preliminaries

Throughout the paper, the cardinality of sets will be denoted by corresponding lower case letters, i.e., $|N| := n$, $|S| := s$, etc. For notational convenience, we will omit braces for singletons, i.e., $S \cup \{i\}$ is written $S \cup i$, etc.

Let $N = \{1, \dots, n\}$ be a fixed and finite set which can be thought as the set of attributes or criteria (in MCDA), players (in cooperative game theory), etc., depending on the domain of application. In this paper, we will mainly focus on MCDA applications.

We suppose that each attribute $i \in N$ takes values in a set L_i, which is supposed to be finite[1] and denoted by $L_i = \{0, 1, \dots, k_i\}$. The alternatives are represented as elements of the Cartesian product $L := L_1 \times \dots \times L_n$. An alternative is thus written as a vector $x = (x_1, \dots, x_n)$ where $x_i \in L_i$ for all $i \in N$. For each $x \in L$, we denote by

[1]The continuous case will be addressed in Section 7.

$S(x) = \{i \in N \mid x_i > 0\}$ the support of x, and by $\mathrm{K}(x) = \{i \in N | x_i = k_i\}$ the kernel of x.

For each $i \in N$, we denote by L_{-i} the set $\times_{j \neq i} L_j$. For each $y_{-i} \in L_{-i}$, and any $\ell \in L_i$, (y_{-i}, ℓ_i) denotes the compound alternative x such that $x_i = \ell$ and $x_j = y_j, \forall j \neq i$. More generally, for any $T \in 2^N \setminus \{\emptyset\}$, for any $x, y \in L$, (x_T, y_{-T}) denotes the compound alternative, while L_T and L_{-T} denote the restricted Cartesian products of attributes.

The vector $0_N = (0, \ldots, 0)$ is the null alternative of L, and $k_N = (k_1, \ldots, k_n)$ is the top element of L. Similarly, we use the notation $0_T, 1_T, k_T$, etc., for any $T \in 2^N \setminus \{\emptyset\}$. With some abuse, we also often make use of $(k-1)_T \in L_T$ as a shorthand for $k_T - 1_T = (k_i - 1)_{i \in T}$. We write $x \leq y$ if $x_i \leq y_i$ for every $i \in N$, $x_T < k_T$ if $x_T \leq (k-1)_T$, $x_T > 0_T$ if $x_T \geq 1_T$ and $x \lneq y$ if $x \leq y$ and $x \neq y$.

The preferences of a Decision Maker (DM) over the alternatives are supposed to be represented by a function $v : L \to \mathbb{R}$. For the sake of generality, we do not make any assumption on v, except that

$$v(0_N) = 0. \tag{1}$$

For convenience, we assume from now on that all attributes have the same number of elements, i.e., $k_i = k$ for every $i \in N$ ($k \in \mathbb{N}$). Note that if this is not the case, we set $k = \max_{i \in N} k_i$, and we extend $v : L \to \mathbb{R}$ to $v' : \{0, \ldots, k\}^N \to \mathbb{R}$ by

$$v'(x) = v(y) \text{ where } y_i = \min(x_i, k_i) \; \forall i \in N.$$

This amounts to duplicating the last element k_i of L_i when $k_i < k$. Under this assumption, we recover well-known concepts.

When $k = 1$, v is a *pseudo-Boolean function* $v : \{0,1\}^N \to \mathbb{R}$ vanishing at 0_N. Equivalently, it can be seen as a set function $v : 2^N \to \mathbb{R}$, with $v(\emptyset) = 0$, which is a *game* in cooperative game theory. A *capacity* (Choquet, 1953) or *fuzzy measure* (Sugeno, 1974) is a monotone game, i.e., satisfying $v(A) \leq v(B)$ whenever $A \subseteq B$. For the general case (when $k \geq 1$), $v : L \to \mathbb{R}$ fulfilling (1) corresponds exactly to the concept of *multichoice game* (Hsiao and Raghavan, 1993), and the numbers $0, 1, \ldots, k$ in L_i are seen as the levels of activity of the players. A *k-ary capacity* (Grabisch and Labreuche, 2003) is a multichoice game v satisfying the monotonicity condition: or each $x, y \in L$ s.t. $x \leq y$, $v(x) \leq v(y)$ and the normalization condition: $v(k, \ldots k) = 1$. Hence, a k-ary capacity represents a preference on L which is increasing with the value of the attributes.

Consider the following example in which decision strategies depend on the values of the attributes.

Example 1 (**System engineers**)**.** Consider an engineering problem with two performance criteria (e.g., the latency of the system and the quality of the system) evaluated on the scale $\{0, 1, 2\}$, where 0 means that the performance is not met, 1 means that the performance is medium, and 2 means that the performance is completely met. Consider the following 2-ary capacity representing the overall satisfaction on the system (the preferences are depicted in Figure 1):

$$\forall x \in \{0,1,2\}^2, \quad v(x) = \begin{cases} x_1 \wedge x_2, & \text{if } x \in \{0,1\}^2, \\ x_1 \vee x_2, & \text{if } x \in \{1,2\}^2, \\ x_1 + x_2 - 1, & \text{otherwise.} \end{cases}$$

For $x \in \{0,1\}^2$, the decision maker is not satisfied at all when one criterion is not met at all, and the other one is at most medium. This corresponds to an intolerant behaviour. On the other hand, for $x \in \{1,2\}^2$, the decision maker is completely satisfied when one criterion is completely met and the other one is at least medium. This corresponds to a tolerant behaviour.

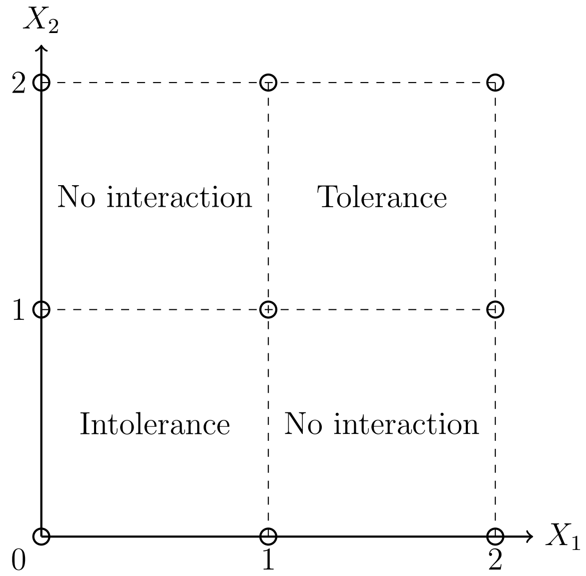

Figure 1: Decision strategies.

Any multichoice game v can be written as:

$$v = \sum_{x \in L \setminus \{0_N\}} m^v(x) u_x, \tag{2}$$

where $m^v(x), x \in L$, is the Möbius transform of v (Rota, 1964) given by

$$m^v(x) = \sum_{\substack{y \leq x \\ x_i - y_i \leq 1, \forall i \in N}} (-1)^{\sum_{i \in N}(x_i - y_i)} v(y),$$

and u_x is a multichoice game (called the unanimity multichoice game) defined by

$$u_x(y) = \begin{cases} 1, & \text{if } y \geq x \\ 0, & \text{otherwise.} \end{cases}$$

Note that the games $u_x, x \in L \setminus \{0_N\}$, are linearly independent, and form a basis of the vector space of multichoice games. We denote by $\mathcal{G}(L)$ the set of multichoice games defined on L, and $\mathcal{G}_+(L)$ the set of multichoice games whose Möbius transform is non-negative. Any multichoice game v can be uniquely decomposed into $v = v^+ - v^-$, with $v^+, v^- \in \mathcal{G}_+(L)$.

Another family of games which form a basis is the family of *Dirac games*, defined for every $x \in L \setminus \{0_N\}$ by

$$\delta_x(y) = \begin{cases} 1, & \text{if } x = y \\ 0, & \text{otherwise.} \end{cases}$$

This yields the decomposition

$$v = \sum_{x \in L \setminus \{0_N\}} v(x)\delta_x. \tag{3}$$

The *derivative* of $v \in \mathcal{G}(L)$ at $x \in L$ w.r.t. $i \in N$ such that $x_i < k$ is defined by

$$\Delta_i v(x) = v(x + 1_i) - v(x).$$

The derivative of $v \in \mathcal{G}(L)$ at $x \in L$ w.r.t. $T \in 2^N \setminus \{\emptyset\}$ such that $\forall i \in T, x_i < k$ is defined recursively as follows,

$$\Delta_T v(x) = \Delta_i(\Delta_{T \setminus i} v(x)).$$

The general expression for the derivative of $v \in \mathcal{G}(L)$ is given by,

$$\Delta_T v(x) = \sum_{A \subseteq T} (-1)^{t-a} v(x + 1_A), \forall T \subseteq N, x_i < k, \forall i \in T.$$

Observe that $\Delta_\emptyset v = v$. For $T \subseteq N$, we denote by $\mathcal{G}_T(L)$ the set of multichoice games whose derivative w.r.t. T and derivative w.r.t. any subset $S \subseteq N \setminus T$ such that $S \neq \emptyset$ are nonnegative.

The following lemma gives a general expression for the derivative in terms of the Möbius transform.

Lemma 1. For any $v \in \mathcal{G}(L)$, any $T \subseteq N$, we have

$$\Delta_T v(x) = \sum_{y_{-T} \leq x_{-T}} m^v(y_{-T}, x_T + 1_T), \forall x \in L, \text{ such that } x_i < k, \forall i \in T.$$

Proof. Let us proceed by induction on $|T|$. For $|T| = 0$, we have

$$\Delta_\emptyset v(x) = v(x) = \sum_{y \leq x} m(y), \forall x \in L.$$

Assume the formula holds for $|T| = \ell$ and let us prove it still holds for $T \cup \{i\}$ with $|T| = \ell$ and $i \in N \setminus T$. We have, for any $x \in L$ such that $\forall j \in T \cup \{i\}, x_j < k$,

$$\begin{aligned}
\Delta_{T \cup i} v(x) &= \Delta_T v(x + 1_i) - \Delta_T v(x) \\
&= \sum_{\substack{y_{-T \cup i} \leq x_{-T \cup i} \\ y_i \leq x_i + 1}} m^v(y_{-T}, x_T + 1_T) - \sum_{\substack{y_{-T \cup i} \leq x_{-T \cup i} \\ y_i \leq x_i}} m^v(y_{-T}, x_T + 1_T) \\
&= \sum_{y_{-T \cup i} \leq x_{-T \cup i}} \Big(\sum_{y_i \leq x_i + 1} m^v(y_{-T}, x_T + 1_T) - \sum_{y_i \leq x_i} m^v(y_{-T}, x_T + 1_T) \Big) \\
&= \sum_{y_{-T \cup i} \leq x_{-T \cup i}} m^v(y_{-T \cup i}, x_{T \cup i} + 1_{T \cup i})
\end{aligned}$$

□

Let us remark that, if $v \in \mathcal{G}_+(L)$ then its derivative w.r.t. any element is nonnegative everywhere.

3 Values and interaction indices

3.1 The case of classical TU-games

In cooperative game theory, the notion of value or power index is one of the most important concepts. A *value* is a function $\phi : \mathcal{G}(2^N) \to \mathbb{R}^N$ which assigns a payoff vector to any game $v \in \mathcal{G}(2^N)$. In MCDA, values are interpreted as importance indices for criteria. The *Shapley value* (Shapley, 1953) of player $i \in N$ is given by

$$\phi_i(v) = \sum_{S \subseteq N \setminus i} \frac{(n-s-1)!s!}{n!} \big(v(S \cup i) - v(S)\big), \forall v \in \mathcal{G}(2^N).$$

The concept of interaction index, which is an extension of that of value, was introduced axiomatically to measure the interaction phenomena among players in cooperative game theory or criteria in multicriteria decision analysis. For a game $v \in \mathcal{G}(2^N)$, the *interaction index* of v is a function $I^v : 2^N \to \mathbb{R}$ that assigns to every coalition $T \subseteq N$ its interaction degree.

Murofushi and Soneda (1993) proposed an interaction index $I(ij)$ for a pair of elements $i, j \in N$ to estimate to which degree i and j interact. Grabisch (1997) defined and extended the interaction index to coalitions containing more than two players. The interaction index (Grabisch, 1997) of a nonempty coalition $S \subseteq N$ in a game $v \in \mathcal{G}(2^N)$ is defined by

$$I^v_{Sh}(S) = \sum_{T \subseteq N \setminus S} \frac{(n-t-s)!t!}{(n-s+1)!} \sum_{K \subseteq S} (-1)^{s-k} v(K \cup T). \tag{4}$$

Note that when $S = \{i\}$, the interaction index coincides with the Shapley value. Moreover that the expression is still valid for $S = \emptyset$, and represents in MCDA the average value of the Choquet integral w.r.t. v over $[0,1]^n$. It is thus closely related to the so-called "*orness*".

