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1. *Introduction*

All in all, measure enters, as such, in Aristotle's *Physics* in no more than in seven different occasions.

Two of them are the object of other papers in the present volume: they respectively concern Aristotle's treatment of time, in IV 10-14, and his argument to prove that a motion cannot but cover a finite space in a finite time and an infinite space (if any) in an infinite time, in VI, 7. Two of the remaining five occasions are strictly connected with this same topic: the first pertains to an argument advanced in VI, 3, apparently for the same purpose, while proving that any continuum is infinitely divisible; the second pertains to an argument advanced in VI, 10, in order to prove that "no point [and] no other indivisible admits being set in motion [οὔτε στιγμὴν οὔτ' ἄλλο ἀδιαίρετον οὐθὲν ἐνδέχεται κινεῖσθαι—241a7]".

The three further occasions are these:

In V, 3, 226b32-34, Aristotle argues that

> Contrary concerning place <is that which is> the most distant in straight line, since the <straight line> is the minimum to be limited and that <which is> limited <is> measure [ἐναντίον δὲ κατὰ τόπον τὸ κατ' εὐθεῖαν ἀπέχον πλεῖστον· ἡ γὰρ ἐλαχίστη πεπέρανται, μέτρον δὲ τὸ πεπερασμένον].

In VIII, 9, 265b8-11, he advances that

> since the circular locomotion is measure of motions, it is necessary that it be the first (for all things are measured by the first), and because <it is> the first, <it> is measure of the other <mouvements> [γὰρ ὅτι μέτρον τῶν κινήσεων ἡ περιφορά, πρώτην ἀναγκαῖον αὐτὴν εἶναι (ἅπαντα γὰρ μετρεῖται τῷ πρώτῳ), καὶ διότι πρώτη, μέτρον ἐστὶν τῶν ἄλλων].

In VIII, 10, 266b21-24, he envisages the possibility of taking

> in <a> finite magnitude same force, the same in kind as that in an infinite magnitude, that <be> as <a> measure of the finite force in the infinite <magnitude> [ληψόμεθα […] τινα δύναμιν τὴν αὐτὴν τῷ γένει τῇ ἐν τῷ ἀπείρῳ μεγέθει, ἐν πεπερασμένῳ μεγέθει οὖσαν, ἣ καταμετρήσει[2] τὴν ἐν τῷ ἀπείρῳ πεπερασμένην δύναμιν].

In none of these three occasions, is measure the main object of Aristotle's attention. Far from that: in all three of these occasions, the appeal to measure plays a quite marginal role. In V 3, the focus is on the relation of being "intermediate [μεταξύ]". In VIII, 9, the focus is on the claim that the first motion is a circular locomotion, which is, in turn, a lemma for the main claim that the first motion is the circular locomotion of the first heaven. In VIII, 10, the focus

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[2] For the translation of the verb 'καταμετρέω', see footnote (5) below.

is on the claim that the first mover has no extension, and the appeal to measure only helps to argue that (266a24-26)

> by what has been said, it is clear that there is no way for an infinite force to be in a finite magnitude [ὅλως οὐκ ἐνδέχεται ἐν πεπερασμένῳ μεγέθει ἄπειρον εἶναι δύναμιν, ἐκ τῶνδε δῆλον].

In the other four occasions, in particular in IV 10-14, the appeal to measure is perhaps not so marginal. Still, there is no doubt that, also in these chapters, the focus of Aristotle's attention is not measure, but time and motion. It seems, then, that Aristotle does not consider measure as a relevant subject to be specifically discussed in his *Physics*.

This marginalisation of measure might surprise us at first glance. Still, in examining the other works devoted to physics, metaphysics and logic in Aristotle's *corpus*, we are brought to a similar conclusion: though the appeal to measure and/or (in)commensurability is quite frequent, Aristotle seems to take these notions as being so well-known to be used, often by analogy, to clarify other notions or points. The case of the incommensurability of the diagonal and the side of a square is emblematic of this attitude. Aristotle makes a myriad of references to it, as a very well-known fact, in order to elucidate, by analogy, other facts or conceptions. However, the point is never discussed as such, whether in its mathematical content, or in connexion to the choice of appropriate notions to be used to account for the physical world.

The only apparent exception comes from *Metaphysics* I, 1, where Aristotle seems to say something just about measure, rather than appealing to measure while dealing with other matters. He says, for example, that (1052b20)

> measure […] is that by which quantity comes to be known [μέτρον […] ἐστιν ᾧ τὸ ποσὸν γιγνώσκεται],

and that 1052b24-27)

> also in the other <genera>, measure is said that by which, firstly, each <thing> comes to be known, and the measure of each <thing is> one, in length, in breadth, in depth, in weight, in speed [καὶ ἐν τοῖς ἄλλοις λέγεται μέτρον τε ᾧ ἕκαστον πρώτῳ γιγνώσκεται, καὶ τὸ μέτρον ἑκάστου ἕν, ἐν μήκει, ἐν πλάτει, ἐν βάθει, ἐν βάρει, ἐν τάχει].

More importantly for my present purpose, he says that (1053a24-27)

> the measure <is> always homogenous: a magnitude, indeed, of magnitudes, and wherever you find length of length, breadth of breadth, sound of sound, weight of weight, unit of units [ἀεὶ δὲ συγγενὲς τὸ μέτρον· μεγεθῶν μὲν γὰρ μέγεθος, καὶ καθ' ἕκαστον μήκους μῆκος, πλάτους πλάτος, φωνῆς φωνή, βάρους βάρος, μονάδων μονάς],

and that (1053a14-15)

> not always <there is only> one measure for a number [οὐκ ἀεὶ δὲ τῷ ἀριθμῷ ἓν τὸ μέτρον],

and even (1053a17-18)

> the diagonal is measured by two <measures>, and <so> the side, and all magnitudes [ἡ διάμετρος δυσὶ μετρεῖται καὶ ἡ πλευρά, καὶ τὰ μεγέθη πάντα].

(I shall return later in § 2 to this claim, which at first glance could appear quite strange)

However, in the context of Aristotle's argumentation, which is essentially devoted to the notion of one, these considerations seem, anew, more as recalls of well-known conceptions, than as original claims. It even seems that Aristotle does not consider it important, or sufficiently useful, to tidily come back to these matters and offer his original insight. What measure is, and how it enters physics and mathematics is, for him, a given, to the extent that appealing to it can help in explaining other things, without needing further clarification or insight.

As surprising as it might appear *prima facie*, it is not so much after a little reflexion. Whatever role measure could have played in Aristotle's time in physics and mathematics, was depending on the content of the available mathematical theories, which is something Aristotle had less to say on, or even about which he never tried to say something. His attitude toward mathematics was always unequivocal: he certainly tried (without much success, by the way, as shown by the quite ambiguous claims of *Metaphysics* M 1-3) to understand its epistemic nature by so continuing, in a quite different direction, a reflexion already initiated by Plato, but he always took it as a datum, as an established portion of knowledge that is not to be contradicted. He never tried to improve or clarify it. What he says about measure seems to confirm this attitude. To clarify this point, a short survey of the mathematical conceptions of measure at his time, and of the way they differ from our present ones, is in order.

2. *What is Measure ? A Short Comparison of Our and Ancient Greek Views*

Among others, one basic feature of measure, which Aristotle passively accepts from mathematics, without any sort of discussion and any effort towards clarification, already emerges quite clearly from the previous quotes: measure pertains to a relation between homogenous *relata*.

This conception will appear quite clearly at work, some decades later, in Euclid's *Elements* (especially in books VII-X, but also in the first two definitions of book V: see below), and will remain at work in the whole *corpus* of ancient and early-modern mathematics. Making this clear, and underlining the difference between this conception and our modern one, is crucial for understanding both Aristotle's claims on this matter and the evolution of mathematics.

There are two basic aspects of this conception that it is convenient to distinguish: the first is that measure pertains to a relation, more precisely to a binary relation; the second is that this is a relation between homogenous *relata*. Distinguishing these points is all the more important being that modern mathematics basically agrees on the former, but disagrees on the latter.

For a modern mathematician, measuring a magnitude is the same as associating it with a number—typically, but not only, with a real number. This is to say that measuring a magnitude is the same as reflecting its rank within a hierarchy of comparable magnitudes into an appropriate numerical order. This reflexion is essential, not only to assign a place to such a magnitude within the relevant hierarchy, but also since, overall, it allows transposing different, possibly imprecisely established hierarchies, into a fixed and well-established linear numerical order—or even, when real numbers are at issue, into a complete ordered field (in the mathematical sense of this term). Among other things, this makes possible to operate on numerical *alias* of the relevant magnitudes according to the usual additive and multiplicative rules.

Strictly speaking, when we say that a flat is bigger than another, what we mean is not just that, but rather that the area of the former is greater than that of the latter. Analogously, when we say that a car goes faster than another, we mean that the speed of the former is greater than that of the latter. Distinguishing between the order of flats, with respect to their extension, and that of their areas, or between the order of cars, with respect to their rapidity, and that of their speeds, is all but a subtlety. Since areas and speeds are numbers, in our language, or they are, at least, expressed by numbers, and the order of numbers is unique and well fixed, once for all, while the order of flats, with respect to their extension, and that of cars, with respect to their rapidity, are not only different to each other, but also quite difficult to establish in themselves. How can we compare, for example, relative to their extension, a flat on three floors with two mezzanine bedrooms with another, all on the same level ? How can we compare, relative to their rapidity, a jeep that is going down from a mule track, with a city car going through downtown ? To do it, we transpose each flat, or better its extension, or each car, or better its rapidity, into an appropriate numerical domain, and we compare the numbers that, through this transposition, we associate with the two flats and the two cars.

When modern mathematicians speak of measure, they are referring to a transposition like these.

In a (not very rigorous) sense, measures are just numbers, namely numbers associated with items with respect to a certain property of such items, so as to reflect their rank within the relevant hierarchy of comparable items, i.e., to flats with respect to their extension, or to cars, with respect to their rapidity. In this sense, we can say that both the area of a flat and the speed of a car are numbers, and they are so just insofar as they are measures of them, with respect to the flat's extension or the car's rapidity. Although, we could also say that what these numbers are measures of, are not properly flats (with respect to their extension), or cars (with respect to their rapidity), but just extension and rapidity themselves. Measures would, then, be numbers associated with properties of items, rather than with the items themselves. Insofar as we take measures to be measures of magnitudes, in conformity with the first parlance, flats and cars are magnitudes when taken with respect to their extension and rapidity, while, in conformity with the second parlance, the magnitudes are extension and rapidity themselves.

