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Bread and Social Justice: Measurement of Social Welfare and Inequalities Using Anthropometrics

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Abstract

We address the question of the measurement of pure health inequalities and achievement in the context of welfare decreasing variables. We adopt a general framework whereby the health variable is reported on an interval, from an optimum level m to a critical survival threshold b. There are two problems that require some departures from the usual framework used to measure inequality and social welfare. Firstly, we show that for welfare decreasing variables, the equally distributed equivalent value is decreasing in progressive transfers (instead of being increasing). Accordingly, appropriate achievement and inequality indices for welfare decreasing variables are introduced. Secondly, because the Lorenz curve and the associated inequality indices are not robust to alternative values of the survival threshold, we argue that the family of translation invariant social welfare functions and related absolute Lorenz curve allow us to undertake inequality comparisons between distributions that are robust to the chosen level of the survival threshold. An illustrative application of the methodology is provided.

Keywords: Health achievement and inequality; welfare decreasing variables; survival thresholds; relative and absolute Lorenz curves.

JEL codes: D63, I14

1. Introduction

The improvement of key health indicators has been a major concern of the development debate for many decades, and remains so today, as formulated for instance in the MDGs (2000-2015) and SDGs (2015-2030). Beyond improving the average value of key indicators, it has increasingly been recognized that the shape of the distribution is also in need of attention. There are several reasons for turning our attention to inequality in the distribution of a health variable. In the case of calorie intake for instance, low levels of nutrition are associated with stunting in infants and certain severe deficiencies for adults (typically, iron, vitamin A and iodine deficiency). High levels of energy intake are also problematic, as they increase the risk of cardiovascular disease and type II diabetes (WHO 2011).

Additionally, there are the usual normative concerns for preferring less to more inequality in health, in relation to two distributions with the same mean value. Thus, greater emphasis and interest by researchers in the last twenty years has placed the measurement of socio-economic inequality and achievement in health at the centre of the development debate (Wagstaff et al 2003, Wagstaff 2002, Erreygers 2013), as opposed to simply improving the aggregate indicators such as life expectancy and the reduction in infant mortality rates. But there are equally important contexts where the focus is on pure rather than socio-economic inequalities in health (e.g. Osmani, 1992, Sen, 2002, Bommier and Stecklov 2002). The measurement of inequalities in the context of self-assessed health (Allison and Foster 2004, Apouey, 2007, Abul Naga and Yalcin 2008, Arrighi et al 2011, Kobus and Milos 2012) is also concerned with quantifying pure health inequalities.

One problem with subjective self-assessed health (SAH) data, however, is that they have been shown to be biased particularly in the context of developing countries, in that they entail a reverse gradient between health and socio-economic status (van Doorslaer and O'Donnell 2011). The research context of our paper is, therefore, the measurement of pure health inequalities and achievement in relation to an objective measure of health. Specifically, we examine the context of welfare non-monotonic health variables in relation to objective health indicators such as sugar level, cholesterol intake, body mass, that exhibit an inverted U relation with health status. Other variables of course exist for which it is accepted that any level of consumption does not improve health and may harm if consumed in significant amounts. Such variables include lead contamination, nicotine intake, dioxin etc. We accommodate the first set of variables by measuring

inequality when the health variable is reported on an interval $(a, m] \cup (m, b)$, where a and b are a lower and a higher critical values (bounds for survival) beyond which survival is no longer likely and m is the optimum level. For instance, in the context of sugar level, the lower bound for survival is $a = 40$ milligrams of glucose per decilitre of blood, and the corresponding critical upper bound is $b = 450$ milligrams per deciliter. In the context of anthropometrics such as body mass, the lower bound is generally taken to be $a = 10$ kilograms per squared meter, b is approximately equal to 60, while m can generally be any values chosen in the interval of 18.5 to 24.9 (WHO, 2004; 2017).

In the context of the second type of variables (welfare decreasing variables), our analysis equally applies by setting the optimum level m to zero. The emphasis of this paper is on the upper tail of the health indicator, the interval from m to b, as the measurement of inequality and achievement is generally well understood in the context of welfare increasing variables (income being the leading example of course). Furthermore, there are interesting contributions in the context of poverty measurement in relation to resource variables that exhibit an inverted U relation with well-being (for instance Apablaza et al, 2016).

The context of inequality measurement *per se* on a welfare decreasing variable should not be seen as problematic: the Hardy, Littlewood and Pólya theorems (Hardy et al, 1952) relate the class of Schur convex functions to progressive transfers that are applied to a distribution of interest. The fact that the utility function is decreasing in a particular health indicator does not invalidate the use of the Lorenz curve or entropy type indices in ranking health distributions: what matters is the Schur-convexity of the inequality measure, or the Schur-concavity of the underlying social welfare function.

There are nonetheless two unresolved problems that require attention. Firstly, we note that a large family of inequality indices are derived in association with a social welfare function. The inequality index is derived *via* a comparison of the mean of a variable with the equally distributed value of the distribution: this is the so called *Atkinson-Kolm-Sen* approach which measures the level of equality as a ratio of the equally distributed equivalent value to the mean of the distribution. Health achievement indices (Wagstaff 2002) are derived also as the equally distributed equivalent (alternatively, the mean scaled by the level of equality in the distribution). We show, however, in the paper that for welfare decreasing variables, the equally distributed equivalent value is a Schur-convex function: that is, the equally distributed equivalent value is decreasing in progressive transfers. This is the opposite of what we should expect of such a summary statistic. In particular,

a naïve computation of (say) an Atkinson (1970) inequality index on a distribution exhibiting some positive level of dispersion will always entail that inequality is smaller than zero.

The second problem that requires attention is that of survival thresholds. When measuring health inequality and achievement, we are concerned with deviations of the individual observations from the critical thresholds. Clinical research can of course inform about sensible values of the threshold. Nonetheless, it remains that the Lorenz curve and scale invariant inequality indices will take different values for different choices of the survival threshold. As it turns out, this second problem in fact reignites the debate regarding rightist versus leftist inequality and social welfare indices (Kolm 1976 a, b). While we do not propose to take sides in this debate, we note that in relation to translation invariant social welfare functions it is possible to derive health inequality and achievement indices that are robust to the choice of survival thresholds. In the same way, the absolute Lorenz curve (Moyes, 1987) allows us to achieve inequality comparisons between distributions that are robust to the chosen level of the survival threshold, whereas the classical Lorenz curve fails to be invariant to the choice of the parameter b. The class of welfare decreasing translation invariant social welfare functions and related inequality and achievement indices thus provide an attractive solution in the context of our research question.

The remaining of the paper is organized as follows. Sections 2 and 3 provide axiomatically derived functions for achievement and inequality indices in the context of welfare decreasing variables, defined on the interval (m, b), and measured in deviation from survival thresholds. Definitions of the underlying relative, generalized, and absolute Lorenz curves are also provided. Sections 4 deals with the effect of the upper survival threshold. For the sake of completeness, and in order to provide a readily applicable methodology, we also provide in Section 5 derivations of the analogue indices in the case of a bounded welfare increasing variables (defined on the interval $(a, m]$). Section 6 presents a hypothetical illustrative example. Section 7 discusses the issue of distributional comparisons in the context of lower and upper tails distributions. Section 8 of the paper illustrates our methodology in the context of health achievement and inequality comparisons in a group of five Arab countries. Section 9 concludes the paper.

2. Framework and notations

We consider anthropometric measures of health such as the body mass index (BMI), which have two defining properties: (i) survival places lower and upper bounds on their domain of variation, and furthermore (ii) they exhibit a non-monotonic, inverted U, relation with health (and well-being more generally). We begin with a presentation of our conceptual framework and notation, and then we turn to these two defining properties of anthropometrics and dwell on their practical consequences in relation to the measurement of achievement and inequality.

Let $\Omega = (\omega_1, \dots, \omega_n)$ be a vector of anthropometric observations where ω_i is defined on one of two intervals: either $w_i \in (a, m]$ or $w_i \in (m, b)$. These intervals divide the anthropometric variable's domain into two distinct subsets: the lower tail defined over the interval $(a, m]$ ranging from critically low values, a, to the optimum, m, and the upper tail defined over the interval (m, b) ranging from the optimum up to critically large values, b. The distribution $\Omega = (\omega_1, \dots, \omega_n)$ can, thus, be partitioned into two separate vectors defined, respectively, over the intervals (m, b) and $(a, m]$.

Let n_1 and $n_2 = n - n_1$ denote, respectively, the number of observations that belong to the two intervals (m, b) and $(a, m]$. We may now define two vectors $Y \coloneqq (y_1, \dots, y_{n_1}) \in (m, b)^{n_1}$ and $X \coloneqq (x_1, \dots, x_{n_2}) \in (a, m]^{n_2}$ such that $\Omega = [X\ Y]$. Consider two individuals i and j with $x_i < m$ and $y_j > m$ such that $x_i - a = b - y_j$. We take the view that despite the fact that $x_i - a = b - y_j$, the welfare levels of two individuals i and j, one being undernourished, and the other over-nourished are not meaningfully comparable. The approach we take in this paper is to measure a particular aspect of the distribution (i.e. achievement and inequality) separately over the two intervals (m, b) and $(a, m]$, leading us to summarize Ω using a two-dimensional vector of achievement and inequality indices. Likewise, when attempting to rank two distributions $\Omega^1 = [X^1\ Y^1]$ and $\Omega^2 = [X^2\ Y^2]$, say, in terms of inequality, we will only conclude that Ω^1 is the more egalitarian distribution, if both X^1 is more egalitarian than X^2 and likewise Y^1 is more egalitarian that Y^2.

In the sections below the focus is initially on the measurements of achievement and inequality indices in the context of welfare decreasing variables, Y. For the sake of completeness, we then report analogue indices in the case of welfare increasing variables, X.