A first axiomatization of the interaction index has been proposed by Grabisch and Roubens (1999), and it is axiomatized in a way similar to the Shapley value. The following axioms have been considered by Grabisch and Roubens :

- Linearity axiom (L): $I^v(S)$ is linear on $\mathcal{G}(2^N)$ for every nonempty $S \subseteq N$.
- Dummy axiom (D): For any $v \in \mathcal{G}(2^N)$, and any $i \in N$ dummy for v, $I^v(S \cup i) = 0, \forall S \subseteq N \setminus i$ ($i \in N$ is said to be *dummy* for v if $\forall S \subseteq N \setminus i, v(S \cup i) = v(S) + v(i)$).
- Symmetry axiom (S) : For any $v \in \mathcal{G}(2^N)$, any permutation σ on N and any nonempty $S \subseteq N$, $I^v(S) = I^{\sigma v}(\sigma S)$.
- Efficiency axiom (E) : For any $v \in \mathcal{G}(2^N)$ and any $i \in N$, $\sum_{i \in N} I^v(i) = v(N)$.
- Recursive axiom (R1): For any $v \in \mathcal{G}(2^N)$ and any $S \subseteq N, s > 1$,

  $$I^v(S) = I^{v^{-j}_{\cup j}}(S \setminus j) - I^{v^{-j}}(S \setminus j), \forall j \in S,$$

  where v^{-j} is the game v restricted to elements in $N \setminus j$, defined by $v^{-j}(S) = v(S), \forall S \subseteq N \setminus j$, and $v^{-j}_{\cup j}$ is the game on $N \setminus j$ in the presence of j defined by $v^{-j}_{\cup j}(S) = v(S \cup j) - v(S), \forall S \subseteq N \setminus j$.

- Recursive axiom (R2): For any $v \in \mathcal{G}(2^N)$ and any $S \subseteq N, s > 1$,

$$I^v_{Sh}(S) = I^{v_{[S]}}([S]) - \sum_{\substack{K \subseteq N \setminus S \\ K \neq \emptyset, S}} I^{v^{-K}}(S \setminus K),$$

  where $v_{[S]}$ is the game where all elements in S are considered as a single element denoted $[S]$, and defined by, for any $K \subseteq N \setminus S$:

$$v_{[S]}(K) = v(K),$$
$$v_{[S]}(K \cup [S]) = v(K \cup S).$$

Axiom (R1) says that the interaction of the players in S is equal to the interaction between the criteria in $S \setminus j$ in the presence of j minus the interaction between the criteria of $S \setminus j$ in the absence of j. Axiom (R2) expresses interaction of S in terms of all successive interactions of subsets. The authors have shown that (R1) and (R2) are equivalent under (L), (D) and (S) axioms.

The following theorem was shown by Grabisch and Roubens (1999).

Theorem 1. The interaction index I_{Sh} given in (4) is the unique interaction index satisfying axioms (L), (D), (D), (E), and ((R1) or (R2)) on $\mathcal{G}(2^N)$.

3.2 The case of games on lattices

Grabisch and Labreuche (2007) generalized the notion of interaction defined for criteria modelled by capacities, by considering functions defined on lattices. The interaction (Grabisch and Labreuche, 2007) is based on the notion of derivative of a function defined on a lattice. For this, they introduced the following definitions:

Let $i = (0_{-j}, i_j)$ with $i_j \in L_j, j \in N$. Let $x, y \in L$ with $y = \vee_{k=1}^n i_k$ and $v \in \mathcal{G}(L)$. The derivative of v w.r.t. i at point $x \in L$ is given by:

$$\Delta_i v(x) = v(x \vee i) - v(x),$$

and the derivative of v w.r.t. y at x is given by:

$$\Delta_y v(x) = \Delta_{i_1}(\Delta_{i_2}(\ldots \Delta_{i_n} v(x) \ldots)).$$

The following definition has been proposed in (Grabisch and Labreuche, 2007) :

Definition 1. Let $J \subseteq N$, and $x = \vee_{j \in J} i_j$, with $i_j = (0_{-j}, \ell_j), \ell_j \in L_j \setminus \{0\}$.

$$I^v(x) = \sum_{y \in A(x)} \alpha^j_{h(y)} \Delta_x v(y),$$

where, $A(x) = \{y \in L | y_j = k \text{ or } 0 \text{ if } j \notin J, y_j = x_j - 1 \text{ else } \}$, $h(y)$ is the number of components of y to k and $\alpha^j_{h(y)} = \frac{(n-j-h(y))!h(y)!}{(n-j+1)!}$.

4 Characterization of the importance index for multichoice games

In this section, we present the importance index (value) for multichoice games defined by Ridaoui et al. (2017b) together with its axiomatization. Let ϕ be a value defined for any $v \in \mathcal{G}(L)$.

Linearity axiom (L) : ϕ is linear on $\mathcal{G}(L)$, i.e., $\forall v, w \in \mathcal{G}(L), \forall \alpha \in \mathbb{R}$,

$$\phi_i(v + \alpha w) = \phi_i(v) + \alpha \phi_i(w), \forall i \in N.$$

Let σ be a permutation on N. For all $x \in L$, we denote $\sigma(x)_{\sigma(i)} = x_i$. For all $v \in \mathcal{G}(L)$, the game $\sigma \circ v$ is defined by $\sigma \circ v(\sigma(x)) = v(x)$.

Symmetry axiom (S): For any permutation σ of N,

$$\phi_{\sigma(i)}(\sigma \circ v) = \phi_i(v), \forall i \in N.$$

Invariance axiom (I): Let us consider two games $v, w \in \mathcal{G}(L)$ such that, for some $i \in N$,

$$\Delta_i v(x) = \Delta_i w(x - 1_i), \quad x_{-i} \in L_{-i}, x_i \in \{0, \dots, k-1\} (\text{mod } k).$$

Then $\phi_i(v) = \phi_i(w)$.

Note that the modulo means that for $x_i = 0$, the condition reads $\Delta_i v(x_{-i}, 0_i) = \Delta_i w(x_{-i}, (k-1)_i)$.

Efficiency axiom (E): For all $v \in \mathcal{G}(L)$,

$$\sum_{i \in N} \phi_i(v) = \sum_{\substack{x \in L \\ x_j < k}} \big(v(x + 1_N) - v(x)\big).$$

Ridaoui et al. (2017a) have shown the following result.

Theorem 2. Let ϕ be a value defined for any $v \in \mathcal{G}(L)$. If ϕ fulfills (L), (I), (S) and (E) then

$$\phi_i(v) = \sum_{x_{-i} \in L_{-i}} \frac{\big(n - \sigma(x_{-i}) - 1\big)! \kappa(x_{-i})!}{\big(n + \kappa(x_{-i}) - \sigma(x_{-i})\big)!} \big(v(x_{-i}, k_i) - v(x_{-i}, 0_i)\big), \forall i \in N \tag{5}$$

5 Axiomatization of the signed interaction index

In this section we intend to define axiomatically an interaction index for multichoice games, which we call *signed interaction index* as it can take positive or negative values. The approach presented here is based on a recursion formula, starting from the importance index (value) defined in Section 4, as in (Grabisch and Roubens, 1999). An interaction index of the k-ary multichoice game $v \in \mathcal{G}(L)$ is a function $I^v : 2^N \setminus \{\emptyset\} \to \mathbb{R}$.

The first axiom (L) is trivially generalized for the interaction index.

Linearity axiom (L) : I^v is linear on $\mathcal{G}(L)$, i.e., $\forall v, w \in \mathcal{G}(L), \forall \alpha \in \mathbb{R}$,

$$I^{v+\alpha w} = I^v + \alpha I^w.$$

Proposition 1. Under **(L)**, for every nonempty $T \subseteq N$, there exist real constants a_x^T, for all $x \in L$, such that for every $v \in \mathcal{G}(L)$

$$I^v(T) = \sum_{x \in L} a_x^T v(x). \tag{6}$$

Proof . It is easy to check that the above formula satisfies the linearity axiom. Conversely, we consider I^v satisfying **(L)**. Using the decomposition with Dirac games (3) and **(L)**, we get

$$I^v(T) = \sum_{x \in L} v(x) I^{\delta_x}(T), \forall T \in 2^N \setminus \{\emptyset\}.$$

Setting $a_x^T = I^{\delta_x}(T), \forall x \in L, \forall T \in 2^N \setminus \{\emptyset\}$, we obtain the wished result. □

Remark 1. Let $i \in N$ be a null criterion for $v \in \mathcal{G}(L)$. We have,

$$\forall T \subseteq N, T \ni i, \Delta_T v(x) = 0, \forall x \in L, x + 1_T \leq k_T.$$

$$\forall T \subseteq N, T \ni i, \Delta_j v(x) = 0, \forall j \in T, \forall x \in L, x + 1_j \leq k.$$

Null axiom (N): If a criterion i is null for $v \in \mathcal{G}(L)$, then for all $T \subseteq N$ such that $T \ni i$, $I^v(T) = 0$.

Proposition 2. Under axioms **(L)** and **(N)**, for every nonempty $T \subseteq N$, there exist real constants b_x^T, for all $x \in L$, with $x + 1_T \leq k_T$, such that for every $v \in \mathcal{G}(L)$

$$I^v(T) = \sum_{\substack{x \in L \\ x_T < k_T}} b_x^T \Delta_T v(x). \tag{7}$$

To prove this result, the following lemmas are useful.

Lemma 2. Let $A \subseteq N$.

$$a_{(x_A, x_{-A})} = \sum_{C \subseteq A} (-1)^{a-c} \sum_{\substack{\ell_C = x_C \\ \ell_{A \setminus C} = (x+1)_{A \setminus C}}}^{k_A} a_{(\ell_C, \ell_{A \setminus C}, x_{-A})}, \forall x_A \in L_A \setminus \{k_A\}.$$

Proof . Let $A \subseteq N$. We proceed by induction on $|A|$. The relation is obviously true for $|A| = 0$. Let us suppose that the relation is true for any set A of at most $l-1$ elements, and try to show it is also true for any set A of l elements. We have , for all $x_A \in L_A \setminus \{k\}^A$,

$$\begin{aligned}
a_{(x_A,x_{-A})} &= a_{(x_{A\setminus i},x_i,x_{-A})} \\
&= \sum_{C\subseteq A\setminus i} (-1)^{a-c-1} \sum_{\substack{\ell_C = x_C \\ \ell_{A\setminus C\cup i}=(x+1)_{A\setminus C\cup i}}}^{k_{A\setminus i}} a_{(\ell_C,\ell_{A\setminus C\cup i},x_i,x_{-A})} \\
&= \sum_{C\subseteq A\setminus i} (-1)^{a-c-1} \sum_{\substack{\ell_C = x_C \\ \ell_{A\setminus C\cup i}=(x+1)_{A\setminus C\cup i}}}^{k_{A\setminus i}} \Big(\sum_{\ell_i=x_i}^{k} a_{(\ell_C,\ell_{A\setminus C\cup i},\ell_i,x_{-A})} - \sum_{\ell_i=x_i+1}^{k} a_{(\ell_C,\ell_{A\setminus C\cup i},\ell_i,x_{-A})}\Big) \\
&= \sum_{C\subseteq A\setminus i} (-1)^{a-c-1} \Big(\sum_{\substack{\ell_{C\cup i} = x_{C\cup i} \\ \ell_{A\setminus C\cup i}=(x+1)_{A\setminus C\cup i}}}^{k_A} a_{(\ell_C,\ell_{A\setminus C\cup i},\ell_i,x_{-A})} - \sum_{\substack{\ell_C = x_C \\ \ell_{A\setminus C}=(x+1)_{A\setminus C}}}^{k_A} a_{(\ell_C,\ell_{A\setminus C\cup i},\ell_i,x_{-A})}\Big) \\
&= \sum_{C\subseteq A\setminus i} \Big((-1)^{a-c-1} \sum_{\substack{\ell_{C\cup i} = x_{C\cup i} \\ \ell_{A\setminus C\cup i}=(x+1)_{A\setminus C\cup i}}}^{k_A} a_{(\ell_C,\ell_{A\setminus C},x_{-A})} + (-1)^{a-c} \sum_{\substack{\ell_C = x_C \\ \ell_{A\setminus C}=(x+1)_{A\setminus C}}}^{k_A} a_{(\ell_C,\ell_{A\setminus C},x_{-A})}\Big) \\
&= \sum_{C\subseteq A} (-1)^{a-c} \sum_{\substack{\ell_C = x_C \\ \ell_{A\setminus C}=(x+1)_{A\setminus C}}}^{k_A} a_{(\ell_C,\ell_{A\setminus C},x_{-A})}
\end{aligned}$$