Still, for a flat—or its extension—to be associated with a unique number reflecting its rank within the hierarchy of flats, with respect to their extension—or of flats' extensions—, an appropriate unit is to be fixed, usually referred to as 'unit of measurement'. The same for a car—or its rapidity—to be associated with a unique number reflecting its rank within the hierarchy of cars, with respect to their rapidity—or of cars' rapidity. If such a unit changes, the numbers does as well. In another (more rigorous) sense, the measure of a flat, with respect to its extension—or of this very extension—or that of a car, with respect to its rapidity—or of this very rapidity—is not such a changing number, but rather something that remains constant under the change of the relevant unit of measurement; this is what we refer to by speaking of measure—and what we also call 'area' (of the relevant flat), or 'speed' (of the relevant car). This is not properly a number in itself, but it is expressed by numbers, a different one for any choice of the unit of measurement. Measuring a magnitude is, still, associating it with a number, but the measure of this magnitude is not properly this number, but rather what this numbers expresses and remains invariant when this number changes as a result of the change of the unit of measurement.

Finally, in a third sense (even more rigorous than the second), what we refer to when speaking of measure, is neither a number nor what it expresses, but rather the very relation between the relevant magnitudes and the relevant numbers or that which numbers express, or, even better, the function associating these magnitudes with these numbers or what they express. If we take this function to associate magnitudes with numbers, then we have to either admit that any choice of a unit of measurement corresponds to a different one-argument function, and, therefore, to a different measure, or to take this function as having two arguments—one for magnitudes and the other for units of measurement—, so as to associate with a number both a magnitude and a unit of measurement, rather than only a magnitude, or only a magnitude, but via a unit of measurement, rather than directly. If we take the relevant function to associate magnitudes with the invariant entity that numbers express, then a unique one-argument function would be at issue for each sort of magnitudes. We could then consider, that to each distinct sort of magnitudes corresponds a different (unique, one-argument) function, namely a different measure, and say, for instance, that areas and speeds are two distinct measures, insofar as they are two distinct functions, respectively pertaining either to extended and moving bodies, or to their extension and rapidity. However, we could also consider that measure is unique, that is, that there is a unique function, generally called 'measure', whose values are associated with distinct sorts of magnitudes in different appropriate ways, and are, of course, invariant under the change of the unit of measurement, though being expressed by distinct numbers for its different choices.

Though quite distinct from each other, these different senses in which we could today speak of measure recover a unique idea: that of conceiving measure as that which is at issue (in a form or another) when non-numerical items, or some of their properties—often called 'intensive'—are associated with numbers, so as to make the order of numbers reflect non-numerical hierarchies, and to provide an indirect way to add and multiply these items to and with each other. What makes these senses different is the possibility of using an imprecise language, as our natural one, to describe different aspects of this basic idea in different ways. Still, however these aspects might be described, one important point remains clear: numbers are not objects of measurement; in none of the foregoing senses, numbers have measures or allow measure, understood as a function defined on them.

Things went in quite a different way for Greek mathematicians. According to them, both magnitudes and numbers had measures. These were understood as different sorts of quantities, continuous and discontinuous, respectively. A number was, for Greek mathematicians, "a multitude composed of unities [ἀριθμὸς δὲ τὸ ἐκ μονάδων συγκείμενον πλῆθος—Euclid, *Elements*, def. VII.1[3]]"—that is, in our language, a natural number greater than one. Its being taken as a discontinuous quantity depended just on its being a multitude, namely on its intrinsically being a compound of distinct elements. A magnitude was, instead, taken to be continuous, just insofar as it was taken to be, intrinsically, a single unitary item, rather than a compound.

I shall come back in §§ 6 and 7 to the conception of continuity giving sense to such an idea, or, at least, on the way Aristotle presents it in his *Physics*, in particular in V 3. For the time being, only two remarks are in order.

The first is that the Greek notion of a magnitude was not at all incompatible with the idea that magnitudes could be items, as distinct from intensive properties of them[4].

The second is that conceiving a magnitude (understood as an item) as a continuous quantity did not forbid Greek mathematicians to consider some parts of it, even if (in conformity with the conception of continuity which I shall describe in §§ 6 and 7) identifying

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[3] Quotes and references to Euclid's *Elements* are relative to Heiberg's critical edition: *Euclidis Elementa*, edidit et latine interpretatus est I. L. Heiberg, Lipsiae, in ædibus B. G. Teubneri, 1883-85 (4 vols.). English translations are Heath's, with some slight amendments from time to time: *The Thirteen Books of Euclid's Elements*, Translated from the text of Heiberg, with Introduction and Commentary by T. L. Heath, Second ed. Revised with Additions, Cambridge 1925 (3 vols.).

[4] This is made clear, among other things, by the way the theory of proportions presented in book V.of the *Elements* (on which I'll shall return pretty soon), explicitly concerned with magnitudes, is applied to geometrical items in book VI. What made Greeks mathematicians open to the idea that magnitudes could be items (as distinct from intensive properties of them), could not be considered here. It will be enough to say that, for them, differently than from us, there was, in fact, no categorical opposition between such a property and the items having it: a length was, for them, for example, if not identical with, at least primarily represented by a segment, in such a way that dealing with the former was just the same as dealing with the latter.

these parts as such breaks the magnitude's continuity. Since identifying these parts did not prevent one from considering the magnitude as a single item in regard to its being so intrinsically, before these parts are identified in it.

This is important since it explains how the essential difference in nature among magnitudes and numbers, due to their respectively being continuous and discontinuous magnitudes, did result, for Greek mathematicians, in no difference concerning their having measures, despite the fact that they intended a measure of a quantity as an aliquot part of it.

More precisely, for them, a measure of a magnitude was another magnitude homogenous to the former, entering this magnitude an exact number of times; a measure of a composite number was, analogously, another number, divisor of the former, while a measure of a prime number was the unity, or better, one of its unities. The measure of a quantity was, therefore, another quantity, homogeneous to the former: an aliquot part of it, as I have just said.

This conception is perfectly alien to any idea of reflection of the rank of an item within a given hierarchy into a fixed order. It rather makes the notion of measure strictly connected to that of part, as Aristotle himself makes clear in *Metaphysics* Z 10, 1034b32-33:

> Part is said in several ways; <in> one of these ways <a part is> that which measures according to the quantity [ἦ πολλαχῶς λέγεται τὸ μέρος, ὧν εἷς μὲν τρόπος τὸ μετροῦν κατὰ τὸ ποσόν.

This quote does not only make the connection of the notions of measure and part explicit. It also suggests that Aristotle was conceiving the former notion as epistemically prior to the latter, as apt to be appealed to in order to elucidate it (rather than vice versa). This is the same posture as Euclid's. Though in the *Elements*, measure is never defined, it is appealed to in order to define:

– parts and multiples of a magnitude, in defs. V.1-2:

> A magnitude is part of a magnitude, the less of the greater, when it measures out the greater [Μέρος ἐστὶ μέγεθος μεγέθους τὸ ἔλασσον τοῦ μείζονος, ὅταν καταμετρῇ[5] τὸ μεῖζον], and

[5] This definition V.1 and the other definitions V.2 and VII.3-5 (mentioned below) apart, the verb 'καταμετρέω' occurs very seldom in the *Elements*: in the πρότασις and the ἔκθεσις of proposition VII.1; in the πρότασις, ἔκθεσις and ἀπόδειξις of proposition X.2; and in the only ἀπόδειξις of proposition X.3. Propositions VII.1 and X.2 are akin, insofar as they both set forth the procedure of ἀνταναίρεσις: in modern terms, two different quantities (either two numbers or two magnitudes) α and β ($\beta<\alpha$) being give, find the natural positive number m such that $\alpha=m\beta+\gamma$ ($0\leq\gamma<\beta$), then the positive natural number n such that $\beta=n\gamma+\delta$ ($0\leq\delta<\gamma$), and so on either indefinitely, or up reaching a null remainder. Proposition X.3 sets forth the procedure to find the maximum common aliquot part (or measure) of two commensurable magnitudes. In all these cases, as well in definitions V.1-2 and VII.3-5, Euclid seems to judge important to emphasize that the relevant whole is completely exhausted when all the relevant parts are added to each other, which means that these parts are aliquot ones. This is, at least, what, Acerbi implicitly suggests, by rendering, in his Italian translation of the *Elements*, 'καταμετρέω' with 'misurare completamente' (Euclide, *Tutte le opere*, a cura di F. Acerbi, Milano, 2007, pp. 975, 1091, 1233, 1235). Still, Euclid also invariably uses the verb 'μετρέω' and its cognates to bring up the fact that a certain quantity is an aliquot part of another, that is, in the same sense as it seems to use the verb 'καταμετρέω'. Possibly, he was conceiving 'καταμετρέω' as a clearer or more stringent version of 'μετρέω', to be used, in some crucial occasions, to emphasize that the parts in question are aliquot ones, by allowing himself to use, then, once this had been made explicit, the more ambiguous verb μετρέω, in the same sense as καταμετρέω (as, for example, already in definitions VII.8-15 and X.1-2, and in other occurrences in the ἀπόδειξις of propositions X.2-3). The two verbs seem, then, to be used by Euclid in the same sense, but with a different emphasis. In my translation of Euclid's definitions, I try to render this slight difference by translating καταμετρέω with 'measure out' and μετρέω and cognates with 'measure' and cognates. I'm far from sure that an akin difference of use of the two verbs should also be attributed to Aristotle. To verify it a much more detailed study than what I'm able to make here would be required. This is why, in my quotes from Aritstotle's treatises, I

multiple, the greater of the less, when it is measure out by the less [Πολλαπλάσιον δὲ τὸ μεῖζον τοῦ ἐλάττονος, ὅταν καταμετρῆται ὑπὸ τοῦ ἐλάττονος];

– numbers that are part and parts of another number, in defs. VII.3-4:

A number is part of a number, the less of the greater, when it measures out the greater [Μέρος ἐστὶν ἀριθμὸς ἀριθμοῦ ὁ ἐλάσσων τοῦ μείζονος, ὅταν καταμετρῇ τὸν μείζονα], and parts when it does not measure it out [Μέρη δέ, ὅταν μὴ καταμετρῇ];

– multiples of a numbers, in def. VII.5:

<A number is> multiple, the greater of the less, when it is measured out by the less [Πολλαπλάσιος δὲ ὁ μείζων τοῦ ἐλάσσονος, ὅταν καταμετρῆται ὑπὸ τοῦ ἐλάσσονος];

– even-times even and even-times odd numbers, in def. VII.8-9:

an even-time even number is that <which is> measured by an even number according to an even number [Ἀρτιάκις ἄρτιος ἀριθμός ἐστιν ὁ ὑπὸ ἀρτίου ἀριθμοῦ μετρούμενος κατὰ ἄρτιον ἀριθμόν], and even-time odd is that <which is> measured by an even number according to an odd number [Ἀρτιάκις δὲ περισσός ἐστιν ὁ ὑπὸ ἀρτίου ἀριθμοῦ μετρούμενος κατὰ περισσὸν ἀριθμόν];

– odd-times even and odd-times odd numbers, in def. VII.10*-11[6]:

and odd-times even is that <which is> measured by an odd number according to an even number [Περισσάκις ἄρτιός ἐστιν ὁ ὑπὸ περισσοῦ ἀριθμοῦ μετρούμενος κατὰ ἄρτιον ἀριθμόν], and odd-times odd is that <which is> measured by an odd number according to an odd number [Περισσάκις δὲ περισσὸς ἀριθμός ἐστιν ὁ ὑπὸ περισσοῦ ἀριθμοῦ μετρούμενος κατὰ περισσὸν ἀριθμόν];

– prime and composite numbers, in def. VII.12 and 14:

a prime number is that <which is> measured only by unity [Πρῶτος ἀριθμός ἐστιν ὁ μονάδι μόνῃ μετρούμενος] […] a composite number is that <which is> measured by some number [Σύνθετος ἀριθμός ἐστιν ὁ ἀριθμῷ τινι μετρούμενος];

– numbers prime and composite to one another, in def. VII.13 and 15:

prime numbers to one another are those <which are> measured only by a unity as common measure [Πρῶτοι πρὸς ἀλλήλους ἀριθμοί εἰσιν οἱ μονάδι μόνῃ μετρούμενοι κοινῷ μέτρῳ], and composite numbers to one another are those <which are> measured by some number as common measure [Σύνθετοι δὲ πρὸς ἀλλήλους ἀριθμοί εἰσιν οἱ ἀριθμῷ τινι μετρούμενοι κοινῷ μέτρῳ];

– commensurable and incommensurable magnitudes, in def. X.1:

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translate both καταμετρέω and cognates and μετρέω and cognates, with 'to measure' and cognates, as it is usually done.