3. Welfare decreasing variables and the measurement of health achievement and inequality

In the *Atkinson-Kolm-Sen* (AKS) approach, the derivation of inequality indices is approached in relation to a family of social welfare function taken to capture society's preferences for greater health achievement, and less health inequality. The inequality index is derived *via* a comparison of the mean of a variable with the equally distributed value of the distribution. Health achievement indices (Wagstaff 2002) are derived also as the equally distributed equivalent (alternatively, the mean scaled by the level of equality in the distribution). One purpose of this section is to show that for welfare decreasing variables, the equally distributed equivalent value is decreasing in progressive transfers (when we would expect the opposite from such a summary statistic). We then dwell further on the implications of this finding for alternative specifications of achievement and AKS inequality indices for welfare decreasing health variables.

Fundamental axioms

Let $W^Y: (m, b)^{n_1} \to \mathbb{R}$ denote a social welfare function in relation to a welfare decreasing health variable. We measure welfare with reference to individuals' position from the upper survival threshold b. We let ι_{n_1} denote an n_1 −dimensional vector of ones, $\iota_{n_1} = (1, \dots, 1)$, and we consider several axioms commonly used for social welfare functions[1]. In what follows therefore $b\iota_{n_1} - Y$ is a compact notation for the vector $(b - y_1, \dots, b - y_{n_1})$. We begin by stating three standard properties ADD, $ANON$ and REP. The first of these, ADD, captures the notion that the social welfare function is the average of welfare levels experienced by individuals. $ANON$ is an anonymity axiom that insures that only the endowment levels y_i matter for the measurement of social welfare. REP is an axiom of invariance of the social welfare function to certain types of population replications.

- ADD (Strong independence principle): The social welfare function is additively separable in the utility functions of the n_1 individuals.
- $ANON$ (Anonymity): For any $n_1 \times n_1$ permutation matrix Π and any distribution $Y \in (m, b)^{n_1}$, there holds $W^Y\left(\left(b\iota_{n_1} - Y\right)\Pi; m\right) = W^Y\left(b\iota_{n_1} - Y; m\right)$.

[1] For a detailed discussion of these axioms, see for instance Kolm (1976 a, b) or Champernowne and Cowell (1998).

- REP (Invariance to population replication): For any distribution $Y \in (m,b)^{n_1}$, replication of the vector Y to a new distribution $(Y,Y) \in (m,b)^{2n_1}$ leaves social welfare unchanged, there holds $W^Y\left(b\iota_{n_1} - Y; m\right) = W^Y\left(b\iota_{2n_1} - (Y,Y); m\right)$.

Our next two axioms formalize the effect of certain transformations of the distribution Y on social welfare. The monotonicity axiom MON requires that social welfare increases when individual endowments y_i are reduced. Preference for greater equality is introduced via the axiom $EQUAL$, requiring that social welfare increases with Pigou-Dalton transfers.

- MON (Monotonicity): The social welfare function W is strictly decreasing $y_1, \dots, y_{n_1}$.
- $EQUAL$ (Social aversion to inequality): $W^Y\left(b\iota_{n_1} - Y; m\right)$ is strictly increasing in Pigou-Dalton transfers.

Finally, we discuss two invariance axioms capturing certain transformations of the data that leave the ordering of distributions by the social welfare function unchanged. The first of these, $SCALINV$, guarantees that social welfare does not change when units of measurement are modified in a particular manner. The second, $TRANSINV$, is used to capture the notion that inequality is invariant to translation shifts of the distribution of resources.

- $SCALINV$ (Scale invariance): For any scalar $\lambda > 0$, and for any pair of distributions $Y_1, Y_2 \in (m,b)^{n_1}$, $W^Y\left(b\iota_{n_1} - Y_1; m\right) \geq W^Y\left(b\iota_{n_1} - Y_2; m\right) \Leftrightarrow W^Y\left(\lambda b\iota_{n_1} - \lambda Y_1; \lambda m\right) \geq W^Y\left(\lambda b\iota_{n_1} - \lambda Y_2; \lambda m\right)$.
- $TRANSINV$ (Translation invariance): For any admissible value λ and for any pair of distributions $Y_1, Y_2 \in (m,b)^{n_1}$, $W^Y\left(b\iota_{n_1} - Y_1; m\right) \geq W^Y\left(b\iota_{n_1} - Y_2; m\right) \Leftrightarrow W^Y\left(b\iota_{n_1} - (Y_1 + \lambda\iota_{n_1}); m\right) \geq W^Y\left(b\iota_{n_1} - (Y_2 + \lambda\iota_{n_1}); m\right)$.

It is a well-known result (see for instance Kolm, 1976a) that together the first three axioms entail a social welfare function of the form

$$W^Y\left(b - y_1, \dots, b - y_{n_1}; m\right) \coloneqq \frac{1}{n_1}\sum_{i=1}^{n_1} \phi(b - y_i) \tag{1}$$

The monotonicity axiom MON restricts the derivative of the function ϕ to have a negative sign on the interval (m,b). On the other hand, the social welfare function satisfies the social aversion to inequality axiom, $EQUAL$, if ϕ is concave on (m,b). Equivalently, following Apablaza et al. (2016), there are two distinct elementary transformations of the distribution Y that contribute to improving social welfare: (i) a Pigou-Dalton transfer on $(m,b)^{n_1}$, and (ii) a decrement

(reduction) of some y_i on $(m,b)^{n_1}$. Formally, let $\delta > 0$, and consider two observations y_l and y_k such that $y_l - \delta \geq y_k + \delta$. A new distribution $Y^* = (y_1^*, \ldots, y_{n_1}^*)$ is obtained from Y via a Pigou-Dalton transfer if $y_l^* = y_l - \delta$, $y_k^* = y_k + \delta$ and $y_i^* = y_i$ for all $i \neq l, k$. The distribution $Y^* = (y_1^*, \ldots, y_{n_1}^*) \in (m,b)^{n_1}$ is obtained from Y via a unique decrement if for some $\delta \leq y_j - m$, $y_j^* = y_j - \delta$ and $y_i^* = y_i$ for all $i \neq j$. Observe that decrements modify the sum total of a distribution, whereas Pigou-Dalton transfers preserve the mean (and sum total) of the original distribution.

Together the axioms ADD, $ANON$, REP, MON, $EQUAL$ and $SCALINV$ restrict the choice of $\phi(b - y_i)$ to the family $u_\beta(b-y)$ of power functions:

$$u_\beta(b-y) = \begin{cases} \dfrac{(b-y)^{1-\beta}}{1-\beta}, & \beta \geq 1, \beta \neq 1 \\ \ln(b-y), & \beta = 1 \end{cases} \tag{2}$$

Accordingly, the family of social welfare functions that satisfies the above six properties is of the form:

$$W_\beta^Y(b - y_1, \ldots, b - y_{n_1}; m) = \frac{1}{n_1}\sum_{i=1}^{n_1} u_\beta(b - y_i) \tag{3}$$

We shall return to our final axiom, $TRANSINV$ in the next section of the paper.

Properties of the relative achievement index

Let $\hat{y} \in (m,b)$ denote the equally distributed health level in the distribution Y such that welfare of $\hat{y}$ is identical to the level of attainment in the current distribution Y. We have that $u_\beta(b - \hat{y}) = W_\beta^Y(b - y_1, \ldots, b - y_{n_1}; m)$. Following Wagstaff (2002), this equally distributed equivalent value is known as the *achievement index* in the field of health economics. In the income inequality literature, the equally distributed equivalent income is increasing in Pigou-Dalton progressive transfers. The context of welfare decreasing health variables produces a subtle difference:

Proposition 1: *Let $u(.)$ denote any strictly decreasing, and concave function that is differentiable on some closed interval $(m^0, b^0) \subseteq (m,b)$. Then, for any distribution $Y \in (m,b)^{n_1}$, with mean $\bar{y}$, the equally distributed equivalent value*

$$\hat{y} = u^{-1}\left(\frac{1}{n_1}\sum_{i=1}^{n_1} u(b - y_i)\right) \tag{4}$$

is decreasing in Pigou-Dalton transfers and satisfies the inequality: $\bar{y} \leq \hat{y}$.

In the context of Eq. 2 and 3, the equally distributed equivalent value is in the form of:

$$\hat{y}_R = g_\beta(Y; m, b) = \begin{cases} b - \left(\frac{1}{n_1}\sum_{i=1}^{n_1}(b - y_i)^{1-\beta}\right)^{1/(1-\beta)}, & \beta \geq 1, \beta \neq 1 \\ b - \exp\left(\frac{1}{n_1}\sum_{i=1}^{n_1}\ln(b - y_i)\right), & \beta = 1 \end{cases} \tag{5}$$

The subscript R in $\hat{y}_R$ is introduced to denote achievement indices that satisfy the scale invariance axiom $SCALINV$. The associated AKS inequality indices are referred to as relative indices in the income inequality literature, and we shall use this convention here also to distinguish the equally distributed equivalent (Eq. 5) from the equally distributed equivalent that satisfies the translation invariance axiom $TRANSINV$, that will be denoted as $\hat{y}_A$ in Section 3 below.

The AKS family of indices and the relative Lorenz curve

It follows from **Proposition 1** that application of the standard AKS inequality index introduced by Atkinson (1970), namely the function $I_R(Y) := 1 - (\hat{y}_R/\bar{y})$ will provide the data user with multiple challenges. Firstly, in the light of the inequality, the index (Eq.4) will usually take on negative values. Secondly, because the index does not depend on the upper threshold b, changes in the units of measurement of y, m and b will change the value taken by the inequality index. More importantly, Pigou-Dalton transfers will increase the value of I_R, suggesting that inequality has increased. It is thus important to adapt the AKS index in the context of welfare decreasing health data, so as to achieve these three desired properties (non-negative property, scale invariance and transfer sensitivity). Consider in particular the following form:

$$\Delta_R^Y(b\iota_{n_1} - Y; m) := 1 - \left(\frac{b - \hat{y}_R}{b - \bar{y}}\right) \tag{6}$$

Because $\hat{y}_R \geq \bar{y}$ in the context of welfare decreasing variables (**Proposition 1**), Δ_R^Y will be a non-negative function. Likewise, Δ_R^Y is now an increasing function of $\hat{y}_R$, as the equally distributed value is a Schur-convex function, decreasing in progressive transfers. Finally, it is clear that the

inequality index (Eq. 6) is invariant to rescaling b, m and the distribution Y by the same constant $\lambda > 0$.