□

Lemma 3.

$$\sum_{\substack{x\in L \\ x<k_N}} b_x \sum_{A\subseteq N} (-1)^{n-a} v(x+1_A) = \sum_{\substack{A\subseteq N \\ 0_A<x_A<k_A}} \sum_{\substack{B\subseteq N\setminus A \\ C\subseteq A}} (-1)^{a+b-c} b_{(x_A-1_C,0_B,(k-1)_{N\setminus A\cup B})} v(x_A,0_B,k_{N\setminus A\cup B})$$

Proof . We shall proceed by induction on n. For simplicity, we denote $N\setminus i$ by S, $(x_A,0_B,k_{S\setminus A\cup B})$ by $x^S_{A,B}$ and $(x_A-1_C,0_B,(k-1)_{S\setminus A\cup B})$ by $x^S_{A,C,B}$, with $C\subseteq A$. The relation is obviously true for $n=1$. Let us suppose that the relation is true for any set of at most $n-1$ elements, and try to show it is also true for any set of n elements. We have

$$\sum_{\substack{x\in L\\ x<k_N}} b_x \sum_{A\subseteq N}(-1)^{n-a}v(x+1_A)$$

$$= \sum_{\substack{x\in L\\ x<k_N}} b_x \sum_{A\subseteq N\setminus i}(-1)^{n-a}\big(v(x+1_A)-v(x+1_{A\cup i})\big)$$

$$= \sum_{x_i<k}\sum_{\substack{x_{-i}\in L_{-i}\\ x_{-i}<k_{-i}}} b_{x_{-i},x_i}\sum_{A\subseteq S}(-1)^{s-a}\big(v(x_{-i}+1_A,x_i+1)-v(x_{-i}+1_A,x_i)\big)$$

$$= \sum_{\substack{A\subseteq S\\ 0_A<x_A<k_A\\ x_i<k}}\ \sum_{\substack{B\subseteq S\setminus A\\ C\subseteq A}}(-1)^{a+b-c}b_{(x^S_{A,C,B},x_i)}\big(v(x^S_{A,B},x_i+1)-v(x^S_{A,B},x_i)\big)$$

$$= \sum_{\substack{A\subseteq S\\ 0_A<x_A<k_A}}\Bigg[\sum_{\substack{B\subseteq S\setminus A\\ C\subseteq A}}\Big((-1)^{a+b-c}b_{(x^S_{A,C,B},k_i-1)}v(x^S_{A,B},k_i)+(-1)^{a+b+1-c}b_{(x^S_{A,C,B},0_i)}v(x^S_{A,B},0_i)\Big)$$

$$+\sum_{\substack{B\subseteq S\setminus A\\ C\subseteq A\\ 0<x_i<k}}\Big((-1)^{a+1+b-c}b_{(x^S_{A,C,B},x_i)}v(x^S_{A,B},x_i)+(-1)^{a+b-c}b_{(x^S_{A,C,B},x_i-1)}v(x^S_{A,B},x_i)\Big)\Bigg]$$

$$= \sum_{\substack{A\subseteq S\\ 0_A<x_A<k_A}}\Bigg[\sum_{\substack{B\subseteq S\setminus A\\ C\subseteq A}}\Big((-1)^{a+b-c}b_{(x^N_{A,C,B})}v(x^N_{A,B})+(-1)^{a+b+1-c}b_{(x^S_{A,C,B\cup i})}v(x^S_{A,B\cup i})\Big)$$

$$+\sum_{\substack{B\subseteq S\setminus A\\ C\subseteq A\\ 0<x_i<k}}\Big((-1)^{a+1+b-c}b_{(x^S_{A\cup i,C,B})}v(x^S_{A\cup i,B})+(-1)^{a+b-c}b_{(x^S_{A\cup i,C\cup i,B})}v(x^S_{A\cup i,B})\Big)\Bigg]$$

$$= \sum_{\substack{A\subseteq S\\ 0_A<x_A<k_A}}\Bigg(\sum_{\substack{B\subseteq N\setminus A\\ C\subseteq A}}(-1)^{a+b-c}b_{(x^N_{A,C,B})}v(x^N_{A,B})+\sum_{\substack{B\subseteq S\setminus A\\ C\subseteq A\cup i\\ 0<x_i<k}}(-1)^{a+1+b-c}b_{(x^S_{A\cup i,C,B})}v(x^S_{A\cup i,B})\Bigg)$$

$$= \sum_{\substack{A\subseteq N\\ 0_A<x_A<k_A}}\ \sum_{\substack{B\subseteq N\setminus A\\ C\subseteq A}}(-1)^{a+b-c}b_{(x_A-1_C,0_B,(k-1)_{N\setminus A\cup B})}v(x_A,0_B,k_{N\setminus A\cup B})$$

which is the desired result. □

We now prove Proposition 2.

Proof . It is easy to check that the formula satisfies the axioms. Conversely, we consider I^v satisfying **(L)** and **(N)**. Let $v\in\mathcal{G}(L)$ and $T\in 2^N\setminus\{\emptyset\}$.
By Proposition 1, there exists $a^T_x\in\mathbb{R}$, for all $x\in L$, such that,

$$I^v(T)=\sum_{x\in L}a^T_x v(x).$$

Then,

$$I^v(T)=\sum_{x_{-i}\in L_{-i}}\sum_{x_i\in L_i}a^T_{(x_{-i},x_i)}v(x_{-i},x_i).$$

Assume now that i is null for v. We have $v(x_{-i}, x_i) = v(x_{-i}, 0_i)$. Hence,

$$I^v(T) = \sum_{x_{-i}\in L_{-i}} \sum_{x_i\in L_i} a^T_{(x_{-i},x_i)} v(x_{-i}, 0_i).$$

By **(N)**, we have, for all $i \in T$ null, and for all $x_{-i} \in L_{-i}$,

$$\sum_{x_i\in L_i} a^T_{(x_{-i},x_i)} = 0.$$

$\forall x_{-T} \in L_{-T}$, let $A \subseteq T, B \subseteq T \setminus A$, and set

$$b^T_{(x_A,0_B,(k-1)_{T\setminus A\cup B},x_{-T})} = (-1)^b \sum_{\ell_A=(x+1)_A}^{k_A} a^T_{(\ell_A,0_B,k_{T\setminus A\cup B},x_{-T})}, \forall x_A \in L_A \setminus \{0, k\}^A.$$

Then, we have, $\forall x_{-T} \in L_{-T}, \forall A \subseteq T, \forall B \subseteq T \setminus A, \forall x_A \in L_A \setminus \{0, k\}^A$,

$$\begin{aligned}
a^T_{(x_A,0_B,k_{T\setminus A\cup B},x_{-T})} &= \sum_{C\subseteq A} (-1)^{a-c} \sum_{\substack{\ell_C=x_C\\ \ell_{A\setminus C}=(x+1)_{A\setminus C}}}^{k_A} a^T_{(\ell_C,\ell_{A\setminus C},0_B,k_{T\setminus A\cup B},x_{-T})} \text{ (using Lemma 2)}\\
&= \sum_{C\subseteq A} (-1)^{a-c} \sum_{\substack{\ell_C=(x_C-1_C)+1_C\\ \ell_{A\setminus C}=x_{A\setminus C}+1_{A\setminus C}}}^{k_A} a^T_{(\ell_C,\ell_{A\setminus C},0_B,k_{T\setminus A\cup B},x_{-T})}\\
&= (-1)^b \sum_{C\subseteq A} (-1)^{a-c}(-1)^b \sum_{y=((x-1_C)+1)_A}^{k_A} a^T_{(y,0_B,k_{T\setminus A\cup B},x_{-T})}\\
&= (-1)^b \sum_{C\subseteq A} (-1)^{a-c} b^T_{(x_A-1_C,0_B,(k-1)_{T\setminus A\cup B},x_{-T})}
\end{aligned}$$

Therefore, it suffices to replace the values of a^T_x in the formula (6), and then the result is established.

$$\begin{aligned}
I^v(T) &= \sum_{x_{-T}\in L_{-T}} \sum_{x_T\in L_T} a^T_{(x_T,x_{-T})} v(x_T, x_{-T})\\
&= \sum_{x_{-T}\in L_{-T}} \sum_{\substack{A\subseteq T\\ 0_A<x_A<k_A}} \sum_{B\subseteq T\setminus A} a^T_{(x_A,0_B,k_{T\setminus A\cup B},x_{-T})} v(x_A, 0_B, k_{T\setminus A\cup B}, x_{-T})\\
&= \sum_{x_{-T}\in L_{-T}} \sum_{\substack{A\subseteq T\\ 0_A<x_A<k_A}} \sum_{B\subseteq T\setminus A} (-1)^b \sum_{C\subseteq A} (-1)^{a-c} b^T_{(x_A-1_C,0_B,(k-1)_{T\setminus A\cup B},x_{-T})} v(x_A, 0_B, k_{T\setminus A\cup B}, x_{-T})\\
&= \sum_{x_{-T}\in L_{-T}} \sum_{\substack{x_T\in L_T\\ x_T<k_T}} b^T_x \sum_{A\subseteq T} (-1)^{t-a} v(x + 1_A) \text{ (using Lemma 3)}\\
&= \sum_{\substack{x\in L\\ x_T<k_T}} b^T_x \sum_{A\subseteq T} (-1)^{t-a} v(x + 1_A).
\end{aligned}$$

□

The next axiom generalizes the invariance axiom introduced for the importance index.

Invariance axiom (I): Let us consider two functions $v, w \in \mathcal{G}(L)$ and a nonempty set $T \subseteq N$ such that, for all $i \in T$,

$$\Delta_i v(x) = \Delta_i w(x - 1_i), x_{-i} \in L_{-i}, x_i \in \{0, \ldots, k-1\} (\text{mod } \ k).$$

Then $I^v(T) = I^w(T)$.

Proposition 3. Under axioms **(L)**, **(N)** and **(I)**, $\forall v \in \mathcal{G}(L), \forall T \in 2^N \setminus \{\emptyset\}$,

$$I^v(T) = \sum_{x_{-T} \in L_{-T}} b^T_{x_{-T}} \sum_{S \subseteq T} (-1)^{t-s} v(0_S, k_{T \setminus S}, x_{-T}).$$

Proof . It is easy to check that the above formula satisfies the axioms. Conversely, we consider I^v satisfying **(L)**, **(N)** and **(I)**. Let $v, w \in \mathcal{G}(L)$ such that v, w satisfy the premise of the Invariance axiom and consider $T \in 2^N \setminus \{\emptyset\}$. By Proposition 2, we have, for any $i \in T$

$$\begin{aligned}
I^v(T) &= \sum_{\substack{x \in L \\ x_T < k_T}} b^T_x \Delta_T v(x) \\
&= \sum_{\substack{x_{-i} \in L_{-i} \\ x_{T \setminus i} < k_{T \setminus i}}} \Big(b^T_{(0_i, x_{-i})} \Delta_{T \setminus i} \Delta_i v(0_i, x_{-i}) + \sum_{\substack{x_i \in L_i \\ x_i \notin \{0,k\}}} b^T_x \Delta_{T \setminus i} \Delta_i v(x) \Big) \\
&= \sum_{\substack{x_{-i} \in L_{-i} \\ x_{T \setminus i} < k_{T \setminus i}}} \Big(b^T_{(0_i, x_{-i})} \Delta_{T \setminus i} \Delta_i w\big((k-1)_i, x_{-i}\big) + \sum_{\substack{x_i \in L_i \\ x_i \notin \{0,k\}}} b^T_x \Delta_{T \setminus i} \Delta_i w(x - 1_i) \Big) \\
&= \sum_{\substack{x_{-i} \in L_{-i} \\ x_{T \setminus i} < k_{T \setminus i}}} \Big(b^T_{(0_i, x_{-i})} \Delta_{T \setminus i} \Delta_i w\big((k-1)_i, x_{-i}\big) + \sum_{\substack{x_i \in L_i \\ x_i < k-1}} b^T_{x_i+1_i, x_{-i}} \Delta_{T \setminus i} \Delta_i w(x) \Big),
\end{aligned}$$

and,

$$I^w(T) = \sum_{\substack{x_{-i} \in L_{-i} \\ x_{T \setminus i} < k_{T \setminus i}}} \Big(b^T_{\big((k-1)_i, x_{-i}\big)} \Delta_{T \setminus i} \Delta_i w\big((k-1)_i, x_{-i}\big) + \sum_{\substack{x_i \in L_i \\ x_i < k-1}} b^T_x \Delta_{T \setminus i} \Delta_i w(x) \Big).$$

Then, by **(I)**, $b^T_{x_i, x_{-i}} = b^T_{x_i+1_i, x_{-i}}, \forall x_{-i} \in L_{-i}, \forall x_i \in L_i \setminus \{k, k-1\}$ and any $i \in T$. Hence, $b^T_{x_T, x_{-T}} = b^T_{(x+1)_T, x_{-T}}$, for all $x_{-T} \in L_{-T}$ and for all $x_T \in L_T$ such that $x_T < k_T$.