[6] In his edition, Heiberg brackets definition ι′ (by considering it interpolated: see I. L. Heiberg, *Litterargeschichtliche Studien über Euklid*, Leipzig 1882, p. 198), and does not translate it in Latin, by translating definition ια′ as definition 10, definition ιβ′ as definition 11, and so on. While Heath follows Heiberg Latin numeration, I follow his Greek one, by referring to definition ι′ as to definition 10*, as in the TLG.

> commensurable magnitudes are said those <magnitudes that are> measured by the same measure, incommensurables, on the contrary, <those> of which it is not possible <that> there be a common measure [σύμμετρα μεγέθη λέγεται τὰ τῷ αὐτῷ μέτρῳ μετρούμενα, ἀσύμμετρα δέ, ὧν μηδὲν ἐνδέχεται κοινὸν μέτρον γενέσθαι];

– segments commensurable and incommensurable by power, in def. X.2:

> <segments of> straight <lines> are commensurable by power when the squares on them are measured by the same surface, and incommensurable when it is not possible <that> a surface <providing> a common measure for the squares on them come out [εὐθεῖαι δυνάμει σύμμετροί εἰσιν, ὅταν τὰ ἀπ᾽ αὐτῶν τετράγωνα τῷ αὐτῷ χωρίῳ μετρῆται, ἀσύμμετροι δέ, ὅταν τοῖς ἀπ᾽ αὐτῶν τετραγώνοις μηδὲν ἐνδέχηται χωρίον κοινὸν μέτρον γενέσθαι].

It seems, then, that, both for Aristotle and for Euclid, that of measure is a primitive notion, with ground on which others can be defined or elucidated, including some which appear to us as much more fundamental, as that of aliquot part or exact divisor of a number. It is, however, quite difficult, for us, to follow them in such an elucidatory process, going from measure to aliquot part, or exact divisor of a number. To clarify their view, it is much easier, for us, to invert the direction of elucidation and using our ideas of part or exact divisor of a number to elucidate their notion of measure.

This makes clear that speaking of measure required, both for Euclid and for Aristotle, as well as for all contemporary mathematicians, having available both an additive operation and an equality relation directly defined on what is to be measured (not only numbers, but also magnitudes), or better on its parts. The reason is that nothing can be conceived as being an aliquot part of a whole without conceiving such a whole are resulting from an additive composition of it with other items (or parts) equal to it in some relevant respect[7].

However, the nature of the additive operation and the equality relation changes, of course, from species to species (adding two segments is, for example, quite another thing that adding two numbers). Moreover, even among the elements of a same specie, one can conceive different additive operations or equality relations: adding two squares so to get another square is, for example, another thing that adding them to get a gnomon, and adding several rectangles whose heights are equal to get a square is another thing than adding several squares to get another square. Hence, under this understanding, the relation of being a measure of is in no way unique in nature (if not for its being a relation between an aliquot part of a whole and this very whole).

The connection between measure and aliquot part makes also clear that a quantity that has a measure has, in fact, several equal measures (at least two) of the same nature, and can even have different measures of the same nature, none of which is a measure of another, since it can not only admit different ways of being break down in aliquot parts, but also different aliquot parts of the same nature, none of which is an aliquot part of another. This is shown by the following example:

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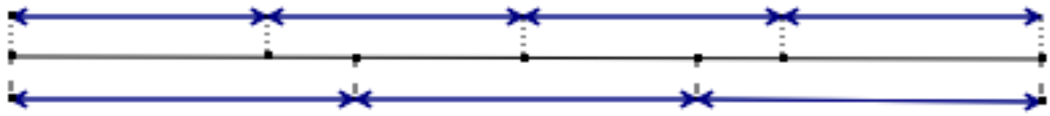
[7] Notice that having this operation and this relation available also makes it easy to define an order relation, since it allows us to state that b is strictly smaller than a if a results from an additive composition of b with some c, as openly suggested by Euclid, in common notion I.8 of the *Elements*: "the whole is greater than the part [τὸ ὅλον τοῦ μέρους μεῖζον ἐστιν]".



as well as from the simple observation that if n and m are whatever natural numbers greater than 1, and $n \times m = p$, then both n and m are aliquot parts of p[8].

Moreover, this connection also makes clear that—while any pair of numbers have a common measure, as implied by Euclid both in stating definition VII.12 an in proving proposition VII.4 of the *Elements*[9]—it is possible to have homogenous magnitudes admitting no common measure, that is, incommensurable magnitudes[10]. The example of the side and the diagonal of a square (that Aristotle pervasively appeals to in his treatises) clearly illustrates this fact. However, of course, from the fact that two magnitudes are incommensurable to one another does in no way follow that one of them has no measure, nor that it has not several different measures, none of which is a measure of another. The example in the previous figure immediately shows, indeed, that any segment has several different measures, none of which is a measure of another, and this is, of course, also the case both for the diagonal and the side of a square, as Aristotle clearly says in the last sentence quoted above from *Metaphysics* I, 1: "the diagonal is measured by two, and <so> the side, and all magnitudes". In light of his conception of measure, this claim is crystal clear, although it has been so many times misinterpreted or considered to be "a complete puzzle"[11]. Again, if a and b are two incommensurable segments, there is certainly a smaller enough part of one of them, let as say of a, which is equal to a part of b (or is even directly a part of it), without being an aliquot part of it, and that, consequently, is not a measure of b, being rather incommensurable with it. Any segment has, then, parts that are incommensurable with it. Moreover, insofar as segments reflect the quantitative relations among whatever sort of homogenous magnitudes—in a way that is similar to that in which real numbers reflect the quantitative relations of homogenous

[8] This is so, of course, also if n and m are prime to each other. But, if this is so, it is immediately clear that none of them is an aliquot part of the other, and, then, a measure of it. Also, the example provided by the figure pertains to mutual primality of the number of the relevant parts. If we take the whole segment to be a, and its aliquot parts represented in the figure (by the two first small segments on the left) to be b and c, respectively, we shall have that $a = 4b = 3c$, and it is enough to observe that 3 and 4 are prime each to another to conclude that b cannot be an aliquot part of c.

[9] This proposition states that "any number is either part or parts of any number, the less of the greater [ἅπας ἀριθμὸς παντὸς ἀριθμοῦ ὁ ἐλάσσων τοῦ μείζονος ἤτοι μέρος ἐστὶν ἢ μέρη]". If definitions VII.3-4 are taken literally (see above), this is an obvious tautology. Still, in proving proposition VII.4, Euclid assumes that two numbers which are not prime to one another either are such that the less measures the greater or have a common measure, and that a number is parts of another if an aliquot part of the former is also an aliquot part of the latter—as, it happens, for instance, with 3 and 5 or with 4 and 6: in both cases, the first number is parts of the second, since the latter results by additive composition of an aliquot part of the former with itself ($5 = 1 + 1 + 1 + 1 + 1$, $6 = 2 + 2 + 2$). This makes the proposition state, in fact, that, for any two distinct numbers, either one of them is an aliquot part of the other, or an aliquot part of one of them is also an aliquot part of the other, which (though being quite easy to be proved) is surely not a tautology. This is what Euclid proves, indeed (although in a so uselessly complex way to raise many suspects about the role and authenticity of this proposition).

[10] Notice that, though being still in use, the adjective 'incommensurable'—literally translating the Greek 'ἀσύμμετρος'—is no more in line with our notion of measure, since, for us, two incommensurable magnitudes are not such to have no common measure (as stated by definition X.1, quoted above), but rather such to have no respective measures (in the first of the three modern senses distinguished above), whose *ratio* is that of two natural numbers.

[11] See T.L. Heath, *Mathematics in Aristotle*, Oxford 1949, p. 218.

magnitudes according to our modern notion of measure—, all this is also the case of any sort of magnitudes, without being, of course, the case of numbers (I shall arrive to this point soon).

It follows that, according to the conception I'm describing, not only both magnitudes and numbers have measures, and the latter are not measures of the former[12], but it also happens that the order of numbers only reflects the order of the measures of a domain of (homogenous and) commensurable magnitudes, but not that of the measures of any sort of magnitudes.

One might argue that this is merely due to the fact that Greek mathematicians had a poorer conception of numbers than our own—that their numbers only corresponded, as I have said above, to our natural numbers greater than one, and, for this reason, were not able to mimic the crucial phenomenon of incommensurability, which is proper to magnitudes. Relative to domains of commensurable magnitudes—one might continue to argue—their numbers worked, in fact, as our (real) ones work relative to any sort of magnitudes, when our notion of measure is at issue. Reasoning in this manner, one would, then, finally conclude that, with respect to such restricted domains, the difference between our and their conception of measure is merely terminological: it merely consists in the fact that they were using the term "measure" to refer to aliquot parts of magnitudes, rather than to evoke the relations connecting these parts to numbers.

This would be, however, a way to inverse the order of factors in the reconstruction of the difference between the two conceptions. Since the reason that made Greek numbers unable to play the role that our numbers play within our conception of measure was just that they were not able to mimic the phenomenon of incommensurability, rather than vice versa. It is, then, because of this last fact that the Greek conception of measure pertains to a relation between homogenous *relata*, rather then, as ours, to a relation between magnitudes and numbers. The phenomenon of incommensurability does not forbid, indeed, that something might play for magnitudes a role structurally akin (though quite different in its specific nature) to the one that (real) numbers play today in connection to measure. I just evoked this fact by observing that segments reflect the quantitative relations among whatever sort of homogenous magnitudes. It is now time to endow it with greater clarity.