In order to derive a new expression for the relative Lorenz curve that is consistent with our framework, it is useful to consider two ordered vectors associated with the distribution Y: firstly the decreasing rearrangement of Y that we denote by the vector $Y\downarrow = (y_{[1]}, \dots, y_{[n]})$, and secondly the increasing rearrangement of Y that we denote by $Y\uparrow = (y_{(1)}, \dots, y_{(n)})$. Clearly, if we want to maintain the well established and meaningful practice of summing resources starting from the worst off individuals, the analogue to summing incomes in increasing order is to sum y values in decreasing order. The conventional Lorenz curve is accordingly modified as follows:

$$RL(j, b\iota_{n_1} - Y) := \frac{1}{n_1(b-\bar{y})}\sum_{i=1}^{j}(b - y_{[i]}), \qquad j = 1, \dots, n_1 \tag{7}$$

In particular if we define the new variable $z := b - y_i$, it then follows that $z_{(i)} = b - y_{[i]}$ and that $\bar{z} = b - \bar{y}$. That is,

$$\frac{1}{n_1(b-\bar{y})}\sum_{i=1}^{j}(b - y_{[i]}) = \frac{1}{n_1\bar{z}}\sum_{i=1}^{j} z_{(i)}, \qquad j = 1, \dots, n_1 \tag{8}$$

and the Lorenz curve $RL(j, b\iota_{n_1} - Y)$ is the classical Lorenz curve formula for the variable z.

The proposition below confirms that the Lorenz curve remains valid for investigating inequality orderings in the present context that takes into account the decreasing nature as well as the survival threshold of the health indicator. In what follows, we first state a result relating social welfare attainment and the relative Lorenz curves of two distributions with identical means. In the subsequent Proposition the comparison is extended to cover distributions with variable sum totals.

Proposition 2: *Let $Y_1, Y_2 \in (m, b)^{n_1}$ denote two distributions of a welfare decreasing health variable of identical sum totals: $\sum_{i=1}^{n_1} y_{1i} = \sum_{i=1}^{n_1} y_{2i}$. The following statements are equivalent:*

(i) $RL(j, b\iota_{n_1} - Y_2) \geq RL(j, b\iota_{n_1} - Y_1)$ for all $j = 1, \dots, n_1$,

(ii) Y_2 is obtained from Y_1 via a finite sequence of Pigou-Dalton transfers,

(iii) $\frac{1}{n_1}\sum_{i=1}^{n_1}\phi(b - y_{2i}) \leq \frac{1}{n_1}\sum_{i=1}^{n_1}\phi(b - y_{1i})$ for any convex function ϕ defined on the interval (m, b).

In the same way, it is possible to rank two distributions of a welfare decreasing health variable defined on the interval (m, b) in terms of social welfare. Define the generalized Lorenz curve $GL\left(j, b\iota_{n_1} - Y\right)$ at the j^{th} ordinate as follows,

$$GL\left(j, b\iota_{n_1} - Y\right) := \frac{1}{n_1}\sum_{i=1}^{j}\left(b - y_{[i]}\right), \qquad j = 1, \dots, n_1 \tag{9}$$

In the same way, if $z_{(i)} = b - y_{[i]}$, the generalized Lorenz curve $GL\left(j, b\iota_{n_1} - Y\right)$ is the classical generalized Lorenz curve formula for the variable z.

The following result – adapted here in the context of welfare decreasing health variables – has been obtained by Shorrocks (2009) in the context of the ordering of distributions of unemployment duration. It is the analogue of **Proposition 2** in the context of distributions of variable sum totals.

Proposition 3: *Let $Y_1, Y_2 \in (m, b)^{n_1}$ denote two distributions of a welfare decreasing health variable. The following statements are equivalent:*

(i) $GL\left(j, b\iota_{n_1} - Y_2\right) \geq GL\left(j, b\iota_{n_1} - Y_1\right)$ *for all* $j = 1, \dots, n_1$,

(ii) $W^Y\left(b\iota_{n_1} - Y_2; m\right) \geq W^Y\left(b\iota_{n_1} - Y_1; m\right)$ *for any social welfare function* W^Y *that satisfies* MON *and* $EQUAL$,

(iii) Y_2 *is obtained from* Y_1 *via a finite sequence of Pigou-Dalton transfers, and or decrements.*

4. The effect of the upper survival threshold

The scale invariance axiom guarantees that changing the units of measurement of y, and the two thresholds m and b does not result in any change in inequality or the Lorenz curve (Eq. 7) introduced in this paper. A separate concern however may have to do with disagreement about the level of the thresholds m and b. The threshold m serves to select observations in our sample. Changing its value will result in a different sample (dropping or adding observations for individuals in good health). It is more challenging however to deal with changes in the upper threshold.

Changing the upper threshold b is of course equivalent to adding an identical amount λ to each person's endowment, that is translating the distribution of resources Y to obtain a new distribution $Y + \lambda\iota_{n_1}$. It is clear that such translational shifts in the distribution of resources will usually modify the level of inequality, and also result in a shift in the relative Lorenz curve. Consider for instance the coefficient of variation $\rho(Y) := \sigma(Y)/\bar{y}$ defined as the ratio of the

standard deviation to the mean. We then can easily observe that the standard deviation is translation invariant, and hence that

$$\rho\left(Y + \lambda\iota_{n_1}\right) = \frac{\sigma(Y)}{\bar{y} + \lambda} \tag{10}$$

It follows therefore that when λ is chosen to be positive, the coefficient of variation falls as a result of a translational shift. In equivalent terms, a reduction in the upper survival threshold results in a reduction of the coefficient of variation. Specifically, we are interested in evaluating the effect of change in b on the relative and generalized Lorenz forms we have introduced in the paper.

By differentiating (Eq. 9) we note that the derivative of the generalized Lorenz curve with respect to b is a constant vector function that is independent of the data[2]. More simply, the generalized Lorenz curve is a linear vector function in b. This however is not the case with the relative Lorenz curve: as the vector derivative of (Eq. 7) with respect to b is a function of the data Y, the ordering of distributions by the relative Lorenz curve is sensitive to the choice of b. Following Kolm (1976) and Moyes (1987), it is however possible to work with inequality indices and Lorenz curves that are invariant to changes in the upper threshold b. As we shall see below, there is however a price to pay, in that the scale invariance property will have to be replaced by a translation invariance axiom.

The Kolm family of absolute indices and the related Lorenz curve

The key to deriving indices that are robust to changes in the upper threshold b is to replace the scale invariance axiom $SCALINV$ by a translation invariance axiom. Specifically, together the axioms ADD, $ANON$, REP, MON, $EQUAL$ and $TRANSINV$ restrict the choice of $\phi(b-y)$ to the family $u_\kappa(b-y)$ of exponential functions (Kolm, 1976 a,b):

$$u_\kappa(b-y) = 1 - \exp\left(-\kappa(b-y)\right), \qquad \kappa > 0 \tag{11}$$

Accordingly, the family of social welfare functions that satisfies the above six axioms is of the form

$$W_\kappa^Y\left(b-y_1, \dots, b-y_{n_1}; m\right) = \frac{1}{n_1}\sum_{i=1}^{n_1} u_\kappa(b-y_i) \tag{12}$$

The equally distributed equivalent value $\hat{y}_A$ pertaining to the above family of social welfare functions satisfies the identity $u_\kappa(b-\hat{y}_A) = \frac{1}{n_1}\sum_{i=1}^{n_1} u_\kappa(b-y_i)$. Specifically,

[2] That is this gradient vector is of the form $(1/n, 2/n, \dots, n/n)$.

$$\hat{y}_A = b + \frac{1}{\kappa}\ln\left(\frac{1}{n_1}\sum_{i=1}^{n_1}\exp\left(-\kappa(b-y_i)\right)\right) \tag{13}$$

By analogy with Wagstaff (2002), $\hat{y}_A$ is the achievement index pertaining to the Kolm family of social welfare functions. Accordingly, we refer to $\hat{y}_A$ as the absolute achievement index.

It is to be noted that $\hat{y}_A$ is invariant to changes in the parameter b. Furthermore, from Proposition 1, the equally distributed equivalent income $\hat{y}_A$ will also be decreasing in Pigou-Dalton transfers, since the axioms MON and $EQUAL$ are satisfied in a relation to the Kolm family of social welfare function.

The Kolm absolute inequality index pertaining to welfare decreasing health variables accordingly is of the form:

$$\Delta_A^Y\left(b\iota_{n_1} - Y; m\right) \coloneqq \hat{y}_A - \bar{y} \tag{14}$$

Following Moyes (1987), the Lorenz curve concept that is invariant to translational shifts of the distribution (i.e. to choices of the upper threshold b) is the absolute Lorenz curve. In the context of welfare decreasing variables, this takes the form

$$AL\left(j, b\iota_{n_1} - Y\right) := \frac{1}{n_1}\sum_{i=1}^{j}\left(\bar{y} - y_{[i]}\right), \qquad j = 1, \dots, n_1 \tag{15}$$

Again, to understand how this formula is obtained, it is easiest once again to perform a change of variable and to define $z_{(i)} = b - y_{[i]}$. It then follows that $z_{(i)} - \bar{z} = \bar{y} - y_{[i]}$ and that $\frac{1}{n_1}\sum_{i=1}^{j}\left(z_{(i)} - \bar{z}\right) = \frac{1}{n_1}\sum_{i=1}^{j}\left(\bar{y} - y_{[i]}\right)$. The absolute Lorenz curve (14) of the welfare decreasing variable y is the Moyes (1987) absolute Lorenz curve, applied to the variable z.

We summarize the above discussion with the following Proposition, that is a corollary to Moyes (1987):

Proposition 4: *Let $Y_1, Y_2 \in (m, b)^{n_1}$ denote two distributions of a welfare decreasing health variable. The following statements are equivalent:*

(i) $AL\left(j, b\iota_{n_1} - Y_2\right) \geq AL\left(j, b\iota_{n_1} - Y_1\right)$ *for all* $j = 1, \dots, n_1$,

(ii) $\Delta_A^Y\left(b\iota_{n_1} - Y_2; m\right) \leq \Delta_A^Y\left(b\iota_{n_1} - Y_1; m\right)$ *for all* $\kappa > 0$ *and for any admissible value of the upper threshold* b .