We conclude that the coefficient $b^T_{x_T, x_{-T}}$ does not depend on x_T.

We set thus $b^T_{x_{-T}} := b^T_{x_T, x_{-T}}$. Hence, for any $v \in \mathcal{G}(L)$, and for any $T \in 2^N \setminus \{\emptyset\}$, we have,

$$\begin{aligned}
I^v(T) &= \sum_{\substack{x \in L \\ x_T < k_T}} b^T_x \Delta_T v(x) \\
&= \sum_{x_{-T} \in L_{-T}} b^T_{x_{-T}} \sum_{\substack{x_T \in L_T \\ x_T < k_T}} \Delta_T v(x) \\
&= \sum_{x_{-T} \in L_{-T}} b^T_{x_{-T}} \sum_{S \subseteq T} (-1)^{t-s} v(0_{T \setminus S}, k_S, x_{-T})
\end{aligned}$$

□

We introduce the Symmetry axiom.

Symmetry axiom (S): For all $v \in \mathcal{G}(L)$, for all permutation σ on N,

$$I^{\sigma \circ v}(\sigma(T)) = I^v(T), \forall T \in 2^N \setminus \{\emptyset\}.$$

Proposition 4. Under axioms **(L)**, **(N)**, **(S)**, $\forall v \in \mathcal{G}(L), \forall T \in 2^N \setminus \{\emptyset\}$,

$$I^v(T) = \sum_{\substack{x \in L \\ x_T < k_T}} b_{x_T; n_0, n_1, \ldots, n_k} \Delta_T v(x), \tag{8}$$

where $b_{x_T; n_0, n_1, \ldots, n_k} \in \mathbb{R}$, and $n_j = |\{\ell \in N \setminus T, x_\ell = j\}|$

Proof . Let $v \in \mathcal{G}(L)$ and let σ be a permutation on N. For every $x \in L$, we put $y = \sigma^{-1}(x)$. From Proposition 2, we have $\forall T \in 2^N \setminus \{\emptyset\}$

$$I^v(T) = \sum_{\substack{y \in L \\ y_T < k_T}} b_y^T \Delta_T v(y),$$

and,

$$\begin{aligned} I^{\sigma \circ v}(\sigma(T)) &= \sum_{\substack{x \in L \\ x_{\sigma(T)} < k_{\sigma(T)}}} b_x^{\sigma(T)} \Delta_{\sigma(T)} \sigma \circ v(x) \\ &= \sum_{\substack{y \in L \\ y_T < k_T}} b_{\sigma(y)}^{\sigma(T)} \Delta_T v(y). \end{aligned}$$

Then, from the symmetry axiom, we have for all $y \in L$ such that $y_T < k_T$: $b_{\sigma(y)}^{\sigma(T)} = b_y^T$.

For every $y \in L$ such that $y_T < k$, we can write,

$$b_{(y_T; y_{-T})}^T = b_y^T = b_{\sigma(y)}^{\sigma(T)} = b_{(\sigma(y)_{\sigma(T)}; \sigma(y)_{-\sigma(T)})}^{\sigma(T)} = b_{(y_T; \sigma(y)_{-\sigma(T)})}^{\sigma(T)}$$

Assuming that $\sigma(T) = T$, then,

$$b_{(y_T; y_{-T})}^T = b_{(y_T; \sigma(y)_{-\sigma(T)})}^T$$

For a fixed $T \neq \emptyset$, $b_{(y_T; \sigma(y)_{-\sigma(T)})}^T$ depends only on $n(y_{-T})$, with $n(y_{-T}) = \{n_0(y_{-T}), \ldots, n_k(y_{-T})\}$, and $n_j(y_{-T}) = |\{\ell \in N \setminus T | y_\ell = j\}|$.

$$p_{y_T; y_{-T}}^T = p_{y_T; n(y_{-T})}^T$$

Suppose now that $\sigma(T) = S$ (with $S \neq T$), and $\sigma(\ell) = \ell, \forall \ell \in N \setminus S \cup T$, then,

$$\begin{aligned} b_{(y_T; n(y_{-T}))}^T &= b_{(y_T; n(\sigma(y)_{-\sigma(T)}))}^{\sigma(T)} \\ &= b_{(y_T; n(y_{-T}))}^{\sigma(T)} \end{aligned}$$

we can conclude that the value $b_{y_T; n(y_{-T})}^T$ does not depend on the exponent T. We denote by $b_{y_T; n(y_{-T})}$ this value. □

Proposition 5. Under axioms **(L)**, **(N)**, **(I)** and **(S)**, for any $v \in \mathcal{G}(L)$ and any nonempty $T \subseteq N$,

$$I^v(T) = \sum_{x_{-T} \in L_{-T}} b_{n(x_{-T})} \sum_{S \subseteq T} (-1)^{t-s} v(0_{T \setminus S}, k_S, x_{-T}), \tag{9}$$

where $b_{n(x_{-T})} \in \mathbb{R}$, $n(x_{-T}) = (n_0, n_1, \ldots, n_k)$ and $n_j = |\{\ell \in N \setminus T, x_\ell = j\}|$

Efficiency axiom (E): For all $v \in \mathcal{G}(L)$,

$$\sum_{i \in N} I^v(i) = \sum_{\substack{x \in L \\ x_j < k}} \big(v(x + 1_N) - v(x)\big).$$

We introduce now the Recursivity axiom which is the exact counterpart of the one for classical games in (Grabisch and Roubens, 1999). For this, we introduce the following definitions.

Let v be a multichoice game in $\mathcal{G}(L)$ and $S \subseteq N$. We introduce the restricted multichoice game v^{-S} of v, which is defined on $N \setminus S$ as follows

$$v^{-S}(x_{-S}) = v(x_{-S}, 0_S), \forall x_{-S} \in L_{-S}.$$

The restriction of v to $N \setminus i$ in the presence of i, denoted by v_i^{-i}, is the multichoice game on L_{-i} defined by

$$v_i^{-i}(x_{-i}) = v(x_{-i}, k_i) - v(0_{-i}, k_i), \forall x_{-i} \in L_{-i}.$$

Recursivity axiom (R): For any $v \in \mathcal{G}(L)$,

$$I^v(T) = I^{v_i^{-i}}(T \setminus i) - I^{v^{-i}}(T \setminus i), \forall T \subseteq N, |T| > 1, \forall i \in T.$$

Lemma 4. Under axioms **(L)**, **(N)**, **(I)** **(S)** and **(R)**, for any $v \in \mathcal{G}(L)$, for any nonempty $T \subseteq N$,

$$I^v(T) = \sum_{\substack{A \subseteq T \\ A \neq \emptyset}} (-1)^{t-a} I^{v_{[A]}^{(-T)\cup[A]}}([A]), \tag{10}$$

with $v_{[A]}^{(-T)\cup[A]}$ is the reduced multichoice game of v to T with respect to A defined on the set $\{0, \ldots, k\}^{(N \setminus T)\cup[A]}$ as follows:

$$v_{[A]}^{(-T)\cup[A]}(x_{-T}, \ell_{[A]}) = v(x_{-T}, \ell_A, 0_{T \setminus A}), \ell \in \{0, \ldots, k\}.$$

Proof . We suppose that the axioms **(L)**, **(N)**, **(I)**, **(S)** and **(R)** are satisfied. We proceed by induction on $|T| =: t$. The formula is true for $t = 1$. Let us assume it is true up to $t = \ell - 1 \geq 1$, and try to prove it for $t = \ell$ elements. By induction assumption we have, for any $v \in \mathcal{G}(L)$, and $i \in T$,

$$I^{v^{-i}}(T \setminus i) = \sum_{\substack{A \subseteq T \setminus i \\ A \neq \emptyset}} (-1)^{t-a-1} I^{v_{[A]}^{(-T)\cup[A]}}([A]),$$

$$I^{v_i^{-i}}(T \setminus i) = \sum_{\substack{A \subseteq T\setminus i \\ A \neq \emptyset}} (-1)^{t-a-1} I^{v_{[A],i}^{(-T)\cup[A]}}([A]).$$

with $v_{[A],i}^{(-T)\cup[A]}(x_{-T}, \ell_{[A]}) = v(x_{-T}, \ell_A, k_i, 0_{T\setminus A\cup i}) - v(0_{-i}, k_i), \ell \in \{0, \dots, k\}$.

Let $A \subseteq T \setminus i$ such that $A \neq \emptyset$. From Proposition 5,

$$\begin{aligned}
I^{v_{[A],i}^{(-T)\cup[A]}}([A]) &= \sum_{x_{-T}\in L_{-T}} b_{n(x_{-T})}\Big(v_{[A],i}^{(-T)\cup[A]}(x_{-T}, k_{[A]}) - v_{[A],i}^{(-T)\cup[A]}(x_{-T}, 0_{[A]})\Big) \\
&= \sum_{x_{-T}\in L_{-T}} b_{n(x_{-T})}\Big(v(x_{-T}, k_{A\cup i}, 0_{T\setminus A\cup i}) - v(x_{-T}, k_i, 0_{T\setminus i})\Big) \\
&= \sum_{x_{-T}\in L_{-T}} b_{n(x_{-T})}\Big(v(x_{-T}, k_{A\cup i}, 0_{T\setminus A\cup i}) - v(x_{-T}, 0_T)\Big) \\
&- \sum_{x_{-T}\in L_{-T}} b_{n(x_{-T})}\Big(v(x_{-T}, k_i, 0_{T\setminus i}) - v(x_{-T}, 0_T)\Big) \\
&= \sum_{x_{-T}\in L_{-T}} b_{n(x_{-T})}\Big(v_{[A\cup i]}^{(-T)\cup[A\cup i]}(x_{-T}, k_{[A\cup i]}) - v_{[A\cup i]}^{(-T)\cup[A\cup i]}(x_{-T}, 0_{[A\cup i]})\Big) \\
&- \sum_{x_{-T}\in L_{-T}} b_{n(x_{-T})}\Big(v^{(-T)\cup i}(x_{-T}, k_i) - v^{(-T)\cup i}(x_{-T}, 0_i)\Big) \\
&= I_{[A\cup i]}^{v^{(-T)\cup[A\cup i]}}([A\cup i]) - I^{v^{(-T)\cup i}}(i).
\end{aligned}$$

By **(R)**, we have

$$\begin{aligned}
I^v(T) &= I^{v_i^{-i}}(T\setminus i) - I^{v^{-i}}(T\setminus i) \\
&= \sum_{\substack{A \subseteq T\setminus i \\ A \neq \emptyset}} (-1)^{t-a-1} I^{v_{[A],i}^{(-T)\cup[A]}}([A]) - \sum_{\substack{A \subseteq T\setminus i \\ A \neq \emptyset}} (-1)^{t-a-1} I^{v_{[A]}^{(-T)\cup[A]}}([A]) \\
&= \sum_{\substack{A \subseteq T\setminus i \\ A \neq \emptyset}} (-1)^{t-a-1}\Big(I_{[A\cup i]}^{v^{(-T)\cup[A\cup i]}}([A\cup i]) - I^{v^{(-T)\cup i}}(i)\Big) - \sum_{\substack{A \subseteq T\setminus i \\ A \neq \emptyset}} (-1)^{t-a-1} I^{v_{[A]}^{(-T)\cup[A]}}([A]) \\
&= \sum_{\substack{A \subseteq T\setminus i \\ A \neq \emptyset}} (-1)^{t-a-1}\Big(I_{[A\cup i]}^{v^{(-T)\cup[A\cup i]}}([A\cup i]) - I^{v_{[A]}^{(-T)\cup[A]}}([A])\Big) - I^{v^{(-T)\cup i}}(i) \sum_{\substack{A \subseteq T\setminus i \\ A \neq \emptyset}} (-1)^{t-a-1} \\
&= \sum_{\substack{A \subseteq T\setminus i \\ A \neq \emptyset}} (-1)^{t-a-1}\Big(I_{[A\cup i]}^{v^{(-T)\cup[A\cup i]}}([A\cup i]) - I^{v_{[A]}^{(-T)\cup[A]}}([A])\Big) + (-1)^{t-1} I^{v_{[i]}^{(-T)\cup[i]}}([i]) \\
&= \sum_{\substack{A \subseteq T \\ A \neq \emptyset}} (-1)^{t-a} I^{v_{[A]}^{(-T)\cup[A]}}([A])
\end{aligned}$$

□

Theorem 3. There is a unique interaction index satisfying **(L)**, **(N)**, **(I)**, **(S)**, **(E)** and **(R)**, which is the signed interaction index given by

$$I_{\mathrm{s}}^{v}(T) := \sum_{x_{-T}\in L_{-T}} \frac{(n-s(x_{-T})-t)!k(x_{-T})!}{(n-s(x_{-T})+k(x_{-T})-t+1)!} \sum_{A\subseteq T}(-1)^{t-a}v(0_{T\setminus A},k_A,x_{-T}),$$

for all $v\in\mathcal{G}(L)$, $T\in 2^N\setminus\{\emptyset\}$.