This depends on Eudoxus's theory of proportion, which though transmitted to us by the 5th book of Euclid's *Elements*, was certainly already in place, in its essential aspects, in Aristotle's time, and which he also takes as a given. This theory permits to establish a four-place relation of proportionality—of being "ἀνάλογον", both in Euclid's and Aristotle's parlance[13]—among four magnitudes whatsoever, provided they be homogenous two by two. This makes the statement that two homogeneous magnitudes whatsoever are in proportion

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[12] Notice that it seems to be so also for the late Pytagorean school. In the fr. 2 of pseudo-Archytas, *De Intellectu*, one reads that "number is measure of multitude, foot of length, balance of poise and counterpoise, rule and plumb line of straightness in both horizontal and vertical direction, <that is the> right angle [πλάθεος μὲν γὰρ μέτρον ἀριθμός, μάκεος δὲ ποῦς, ῥοπᾶς δὲ καὶ σταθμοῦ ζυγόν, ὀρθότατος δὲ καὶ εὐθύτατος κανὼν καὶ στάθμα, ὀρθὰ γωνία]" (= 37,16-18 Thesleff). If numbers are taken to be "multitude[s] composed of unities", as for Euclid (see above), this implies that numbers are measure of numbers but not of other quantities.

[13] Concerning Aristotle, and limited to the *Physics*, see: IV.8, 215b3, 29 and 216a7, VII, 5, 250a4, 8, 14, 28, VIII, 10, 266b 2, 19. I only quote, as an example, the passage including 250a4: "Let, then, A be the mover, B the moved, Γ the quantity of the distance [on which the moved is] moved, and Δ the time in which [it is so]. In the same time [Δ], the same mover A will move half of B twice Γ, and on [the whole] Γ in the half of Δ, so that there will be proportion [εἰ δὴ τὸ μὲν A τὸ κινοῦν, τὸ δὲ B τὸ κινούμενον, ὅσον δὲ κεκίνηται μῆκος τὸ Γ, ἐν ὅσῳ δέ, ὁ χρόνος, ἐφ'οὗ τὸ Δ, ἐν δὴ τῷ ἴσῳ χρόνῳ ἡ ἴση δύναμις ἡ ἐφ'οὗ τὸ A τὸ ἥμισυ τοῦ B διπλασίαν τῆς Γ κινήσει, τὴν δὲ τὸ Γ ἐν τῷ ἡμίσει τοῦ Δ· οὕτω γὰρ ἀνάλογον ἔσται] (249b30—250a4)". Concerning Euclid, it will be enough to mention the def. V.6 of the *Elements*: "end let the magnitudes having in the same ratio be said in proportion [τὰ δὲ τὸν αὐτὸν ἔχοντα λόγον μεγέθη ἀνάλογον καλείσθω]". What Euclid means by four magnitudes to have the same ratio is made clear in definition V.5. The notoriety, ambiguity and complexity of this last definition suggest me to avoid quoting and discussing it, here. It will be enough to say that none of the ambiguities and difficulties concerning with it affects what I'm arguing about the notion of measure.

with two segments sensible. Even if this does not makes these two segments easy to identify in any case, this is enough, together with the provable existence of the fourth proportional of any triad of segments[14], to explain what it means that segments reflect, as I have said, the quantitative relations among whatever sort of homogenous magnitudes: for any two homogenous magnitudes α and β of whatever sort and any given segment *a*, there will be a second segment *b*, such that $\alpha : \beta = a : b$. Still, segments are magnitudes and, though they easily admit an additive operation, they do not admit a multiplicative one—that is, mathematically speaking, they form an ordered (and complete) additive group, but certainly not a ring, *a fortiori* not a field. This is a fundamental, structural difference between the way (real) numbers work, for us, in connection to measure, and the way segments reflect the quantitative relations among whatever sort of homogenous magnitudes. Another pertains to the fact that the latter way depends on the fourth-place relation of proportionality, rather than on the binary one of measure (in the third of the modern senses distinguished above).

These conceptions—so different from ours, as I hope to have shown—fashioned mathematics from the discovery of the incommensurability of the side and the diagonal, at least up to the 17th century. They called for a radical separation of arithmetic and geometry, which was definitively overcome only in the 19th century. It seems to me that all that Aristotle takes for granted about measure, in relation to mathematics and physics, can be understood only by switching from our own to this conception of measure.

3. *Physics VIII, 9 and VIII, 10*

This does not only concern what Aristotle explicitly says about measure, in relation to mathematics and physics, but also, overall, what he does not say about it.

In what follows I shall mainly focus on what he does and does not say in the passage of *Physics* V 3 mentioned above. Before coming to that, however, let me take some more space to consider something else he says.

I'd like in particular to consider the two other passages from the *Physics* also mentioned above, from VIII, 9, and VIII, 10, respectively.

3a. *The passage from VIII, 10*

The latter is simpler to understand. Aristotle considers the possibility of taking, within a finite magnitude, a force homogenous to another finite one that is supposed to reside in an infinite magnitude, in such a way that the former be a measure of the latter.

This supposition would simply be nonsensical if it were to be intended in light of our notion of measure, since, for us, the measure of a force is certainly not a force. It would also be nonsensical, however, if it were not established that a given force, as big as it might be, could have a measure as small as required, so small as to reside within any given magnitude, as small as it might be—since Aristotle does not seem to make any requirement on the sizes of the relevant forces and of the finite magnitude[15]. Insofar as the reference seems here to be to extended physical magnitudes, this depends on the assumption that both forces and extended physical magnitudes can be infinitely divided. This is reminiscent of his conclusion,

[14] See, *Elements*, prop. VI.12: "To find the fourth proportional to three given [segments of] straight [line] [τριῶν δοθεισῶν εὐθειῶν τετάρτην ἀνάλογον προσευρεῖν]". The construction is quite simple: if *a*, *b*, and *c* are the three given segments, it is enough to get two (long enough) non-collinear segments AF and AG, sharing the extremity A, and cut two segments AB and BD, respectively equal to *a* equal to *b*, on the former, and another segment AC equal to *c*, on the latter, then draw the segment BC and the parallel to it through D; if this parallel cut AG in E, then CE is the required fourth proportional, since AB : BD = AC : CE.

[15] Notice that the point I'm making is independent of what it might mean for a force to reside in a magnitude: an interesting question, which is however not relevant for my present purpose.

in *Physics* VI, 2, about the infinite divisibility both of time and extended magnitudes—one of the crucial achievements of his whole treatise. The effort to reach such a conclusion, there, and its application, here (under the replacement of time with force), while speaking of measure, illustrate an important point: though mathematics, in particular the notion of measure coming from it, is taken as a given, by Aristotle, its applicability to physical matters is not; it rather requires to be established. This is one of the major concerns of his treatise, in my mind.

3b. *The passage from VIII, 9*

The passage from VIII, 9 is more problematic. Aristotle seems to argue both that circular locomotion is the first motion since it is the measure of all other motions, and that it is their measure because it is the first. Here my 'since' renders Aristotle's 'γὰρ', my 'because' his 'διότι'. If both are intended to indicate a causal relation, the problem is evident, since nothing can be the cause of its own cause. So, either Aristotle's argument is a paralogism, or at least one of these particles does not have a causal force.

The argumentative context of the passage—which is part of an argument aiming to establish that circular locomotion is the first motion—is not enough to conclude that the former has such a force (and, then, not the latter, if any paralogism is to be avoided)[16], since nothing ensures that Aristotle's argument is intended to mirror what he was taking to be the causal order of things[17]. But, even if it were not—that is, even if Aristotle were maintaining that the circular locomotion being the first motion makes it be the measure of all other ones[18]—, his argument would still require establishing, on independent bases, that circular locomotion is the measure of all other motions.

Furthermore, this would also be so if Aristotle's claim were merely intended to mean that circular locomotion is the measure of all other motions if and only if it is the first. Since the argument for the primality of circular locomotion would, then, be based on this double implication and on the other premise that this motion is the measure of all other motions, to conclude, by *modus ponens*, without involving any causal relation, that it is the first motion. This seems to be the Aquinas's interpretation (*In Phys*., lib. VIII, cap. IX, lect. XX, n. 2).

According to Aquinas, that circular locomotion is the measure of all other motions is just what Aristotle would have proven in IV 14, 223b18-21[19]. This reconnects the point to that discussed in IV 10-14, namely time as number and/or measure of motion. The question is discussed by Sylvain Delcomminette in his paper on this same volume, and I cannot but refer to what he says there. There is, here, only room for a few observations.

In 223b18-21, Aristotle claims this:

> If, then, the first <is the> measure of all the homogenous, uniform locomotion in circle <is> above all measure, since the number of this is best known; <and> neither alteration, nor increasing, nor generation are uniform, but locomotion is <so> [εἰ οὖν τὸ πρῶτον μέτρον πάντων τῶν συγγενῶν, ἡ κυκλοφορία ἡ ὁμαλὴς μέτρον μάλιστα, ὅτι ὁ ἀριθμὸς ὁ ταύτης γνωριμώτατος. ἀλλοίωσις μὲν οὖν οὐδὲ αὔξησις οὐδὲ γένεσις οὐκ εἰσὶν ὁμαλεῖς, φορὰ δ' ἔστιν].

[16] Under this reading, the circular locomotion being the first motion would merely entail that it is the measure of all other motions, without making it so, while its being such a measure makes it the first motion.

[17] Thanks to Sylvain Delcomminette for suggesting this to me.

[18] While its being such measure merely entails that it is the first motion.

[19] According to Ross (*Aristotle's Physics*, A Revised Text with Introduction and Commentary by W.D. Ross, Oxford, 1936, p. 719) for Aristotle this would be, instead, a "fact […] obvious in itself", that he would have merely "noted" in 223b19.

Though the claim, here, is not plainly that circular locomotion is the measure of all other motions insofar as it is the first, it does not seem to be able to provide the argument—allowing to establish that the circular locomotion is the measure of all other motions independently of the premise that it is the first one—which is required to license the other argument of 265b8-11. All that Aristotle seems to say here, which is not also said in this last passage, is that circular locomotion is the first motion insofar as its number is best known, and that it is uniform, while alteration, increasing, and generation are not (from which he takes to follow, by *modus ponens*, that it is the measure of all other motions, provided that what is the first among homogeneous is the measure of them). Unless one admitted that the appeal to the fact that that circular locomotion is the first motion is here immaterial—namely that Aristotle is merely arguing, in fact, that circular locomotion is the measure of all other motion insofar as it is uniform, while alteration, increasing, and generation are not, and its number is best known—it seems, then, hard not to see some sort of *petitio principi* in the conjunction of 223b18-21 and 265b8-11. Aristotle's point seems, indeed, that circular locomotion is the first and it is the measure of all other motions (rather that it is the former insofar as it is the latter or vice versa), as shown by the fact that it is uniform (while no other sort of motion is, better can be, so), and its number is best known.