5. Welfare increasing variables and the measurement of health achievement and inequality

In this section, we derive the analogue functions of achievement and inequality indices for the case of a bounded welfare increasing variable, that is the lower tail of the distribution, Ω.

Let $W^X: (a,m]^{n_2} \to \mathbb{R}$ denote a social welfare function in relation to a welfare increasing health variable and let $X - a\iota_{n_2}$ be a compact notation for the vector $(x_1 - a, \dots, x_{n_2} - a)$. We measure welfare with reference to individuals' position from the lower survival threshold a. W^X satisfies the standard axioms ADD, $ANON$, REP, MON and $EQUAL$. Together with $SCALINV$, the standard axioms restrict the choice of a function $\phi(x_i - a)$ that is increasing on $(a,m]$ to the family $u_\alpha(x-a)$ of power functions:

$$u_\alpha(x-a) = \begin{cases} \dfrac{(x-a)^{1-\alpha}}{1-\alpha}, & \alpha \geq 1, \alpha \neq 1 \\ \ln(x-a), & \alpha = 1 \end{cases} \tag{16}$$

Accordingly, the family of social welfare functions that satisfies the above six axioms is of the form:

$$W_\alpha^X\left(x_1 - a, \dots, x_{n_2} - a; m\right) = \frac{1}{n_2}\sum_{i=1}^{n_2} u_\alpha(x_i - a) \tag{17}$$

Let $\hat{x} \in (a,m]$ denote the *achievement index* in the distribution X such that $u_\alpha(x-a) = W_\alpha^X\left(x_1 - a, \dots, x_{n_2} - a; m\right)$, then

$$\hat{x}_R = g_\alpha(X; a, m) = \begin{cases} a + \left(\dfrac{1}{n_2}\displaystyle\sum_{i=1}^{n_2}(x_i - a)^{1-\alpha}\right)^{1/(1-\alpha)}, & \alpha \geq 1, \alpha \neq 1 \\ a + \exp\left(\dfrac{1}{n_2}\displaystyle\sum_{i=1}^{n_2}\ln(x_i - a)\right), & \alpha = 1 \end{cases} \tag{18}$$

is increasing in Pigou-Dalton transfers and satisfies the inequality: $\bar{x} \geq \hat{x}$.

Similarly, the AKS index will also be adapted to achieve the properties of non-negative property, scale invariance and transfer sensitivity such that

$$\Delta_R^X\left(X - a\iota_{n_2}; m\right) \coloneqq 1 - \left(\frac{\hat{x}_R - a}{\bar{x} - a}\right) \tag{19}$$

It is also clear that the Δ_R^X is invariant to rescaling a, m and the distribution X by a constant $\lambda > 0$.

The corresponding Lorenz curve and generalized Lorenz curve for the ordered vector $X\uparrow=\left(x_{(1)},\ldots,x_{(n)}\right)$ are:

$$RL\left(j,X-a\iota_{n_2}\right):=\frac{1}{n_2(\bar{x}-a)}\sum_{i=1}^{j}\left(x_{(i)}-a\right),\qquad j=1,\ldots,n_2 \tag{20}$$

$$GL\left(j,X-a\iota_{n_2}\right):=\frac{1}{n_2}\sum_{i=1}^{j}\left(x_{(i)}-a\right),\qquad j=1,\ldots,n_2 \tag{21}$$

Akin to the case of a welfare decreasing variable, the relative Lorenz curve of a bounded welfare increasing variable is also sensitive to the choice of the threshold a. It is possible to work with inequality indices and Lorenz curves that are invariant to changes in the threshold a. This requires specifying a family of welfare functions that satisfy, in addition to the standard axioms, the translation invariance axiom, $TRANSINV$. The choice of $\phi(x-a)$ is thus restricted to the family $u_\kappa(x-a)$ of exponential functions

$$u_\kappa(x-a)=1-\exp\left(-\kappa(x-a)\right),\qquad \kappa>0 \tag{22}$$

Accordingly, the equally distributed equivalent value $\hat{x}_A$ and the Kolm absolute inequality index pertaining to the above family of social welfare functions are

$$\hat{x}_A=a-\frac{1}{\kappa}\ln\left(\frac{1}{n_2}\sum_{i=1}^{n_2}\exp\left(-\kappa(x_i-a)\right)\right) \tag{23}$$

$$\Delta_A^X\left(X-a\iota_{n_2};m\right):=\bar{x}-\hat{x}_A \tag{24}$$

In the context of welfare increasing variables, the absolute Lorenz curve takes the form

$$AL\left(j,X-a\iota_{n_2}\right):=\frac{1}{n_2}\sum_{i=1}^{j}\left(x_{(i)}-\bar{x}\right),\qquad j=1,\ldots,n_2 \tag{25}$$

6. An illustrative hypothetical example

The purpose of this section is to illustrate the above methodology using a hypothetical example (illustrated in Table 1 and 2).

Consider $n_1=n_2=10$ individuals, with critical values $a=10$ and $b=60$ and optimum value $m=24.9$. Assume that the mean level associated with the set of upper tail distributions, Υ_β, is $\bar{y}=42.45$, and the mean level associated with the set of the lower tail distributions, χ_α, is $\bar{x}=17.45$. The least egalitarian distribution associated with any distribution belongs to Υ_β is given by the distribution Y_0 with five persons having resources $b=60$, and another five persons having

resources $m = 24.9$. Each of the distributions Y_1 to Y_5 are obtained by taking two extreme values $(60, 24.9)$ and replacing them by their mean i.e. $(42.45, 42.45)$. The final distribution Y_5 is equal to $\perp \Upsilon_\beta$, the least inegalitarian distribution of the upper tail.

Table 1: Relative achievement and inequality indices in the upper tail of BMI distribution

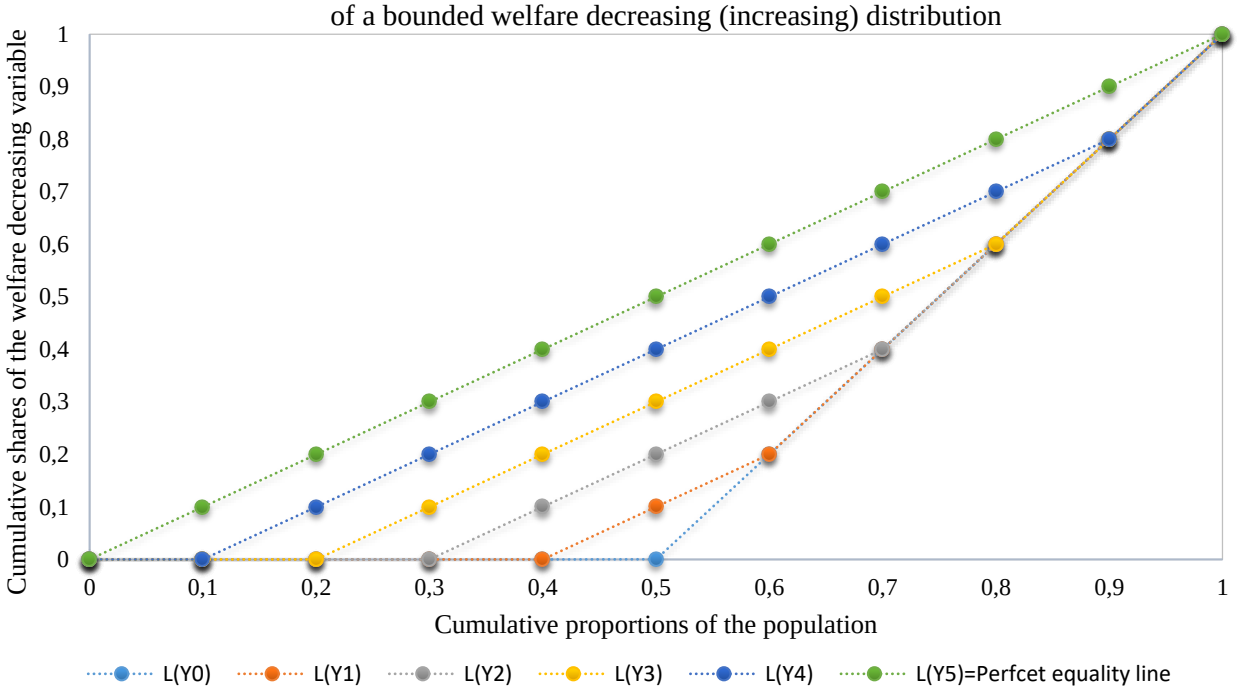
			Y: from the least to the most egalitarian distribution					
		i	Y_0	Y_1	Y_2	Y_3	Y_4	Y_5
		1	60	60	60	60	60	42.45
		2	60	60	60	60	42.45	42.45
		3	60	60	60	42.45	42.45	42.45
		4	60	60	42.45	42.45	42.45	42.45
		5	60	42.45	42.45	42.45	42.45	42.45
		6	24.9	42.45	42.45	42.45	42.45	42.45
		7	24.9	24.9	42.45	42.45	42.45	42.45
		8	24.9	24.9	24.9	42.45	42.45	42.45
		9	24.9	24.9	24.9	24.9	42.45	42.45
		10	24.9	24.9	24.9	24.9	24.9	42.45
Inequality-aversion parameter	β	Total	424.5	424.5	424.5	424.5	424.5	424.5
		Mean. $\bar{y}$	42.45	42.45	42.45	42.45	42.45	42.45
	0.5	Achievement index. $\hat{y}_R$	50.926	49.452	47.867	46.170	44.363	42.445
		AKS inequality index. Δ_R^Y	0.483	0.399	0.309	0.212	0.109	0
	1	Achievement index. $\hat{y}_R$	59.408	58.833	57.702	55.474	51.086	42.445
		AKS inequality index. Δ_R^Y	0.9663	0.9335	0.8691	0.7422	0.4922	0
	2	Achievement index. $\hat{y}_R$	59.980	59.975	59.967	59.950	59.900	42.445
		AKS inequality index. Δ_R^Y	0.9989	0.9986	0.9981	0.9972	0.9943	0

Table 2: Relative achievement and inequality indices in the lower tail of BMI distribution