Proof . Let $v\in\mathcal{G}(L)$, and $T\in 2^N\setminus\{\emptyset\}$. By axioms **(L)**, **(N)**, **(I)**, **(S)** and **(E)**, we have

$$I^{v_{[A]}^{(-T)\cup[A]}}([A]) = \sum_{x_{-T}\in L_{-T}} b_{n(x_{-T})}\big(v_{[A]}^{(-T)\cup[A]}(x_{-T},k_{[A]}) - v_{[A]}^{(-T)\cup[A]}(x_{-T},0_{[A]})\big),$$

with $b_{n(x_{-T})} = \dfrac{\big(n-t-s(x_{-T})\big)!k(x_{-T})!}{\big(n-t+1+k(x_{-T})-s(x_{-T})\big)!}$.

By Lemma 4, we have

$$\begin{aligned}
I^{v}(T) &= \sum_{\substack{A\subseteq T\\ A\neq\emptyset}}(-1)^{t-a} I^{v_{[A]}^{(-T)\cup[A]}}([A])\\
&= \sum_{\substack{A\subseteq T\\ A\neq\emptyset}}(-1)^{t-a} \sum_{x_{-T}\in L_{-T}} b_{n(x_{-T})}\big(v(x_{-T},k_A,0_{T\setminus A}) - v(x_{-T},0_T)\big)\\
&= \sum_{x_{-T}\in L_{-T}} b_{n(x_{-T})} \sum_{\substack{A\subseteq T\\ A\neq\emptyset}}(-1)^{t-a}\big(v(x_{-T},k_A,0_{T\setminus A}) - v(x_{-T},0_T)\big)\\
&= \sum_{x_{-T}\in L_{-T}} b_{n(x_{-T})} \sum_{A\subseteq T}(-1)^{t-a} v(k_A,0_{T\setminus A},x_{-T}).
\end{aligned}$$

□

Observe that as for the interaction index for classical games, the formula is still valid for $T=\emptyset$.

Example 2 (**System engineers (continued)**)**.** Consider the 2-ary capacity v given in Example 1. The computation of the signed interaction index w.r.t. v gives

$$I_{\mathrm{s}}^{v}(\{1,2\}) = v(2,2) - v(0,2) - v(2,0) + v(0,0) = 2-1-1+0 = 0.$$

The signed interaction index indicates that there is no interaction among criteria. Evidently, this is counterintuitive because the two criteria (latency of the system and quality of the system) interact with one another, as depicted in Figure 1. This is because the interaction effects existing in the subdomain $\{1,2\}^2$ of L cancel the interaction effects in the subdomain $\{0,1\}^2$: interaction is positive in the latter, and negative in the former.

The above example motivates the introduction of another interaction index which does not permit cancellation between positive and negative interaction, just by considering absolute interaction effects. This is the object of the next section.

6 Axiomatization of the absolute interaction index

In this section, we propose an axiomatic approach to define an interaction index which has the following general form:

$$I^v(T) = \sum_{\substack{x \in L \\ x_T < k_T}} b_x^T |\Delta_T v(x)|,$$

where, for any nonempty $T \subseteq N$, $\{b_x^T \mid x \in L, x_T < k_T\}$ is a family of real coefficients. As this index is not linear, the axiomatization scheme cannot be based on linearity as the previous one, and we follow the scheme taken in (Ridaoui et al., 2017a) for the absolute importance index. The two first axioms are a kind a replacement of linearity.

Conic Combination axiom (CC) : $\forall v, w \in \mathcal{G}_+(L), \forall \alpha \in \mathbb{R}_+$,

$$I^{(v+\alpha w)} = I^v + \alpha I^w.$$

Decomposition axiom (D): If $v, w \in \mathcal{G}_+(L)$ and $v - w \in \mathcal{G}_T(L)$ for some $T \subseteq N$,

$$I^{v-w}(T) = I^v(T) - I^w(T).$$

Proposition 6. Under axioms **(CC)** and **(D)**, for all nonempty $T \subseteq N$, there exists constants $a_x^T \in \mathbb{R}$, for all $x \in L$, such that for all $v \in \mathcal{G}_T(L)$,

$$I^v(T) = \sum_{x \in L} a_x^T v(x). \tag{11}$$

Proof. Let $v \in \mathcal{G}_+(L)$. By using the basis of unanimity multichoice games and applying **(CC)** we find for every nonempty $T \subseteq N$,

$$I^v(T) = \sum_{x \in L} m^v(x) b_x^T, \text{ with } b_x^T = I^{u_x}(T).$$

Since $m^v(x)$ is a linear combination of all coefficients $v(y), y \in L$, a rearrangement of terms leads to the following formula:

$$I^v(T) = \sum_{y \in L} a_y^T v(y), \tag{12}$$

where a_y^T are real constants independent from v. We have for every $T \in 2^N \setminus \{\emptyset\}$ and $v \in \mathcal{G}_T(L)$,

$$\begin{aligned}
I^v(T) &= I^{v^+ - v^-}(T), \quad v^+, v^- \in \mathcal{G}_+(L) \\
&= I^{v^+}(T) - I^{v^-}(T), \quad \text{by } \textbf{(D)} \\
&= \sum_{x \in L} a_x^T v^+(x) - \sum_{x \in L} a_x^T v^-(x), \quad \text{by (12)} \\
&= \sum_{x \in L} a_x^T v(x).
\end{aligned}$$

$\square$

Marginal contribution axiom (MC): Let $T \subseteq N, T \neq \emptyset$ and $v, w \in \mathcal{G}(L)$ such that

$$|\Delta_T v(x)| = |\Delta_T w(x)|, \forall x \in L, x_T < k_T,$$

then $I^v(T) = I^w(T)$.

Remark 2. The Null Axiom **(N)** is implied by **(CC)** and **(MC)** (or **(D)** and **(MC)**). Indeed, the **(CC)** axiom or **(D)** implies $I^{\mathbf{0}}(T) = 0$ for any $T \subseteq N$, where $\mathbf{0}$ is the null game, and if i is a null criterion for a multichoice game v such that $T \ni i$, then $\Delta_T v(x) = 0 = \Delta_T \mathbf{0}(x)$, for every $x \in L$ such that $x_T < k_T$. Hence, by **(MC)**, $I^v(T) = 0$.

Proposition 7. Under axioms **(CC)**, **(D)** and **(MC)**, for all nonempty $T \subseteq N$, there exists constants $b_x^T \in \mathbb{R}$, for all $x \in L, x_T < k_T$, such that for all $v \in \mathcal{G}(L)$,

$$I^v(T) = \sum_{\substack{x \in L \\ x_T < k_T}} b_x^T |\Delta_T v(x)|. \tag{13}$$

To prove this proposition, the following Lemma is useful.

Lemma 5. Let $T \subseteq N, T \neq \emptyset$. For every $v \in \mathcal{G}(L)$, there exists $w \in \mathcal{G}_T(L)$, such that

$$|\Delta_T v| = |\Delta_T w|.$$

Proof . Let $v \in \mathcal{G}(L)$ and $T \subseteq N, T \neq \emptyset$. For any $x \in L$ such that $x_T < k_T$, we set

$$D(x) = \begin{cases} \sum\limits_{\substack{z_T \in L_T \\ \forall i \in T, z_i < x_i}} |\Delta_T v(z_T, x_{-T})|, & \text{if } x_i > 0, \forall i \in T \\ 0, & \text{otherwise.} \end{cases}$$

Define inductively the following multichoice game w:

- $w(0, \dots, 0) = 0$,
- for every $x_{-T} \in L_{-T}$, $x_{-T} = (0_A, x_{-T\cup A})$, with $A \subseteq N \setminus T \cup A$ and $x_{-T\cup A} > 0_{-T\cup A}$,

$$w(0_{T\cup A}, x_{-T\cup A}) = \max_{\substack{\emptyset \neq B \subseteq N \setminus T \cup A \\ y_{-A} \in L_{-A} \\ 0_B < y_B \leq x_B}} \Bigg[\sum_{\substack{z_B \lneq y_B \\ z_B \geq y_B - 1_B}} (-1)^{1+\sum_{i\in B} y_i - z_i} \Big(w(0_{T\cup A}, z_B, x_{-T\cup A\cup B}) + D(y_T, 0_A, z_B, x_{-T\cup A\cup B}) \Big) - D(y_T, 0_A, x_{-T\cup A}) \Bigg],$$

- for every $x \in L$, such that $x_T \neq 0_T$,

  $w(x) = w(0_T, x_{-T}) + D(x)$.

By construction, we have for every $x \in L \setminus \{0_T\}$ such that $x_T < k_T$,

$$\begin{aligned}
\Delta_T w(x) &= \sum_{A \subseteq T} (-1)^{t-a} w(x + 1_A) \\
&= \sum_{A \subseteq T} (-1)^{t-a} \big(w(0_T, x_{-T}) + D(x + 1_A) \big) \\
&= \sum_{A \subseteq T} (-1)^{t-a} D(x + 1_A) \\
&= \sum_{A \subseteq T} (-1)^{t-a} \sum_{\substack{\forall i \in A, z_i \leq x_i \\ \forall i \in T \setminus A, z_i < x_i}} |\Delta_T v(z_A, z_{T \setminus A}, x_{-T})| \\
&= |\Delta_T v(x)|,
\end{aligned}$$

and if $x = (0_T, x_{-T})$, we have

$$\begin{aligned}
\Delta_T w(x) &= (-1)^{t-a} w(0_{-T}, x_{-T}) + \sum_{\emptyset \neq A \subseteq T} (-1)^{t-a} w(x + 1_A) \\
&= \sum_{\emptyset \neq A \subseteq T} (-1)^{t-a} D(x + 1_A) \\
&= \sum_{A \subseteq T} (-1)^{t-a} D(x + 1_A).
\end{aligned}$$

Hence, w is a multichoice game such that $\Delta_T w(x) \geq 0$ and $|\Delta_T v| = |\Delta_T w|$.

Let now $S \subseteq N \setminus T, S \neq \emptyset$, and $x \in L$ such that $x_S + 1_S \leq k_S$. Let $A \subseteq N \setminus T \cup S$ such that $x_{-T \cup S} = (0_A, x_{-T \cup S \cup A})$ with $x_{-T \cup S \cup A} > 0$, and we set $\ell_{-T \cup A} = (x_S + 1_S, x_{-T \cup A \cup S})$. We distinguish the two following cases:

- if $x_T = 0_T$, for every $B \subseteq N \setminus T \cup A, B \neq \emptyset, y_{-A} \in L_{-A}, 0_B < y_B \leq \ell_B$, we have

$$\begin{aligned}
w(0_{T \cup A}, \ell_{-T \cup A}) \geq \sum_{\substack{z_B \lneq y_B \\ z_B \geq y_B - 1_B}} &(-1)^{1 + \sum_{i \in B} y_i - z_i} \Big(w(0_{T \cup A}, z_B, \ell_{-T \cup A \cup B}) \\
&+ D(y_T, 0_A, z_B, \ell_{-T \cup A \cup B}) \Big) - D(y_T, 0_A, \ell_{-T \cup A}),
\end{aligned}$$

  we take $B = S, y_T = 0_T$ and $y_S = x_S + 1_S$, we obtain

$$w(0_{T \cup A}, x_S + 1_S, x_{-T \cup A \cup S}) \geq \sum_{\substack{z_S \lneq x_S + 1_S \\ z_S \geq x_S}} (-1)^{1 + \sum_{i \in S} x_i + 1 - z_i} w(0_{T \cup A}, z_S, x_{-T \cup A \cup S})$$

- if $x_T \neq 0_T$, we have

$$w(x_T, x_{-T} + 1_S) = w(0_T, x_{-T} + 1_S) + D(x_T, x_{-T} + 1_S),$$

  and for every $B \subseteq N \setminus T \cup A, B \neq \emptyset, y_{-A} \in L_{-A}, 0_B < y_B \leq \ell_B$, we have

$$\begin{aligned}
w(0_{T \cup A}, \ell_{-T \cup A}) \geq \sum_{\substack{z_B \lneq y_B \\ z_B \geq y_B - 1}} &(-1)^{1 + \sum_{i \in B} y_i - z_i} \Big(w(0_{T \cup A}, z_S, \ell_{-T \cup A \cup B}) \\
&+ D(y_T, 0_A, z_B, \ell_{-T \cup A \cup B}) \Big) - D(y_T, x_{-T} + 1_S),
\end{aligned}$$

we take $B = S, y_T = x_T$ and $y_S = x_S + 1_S$, we obtain

$$\begin{aligned} w(x_T, x_{-T} + 1_S) \geq & \sum_{\substack{z_S \lneq x_S + 1_S \\ z_S \geq x_S}} (-1)^{1+\sum_{i\in S} x_i + 1 - z_i} \Big(w(0_{T\cup A}, z_S, x_{-T\cup A\cup S}) \\ & \qquad + D(x_T, 0_A, z_S, x_{-T\cup A\cup S}) \Big) \\ \geq & \sum_{\substack{z_S \lneq x_S + 1_S \\ z_S \geq x_S}} (-1)^{1+\sum_{i\in S} x_i + 1 - z_i} w(x_T, 0_A, z_S, x_{-T\cup A\cup S}). \end{aligned}$$

Then, $\Delta_S v(x) \geq 0$.