A way to avoid the problem might be trusting what Aristotle says just before, namely that (223b16-18)

> <it is> by a motion limited in time <that> the quantity of motion and time is measured [ὑπὸ τῆς ὡρισμένης κινήσεως χρόνῳ μετρεῖται τῆς τε κινήσεως τὸ ποσὸν καὶ τοῦ χρόνου].

Any logical subtlety apart, Aristotle's line of argument might, then, be the following: the circular and uniform locomotion of the first heaven is the measure of any other motion, since it is made manifest by the alternation of day and night, and the constant total time of a day and a night can be taken, in turn, as a privileged measure of time, and motions stay to each others as the times to complete them (provided that the same change be produced by them). This argument seems independent of the assumption that circular locomotion is the first motion, and leads to the conclusion that the circular locomotion of the first heaven is the measure of any other motion. If this were considered enough for concluding, more generally, that circular locomotion is the measure of any other motion, we would then have an argument for this conclusion that is independent of this assumption, and it is, therefore, appropriate for licensing the other argument of 265b8-11.

Now, the former argument depends on Eudoxus's theory of proportion, in particular on the possibility that this theory concedes to make two homogenous magnitudes, in this case two times, form a proportion with two other homogenous magnitudes of a different species, in this case two motions (producing the same change). Once again, mathematics would be taken for granted, rather than discussed, and would be used to prove what is to be proved, namely that circular locomotion is the measure of all other motions.

This interpretation seems to me all the more convenient that the passage from this locomotion to time and from time to any other motion makes clear the sense in which Aristotle could have taken circular locomotion to be a measure of other motions, including rectilinear locomotion, despite his taking straight lines and circles "not comparable [οὐ συμβλητά]" to each others (248b4-7)[20].

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[20] This would also explain how Aristotle can argue that time measures motion, though time is not homogenous with motion. Indeed, as rightly observed by Delcomminette in his paper included in this same volume (to which I refer for the indication of the relevant passages), "en réalité, c'est toujours un mouvement qui

4. *What Aristotle Says in V 3 About Measure*

I can, now, come to what Aristotle says in V 3, 226b32-34, which is not a small thing. Indeed, he says at least four things:

i) that two places are contrary each to another if they are "the most distant in straight line";

ii) that straight line is "the minimum to be limited", namely, if I understand well, the minimal line connecting two given places, or, better, points;

iii) that "that <which is> limited is measure", namely—if I once again understand correctly—that only something limited can be a measure;

iv) that (*ii*) and (*iii*) entail (*i*)[21].

If the minimal line is intended to be minimal among all possible polygonal chains, claim (*ii*) is an immediate consequence of proposition I.20 of Euclid's *Elements*[22]. In all its generality, it cannot, of course, be proved in the setting of Euclid's geometry, which concerns no other line than straight lines and circumferences. Still, under a natural notion of a line, it is easy to imagine a proof of it by exhaustion founded on visual evidence, which any Greek mathematician would have accepted. Archimedes takes it to be a liminal assumption (or "λαμβανόμενος", in its language) of his *On the Sphere and Cylinder*[23]. One can then say, rigorously enough, that this was an accepted truth in Greek geometry.

Claim (*iii*) immediately follows from the very notion of a measure as aliquot part, at least when this is applied to extended magnitudes, like lines, as it is only required in the context of Aristotle's argument.

Claim (*iv*) can, of course, be contested, but, with a quite natural understating of the notion of what counts as contrary with respect to place (in Aristotle's sense), it is much more than plausible[24].

Finally, if claims (*ii*)-(*iv*) are accepted, claim (*i*) then, of course, follows.

Hence, though Aristotle says much, here, this seems to be both plain enough and in line with mathematics of his time.

This does not mean, however, that it is trivial, since it has the notable merit of explaining quite clearly, though implicitly, in which sense aliquot parts are measures; they are so since they allow for the use of counting to establish how big that which is measured is. In the particular case at issue, it allows for the use of counting to establish how far two places are, which requires, of course, the choice of a single canonical path on which the aliquot parts that are to be counted have to be taken.

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mesure un mouvement, et un temps un temps; mais les deux sont corrélatifs, puisque le temps est " quelque chose du mouvement ", à savoir son nombre, et donc précisément *ce qui peut* être délimité en lui".

[21] This point (*iv*) depends, of course, on assigning an implicative force to the particle 'γάρ' occurring in this passage. But this particle could also be taken as such to introduce an explanation. In this case (*ii*) and (*iii*) would be intended to explain, rather that to entail, (*i*). In her paper included in this same volume (§ 3), Giovanna Giardina takes this to be the case for (*ii*), while she takes (*iii*) to be a mere addition. The main conclusions that I shall argue for here, concerning these claims—namely that they are in line with mathematics of Aristotle's time and provide an implicit explanation of the sense in which aliquot parts are measures—do not essentially depend, however, on admitting my implicative reading against Giardina's (and others') explicative one. Thanks both to Giovanna Giardina herself and Sylvain Delcomminette for having attracted my attention on this point.

[22] Since this proposition states that "two sides of any triangle, taken together in whatever way are greater than the remaining one [παντὸς τριγώνου αἱ δύο πλευραὶ τῆς λοιπῆς μείζονές εἰσι πάντῃ μεταλαμβανόμεναι]".

[23] This is the first one of book I: "Of the lines having the same extremities, let the straight [one] be the smallest [τῶν τὰ αὐτὰ πέρατα ἐχουσῶν γραμμῶν ἐλαχίστην εἶναι τὴν εὐθεῖαν]".

[24] This seems to me to be so, also if point (*iv*) is intended as I have said in footnote (21), above.

Of course, under the conception described above, a straight line, namely the trajectory of this path, has, as Aristotle argues, infinitely many possible measures (in fact, uncountably many, as we know today), and Aristotle, as any mathematician of his time, was certainly aware of this (their uncountability apart, of course). Hence, what is relevant here is not, as it were, absolute counting, that is counting how many times a prefixed unity of measure enters a certain magnitude, but the constant ratio[25] between the numbers resulting from counting any common aliquot part (or measure) of two given magnitudes: let these magnitudes be a and b, and c and c^* whatever two common aliquot parts of them, such that $a = mc = m^*c^*$ and $b = nc = n^*c^*$, then $m : n = m^* : n^* = a : b$.

This displays a crucial difference, which I have already accounted for in § 2 above, among the use of counting—namely of integer positive numbers—to identify the quantitative relations among two magnitudes, and our way to measure through numbers: the former only apply when common measures (in the Greek sense) are available, that is, only to commensurable magnitudes. Commensurability appears, then, as a condition of applicability of the Pythagorean program of numeralisation of nature. Although the discovery of the incommensurability of the side and the diagonal of a square makes clear that this program cannot be applied in general, commensurability discerns the portions of the word to which it can be applied. This should be enough to make clear the philosophical import that both the notion of measure and that of incommensurability—in particular the paradigmatic case of the incommensurability of the side and the diagonal of a square—had for a post-Pythagorean philosopher, as Aristotle was.

This import pertains, however, to two purely mathematical notions and the basic results concerned with them. What such a post-Pythagorean philosopher could have done, concerning them, is just what Aristotle does here, though implicitly and in a quite particular case: not discussing, clarifying or amending them, but rather making this import clear through an enquiry regarding the way these notions can be used to account for the physical world.

5. *What Aristotle does not Say in V 3 About Measure*

However, one could retort: if this is in line with Aristotle's passive approach to mathematics, it is, still, much more than nothing. It both responds to a pedagogical function that philosophy cannot refuse to have, and complies with a genuine philosophical mission, like that of searching for the conceptual frame of such an account—a mission that Aristotle considered to be both decisive and compelling. Consequently, if this is so, in which sense does 226b32-34 manifest some sort of lack in philosophical elaboration, up to the point of making what Aristotle does not say in it relevant ?

The answer depends on the context of this passage, or, better, on the comparison between this context and its content, which I have just tried to clarify.

This context pertains, as I have said above, to Aristotle's elucidation, possibly even to his definition, of the relation of being intermediate. This elucidation, in turn, involves the notion of contrary, or, better, more generally, those of *terminus a quo* and *terminus ad quem* of a motion. In this context, Aristotle felt appropriate to make clear what being contrary means for places, and, consequently, for items having a place. Strictly speaking, this is not required by the internal logic of his elucidation, but it appears to be quite natural in light of the fact that this elucidation is part of a chain of similar elucidations concerning six other notions, all of which have a relevant application to spatial items, though not being exclusively attached to

[25] See footnote (13), above.

them: the notions of being "together [ἅμα]", "apart [χωρίς]", "joined [ἁπτόμενον]", "consecutive [ἐφεξῆς]", "contiguous [ἐχόμενον]" and "continuous [συνεχὲς]".

This, again, does not reveal what is lacking in 226b32-34. Rather, it makes us see what pushed Aristotle to insert the previous passage about contraries relative to place and measure.

What is lacking becomes clear when one reflects on the last notion in the foregoing list: that of being continuous, or, better, to stay closer to Aristotle's parlance, that of "the continuum [τὸ συνεχὲς]". For is it not obvious, for us, that measure and continuity are strictly connected notions ? That elucidating one should be required for—or go, at least, naturally together with—elucidating the other ? What one is left wondering, then, is just this: that, despite his inserting this passage in a (quite complex) argumentation aiming at the elucidation of the notion of continuity (or the continuum), Aristotle does not take the occasion for establishing some conceptual link among measure and continuity, and leaves, rather, this passage essentially isolated in its context. Aristotle does not say which link connects measure with continuity. Why ? Is this lack only a lack for us, or is it, in a sense or another, also a lack for him ?

Responding to these questions is the aim of the following part of my paper.

To begin with, let us ask ourselves why we are accustomed to strictly connecting measure and continuity. The answer naturally results from what I have said about our notion of measure in § 2: when speaking of measure, we naturally think to measure of magnitudes by real numbers, and we consider that the ordered set formed by them is the continuum. More than that, we think of real numbers as being strictly connected to measure of magnitudes, either because this is, for us, their most important and natural application, or even because some of us could be willing to follow Frege[26] (or others, like Hölder[27], and, much more recently, Hale[28]) in arguing that real numbers should be defined as measures of magnitudes.

The former motivation is, by far, more commonplace than the latter. It makes it for us natural, in teaching, to introduce real numbers by starting from natural ones, then passing to rationals and showing that these latter numbers cannot provide a measure of magnitudes that we are easily able to deal with—by appealing, for this purpose, to the example of the side and diagonal of a square—and finally proving that real numbers (conveniently defined starting from rational ones), not only allow us to express the ratio of the side and the diagonal of a square, but are also in biunivocal correspondence with the points that can be taken on an oriented straight line as the extremities of a segment originated in a fixed point of it, which makes possible to express the ratio between any two pairs of segments, and, consequently, convey geometrical continuity in an arithmetical way.