			X: from the least to the most egalitarian distribution					
		i	X_0	X_1	X_2	X_3	X_4	X_5
		1	10	10	10	10	10	17.45
		2	10	10	10	10	17.45	17.45
		3	10	10	10	17.45	17.45	17.45
		4	10	10	17.45	17.45	17.45	17.45
		5	10	17.45	17.45	17.45	17.45	17.45
		6	24.9	17.45	17.45	17.45	17.45	17.45
		7	24.9	24.9	17.45	17.45	17.45	17.45
		8	24.9	24.9	24.9	17.45	17.45	17.45
		9	24.9	24.9	24.9	24.9	17.45	17.45
		10	24.9	24.9	24.9	24.9	24.9	17.45
Inequality-aversion parameter	α	Total	174.5	174.5	174.5	174.5	174.5	174.5
		Mean, $\bar{x}$	17.45	17.45	17.45	17.45	17.45	17.45
	0.5	Achievement index, $\hat{x}_R$	13.921	14.537	15.199	15.906	16.658	17.455
		AKS inequality index, Δ_R^X	0.474	0.391	0.303	0.208	0.107	0.000
	1	Achievement index, $\hat{x}_R$	10.386	10.698	11.262	12.281	14.124	17.455
		AKS inequality index, Δ_R^X	0.948	0.906	0.831	0.694	0.447	0.000
	2	Achievement index, $\hat{x}_R$	10.020	10.025	10.033	10.050	10.099	17.455
		AKS inequality index, Δ_R^X	0.997	0.997	0.996	0.993	0.987	0.000

The Lorenz curves pertaining to this sequence of distributions are sketched in Figure 1. Similarly, X_0, is the least egalitarian distribution associated with any distribution belongs to χ_α with five individuals having resources $a = 10$, and five individuals having resources $m = 24.9$. The final distribution X_5 is the least inegalitarian distribution of the lower tail. The corresponding Lorenz curves of this sequence of distributions are similar to that sketched in Figure 1.

Figure 1: An illustration of a sequence of relative Lorenz curves of a bounded welfare decreasing (increasing) distribution



There are three main points that the above simple examples serve to illustrate. First, similar to the case of a welfare increasing variable, the Lorenz curves of a bounded welfare decreasing variable lie below the equality line. Second, successive Pigou-Dalton transfers displace the Lorenz curves upwards in the direction of the equality line. Third, unlike the standard case of a welfare increasing variable, the equally distributed equivalent value $\hat{y}_A$ is generally larger than the mean of the distribution, and is decreasing in Pigou-Dalton transfers. Taking for instance $\beta = 0.5$, as shown in Table 1, the equally distributed equivalent value ranges from 50.9 for the least egalitarian distribution Y_0, and decreases with Pigou-Dalton transfers, until it equals the mean at Y_5, the most egalitarian distribution. By contrast, Table 2 shows that, for $\alpha = 0.5$, the equally distributed equivalent value ranges from 13.9 for the least egalitarian distribution X_0, and decreases with Pigou-Dalton transfers, until it equals the mean at the most egalitarian distribution. The sensitivity of the relative achievement and inequality indices are illustrated in the empirical application below.

7. From the lower and upper tails to the distributional comparisons

A remark is due before we turn to our empirical examination of inequality and social welfare in a group of Arab countries. We have thus far discussed separately the question of distributional comparisons in the X and Y domains that are below and above the optimum value, m. As argued earlier we do not assume that well-being is meaningfully comparable below and above the optimum value, m, in the context of anthropometrics. In turn, this entails that when we cardinalize social preferences with a given pair of (α, β) values, any given social welfare or inequality vector does not entail a complete ordering of distributions $\Omega = [X\ Y]$.

To observe that the resulting order is incomplete, consider two hypothetical distributions Ω^A and Ω^B such that the welfare of those below the optimum is higher in distribution Ω^A, but the welfare of those above the optimum is higher in Ω^B. Such patterns do not allow us to rank the distributions Ω^A and Ω^B in terms of welfare. These patterns will be encountered in the section below when we compare the distribution of body mass pertaining to Egypt and Yemen with those of other countries. For this reason, if it is deemed necessary to compare the entire distribution of well-being across two countries, we are led to construct vectors of social welfare and inequality indices. Formally, this would lead us to conclude that welfare is higher in country A than in country B if and only if the generalized Lorenz curve X^A lies above the GLC of X^B and the concave GLC of Y^A lies below that of GLC of Y^B.

8. An empirical application: Health achievement and inequality in five Arab countries

The purpose of this section is to assess health achievement and inequality using data pertaining to five Arab countries. Specifically, we calculate the AKS relative inequality and achievement indices for welfare decreasing health variables and the related generalized Lorenz curves. As the relative indices and Lorenz curves are sensitive to the value assigned to the upper survival threshold b, we then calculate the *absolute inequality* and *achievement indices* as well as the related absolute Lorenz curves. For the sake of completeness, we present in the last-sub-section relative and absolute inequality and achievement indices for welfare increasing health variables. Our application makes use of anthropometric data on adult (non-pregnant) women of reproductive age (15 to 49). The analysis is performed using data from the latest available Demographic and Health Surveys (DHS) conducted in five Arab countries: Egypt (2015), Yemen (2013), Jordan (2012), Comoros (2012) and Morocco (2004). The anthropometric indicator of interest here is

taken as body mass index (BMI), calculated by the authors as the weight in kilograms divided by the square of height measured in meters. The implementation of the above methodology necessitates that we assign values for the survival thresholds a and b and the optimum value m. In line with the guidelines of the World Health Organization (2004; 2017), for the purpose of the present analysis, we set the value of a to be equal to 10 and b to be equal to 60 , while the cut-off value m is fixed at 24.90[3]. After cleaning the data for missing and miscoded values on the variable of interest, the respective sample sizes are as follows: Egypt ($n_1 = 5226, n_2 = 1962$), Jordan ($n_1 = 6336, n_2 = 4740$), Morocco ($n_1 = 6239, n_2 = 10677$), Yemen ($n_1 = 5669, n_2 = 17276$) and Comoros ($n_1 = 1927, n_2 = 3156$).

Relative health achievement and inequality of the upper tail of the BMI distribution

We begin by examining the relative inequality indices $\Delta_R^Y = \left(b\iota_{n_1} - Y; m\right)$ and the related achievement indices $\hat{y}_R$ as well as the corresponding generalized Lorenz curves $GL = \left(j, b\iota_{n_1} - Y\right)$ in the five countries. We report in Table 3 calculations pertaining to inequality and achievement indices in relation to two values for the inequality aversion parameter: $\beta = 0.5, 1$ and 3. To begin with, it is worth noting that the mean of the distribution is highest in Egypt (32.4) and lowest in Morocco (28.8). Consider first the results pertaining to $\beta = 1$. Recalling that achievement (welfare) is decreasing in y, we find that the anthropometric achievement index $\hat{y}_R$ ranks social welfare as lowest in Egypt ($\hat{y}_R = 33.2$) followed by Jordan ($\hat{y}_R = 31.6$), Comoros ($\hat{y}_R = 29.7$),Yemen ($\hat{y}_R = 29.5$), while it is highest in Morocco ($\hat{y}_R = 29.1$). Increasing the social aversion to inequality ($\beta = 3$), results in lower health achievement (that is, higher values) in all countries. We note nonetheless that this does not change the ranking order of the countries.

[3] For most anthropometrics, m could be defined as a set or a range of values. The WHO guidelines for non-pregnant women define an optimum interval with values of m ranging between 18.5 and 24.9. As this does not raise unresolved conceptual problems in the context of our paper, we shall simplify our exposition by assuming that m is a single point (i.e., unique optimum health level, which is the upper bound of the optimum range. The survival lower bound is at least 10, while the fatal upper range involves any body mass value in excess of 60.

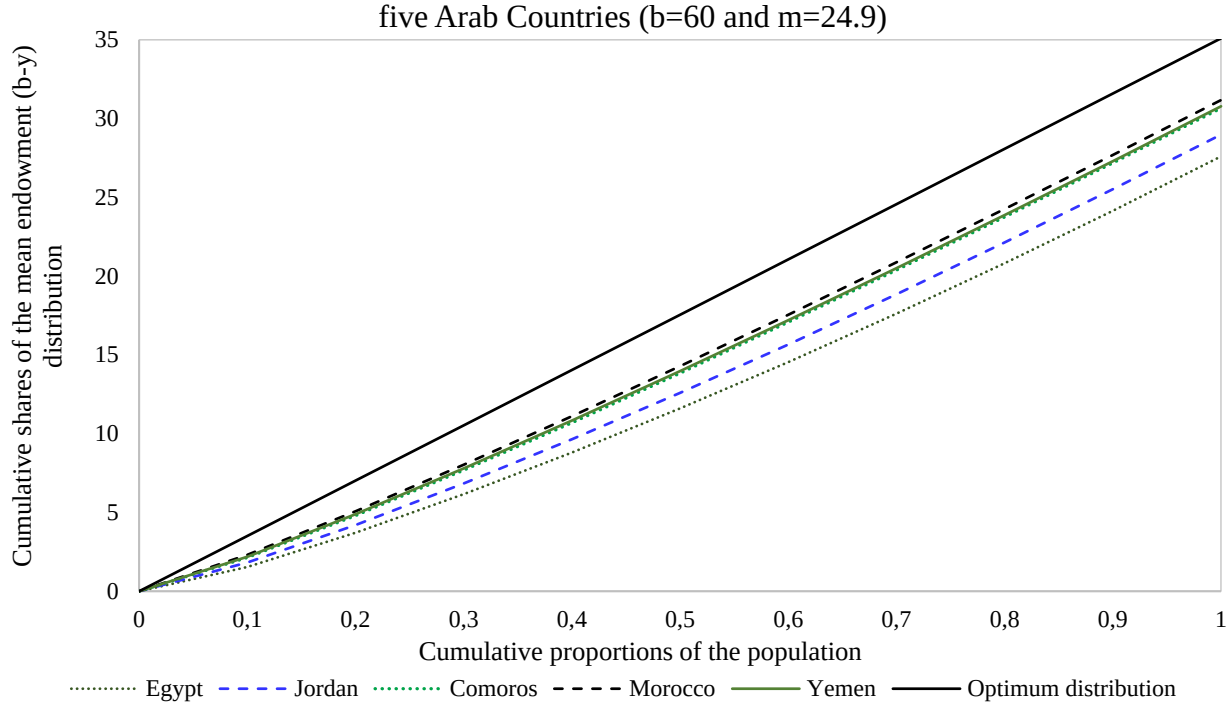
Table 3: Relative health achievement and inequality in five Arab countries:
The upper tail of the BMI distribution ($m = 24.9$, $b = 60$)