□

We now prove Proposition 7.

Proof. Let $v \in \mathcal{G}(L)$ and $T \subseteq N, T \neq \emptyset$.

It is clear that Formula (13) satisfies **(MC)**. Let us check that it satisfies **(CC)** and **(D)**. Let $v, w \in \mathcal{G}_+(L)$, and $T \in 2^N \setminus \{\emptyset\}$. For any $\alpha \in \mathbb{R}_+$, we have

$$\begin{aligned} I^{(v+\alpha w)}(T) &= \sum_{\substack{x \in L \\ x_T < k_T}} b_x^T |\Delta_T (v + \alpha w)(x)| \\ &= \sum_{\substack{x \in L \\ x_T < k_T}} b_x^T |\Delta_T v(x) + \alpha \Delta_T w(x)| \\ &= \sum_{\substack{x \in L \\ x_T < k_T}} b_x^T \big(|\Delta_T v(x)| + \alpha |\Delta_T w(x)| \big) \text{ (by Lemma 1).} \end{aligned}$$

Thus

$$I^{(v+\alpha w)}(T) = I^v(T) + \alpha I^w(T),$$

and the **(CC)** axiom is satisfied. If now $v - w \in \mathcal{G}_T(L)$, we have $\Delta_T(v - w) \geq 0$, and then

$$\begin{aligned} I^{(v-w)}(T) &= \sum_{\substack{x \in L \\ x_T < k_T}} b_x^T |\Delta_T (v - w)(x)| \\ &= \sum_{\substack{x \in L \\ x_T < k_T}} b_x^T \Delta_T (v - w)(x) \\ &= \sum_{\substack{x \in L \\ x_T < k_T}} b_x^T \big(|\Delta_T (v)(x)| - |\Delta_T w(x)| \big). \end{aligned}$$

Hence, **(D)** is satisfied. Conversely, we consider I satisfying the axioms **(CC)**, **(D)** and **(MC)**.

Let $v \in \mathcal{G}(L)$ and $T \subseteq N, T \neq \emptyset$, by Lemma 5 there exists $w \in \mathcal{G}_T(L)$, such that $|\Delta_T v(x)| = |\Delta_T w(x)|, x \in L, x_T < k_T$. By Remark 2 , I satisfies the Null axiom for any multichoice game. Then we deduce by (11) and Proposition 2

$$I^w(T) = \sum_{\substack{x \in L \\ x_T < k_T}} b_x^T \Delta_T w(x),$$

And by axiom **(MC)** we have,

$$\begin{aligned} I^v(T) &= I^w(T) \\ &= \sum_{\substack{x \in L \\ x_T < k_T}} b_x^T \Delta_T w(x) \\ &= \sum_{\substack{x \in L \\ x_T < k_T}} b_x^T |\Delta_T v(x)|. \end{aligned}$$

□

Proposition 8. Under axioms **(CC)**, **(D)**, **(MC)** and **(I)**, for all $v \in \mathcal{G}(L)$,

$$I^v(T) = \sum_{\substack{x \in L \\ x_T < k_T}} b_{x_{-T}}^T |\Delta_T v(x)|, T \subseteq N, T \neq \emptyset.$$

(the proof works as in the case of Proposition 3)

Symmetry for Dirac games (SD): Let $x \in L \setminus \{0_N\}$. $\forall T \subseteq \mathrm{K}(x) \cup \big(N \setminus \Sigma(x)\big)$, with $T \neq \mathrm{K}(x)$ and $T \neq N \setminus \Sigma(x)$,

$$\begin{aligned} I^{\delta_x}(T \cup i) &= I^{\delta_x}(T \cup j), \forall i, j \in \mathrm{K}(x), i, j \notin T, \\ I^{\delta_x}(T \cup i) &= I^{\delta_x}(T \cup j), \forall i, j \in N \setminus \Sigma(x), i, j \notin T. \end{aligned}$$

This axiom says that $i, j \in N$ are symmetric for the Dirac games, if they are in the kernel, or outside the support. It means that the interaction for Dirac games does not depend on the labelling of the players in the kernel, or outside the support.

Take now $x \in L \setminus \{0_N\}$ such that $\kappa(x) \neq 0$ and remark that $\sum_{i \in N \setminus T} I_{\mathrm{s}}^{\delta_x}(T \cup i) = 0$ for any $T \subsetneq \mathrm{K}(x) \cup \big(N \setminus \Sigma(x)\big)$, which implies

$$\sum_{\substack{i \in \mathrm{K}(x) \\ i \notin T}} I_{\mathrm{s}}^{\delta_x}(T \cup i) = - \sum_{\substack{i \in N \setminus \Sigma(x) \\ i \notin T}} I_{\mathrm{s}}^{\delta_x}(T \cup i).$$

We also note that $I_{\mathrm{s}}^{\delta_x}(T \cup i)$ is positive for $i \in K(x), i \notin T$, and $I_{\mathrm{s}}^{\delta_x}(T \cup i)$ is negative for $i \in N \setminus \Sigma(x), i \notin T$. Hence

$$\sum_{\substack{i \in \mathrm{K}(x) \\ i \notin T}} |I_{\mathrm{s}}^{\delta_x}(T \cup i)| = \sum_{\substack{i \in N \setminus \Sigma(x) \\ i \notin T}} |I_{\mathrm{s}}^{\delta_x}(T \cup i)|.$$

This can be interpreted as the total interaction of elements in $K(x) \setminus T$ is equal to the total interaction of terms in $N \setminus \Sigma(x) \cup T$. We then propose the following axiom.

Absolute Efficiency (AE): Let $x \in L \setminus \{0_N\}$ such that $\mathrm{K}(x) \neq 0$.

$$\forall T \subsetneq \mathrm{K}(x) \cup \big(N \setminus \Sigma(x)\big), \sum_{\substack{i \in \mathrm{K}(x)\\ i \notin T}} I^{\delta_x}(T \cup i) = \sum_{\substack{i \in N \setminus \Sigma(x)\\ i \notin T}} I^{\delta_x}(T \cup i).$$

In all previous axioms, the Interaction is given up to a dilation coefficient. In order to fix this coefficient, we introduce a calibration axiom taking the previous case when $\Sigma(x) = N$.

Calibration (C): For every $x \in L$, such that $\Sigma(x) = N$,

$$\sum_{\substack{i \in \mathrm{K}(x)\\ i \notin T}} I^{\delta_x}(T \cup i) = 1, \forall T \subsetneq \mathrm{K}(x).$$

Theorem 4. There is a unique interaction index satisfying **(CC)**, **(D)**, **(I)**, **(MC)**, **(SD)**, **(AE)** and **(C)**, which is the absolute interaction index given by

$$I^v_{\mathrm{a}}(T) := \sum_{\substack{x \in L\\ x_T < k_T}} \frac{(n - \sigma(x_{-T}) - t)!\kappa(x_{-T})!}{(n - \sigma(x_{-T}) + \kappa(x_{-T}) - t + 1)!} |\Delta_T v(x)|,$$

for all $v \in \mathcal{G}(L)$, $T \in 2^N \setminus \{\emptyset\}$.

Proof. It is clear that the above formula satisfies **(CC)**, **(D)**, **(I)**, **(MC)**. We show that I_{a} satisfies **(SD)**, **(AE)** and **(C)**. Let $x \in L \setminus \{0_N\}$ and $T \subseteq N, T \neq \emptyset$, and for any $i \in N \setminus T$, we set $p^{x_{-T \cup i}} = \frac{(n-\sigma(x_{-T\cup i})-t-1)!\kappa(x_{-T\cup i})!}{(n-\sigma(x_{-T\cup i})+\kappa(x_{-T\cup i})-t)!}$.

- Suppose $T \subseteq \mathrm{K}(x) \cup \big(N \setminus \Sigma(x)\big)$ with $T \neq \mathrm{K}(x)$ and $T \neq N \setminus \Sigma(x)$. For any $i, j \in \mathrm{K}(x)$ such that $i, j \notin T$, we have

$$\begin{aligned}
I^{\delta_x}_{\mathrm{a}}(T \cup i) &= \sum_{\substack{y \in L\\ y_{T \cup i} < k_{T \cup i}}} p^{y_{-T \cup i}} |\Delta_{T \cup i} \delta_x(y)| \\
&= p^{x_{-T \cup i}} \sum_{y_T < k_T} \Big| \sum_{A \subseteq T} (-1)^{t-a} \delta_x(y_T + 1_A, k_i, x_{-T \cup i}) \Big| \\
&= p^{x_{-T \cup j}} \sum_{y_T < k_T} \Big| \sum_{A \subseteq T} (-1)^{t-a} \delta_x(y_T + 1_A, k_j, x_{-T \cup j}) \Big| \\
&= \sum_{\substack{y \in L\\ y_{T \cup j} < k_{T \cup j}}} p^{y_{-T \cup j}} |\Delta_{T \cup j} \delta_x(y)| \\
&= I^{\delta_x}_{\mathrm{a}}(T \cup j),
\end{aligned}$$

and for any $i, j \in N \setminus \Sigma(x)$, we have

$$\begin{aligned}
I_{\mathrm{a}}^{\delta_x}(T \cup i) &= p^{x_{-T\cup i}} \sum_{y_T < k_T} \Big| \sum_{A \subseteq T} (-1)^{t-a} \delta_x(y_T + 1_A, 0_i, x_{-T\cup i}) \Big| \\
&= p^{x_{-T\cup j}} \sum_{y_T < k_T} \Big| \sum_{A \subseteq T} (-1)^{t-a} \delta_x(y_T + 1_A, 0_j, x_{-T\cup j}) \Big| \\
&= \sum_{\substack{y \in L \\ y_{T\cup j} < k_{T\cup j}}} p^{y_{-T\cup j}} |\Delta_{T\cup j} \delta_x(y)| \\
&= I_{\mathrm{a}}^{\delta_x}(T \cup j).
\end{aligned}$$

Hence, **(SD)** is satisfied.

- If $\kappa(x) \neq 0$, and $T \subsetneq \mathrm{K}(x) \cup \big(N \setminus \Sigma(x)\big)$, we have

$$\begin{aligned}
I_{\mathrm{a}}^{\delta_{(k_i, x_{-i})}}(T \cup i) &= p^{x_{-T\cup i}}, \forall i \in \mathrm{K}(x), i \notin T \\
I_{\mathrm{a}}^{\delta_{(0_i, x_{-i})}}(T \cup i) &= p^{x_{-T\cup i}}, \forall i \in N \setminus \Sigma(x), i \notin T.
\end{aligned}$$

We set $S = \Sigma(x) \setminus \mathrm{K}(x)$, and $x_T = (0_A, k_{T\setminus A})$, with $A \subseteq T$. We have

$$\begin{aligned}
\sum_{\substack{i \in \mathrm{K}(x) \\ i \notin T}} I_{\mathrm{a}}^{\delta_{(k_i, x_{-i})}}(T \cup i) &= \sum_{\substack{i \in \mathrm{K}(x) \\ i \notin T}} \frac{(n - \kappa(x) - a - s)!(\kappa(x) + a - t - 1)!}{(n - s - t)!} \\
&= \frac{(n - \kappa(x) - a - s)!(\kappa(x) + a - t)!}{(n - s - t)!}, \\
&= \sum_{\substack{i \in N \setminus \Sigma(x) \\ i \notin T}} \frac{(n - \kappa(x) - a - s - 1)!(\kappa(x) + a - t)!}{(n - s - t)!} \\
&= \sum_{\substack{i \in N \setminus \Sigma(x) \\ i \notin T}} I_{\mathrm{a}}^{\delta_{(0_i, x_{-i})}}(T \cup i).
\end{aligned}$$

Then, **(AE)** is satisfied.