This suggests that what appears to us as an evident lack in Aristotle's treatment of continuity and measure is such only for us, provided that the way we naturally connect these notions requires tools that were perfectly unavailable to him.

This provides a quite simple answer to our questions. However, though certainly correct, this answer is too simple—for our natural passage from real numbers in connecting measure to continuity is nothing but the reflex of a deeper and direct connection among these notions, which depends in no way on real numbers, but is rather the consequence of the phenomenon of incommensurability, conceived in the Greek sense as absence of a common measure (intended as a common aliquot part). To see this, it is enough to reflect on the fact that the incommensurability of the side and diagonal of a square is enough to show that geometrical

[26] See G. Frege, *Grundgesetze der Arithmetik*, H. Pohle, Jena 1893-1903 (2 vols), part III, vol. 2, §§ 55-245.

[27] See O. Hölder, "Die Axiome der Quantitat und die Lehre vom Mass", *Berichte über die Verhandlungen der Königlich Sächsischen Gesellschaft der Wissenschaften zu Leipzig: mathematische und physikalische Klasse*, 53 (1901), pp. 1-64.

[28] See B. Hale "Reals by Abstraction", *Philosophia Mathematica*, 3rd series, 8, 2000, pp. 100-123.

construction by rule and compass—the simplest form of construction involved in Greek geometry—provides us with pairs of incommensurable segments (in this sense), which is, in turn, enough to show that the continuity of segments makes some of them have no common measure (in the Greek sense).

Our first answer requires, therefore, to be completed. To see how, it is important to observe that the notion of continuity enters the previous remark in a quite vague sense, to convey the visual intuition of an absence of gaps. But is this the way Aristotle was conceiving continuity ?

The answer is clearly negative. His conception is much more complex and goes far away from geometry, and, more generally, mathematics, by rather providing a cornerstone of his own physics. When we look closely at this conception, in the way it is presented in V 3, we see that not only does it require no direct connection with measure, but rather makes continuity dependent on the absence of (actual) parts, and, then, also of aliquot (such) parts, which makes it perfectly orthogonal to measure, as conceived of by Aristotle.

To justify this claim, I have to parse the whole passage of V 3 that 226b32-34 belongs to[29].

6. *The Continuum*

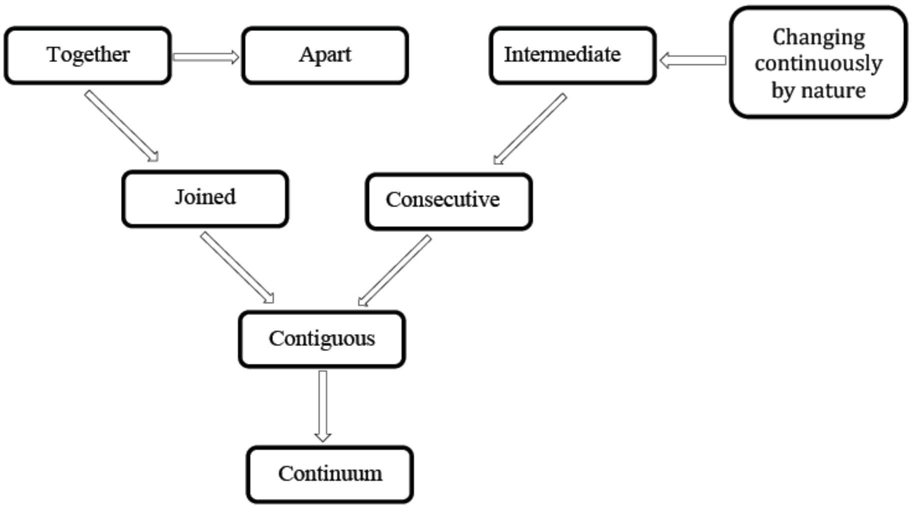
Here is this passage, including—though, as said, not indispensably at all—the passage quoted above about contraries in relation to space and measure (226b21-227a17):

> I say, therefore, to be together those <things> according to place, insomuch as <they> are in one primary place, apart, instead, insomuch as <they are> in another <place>, while joined <those whose> extremities are together. | [However, since every change <happens> in the opposites, but the opposites are both the contraries and by contradiction, <and> in contradiction <there is> nothing in the middle, <it is> clear that the intermediate will be in the contraries. | To the minimum, the intermediate is in three <items>, for the contrary is the extreme of change,] while intermediate is <the thing> that what changes naturally reaches before <arriving at> the last towards which it changes, while changing continuously by nature. [To the minimum … of change]. It, then, moves continuously <that which> leaves nothing or very little of the thing—not of the time (for nothing prevents it from leaving <something> and that immediately after the hypate, the nete sounds) but of the things in which it moves. This is clear both in changes concerning place and in the others. Contrary concerning place <is> the most distant in straight line, since the <straight line> is the minimum to be limited and that <which is> limited <is> measure. Next, consecutive <is> that of which—by being after the beginning in position, in form or in some other respect <and being> so determined—there is no intermediate of the same kind of that <which> it is consecutive <of>. (I say, for example, that a line or lines <are consecutive> to a line, or a unit or units to a unit, or a house to a house; but nothing prevents that there be an intermediate of another <kind>). For the consecutive <is> consecutive to something and posterior to something; for one <is> not consecutive to two, nor the first day of the month <is> consecutive to the second, but the latter to the former. Contiguous, again, is that <which,> being consecutive, will be joined. [However … in the contraries]. Finally, the continuum is just some contiguous; I say it is continuous when the limit in which each of two <parts> join each other becomes the same and one, and, as the name points out, hold with. But this <is> not so, if the extremes are two. This having been laid down <it is> clear that the continuum is in those things from which, by nature,

[29] In doing that, I shall base on some material drawn from a joint working paper with Giovanna Giardina and Chiara Martini, who I thank for allowing me to use it.

some one is generated by contact. And as, once generated, that <which> holds with <becomes> one, so also the whole will be one, for example, <by a> bolt or glue or junction, or organic union. [ἅμα μὲν οὖν λέγω ταῦτ' εἶναι κατὰ τόπον, ὅσα ἐν ἑνὶ τόπῳ ἐστὶ πρώτῳ, χωρὶς δὲ ὅσα ἐν ἑτέρῳ, ἅπτεσθαι δὲ ὧν τὰ ἄκρα ἅμα. [ἐπεὶ δὲ πᾶσα μεταβολὴ ἐν τοῖς ἀντικειμένοις, τὰ δ' ἀντικείμενα τά τε ἐναντία καὶ τὰ κατὰ ἀντίφασιν, ἀντιφάσεως δ' οὐδὲν ἀνὰ μέσον, φανερὸν ὅτι ἐν τοῖς ἐναντίοις ἔσται τὸ μεταξύ. | ἐν ἐλαχίστοις δ' ἐστὶ τὸ μεταξὺ τρισίν· ἔσχατον μὲν γάρ ἐστι τῆς μεταβολῆς τὸ ἐναντίον,] μεταξὺ δὲ εἰς ὃ πέφυκε πρότερον ἀφικνεῖσθαι τὸ μεταβάλλον ἢ εἰς ὃ ἔσχατον μεταβάλλει κατὰ φύσιν συνεχῶς μεταβάλλον. [ἐν ... εναντιον.] συνεχῶς δὲ κινεῖται τὸ μηθὲν ἢ ὅτι ὀλίγιστον διαλεῖπον τοῦ πράγματος, μὴ τοῦ χρόνου (οὐδὲν γὰρ κωλύει διαλείποντα, καὶ εὐθὺς δὲ μετὰ τὴν ὑπάτην φθέγξασθαι τὴν νεάτην) ἀλλὰ τοῦ πράγματος ἐν ᾧ κινεῖται. τοῦτο δὲ ἔν τε ταῖς κατὰ τόπον καὶ ἐν ταῖς ἄλλαις μεταβολαῖς φανερόν. ἐναντίον δὲ κατὰ τόπον τὸ κατ'εὐθεῖαν ἀπέχον πλεῖστον· ἡ γὰρ ἐλαχίστη πεπέρανται, μέτρον δὲ τὸ πεπερασμένον. ἐφεξῆς δὲ οὗ μετὰ τὴν ἀρχὴν ὄντος ἢ θέσει ἢ εἴδει ἢ ἄλλῳ τινὶ οὕτως ἀφορισθέντος μηδὲν μεταξύ ἐστι τῶν ἐν ταὐτῷ γένει καὶ οὗ ἐφεξῆς ἐστιν (λέγω δ' οἷον γραμμὴ γραμμῆς ἢ γραμμαί, ἢ μονάδος μονὰς ἢ μονάδες, ἢ οἰκίας οἰκία· ἄλλο δ' οὐδὲν κωλύει μεταξὺ εἶναι). τὸ γὰρ ἐφεξῆς τινὶ ἐφεξῆς καὶ ὕστερόν τι· οὐ γὰρ τὸ ἓν ἐφεξῆς τοῖν δυοῖν, οὐδ' ἡ νουμηνία τῇ δευτέρᾳ ἐφεξῆς, ἀλλὰ ταῦτ' ἐκείνοις. ἐχόμενον δὲ ὃ ἂν ἐφεξῆς ὂν ἅπτηται. [ἐπεὶ ... μεταξύ.] τὸ δὲ συνεχὲς ἔστι μὲν ὅπερ ἐχόμενόν τι, λέγω δ' εἶναι συνεχὲς ὅταν ταὐτὸ γένηται καὶ ἓν τὸ ἑκατέρου πέρας οἷς ἅπτονται, καὶ ὥσπερ σημαίνει τοὔνομα, συνέχηται. τοῦτο δ' οὐχ οἷόν τε δυοῖν ὄντοιν εἶναι τοῖν ἐσχάτοιν. τούτου δὲ διωρισμένου φανερὸν ὅτι ἐν τούτοις ἐστὶ τὸ συνεχές, ἐξ ὧν ἕν τι πέφυκε γίγνεσθαι κατὰ τὴν σύναψιν. καὶ ὥς ποτε γίγνεται τὸ συνέχον ἕν, οὕτω καὶ τὸ ὅλον ἔσται ἕν, οἷον ἢ γόμφῳ ἢ κόλλῃ ἢ ἁφῇ ἢ προσφύσει.]

A cursory scrutiny of this passage is enough to realize that the statement concerning the continuum appeals, either directly or indirectly, to five of the six relations previously introduced—those of being together, joined, intermediate, consecutive, and contiguous—and that that of being intermediate is introduced, in turn, by appealing to the property of "changing continuously by nature [μεταβάλλω κατὰ φύσιν συνεχῶς]". Here is, indeed, the tree of dependences that this passage displays:



If the elucidation of the property of naturally changing continuously required that of the continuum, Aristotle's statement would be inescapably circular. This is not so, however, since Aristotle explains what he means with "moving continuously [συνεχῶς δὲ κινεῖται]" without any appeal to the continuum; and it seems quite natural to consider that he is intending to apply this explanation to the property of "changing continuously by nature". This allows taking Aristotle's elucidations of the relations of being together, intermediate, joined, consecutive, contiguous and of the property of naturally changing continuously as genuine definitions, free of circularity and preliminary to the statement about the continuum.