Inequality-aversion parameter	β	Countries	Egypt	Jordan	Comoros	Yemen	Morocco
		Sample sizes, n_1	5226	6336	1927	5669	6239
		Mean, $\bar{y}$	32.40	31.02	29.34	29.22	28.82
	0.5	Achievement index, $\hat{y}_R$	32.76	31.27	29.51	29.37	28.94
		AKS inequality index, Δ_R^Y	0.0131	0.0087	0.0056	0.0048	0.0038
	1	Achievement index, $\hat{y}_R$	33.20	31.56	29.71	29.54	29.07
		AKS inequality index, Δ_R^Y	0.0291	0.0186	0.0121	0.0104	0.0079
	3	Achievement index, $\hat{y}_R$	40.86	33.60	31.20	30.73	29.78
		AKS inequality index, Δ_R^Y	0.3067	0.0890	0.0606	0.0490	0.0309

Turning now to inequality, we find that the AKS index for welfare decreasing variables, $\Delta_R^Y = \left(b\iota_{n_1} - Y; m\right)$, for $\beta = 1$, takes the highest value in Egypt: 2.9%. In contrast, this figure is the lowest in Morocco 0.8%, while inequality is between these two values in the context of the other three countries. Because the mean and dispersion of the distribution are highest in Egypt ($\bar{y} = 32.4, \Delta_R^Y = 2.9\%$), health achievement is lowest in Egypt. Once again, the magnitudes of inequalities increase with the inequality aversion parameter. For instance, for $\beta = 3$, inequality in Egypt, is the highest at approximately 30.6%. This is followed by 8.9% in Jordan, 6.1% in the Comoros and 4.9% in Yemen, while it is the lowest in Morocco (about 3.1%).

To investigate systematically the welfare ordering of these five countries the generalized Lorenz curves pertaining to the upper tail of the BMI distributions are plotted in Figure 2. The generalized Lorenz curve of a hypothetical optimum health distribution $Y^* = (m, \ldots, m)$, would take the form of a straight line starting at zero with a slope equal to $b - m$. This would entail the mean of the distribution $\bar{y}$ approaching $m = 24.90$, the anthropometric achievement index, $\hat{y}_R$, approaching $\bar{y}$ and the inequality index, $\Delta_R^Y = \left(b\iota_{n_1} - Y; m\right)$, approaching zero. We can view the process of improvement in the distribution of body mass as one where the generalized Lorenz curves approach from below the generalized Lorenz curve of the optimum distribution. In accordance with Proposition 3, the generalized Lorenz curve of Egypt lying below the other four curves, (see Figure 2) and that of Morocco being closest to the straight line, entails that the above welfare ordering of the five countries is robust to the choice of value assigned to the inequality aversion parameter β.

Figure 2: Generalized Lorenz curves of the upper tail BMI distribution in five Arab Countries (b=60 and m=24.9)

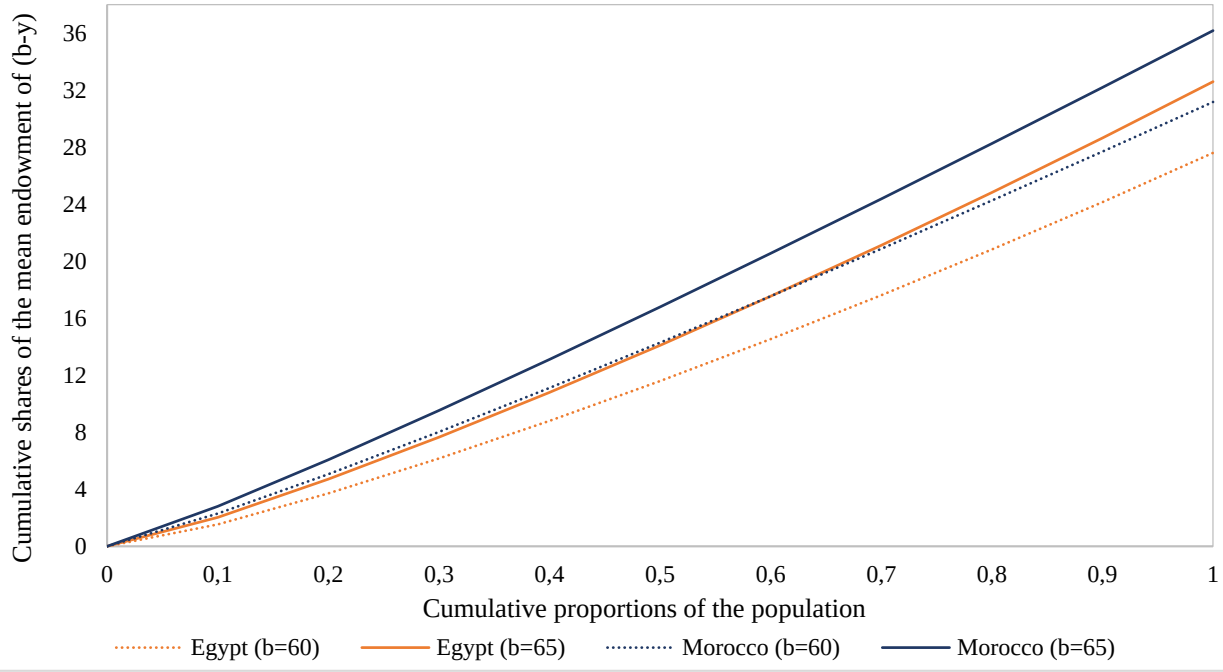


As argued in Section 4 nonetheless, these findings are not necessarily robust to the choice of the survival threshold b. For instance, increasing the survival threshold to $b = 65$, and setting the inequality aversion parameter to $\beta = 1$, the value of inequality for Egypt decreases to 1.9% (compared with 2.9%) while that for Morocco falls to 0.6% (compared with 0.8%) (see Table 4). We also illustrate the relative Lorenz curves pertaining to Egypt and Morocco in relation to the two survival thresholds $b = 60$ and $b = 65$. As shown in Figure 3, we observe that the curvature of the generalised Lorenz curve is altered by changes in b.

Table 4: Sensitivity of the relative health achievement and inequality indices to changes in the value of the survival thresholds ($m = 24.9$, $\beta = 1$)

		Countries	Egypt	Jordan	Comoros	Yemen	Morocco
Upper thresholds	b	Sample sizes, n_1	5226	6336	1927	5669	6239
		Mean, $\bar{y}$	32.40	31.02	29.34	29.22	28.82
	60	Achievement index, $\hat{y}_R$	33.20	31.56	29.71	29.54	29.07
		AKS inequality index, Δ_R^Y	0.0291	0.0186	0.0121	0.0104	0.0079
	65	Achievement index, $\hat{y}_R$	33.02	31.45	29.64	29.48	29.02
		AKS inequality index, Δ_R^Y	0.0190	0.0128	0.0084	0.0073	0.0057

Figure 3: Generalized Lorenz curves of the upper tail BMI distribution of Egypt and Morocco for different survival threshold values (b)

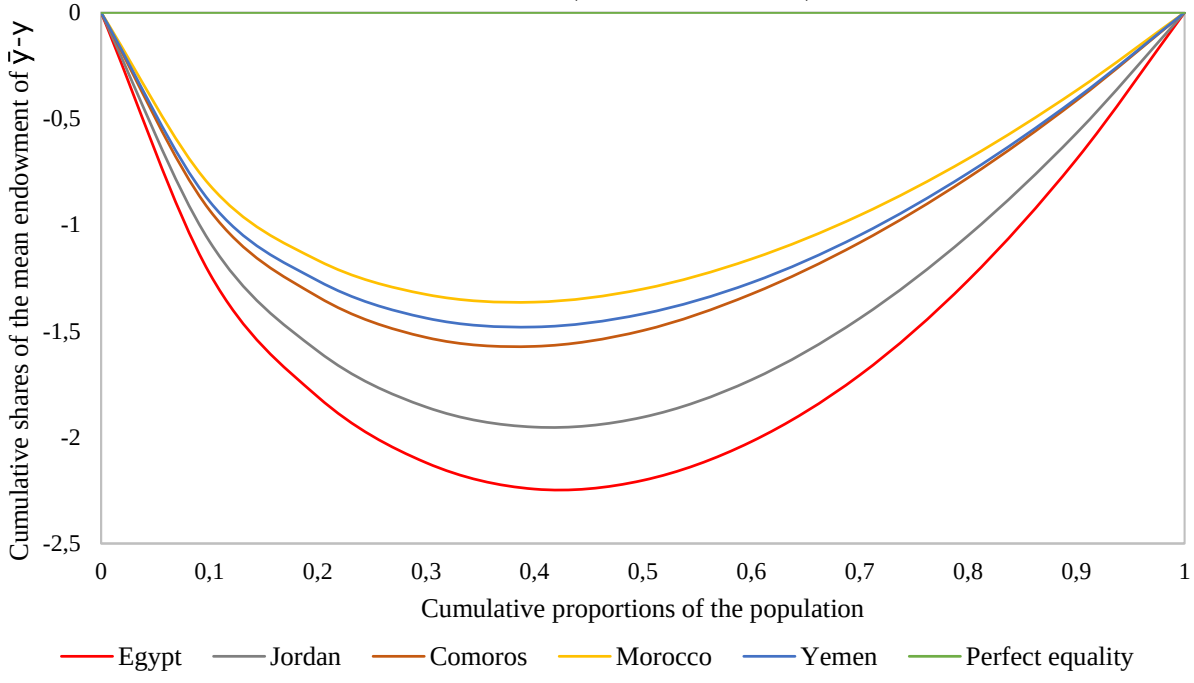


As the results discussed above are sensitive to the choice of the survival threshold, in the following subsection we report findings pertaining to the Kolm family of *absolute inequality* and *achievement indices* and *related absolute Lorenz curve*.