- Let us take $\Sigma(x) = N$. For any $T \subsetneq \mathrm{K}(x)$, we have

$$\begin{aligned}
\sum_{\substack{i \in \mathrm{K}(x) \\ i \notin T}} I_{\mathrm{a}}^{\delta_{(k_i, x_{-i})}}(T \cup i) &= \sum_{\substack{i \in \mathrm{K}(x) \\ i \notin T}} p^{x_{-T\cup i}} \\
&= \sum_{\substack{i \in \mathrm{K}(x) \\ i \notin T}} \frac{1}{k(x_{-T\cup i}) + 1} \\
&= 1.
\end{aligned}$$

Hence, **(C)** is satisfied.

Conversely, we consider I satisfying the axioms **(CC)**, **(D)**, **(I)**, **(MC)**, **(SD)**, **(AE)** and **(C)**. Let $x = (0_{N\setminus S\cup M}, x_S, k_M) \in L$ with $x_S \in L_S \setminus \{0,k\}^S$, $S = \Sigma(x) \setminus \mathrm{K}(x)$ and $M = \mathrm{K}(x)$.

By Proposition 8, we have for any $T \subseteq N, T \neq \emptyset$

$$I^{\delta_x}(T) = b^T_{x_{-T}} \sum_{\substack{y_T \in L_T \\ y_T < k_T}} |\Delta_T \delta_x(x_{-T}, y_T)|.$$

Then we obtain,

- if $T \cap S = \emptyset$,

$$I^{\delta_{(x_{-T},k_T)}}(T) = I^{\delta_{(x_{-T},0_T)}}(T) = b^T_{x_{-T}} = I^{\delta_{(x_{-T},\ell_T)}}(T), \forall \ell \in \{0,k\}, \quad (14)$$

- if $T \cap S \neq \emptyset$,

$$I^{\delta_x}(T) = 2^{|T\cap S|} I^{\delta_{(x_{-T},k_T)}}(T). \quad (15)$$

Then it suffices to determine $I^{\delta_{(x_{-T},k_T)}}(T)$, with $T \subseteq M$, $T \neq \emptyset$, and $x_{-T} = (0_{N\setminus S\cup M}, x_S, k_{M\setminus T})$.

- If $S \cup M = N$, by axioms (SD) and (C), we have

$$I^{\delta_x}(T) = \frac{1}{m-t+1}, \forall T \subseteq M, T \neq \emptyset. \quad (16)$$

- If $S \cup M \neq N$, by axioms (SD) and (AE), we have, $\forall t \in \{0,\dots,m-1\}$ and $\forall m \in \{1,\dots,n-1\}, \forall s \in \{1,\dots,n-2\}$ with $s+m<n$,

$$(m-t) I^{\delta_{(0_{N\setminus M\cup S},x_S,k_{M\setminus T\cup i},k_{T\cup i})}}(T\cup i) = (n-s-m) I^{\delta_{(0_j,0_{N\setminus M\cup S\cup j},x_S,k_{M\setminus T},k_T)}}(T\cup j)$$

  and by (14), we obtain

$$I^{\delta_{(0_{N\setminus M\cup S},x_S,k_{M\setminus T\cup i},k_{T\cup i})}}(T\cup i) = \frac{n-s-m}{m-t} I^{\delta_{(0_{N\setminus M\cup S\cup j},x_S,k_{M\setminus T},k_{T\cup j})}}(T\cup j)$$

  Thus, recursively,

$$I^{\delta_{(0_{-M\cup S},x_S,k_{M\setminus T\cup i},k_{T\cup i})}}(T\cup i) = \frac{n-s-m}{m-t} I^{\delta_{(0_{-M_1\cup S},x_S,k_{M_1\setminus T\cup j},k_{T\cup j})}}(T\cup j), M_1 = M\cup j$$

$$I^{\delta_{(0_{-M_1\cup S},x_S,k_{M_1\setminus T\cup i},k_{T\cup i})}}(T\cup i) = \frac{n-s-m-1}{m+1-t} I^{\delta_{(0_{-M_2\cup S},x_S,k_{M_2\setminus T\cup j},k_{T\cup j})}}(T\cup j), M_2 = M_1\cup j$$

$$I^{\delta_{(0_{-M_2\cup S},x_S,k_{M_2\setminus T\cup i},k_{T\cup i})}}(T\cup i) = \frac{n-s-m-2}{m+2-t} I^{\delta_{(0_{-M_3\cup S},x_S,k_{M_3\setminus T\cup j},k_{T\cup j})}}(T\cup j), M_3 = M_2\cup j$$

$$I^{\delta_{(0_{-M_3\cup S},x_S,k_{M_3\setminus T\cup i},k_{T\cup i})}}(T\cup i) = \frac{n-s-m-3}{m+3-t} I^{\delta_{(0_{-M_4\cup S},x_S,k_{M_4\setminus T\cup j},k_{T\cup j})}}(T\cup j), M_4 = M_3\cup j$$

$$\vdots$$

$$I^{\delta_{(0_j,x_S,k_{M_{n-s-1}\setminus T\cup i},k_{T\cup i})}}(T\cup i) = \frac{1}{n-s-1-t} I^{\delta_{(x_S,k_{M_{n-s}\setminus T\cup j},k_{T\cup j})}}(T\cup j), M_{n-s} = M_{n-s-1}\cup j$$

Then by (16), we obtain

$$I^{\delta_{(0_{-M\cup S}, x_S, k_{M\setminus T\cup i}, k_{T\cup i})}}(T\cup i) = \frac{(n-s-m)!(m-t-1)!}{(n-s-t)!},$$

therefore, for any $T \subseteq M, T \neq \emptyset$,

$$\begin{aligned} I^{\delta_x}(T) &= \frac{(n-s-m)!(m-t)!}{(n-s-t+1)!} \\ &= \frac{(n-\sigma(x))!(\kappa(x)-\kappa(x_T))!}{(n-\sigma(x)+\kappa(x)-t+1)!} \\ &= \frac{(n-t-\sigma(x_{-T}))!(\kappa(x_{-T}))!}{(n-\sigma(x_{-T})+\kappa(x_{-T})-t+1)!} \end{aligned}$$

□

Note that, alike the signed interaction index, the expression is still valid for $T = \emptyset$.

Example 3 (**System engineers (continued)**)**.** Consider the 2-ary capacity v given in Example 1. The computation of the absolute interaction index w.r.t. v gives

$$I^v_{\text{a}}(\{1,2\}) = |\Delta_{\{1,2\}}v(0,0)|+|\Delta_{\{1,2\}}v(0,1)|+|\Delta_{\{1,2\}}v(1,0)|+|\Delta_{\{1,2\}}v(1,1)| = 1+0+0+1 = 2.$$

The absolute interaction index shows that there is a synergy of information between criteria, which was not visible using the signed interaction index.

7 Interaction indices for the Choquet integral

We propose in this section an interpretation of the interaction in continuous spaces, that is, after extending v to the continuous domain $[0,k]^N$. The most usual extension of v on $[0,k]^N$ is the Choquet integral with respect to k-ary capacities (Grabisch and Labreuche, 2003).

Let $z \in [0,k]^N$. For any $i \in N$, we define $\overline{z} \in L$ by $\overline{z}_i \leq z_i < \overline{z}_i + 1$ if $\overline{z}_i < k$ and $z_i = k-1$ otherwise. The Choquet integral w.r.t. a k-ary capacity v at point z is defined by

$$\mathcal{C}_v(z) = v(\overline{z}) + C_{\mu_{\overline{z}}}(z-\overline{z}),$$

where $\mu_{\overline{z}}$ is a capacity given by

$$\mu_{\overline{z}}(A) = v((\overline{z}+1)_A, \overline{z}_{-A}) - v(\overline{z}), \forall A \subset N.$$

For any $T \subseteq N$, we define inductively the variation of $\mathcal{C}_v$ at z w.r.t. T by

$$\begin{aligned} \Delta_{\emptyset}\mathcal{C}_v(z) &= \mathcal{C}_v(z) \\ \Delta_i\mathcal{C}_v(z) &= \mathcal{C}_v(k_i, z_{-i}) - \mathcal{C}_v(0_i, z_{-i}), \forall i \in N \\ \Delta_{T\cup i}\mathcal{C}_v(z) &= \Delta_i(\Delta_T\mathcal{C}_v(z)), \forall i \in N \setminus T. \end{aligned}$$

Generalizing the above definitions, the variation of $\mathcal{C}_v$ at z w.r.t. T is

$$\Delta_T \mathcal{C}_v(z) = \sum_{S\subseteq T} (-1)^s \mathcal{C}_v(0_S, k_{T\setminus S}, z_{-T}).$$

We begin by giving a formula relating the (signed) interaction index to the interaction index for classical games. In the formula below, as well as in the whole section, we consider the expression of the interaction index extended to 2^N.

Proposition 9. For every $v \in \mathcal{G}(L)$,

$$I^v_{\mathrm{s}}(T) = \sum_{x\in\{0,\dots,k-1\}^N} I^{\mu_x}_{Sh}(T), \forall T \subseteq N,$$

where I_{Sh} is the interaction index for classical games given in (4).

To prove this result, the following combinatorial result is useful.

Lemma 6.

$$\sum_{S\in[A,B]} \frac{(n-s-1)!s!}{n!} = \frac{(n-b-1)!a!}{(n-b+a)!}, \forall A, B \subseteq N, A \subseteq B,$$

where $[A, B] = \{C \subseteq N : A \subseteq C \subseteq B\}$.

Proof . Let $A, B \subseteq N$, such that $A \subseteq B$,

$$\begin{aligned}
\sum_{S\subseteq[A,B]} \frac{(n-s-1)!s!}{n!} &= \sum_{S\subseteq[\emptyset,B\setminus A]} \frac{(n-s-a-1)!(s+a)!}{n!} \\
&= \sum_{s=0}^{b-a} \binom{b-a}{s} \frac{(n-s-a-1)!(s+a)!}{n!} \\
&= \sum_{s=0}^{b-a} \binom{b-a}{s} \int_0^1 x^{n-s-a-1}(1-x)^{s+a}\,\mathrm{d}x \\
&= \int_0^1 x^{n-b-1}(1-x)^a \sum_{s=0}^{b-a} \binom{b-a}{s} x^{b-a-s}(1-x)^s\,\mathrm{d}x \\
&= \int_0^1 x^{n-b-1}(1-x)^a\,\mathrm{d}x \\
&= \frac{(n-b-1)!a!}{(n-b+a)!}
\end{aligned}$$

□

We now prove Proposition 9.

Proof . Let $T \subseteq N$.

$$\begin{aligned}
\sum_{x\in\{0,\dots,k-1\}^N} I_{Sh}^{\mu_x}(T) &= \sum_{x\in\{0,\dots,k-1\}^N} \sum_{S\subseteq N\setminus T} \frac{(n-s-t)!s!}{(n-t+1)!}\Delta_T\mu_x(S)\\
&= \sum_{x\in\{0,\dots,k-1\}^N} \sum_{S\subseteq N\setminus T} \frac{(n-s-t)!s!}{(n-t+1)!}\Delta_T v(x+1_S)\\
&= \sum_{\substack{z\in L\\ z_T<k_T}} \Delta_T v(z) \sum_{\substack{S\subseteq N\setminus T\\ \forall j\in S, z_j>0\\ \forall j\in N\setminus S, z_j<k}} \frac{(n-s-t)!s!}{(n-t+1)!}\\
&= \sum_{\substack{z\in L\\ z_T<k_T}} \Delta_T v(z) \sum_{\substack{S\subseteq N\setminus T\cap S(z_{-T})\\ S\supseteq K(z_{-T})}} \frac{(n-s-t)!s!}{(n-t+1)!}\\
&= \sum_{\substack{z\in L\\ z_T<k_T}} \Delta_T v(z) \sum_{\substack{S\subseteq S(z_{-T})\\ S\supseteq K(z_{-T})}} \frac{(n-s-t)!s!}{(n-t+1)!}\\
&= \sum_{\substack{z\in L\\ z_T<k_T}} \frac{(n-s(z_{-T})-t)!k(z_{-T})!}{(n-s(z_{-T})+k(z_{-T})-t+1)!}\Delta_T v(z).
\end{aligned}$$

□

Thus, the signed interaction index on L takes the form of the sum over all the cells of the grid L, where in each cell the interaction index for the local (classical) game is taken.