By going back along the tree, we might resume these definitions as follows:

two items are contiguous if (and only if) they are both consecutive and joined, namely they admit no intermediate of the same kind (which makes them consecutive), and have extremities that are together (which makes them joined), provided that an item, say σ, is intermediate between two other items, say τ and θ, if what naturally changes continuously by nature and comes from τ reaches σ before reaching θ, and that two items are together if they are in one single primary place.

6a. *The Right Branch of the Tree*

Ross[30] has observed that later, in V 3, 227a18, "Aristotle implies […] that ἐφεξῆς is a wider term including ἁπτόμενον", which would make "ἐχόμενον […] a mere synonym of ἁπτόμενον". Though he is right concerning 227a18[31], it remains that, in the passage quoted above, Aristotle presents being joined and being contiguous as distinct relations. This is not only because their definitions appeal to two distinct webs of notions, but also because the definition of being consecutive and the dependency of that of being contiguous on it seem to imply that two items can bear both relations each to another only with respect to a certain change and only if they are of the same kind: two conditions which the relation of being joined is in no way submitted to. Let us consider these two conditions at once.

Let us begin with the former. It makes an item not consecutive to another *simpliciter*, but in relation to a certain change, or, more generally, to a certain order. This is because Aristotle defines the relation of being intermediate, by appealing to the property of changing continuously by nature. If we insert Aristotle's definition of moving continuously within that of being intermediate and take it as apt to define the property of changing continuously by nature, we get, indeed, the result that σ is intermediate between τ and θ if what changes in such a way as to leave "nothing or very little of the thing—not of the time" and comes from τ reaches σ before reaching θ. What exactly does it mean ?

Though the definition of being consecutive comes after, and depends on that of being intermediate, understanding the former can help in understanding the latter. The broad sense of the definition of being consecutive is clear: two items are consecutive if they admit no intermediate of the same kind. Its details are, instead, far from being so. In stating this definition, Aristotle refers to

> that of which—by being after the beginning in position, in form or in some other respect <and being> so determined—there is no intermediate of the same kind of which it is consecutive.

Moreover, in defining the property of naturally changing continuously, Aristotle alternatively uses the verbs 'μεταβάλλω' and 'κινέω'. This suggests that we should take τ and θ (as well as σ) as liable to have different natures in different cases, namely according to whether the change concerns substance, quantity, quality or place. The possibility that there be some intermediate forces us, however, to discard change concerning substance, that is, generation or corruption. We should, then, apparently, take τ and θ, as well as σ, to be stages of an alteration, an augmentation or diminution, or a locomotion.

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[30] See *Aristotle's Physics*, cit., p. 626.

[31] The reference should be, in fact, to 227a18-19: "Since, that <which is> joined is, indeed, necessarily consecutive, <while> that <which is> consecutive <is> not all joined [τὸ μὲν γὰρ ἁπτόμενον ἐφεξῆς ἀνάγκη εἶναι, τὸ δ' ἐφεξῆς οὐ πᾶν ἅπτεσθαι]".

The "very little of the thing—not of time" that might be left by changing continuously by nature from τ to θ should, then, be intended as a gap occurring in one of these processes, so that something could so change by still leaving some non-temporal gaps. This is made quite clear by Aquinas's comments (*in Ph.*: *Lib*. V, Cap. III, *Lect*. 4). He appeals to the example of "crossing a road, in which the stones are placed at a small distance from one other, through which a man crosses from one part of the road to another with continuous motion [*transitibus viarum, in quibus ponuntur lapides modicum ab invicem distantes, per quos homo transit de una parte viae ad aliam, motu continuo*]".

Both this and Aristotle's example of the hypathe and the nete suggest a quite plausible interpretation. The appropriate way in which someone can cross a road using stepping stones is just by jumping from a stone to another, not by going down one, then going up another to avoid jumping. Analogously, the appropriate way for a sitarist to play a melody so conceived to pass from the lowest note to the highest, is just sounding the nete immediately after the hypate. This suggests that something changes continuously by nature by leaving some gaps in the thing, when leaving these gaps is required by the very particular nature of the change itself, namely when leaving these gaps makes the relevant subject pass from any given stage to that which is required by the order imposed by this nature. In other terms, Aristotle seems to take something to change continuously by nature when its change cannot be, by its nature, so refined to present more stages than those it passes through.

It follows that τ and θ are consecutive with respect to a certain change if (and only if) the very nature of this change makes the relevant subject passes from the former to the latter without passing from any σ while achieving this change. Analogously, σ is intermediate between τ and θ if, by achieving this change, this subject passes from τ to it before reaching θ.

Clearly, both conditions are independent of how τ, θ and, possibly, σ stand each to another with respect to some relevant substratum. Hence Aristotle's definitions allow two bodies to not only be consecutive, with respect to a certain locomotion, without being joined, and, then, contiguous, but also to be joined without being consecutive, with respect to this locomotion, and, then, contiguous, again. This makes also clear why, according to Aristotle's definitions, the relations of being consecutive and contiguous are relative to a certain change, while that of being joined is not.

That these relations be not equivalent is also suggested by the second of the two conditions mentioned above, according to which two items can be consecutive, and, then, contiguous, only if they are of the same kind.

Aristotle implies this by defining the consecutive as "that of which—by being after the beginning […] and being so determined—there is no intermediate of the same kind of that <which> it is consecutive <of>". Though the exact interpretation of this passage is far from easy, it seems that Aristotle is taking, here, the "same kind" to be that of that to (or, better, of, in his parlance) which the consecutive is so, that is of the *terminus a quo* of the relevant change, or τ, in the previous notation. Although, this *terminus* is *a quo* as opposite to a *terminus ad quem* of this same change, namely θ, which is said, here, to be "determined" by its "being after the beginning", namely just by its being the *terminus ad quem* of this change. The kind of both θ and τ seems, then, to be taken by Aristotle as being determined by this change, and being, then, the same. The appropriate way to understand the definition seems, then, to be this: θ is consecutive to (or of) τ if (and only if) τ and θ are homogeneous and admit no intermediate of their same kind.

6b. *The Left Branch of the Tree*

We should now look at the other branch of the tree and consider the definitions of being together and being joined. Focusing on them independently of the statement about the continuous would be quite artificial, however, since, despite its apparent obscurity, a quick look to this statement is enough to notice a relevant analogy with these last definitions: whereas two items are said to be "joined" if their "extremities [ἄκρα]" are "together", a continuum is said to obtain when the "limit [πέρας] in which each of two <parts> join each other [οἷς ἅπτονται] becomes the same and one [ταὐτὸ γένηται καὶ ἓν]", and to be "in those things from which, by nature, some one [ἕν τι] is generated by contact [σύναψιν]". The latter claim is echoed in VI, 1, 231a22: "continuous <is that> whose extremes <are> one [συνεχῆ μὲν ὧν τὰ ἔσχατα ἕν]".

This suggests paraphrasing the relevant aspect of Aristotle's statement as follows: for a continuum to obtain the limits of the contiguous parts of the relevant item are not merely to be ἅμα but are rather to be ἕν.

This arouses, however, a delicate problem: in light of the definitions of the relation of being together—according to which two items are together, "insomuch as <they> are in one primary place [ὅσα ἐν ἑνὶ τόπῳ ἐστὶ πρώτῳ]"—it is natural to wonder what it could mean, for two things, to be ἅμα without being ἕν, namely how we can conceive of being one as an additional feature in respect to being together.

The difficulty becomes obvious when the definition of the relation of being together is compared to that of primary place (IV 4, 210b34-211a2), which openly suggests that being in the same primary place be the same as having exactly the same place, as being spatially indiscernable[32]. Provided that it would be odd to appeal to temporal discernibility to explain how the primary place of two distinct items could be the same, the question arises, then, of understating as it might happen that two distinct items be together. This requires understanding how the two notions of being ἅμα and being ἕν should be conceived without conflating them.

Another difficulty depends on Aristotle's appeal to the notion of place, since, if it is admitted that being together amounts to being together in place, it becomes difficult to see how the relations of being joined and contiguous can apply to items which, by their own nature, have no place; and this, in turn, makes it difficult to see how Aristotle's statement about the continuum might pertain to such sorts of items.

There is still more, since if being together amounts to being together in place, it also becomes difficult to see how extremities could be together, and, then, according to Aristotle's definitions, how two items—even those having a place, including physical bodies—could be joined and, then, contiguous and, possibly, form a continuum. For, by restating his definition of place, he claims that this is "the limit of the encompassing body, insofar as it holds with the encompassed [τὸ πέρας τοῦ περιέχοντος σώματος <καθ' ὃ συνάπτει τῷ περιεχομένῳ>]" (IV 4, 212a6-6a), thus implying that extremities have no place, provided that they, in turn, have no limits.

Some help comes from what Aristotle says in IV 10, 218a25-27 and IV 11, 219a14-19, namely that

> being together according to the time and being neither prior nor posterior is being in one and the same now [τὸ ἅμα εἶναι κατὰ χρόνον καὶ μήτε πρότερον μήτε ὕστερον τὸ ἐν τῷ αὐτῷ εἶναι καὶ ἑνὶ [τῷ] νῦν ἐστιν],

and that

[32] Here is Aristotle's statement: "we claim, indeed, that the place is firstly <that which> encompasses that of which it is place, and <is> nothing of the thing, and the primary is neither smaller, nor greater <than it> [ἀξιοῦμεν δὴ τὸν τόπον εἶναι πρῶτον μὲν περιέχον ἐκεῖνο οὗ τόπος ἐστί, καὶ μηδὲν τοῦ πράγματος, ἔτι τὸν πρῶτον μήτ'ἐλάττω μήτε μείζω]".

> prior and posterior are primarily in place [τὸ […] πρότερον καὶ ὕστερον ἐν τόπῳ πρῶτόν ἐστιν]", since they are "in magnitude [ἐν τῷ μεγέθει],

and then "in motion [ἐν κινήσει]", and, because of that, "in time [ἐν χρόνῳ]". This suggests that being together in place is, for Aristotle, a basic relation, in terms of which the more general relation of being together *tout court* is to be defined. One could, then, consider that Aristotle's aim was only that of defining such a relation in a basic, and possibly paradigmatic case, though tacitly leaving the possibility open for a generalisation to be made when the definition would have been appealed later. If this is so, the second difficulty dissolves.

However the problem remains of understanding how Aristotle's definition of being together (in place) is to be translated into a relation between items that have no place. This is a difficult question to be answered in general. But we can suppose that Aristotle was hoping that taking his definitions of being together and joined *cum grano salis* would have been enough for making them clearly suggest whether these relations obtain or not, in some relevant paradigmatic cases concerned with items having a place, and that this was sufficient for his purpose.