Absolute health achievement and inequality of the upper tail of the BMI distribution

To explore systematically an inequality ordering of countries that is robust to the choice of upper threshold values, we depict in Figure 4 absolute Lorenz curves for the five countries of interest. To read these findings, we can observe that the perfect equality line of the absolute Lorenz curve coincides with the horizontal axis. In our context, this line indicates a distribution where everyone has the same body mass value. The further an absolute Lorenz curve dips from the perfect equality line, the higher the level of inequality. As the five absolute Lorenz curves do not intersect, we conclude, in accordance with Proposition 4, that absolute inequality is lowest in Morocco and highest in Egypt. The result is of interest, as it reveals that for all inequality aversion parameter $\kappa > 0$, absolute inequality indices for welfare decreasing variables (Eq. 14) will order the countries in the same way as absolute Lorenz curve criterion.

Figure 4: Absolute Lorenz curves of the upper tail BMI distributions of five Arab countries (b=60 and m=24.9)



Computations of alternative absolute achievement, $\hat{y}_A$, and inequality indices, $\Delta_A^Y(b\iota_{n_1} - Y_2; m)$, are reported in Table 5. We discuss briefly the findings related to $\kappa = 1$. In this regard, it is to be noted that the welfare ranking of the countries, as measured by the absolute achievement and inequality indices remains the same as the one observed using the relative indices. Health achievement remains the lowest in Egypt ($\hat{y}_A = 51.7$) and the highest in Morocco ($\hat{y}_A = 44.8$). Similarly, inequality remains highest in Egypt ($\hat{y}_A = 19.3$) and lowest in Morocco ($\hat{y}_A = 16.0$). Interestingly, the values of these absolute indices are invariant to the choice of the survival threshold.

Table 5: Absolute health achievement and inequality in five Arab countries: The upper tail of the BMI distribution ($m = 24.9$, $b = 60$)

		Countries	Egypt	Jordan	Comoros	Yemen	Morocco
Inequality-aversion parameter	κ	Sample sizes, n_1	5226	6336	1927	5669	6239
		Mean, $\bar{y}$	32.40	31.02	29.34	29.22	28.82
	0.5	Achievement index, $\hat{y}_A$	45.87	42.97	41.79	40.94	37.79
		Kolm inequality index, Δ_A^Y	13.47	11,95	12,45	11,71	8.97
	1	Achievement index, $\hat{y}_A$	51.69	49.15	47.75	47.63	44.83
		Kolm inequality index, Δ_A^Y	19.29	18.13	18.41	18.41	16.01
	3	Achievement index, $\hat{y}_A$	55.33	52.75	51.22	51.49	49.05
		Kolm inequality index, Δ_A^Y	22.93	21.73	21.88	22.27	20.23

Relative and absolute health achievement and inequality of the lower tail of the BMI

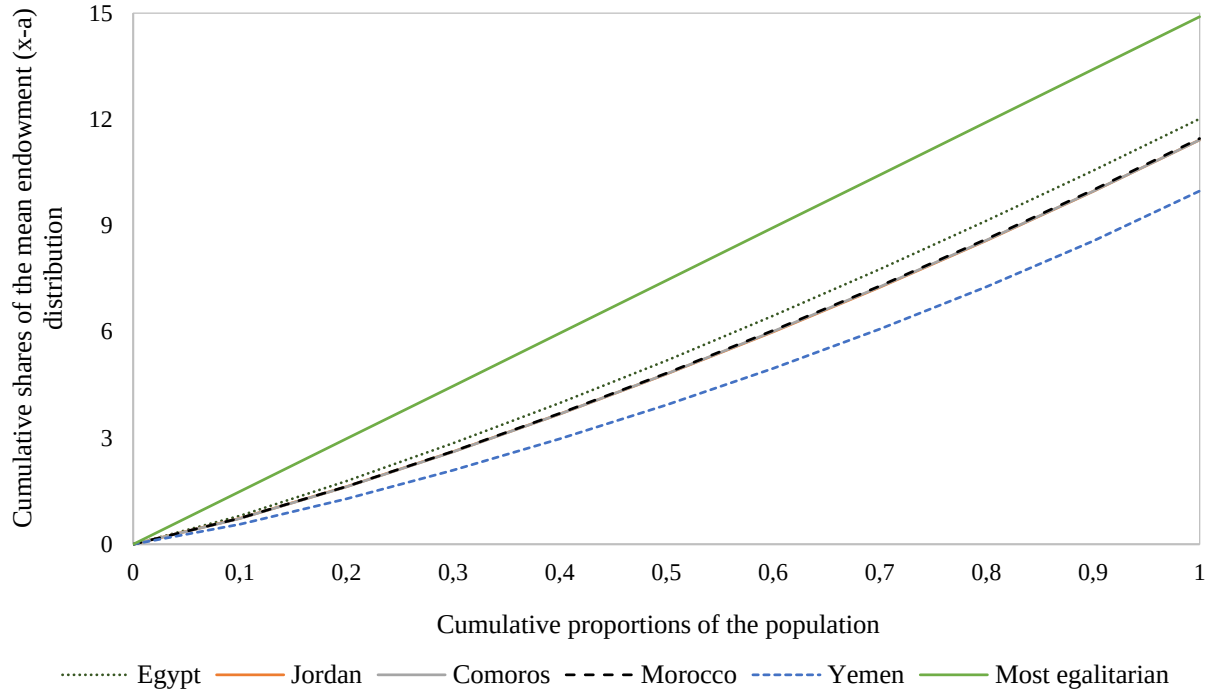
We report in Table 6 and 7 results pertaining to achievement and inequality in relation to the lower tail of the BMI distribution in the five Arab countries. For instance, health achievement is lowest in Yemen ($\hat{x}_R = 18.3$) because the mean of the distribution is lowest in Yemen ($\bar{x} = 20$) while the dispersion is the highest ($\Delta_R^X = 16.9\%$) for $\alpha = 3$. By contrast, health achievement appears to be the highest in Egypt ($\hat{x}_R = 21.36$). Once again, the magnitudes of health achievement decrease with the inequality aversion parameter while inequality increases.

Table 6: Relative health achievement and inequality in five Arab countries:
The lower tail of the BMI distribution ($a = 10$, $m = 24.9$)

		Countries	Yemen	Comoros	Jordan	Morocco	Egypt
Inequality-aversion parameter	α	Sample sizes, n_2	17276	3156	4740	10677	1962
		Mean, $\bar{x}$	19.98	21.42	21.42	21.46	22.02
	0.5	Achievement index, $\hat{x}_R$	19.80	21.30	21.30	21.35	21.92
		AKS inequality index, Δ_R^X	0.0174	0.0101	0.0099	0.0098	0.0077
	1	Achievement index, $\hat{x}_R$	19.62	21.18	21.18	21.23	21.83
		AKS inequality index, Δ_R^X	0.0360	0.0209	0.0205	0.0201	0.0158
	3	Achievement index, $\hat{x}_R$	18.30	20.58	20.60	20.66	21.36
		AKS inequality index, Δ_R^X	0.1685	0.0739	0.0721	0.0698	0.0546

An inspection of generalized Lorenz curves is useful to investigate the robustness of our findings. In accordance with the above results, the generalized Lorenz curve of Yemen lies below the other four curves, while that of Egypt dominates all other curves (see Figure 5). The remaining three generalized Lorenz curves coincide reflecting the similar values of achievement and inequality indices reported in Table 6.

Figure 5: Generalized Lorenz curves of the lower tail BMI distribution in five Arab Countries (a = 10 and m = 24.9)



Given the sensitivity of the relative achievement and inequality indices to the choice of survival threshold a, we also compute the alternative absolute achievement, $\hat{x}_A$, and inequality indices, Δ_A^X. Results, which are reported in Table 7, confirm the above ranking with Yemen having the least achievement and the highest inequalities in the lower tail distribution of the BMI.

Table 7: Absolute health achievement and inequality in five Arab countries: The lower tail of the BMI distribution ($a = 10, m = 24.9$)

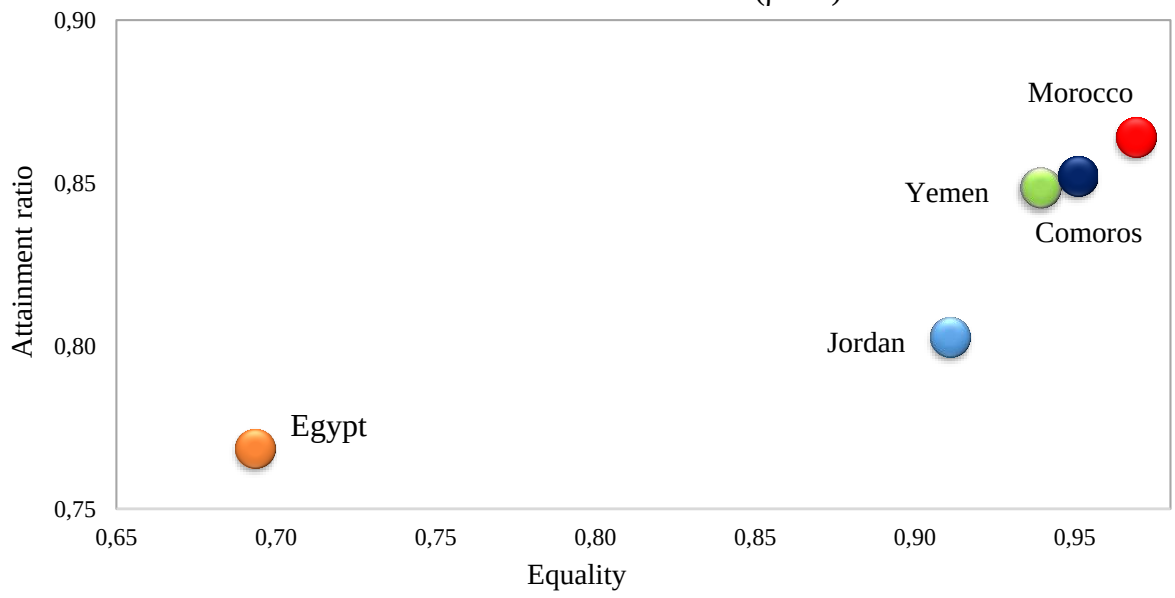
		Countries	Yemen	Comoros	Jordan	Morocco	Egypt
Inequality-aversion parameter	κ	Sample sizes, n_2	17276	3156	4740	10677	1962
		Mean, $\bar{x}$	19.98	21.42	21.42	21.46	22.02
	0.5	Achievement index, $\hat{x}_A$	18.40	20.03	20.09	20.14	20.80
		Kolm inequality index, Δ_A^X	1.58	1.39	1.33	1.32	1.21
	1	Achievement index, $\hat{x}_A$	16.90	18.54	18.62	18.73	19.48
		Kolm inequality index, Δ_A^X	3.08	2.88	2.80	2.73	2.54
	3	Achievement index, $\hat{x}_A$	14.39	16.35	15.83	16.14	17.65
		Kolm inequality index, Δ_A^X	5.60	5.07	5.58	5.32	4.36

A tentative ranking

It is tempting to rank the five countries under consideration in terms of both the level of equality (i.e. one minus the AKS inequality index) and the *level of health attainment*. The latter is defined as the ratio of the optimum value to the mean ($m/\bar{y}$) in the case of the upper tail distribution, while it takes the reciprocal form $\bar{x}/m$ for the case of the lower tail distribution[4].