Theorem 5. Let v be a k-ary capacity.

$$I_{\mathrm{s}}^v(T) = \frac{1}{k^t}\int_{[0,k]^n} \Delta_T \mathcal{C}_v(z)\,\mathrm{d}z, \forall T\subseteq N.$$

To prove this result, the following lemmas are useful.

Lemma 7. The Choquet integral at point $z\in[0,k]^N$ w.r.t. a unanimity multichoice game u_x, $x\in L$ is given by

$$\mathcal{C}_{u_x}(z) = \begin{cases} 1, & \text{if } z\geq x\\ \bigwedge_{i\in N: z_i<x_i} (z_i-\overline{z}_i), & \text{if } \exists i\in N: z_i<x_i, z_{-i}\geq x_{-i}-1_{-i}\\ 0, & \text{otherwise}\end{cases}$$

Proof. Let $u_x, x\in L$ a unanimity multichoice game, and let $z\in[0,k]^N$. There are three different cases,

- Case 1 : $x\leq z$,

  we have $v_x(\overline{z})=1$, and $\mu_{\overline{z}}(A)=0$ for any $A\subset N$, then $\mathcal{C}_{v_x}(z)=1$.
- Case 2 : $\exists i\in N$ such that $x_i>z_i+1$,

  we have $v_x(\overline{z})=0$ et $\mu_{\overline{z}}(A)=0$ for any $A\subset N$, then $\mathcal{C}_{v_x}(z)=0$.

- Case 3 : $\exists i \in N$ such that $x_i > z_i$,

  we have $v_x(\overline{z}) = 0$ and $\mu_{\overline{z}}(A) = \mu_S(A)$ for any $A \subset N$, with μ_S a unanimity capacity and $S = \{i \in N : x_i > z_i\}$, then $\mathcal{C}_{v_x}(z) = \bigwedge_{i \in S}(z_i - \overline{z}_i)$.

□

Lemma 8. For any $v \in \mathcal{G}(L)$, any $T \subseteq N$, we have

$$\Delta_T \mathcal{C}_v(z) = \sum_{S \subseteq N \setminus T} \sum_{\substack{x \in L, x_T \geq 1_T \\ x_S = (\overline{z}+1)_S \\ x_{N \setminus T \cup S} \leq \overline{z}_{N \setminus T \cup S}}} m^v(x) \bigwedge_{i \in S}(z_i - \overline{z}_i).$$

Proof. Let $z \in [0,k]^N$, $\forall T \subseteq N$ and $\forall L \subseteq T$, we have by linearity of the Choquet integral w.r.t. games, by Lemma 7 and (2),

$$\mathcal{C}_v(0_L, k_{T \setminus L}, z_{-T}) = \sum_{\substack{x \in L \\ x_L = 0_L \\ x_{-T} \leq z_{-T}}} m^v(x) + \sum_{\substack{x \in L \\ x_L \leq 1_L \\ x_{-T} \leq (z+1)_{-T} \\ \exists i \in N \setminus (T \setminus L): \\ x_i > (0_L, z_{-T})_i}} m^v(x) \bigwedge_{\substack{i \in N \setminus (T \setminus L): \\ x_i > (0_L, z_{-T})_i}} (0_T, z_{-T} - \overline{z}_{-T})_i.$$

Letting $\phi_i = (0_T, z_{-T} - \overline{z}_{-T})_i$, we have

$$\begin{aligned}
\sum_{\substack{x \in L \\ x_L \leq 1_L \\ x_{-T} \leq (z+1)_{-T} \\ \exists i \in N \setminus (T \setminus L): \\ x_i > (0_L, z_{-T})_i}} m^v(x) \bigwedge_{\substack{i \in N \setminus T: \\ x_i > z_i}} \phi_i &= \sum_{\substack{x \in L \\ x_L = 0_L \\ x_{-T} \leq (z+1)_{-T} \\ \exists i \in N \setminus T : x_i > z_i}} m^v(x) \bigwedge_{\substack{i \in N \setminus T: \\ x_i > z_i}} \phi_i + \sum_{\substack{x \in L \\ x_L \leq 1_L \\ x_L \neq 0_L \\ x_{-T} \leq (z+1)_{-T}}} m^v(x) \bigwedge_{\substack{i \in N \setminus T: \\ x_i > z_i}} \phi_i \wedge \bigwedge_{j \in L} \phi_j \\
&= \sum_{\substack{x \in L \\ x_L = 0_L \\ x_{-T} \leq (z+1)_{-T} \\ \exists i \in N \setminus T : x_i > z_i}} m^v(x) \bigwedge_{i \in N \setminus T : x_i > z_i} (z_i - \overline{z}_i).
\end{aligned}$$

Then,

$$\mathcal{C}_v(0_L, k_{T \setminus L}, z_{-T}) = \sum_{\substack{x \in L \\ x_L = 0_L \\ x_{-T} \leq z_{-T}}} m^v(x) + \sum_{\substack{x \in L \\ x_L = 0_L \\ x_{-T} \leq (z+1)_{-T} \\ \exists i \in N \setminus T : x_i > z_i}} m^v(x) \bigwedge_{\substack{i \in N \setminus T: \\ x_i > z_i}} (z_i - \overline{z}_i).$$

We use the following expression

$$\forall T \subseteq N, \sum_{A \subseteq T} (-1)^{|A|} \sum_{\substack{x_A = 0_A \\ x_{T \setminus A} \in L_{T \setminus A}}} m(x) = \sum_{\substack{x_T \in L_T \\ x_T \geq 1_T}} m(x), \forall x_{-T} \in L_{-T},$$

and the definition of $\Delta_T \mathcal{C}_v$. We get:

$$\begin{aligned}\Delta_T \mathcal{C}_v(z) &= \sum_{\substack{x \in L \\ x_T \geq 1_T \\ x_{-T} \leq z_{-T}}} m^v(x) + \sum_{\substack{x \in L \\ x_T \geq 1_T \\ x_{-T} \leq (z+1)_{-T} \\ \exists i \in N \setminus T : x_i > z_i}} m^v(x) \bigwedge_{\substack{i \in N \setminus T: \\ x_i > z_i}} (z_i - \overline{z}_i) \\ &= \sum_{\substack{x \in L \\ x_T \geq 1_T \\ x_{-T} \leq \overline{z}_{-T}}} m^v(x) + \sum_{\substack{S \subseteq N \setminus T \\ S \neq \emptyset}} \sum_{\substack{x \in L, x_T \geq 1_T \\ x_S = (\overline{z}+1)_S \\ x_{N \setminus T \cup S} \leq \overline{z}_{N \setminus T \cup S}}} m^v(x) \bigwedge_{i \in S} (z_i - \overline{z}_i),\end{aligned}$$

the lemma is proved. □

We now prove Theorem 5.

Proof . Let v a k-ary capacity. For every $T \subseteq N$, we have,

$$\begin{aligned}
\int_{[0,k]^n} \Delta_T \mathcal{C}_v(z)\,\mathrm{d}z &= \sum_{\substack{x\in L\\ \forall i\in N, x_i<k}} \sum_{S\subseteq N\setminus T} \sum_{\substack{y\in L, y_T\geq 1_T\\ y_S=x_S+1_S\\ y_{N\setminus T\cup S}\leq x_{N\setminus T\cup S}}} m^v(y) \int_{[x,x+1_N]^n} \bigwedge_{i\in S}(z_i-x_i)\,\mathrm{d}z\\
&= \sum_{\substack{x\in L\\ \forall i\in N, x_i<k}} \sum_{S\subseteq N\setminus T} \sum_{\substack{y\in L, y_T\geq 1_T\\ y_S=x_S+1_S\\ y_{N\setminus T\cup S}\leq x_{N\setminus T\cup S}}} m^v(y) \int_{[0,1]^n} \bigwedge_{i\in S} z_i\,\mathrm{d}z\\
&= \sum_{\substack{x\in L\\ \forall i\in N, x_i<k}} \sum_{S\subseteq N\setminus T} \sum_{\substack{y_{-S}\in L_{-S}\\ y_T\geq 1_T\\ y_{N\setminus T\cup S}\leq x_{-T\cup S}}} \frac{m^v(x_S+1_S, y_T, y_{-T\cup S})}{s+1}\\
&= k^t \sum_{\substack{x_{-T}\in L_{-T}\\ \forall i\in N\setminus T, x_i<k}} \sum_{S\subseteq N\setminus T} \sum_{\substack{y_{-S}\in L_{-S}\\ y_T\geq 1_T\\ y_{N\setminus T\cup S}\leq x_{-T\cup S}}} \frac{m^v(x_S+1_S, y_T, y_{-T\cup S})}{s+1}\\
&= k^t \sum_{\substack{x\in L\\ \forall i\in N, x_i<k}} \sum_{S\subseteq N\setminus T} \sum_{\substack{y_{-T\cup S}\in L_{-T\cup S}\\ y_{-T\cup S}\leq x_{-T\cup S}}} \frac{m^v(x_{T\cup S}+1_{T\cup S}, y_{-T\cup S})}{s+1}\\
&= k^t \sum_{\substack{x\in L\\ \forall i\in N, x_i<k}} \sum_{S\subseteq N\setminus T} \frac{\Delta_{T\cup S} v(x)}{s+1} \quad \text{(by Lemma (1))}\\
&= k^t \sum_{\substack{x\in L\\ \forall i\in N, x_i<k}} \sum_{S\subseteq N\setminus T} \sum_{A\subseteq T\cup S} \frac{(-1)^{t+s-a} v(x+1_A)}{s+1}\\
&= k^t \sum_{\substack{x\in L\\ \forall i\in N, x_i<k}} \sum_{S\subseteq N\setminus T} \sum_{A\subseteq T\cup S} \frac{(-1)^{t+s-a} v(x+1_A) - (-1)^{t+s-a} v(x)}{s+1}\\
&= k^t \sum_{\substack{x\in L\\ \forall i\in N, x_i<k}} \sum_{S\subseteq N\setminus T} \sum_{A\subseteq T\cup S} \frac{(-1)^{t+s-a} \mu_x(A)}{s+1}\\
&= k^t \sum_{\substack{x\in L\\ \forall i\in N, x_i<k}} \sum_{S\subseteq N\setminus T} \frac{1}{s+1} m^{\mu_x}(T\cup S).
\end{aligned}$$

We use the expression of interaction index for capacities in terms of the Mobius representation (Grabisch et al., 2000), and by Proposition (9), we get

$$\int_{[0,k]^n} \Delta_T \mathcal{C}_v(z)\,\mathrm{d}z = k^t I_s^v(T).$$

□

The interaction index on continous domain appears as the mean of relative amplitude of the range of $\mathcal{C}_v$ w.r.t. T, when the remaining variables take uniformly random values. The total variation is the local interaction of $\mathcal{C}_v$ at point z.

Example 4 (**System engineers (continued)**)**.** Consider the 2-ary capacity v given in Example 1. By applying the formula of Theorem 5 we find 0, and therefore the theorem is satisfied in the example.

Remark 3. It is interesting to compare the expression of the interaction index obtained in Theorem 5 with the general expression of the interaction index for an arbitrary aggregation function F defined on some domain $[a,b]^n$ and taking value in $[a,b]$ (see (Grabisch et al., 2009, Sec. 10.4)):

$$I_T(F) = \frac{1}{(b-a)^n}\int_{[a,b]^n} \frac{\Delta_T F(x)}{b-a}\,\mathrm{d}x,$$

where $T \subseteq N$ and $\Delta_T F(x)$ is the total variation of F w.r.t. T at x, defined by

$$\Delta_T F(x) = \sum_{S\subseteq T}(-1)^s F(a_S, b_{T\setminus S}, x_{-T}).$$

Applying this formula to $F = \mathcal{C}_v$ on $[0,k]^n$ we find:

$$I_T(\mathcal{C}_v) = \frac{1}{k^{n+1}}\int \Delta_T \mathcal{C}_v(x)\,\mathrm{d}x = \frac{1}{k^{n-t+1}} I^v_{\mathrm{s}}(T).$$

The difference in the normalizing coefficient comes from the axioms we have chosen, essentially the efficiency axiom. Indeed, it is possible to recover exactly $I_T(\mathcal{C}_v)$ if in the efficiency axiom (E), we divide the right hand side of the equality by k^n (this would express an efficiency or *average* variation per cell in the grid, compared to a *total* variation on the grid), and in the recursivity axiom (R), we multiply by k the right hand side (since it concerns games with $n-1$ players).

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