We remain, then, with the problem of understanding the distinction between being ἅμα (in a place) and of being ἕν. Ross[33] suggests two ways of solving the problem. He wonders how "two [distinct] things can[…] [be] ἅμα κατὰ τόπον", provided that "the place which contains nothing but A cannot contain nothing but B". Here are two ways of how being ἅμα κατὰ τόπον might be understood, according to him, in order to overcome the difficulty:

(1) Whatever two items having a place are ἅμα if they are the same item that "discharges two functions", namely, by using another parlance than Ross's, if they are intensionally two, though being extensionally one; it would follow, if I understand correctly, that these items would be ἕν if they also discharge the same function, namely they are also intensionally one.

(2) Whatever item having a place are ἅμα if they are distinct items but "they are in one place which contains nothing but *the two*, i.e. where there is nothing between them", namely, in he same parlance as above, if they are extensionally two, though occupying the same (primary) place; it would follow, if I understand correctly again, that these items are ἕν if they are the same item, namely they are extensionally one, though possibly discharging two functions, namely though possibly being intensionally two.

Ross discards (1), but his reasons for that are far from convincing, at least to me[34]. Moreover, answer (2) does not match at all with the theory of places expounded in book IV.

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[33] See *Aristotle's Physics*, cit., p. 627.

[34] Here is his argument (*ibidem*); "since ἅμα is used in the definition not of continuity but of the less close relation of contact [i.e. of being joined, in my translation] (226b23), and the unity of two ἄκρα is expressly distinguished from their being ἅμα (227a22-3), it is evident that Aristotle's meaning is not […][(1)]". Both reasons seems circular to me. As for the first, consider that supposing that the meaning of being ἅμα is that conveyed in (1) has, as a consequence, that two items (having a place) are contiguous if one extremity on each of them is the same as one of the other. This would work against interpretation (1), as required by Ross, only if it were granted that Aristotle was taking two such items are continuous (taking, then, continuity to be a binary relation). But admitting (1) just leads, as we shall see, to deny that this is so, whence the circularity. As for the second reason, consider what Aristotle says in 227a21-23, "indeed if [something is] continuum [it is] necessary [it] be joined, but if joined [it is] not [necessary it be] continuous, since [it is] not necessary [that] the extremities of [the things] themselves be one if they are together [εἰ μὲν συνεχές, ἀνάγκη ἅπτεσθαι, εἰ δ' ἅπτεται, οὔπω συνεχές· οὐ γὰρ ἀνάγκη ἓν εἶναι αὐτῶν τὰ ἄκρα, εἰ ἅμα εἶεν]". This would undermine interpretation (1) only if being one were understood as being extensionally one, rather than also intensionally, as required by this interpretation. According to it, what Aristotle is saying here is merely that it is possible that one same extremity discharges two functions, and counts, then, as two items that are merely ἅμα, without being also ἕν, that is, intensionally one. Hence this would be an evidence against this interpretation, only if this interpretation were rejected, whence the circularity, anew.

Indeed, the idea that a single primary place could contain two distinct items, extensionally speaking, is hardly compatible with the definition of proper place mentioned above.

To avoid this unpleasant conclusion, one could consider that Ross's point is not that this is possible, but rather that two items are ἅμα if the primary place of their compound (or mereological union, as we would say today) contains nothing but them[35]. If it were admitted that Aristotle considered this compound as a genuine (single) item having a place (which could be hardly granted, it seems to me), this could not contradict Aristotle's theory of place. Whereas, if it did not, then, it would make his definition of being ἅμα such to make ἅμα any two items whatsoever having a place, as spatially distant as they might be. Unless the definition were made more precise by maintaining that two such items are in one primary place if the place of their compound contained only them, either in the sense that it did not include room for any possible other item—which would be, anew, incompatible avec Aristotle's theory of place (by conveying the idea that place is nothing but a portion of space, understood as an external and fixed container)—or in the sense that they admit no intermediary among them—which would, in turn, either make the definition of being ἅμα conflate with that of being joined, or depend on a definition of being intermediary alternative to that Aristotle provides in V 3 (which would be, by the way, quite hard to imagine, at least for me).

This leads me to favour option (1).

6c. *Two Orthogonal Notions*

One thing to be noticed about this option is this: despite Ross's presenting it as being concerned with items having a place, according to it (as well according to option (2), by the way), requiring that, for a continuum to obtain, the limits of the relevant parts of some items be, not merely ἅμα, but rather ἕν, renders *ipso facto* the question of whether these parts or items have a place immaterial, since both the condition of being intensionally two but extensionally one, and that of being intensionally—and, then, also extensionally—one, are perfectly intelligible independently of any sort of consideration concerning whatever sort of place.

Hence this answer suggests a way to understand Aristotle's statement about the continuum that makes it free from any restriction to items having a place. Here it is:

(1*a*) an item which has homogenous parts that, in turn, have extremities is continuous (or a continuum), relative to an appropriate change making these parts consecutive two by two under the order it induces, if (and only if) the extremities of each pair of consecutive parts cannot be distinguished functionally, namely are intensionally one.

The clause about consecutiveness could be avoided so as to make the condition independent of the reference to a certain appropriate change or order. This would result in the following condition:

(1*b*) an item which can be divided into homogenous parts having extremities is continuous (or a continuum), if (and only if) whatever parts into which it could be divided might be made consecutive two by two under the order induced by an appropriate change in such a way that the extremities of each pair of consecutive parts could not be distinguished functionally, namely are intensionally one (by possibly requiring that this obtain for whatever such order).

If, moreover, it were conceded to disregard the dependency of Aristotle's statement from the definition of the relation of being consecutive, one could finally state that:

(1*c*) an item which can be divided into parts having extremities is continuous (or a continuum), if (and only if) whatever parts into which it could be divided might be made to

[35] I thank anew Sylvain Delcomminette for suggesting to me this possible interpretation of Ross's point.

have extremities which could not be distinguished functionally two by two, namely that could be intensionally one two by two.

At first glance, these conditions have all the highly unsuitable consequence that, according to them, no item could be continuous (or be a continuum), since the extremities of whatever two distinct parts of whatever item (divisible into parts) cannot but be functionally distinct, or intensionally two. Even if two parts shared, indeed, an extremity, this might always be considered both as an extremity of one of these parts and as an extremity of the other. No matter how unsuitable as this consequence might be considered to be, this last remark is the key to the interpretation I favour, since the very simple remark that such a consequence is openly unsuitable suggests reading Aristotle's statement as supplying a sort of description by *reductio to absurdum*.

In a nutshell, the point is that this consequence follows only if the relevant parts are taken to be actual. One could retort that both conditions (1*b*) and (1*c*) concern, in fact, potential, not actual, parts. However, this is in fact wrong. What is true is that these conditions concern items that can be divided into appropriate parts by appealing to a circumstance that is required to obtain (though it cannot obtain) in case such a division were realised. Their consequence is, then, not that no item divisible into parts could be continuous (or be a continuum), but that it could not be so once the division were realised.

This is not only far from unsuitable, but also far from surprising, and it is rather perfectly in line with the further arguments Aristotle advances in his treatise about continuity, since it simply entails that any sort of division of whatever item into (actual) parts breaks the continuity. This suggests the following rephrasing of conditions (1*a*-*c*):

(1*a**) an item is continuous (or a continuum) relative to an appropriate change if (and only if) it has no actual parts but is liable to be divided into homogenous parts that could be made consecutive two by two under the order induced by this change.

(1*b**) an item is continuous (or a continuum) if (and only if) it has no actual parts but is liable to be divided into homogenous parts in such a way that, however this division be realized, the resulting parts can be made consecutive two by two under the order induced by an appropriate change.

(1*c**) an item is continuous (or a continuum) if (and only if) it has no actual parts but is liable to be divided into parts.

By reinserting in condition (1*c**) part of what is suggested by the dependence of Aristotle's statement from the definition of the relation of being consecutive, one would finally get the following condition:

(1*c***) an item is continuous (or a continuum) if (and only if) it has no actual parts but is liable to be divided into homogenous parts.

Though quite different from each other, conditions (1*a**-*c**) all convey the same basic idea: what Aristotle requires for an item to be continuous (or to be a continuum) is that it have no actual parts, but only potential ones, namely that it be "actually indivisible [ἀδιαίρετον […] ἢ ἐνεργείᾳ]", but not "potentially indivisible [ἀδιαίρετον […] ἢ δυνάμει]" (*De anima*, III, 6, 430b6), or, even better, that it be intrinsically one, though divisible, since, as rightly observed by Alexander (Simplicius, *In Phys*. 570,6), "even the one"—understood as the single extremity that two parts would have in common if their extremities were merely ἅμα but not ἕν (according to the former of the two interpretations of this distinction presented above)—"is destroyed on account of being continuous", namely when the parts come to coalesce in a single (continuous) item, or are there only potentially. The last sentence of Aristotle's statement should, then, be intended to make clear that intrinsic unity, namely continuity, is not the same as original unity, since something could be made intrinsically one, and thus continuous (or a continuum), "by a bolt or glue or junction, or organic union", that is, shortly speaking, by an appropriate coalescence. In other terms, what this passage would argue for, would be that continuity can be generated or restored, which fits with what Aristotle says just above: "the continuum is in those things from which, by nature, some one is generated by contact".

7. *Conclusions*

As telling as they might be, neither this basic idea of continuity nor its implementation in conditions (1*a**-*c***), are clear-cut enough to unequivocally fix the property of being continuous (or a continuum) for an indisputable general use. None of them is, for example, able to make clear how an item should be in order to have no actual parts.

Far from being a deceptive result of my analysis, this confirms the claim I made above: Aristotle was not taking continuity as a mathematical notion; his elucidating was not at all, for him, a way to contribute to mathematics; rather, he conceived it as a primitive and salient property of physical bodies and natural changes that, in agreement to Aristotle's conception of the nature of μαθηματικά exposed in *Metaphysics* M 1-3, can be extended to them, without, however, properly admitting a definition, a mere description of this property being enough.

If my interpretation is correct, it explains in a quite satisfactory way, I think, the lack of any account of the connection between measure and continuity in V 3. Not only, was such a connection, which appears quite natural to us, not at all so for him, because of the absence of any notion similar to that of real number (which provides, of us, a bridge between them). More deeply, it were also not so, for him, since his conceptions of measure and continuity, so different than ours, made these two notions orthogonal to each other: while a measure of a quantity—a number or a magnitude—was, for him, an aliquot part of it, the continuity was, for him, dependent on the absence of actual parts. Hence, according to Aristotle's view, one could have identified a measure of any continuous item counting as a magnitude, only by breaking its continuity[36].

[36] What makes still possible, in light of this notion of continuity, for a magnitude, understood as a continuous quantity, to have measures has been explained in § 2, and there should be no need to return to it here.