Figure 6 shows that Yemen, the Comoros, Morocco and Jordan display similar distributional patterns for the upper tail BMI with the attainment ratio being in the range of 80% to 86% and the level of equality being in the range of 91 to 97%. It is interesting to note that Egypt stands well apart from these countries. Specifically, Egypt has a considerably lower level of equality (70%) and a lower health attainment ratio (77%).

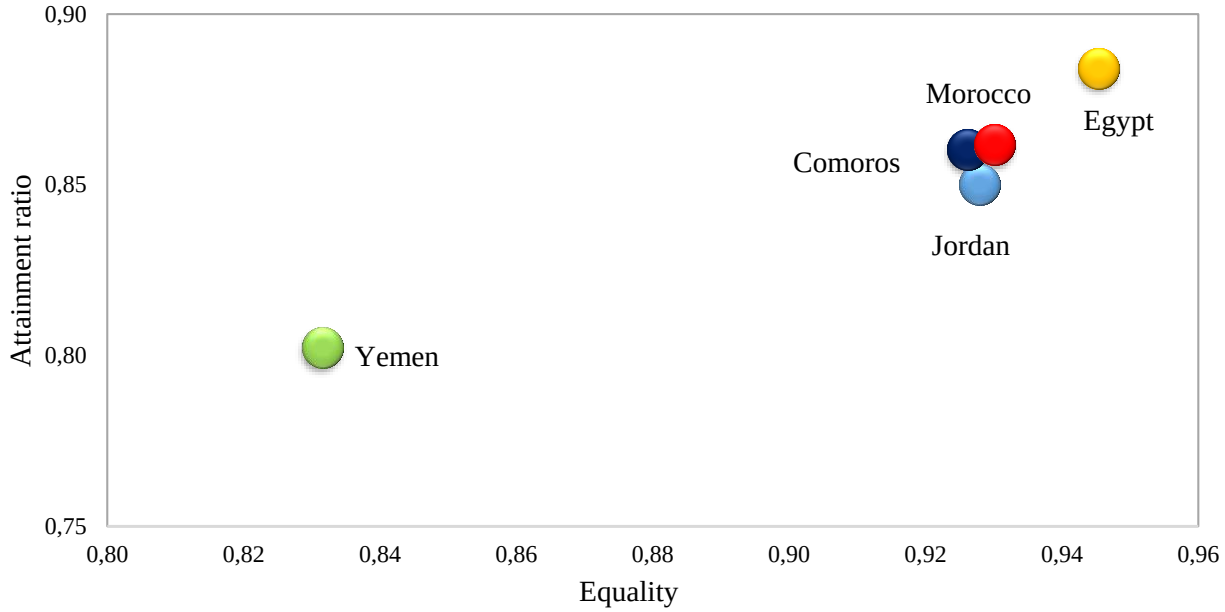
Figure 6: Health attainement *vs*. equality of the upper tail BMI distribution in the five Arab countries (β = 3)



Results on the lower tail, presented in Figure 7, reveal that Yemen and Egypt stand well apart from the three other countries. Yemen exhibits lower levels of both equality (83%) and health attainment (80%) while the opposite is observed for Egypt, moving from the bottom to top in terms of both the equality and attainment dimensions (94% and 88%, respectively).

[4] Alternatively, we can plot on the vertical axis the ratio of optimum value to achievement ($m/\hat{y}$) for the case of the upper tail or the reciprocal of this ratio ($\hat{x}/m$) in the case of lower tail.

Figure 7: Health attainement vs. equality of the lower tail BMI distribution in the five Arab countries (α=3)



Overall, we have not found any *pairwise comparisons* of countries where country A had a higher level of inequality and attainment in the upper and lower tails of the distributions. Nonetheless, we note that the cardinal welfare rankings of the five countries in Figures 6 and 7 are in broad agreement with the ordinal rankings of the countries in terms of the generalized Lorenz curves.

9. Conclusion

The last two decades have placed the measurement of inequality and achievement in health at the centre of the development debate. The purpose of our paper was to address the question of the measurement of pure health inequalities and achievement in the context of welfare decreasing variables. We were thus led to adopt a general framework whereby the health variable is reported on an interval, from an optimum level m to a critical level b, beyond which survival was no longer assured.

We have noted in our discussion above that the context of inequality measurement *per se* on a welfare decreasing variable was not the problem. Specifically, we have argued that as the utility function was Schur-concave (be it increasing or decreasing in the underlying health indicator) and

the associated inequality index was Schur-convex, the Lorenz curve could be used to order health distributions in the same way that it was applied to order income distributions.

There were however two significant problems that required some departures from the usual framework used to measure inequality and social welfare in relation to income distributions. Firstly, we have shown in Proposition 1 that for welfare decreasing variables, the equally distributed equivalent value – the summary statistic used to derive health achievement indices – is a Schur-convex function: that is, a function that is decreasing in progressive transfers. This is the opposite of what we should expect of such a summary statistic. This has meant that the relative Atkinson-Kolm-Sen inequality indices available from the income inequality literature required some adaptation in the context of welfare decreasing variables. Accordingly, appropriate achievement and inequality indices for welfare decreasing variables were introduced in Sections 2 of the paper.

The second problem that required attention was that of survival thresholds, a property inherent to many health indicators. We have acknowledged that clinical research informs about sensible values of the survival threshold b. Nonetheless, it remained that the Lorenz curve and the associated scale invariant inequality indices were not robust to alternative values of the survival threshold. For this second problem we have argued that the family of translation invariant social welfare functions introduced by Kolm (1976 a, b) and related absolute Lorenz curve (Moyes, 1987) allowed us to undertake inequality comparisons between distributions that are robust to the chosen level of the survival threshold. Translation invariant achievement and inequality indices in the context of welfare decreasing variables, were accordingly introduced in Section 4 and an illustrative application of the methodology was provided in Section 8 of the paper.

One important extension of our framework would consist in deriving achievement and inequality indices of welfare decreasing variables in the context of the socio-economic disparities in health. This would complement the readily available normative framework existing in this literature in relation to welfare increasing health indicators (Wagstaff 2002, Erreygers 2013). Another possible extension of the analysis could consist in remaining centred in the context of disparities in pure health, but adopting a multidimensional perspective on the measurement of achievement and inequality, where the health variable is welfare decreasing.

When applied to a group of five Arab countries (Egypt, Morocco, Jordan, Comoros and Yemen), the findings of the paper provided some support for the hypothesis that lack of bread and

social justice may have contributed to the Yemeni revolt of 2011 (Zurayk and Gough, 2014). In Egypt on the other hand, it would appear to have been low levels of social justice rather than lack of bread that may have contributed to the outbreak of the Arab uprising (Kadri, 2014).

Appendix

Proof of Proposition 1: Because, by assumption, $u(.)$ is a strictly decreasing and differentiable function, it follows, that u^{-1} exists, is strictly decreasing and differentiable on the interval $[u(b^0 - b), u(m^0 - b)]$. Let $t \coloneqq \frac{1}{n_1}\sum_{i=1}^{n_1} u(b - y_i)$ be an element of the interval $[u(b^0 - b), u(m^0 - b)]$, and define the function $h(y_1, \dots, y_n) \coloneqq u^{-1}\left(\frac{1}{n_1}\sum_{i=1}^{n_1} u(b - y_i)\right) = u^{-1}(t)$, so that $\hat{y} = h(y_1, \dots, y_n)$.

Our next task is to show that $h(y_1, \dots, y_n)$ is a Schur-convex function. On the basis of Remark $3.A.5$ of Marshall et al (2011 p. 85), in showing this, without loss of generality we may readily consider the case of $n_1 = 2$ individuals with endowments $y_i > y_j$. Because $u(.)$ is concave, the social welfare function W is Schur-concave and

$$(y_i - y_j)\left(\frac{\partial u}{\partial y_i} - \frac{\partial u}{\partial y_j}\right) \leq 0$$

On the other hand,

$$(y_i - y_j)\left(\frac{\partial h}{\partial y_i} - \frac{\partial h}{\partial y_j}\right) = (y_i - y_j)\frac{\partial u^{-1}}{\partial t}\left(\frac{\partial u}{\partial y_i} - \frac{\partial u}{\partial y_j}\right) \geq 0$$

since the inverse function $u^{-1}(.)$ is strictly decreasing and therefore $\partial u^{-1}/\partial t < 0$. It follows therefore that $\hat{y} = h(y_1, \dots, y_n)$ is a Schur-convex function, that is a function decreasing in Pigou-Dalton transfers.

Because $u(.)$ is decreasing, we have furthermore that $u(b - \bar{y}) \geq u(b - \hat{y}) \Leftrightarrow \hat{y} \geq \bar{y}$.

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