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Abstract

We study the issue of assigning weights to players that identify winning coalitions in plurality voting democracies. For this, we consider *plurality games* which are simple games in partition function form such that in every partition there is at least one winning coalition. Such a game is said to be precisely supportive if it is possible to assign weights to players in such a way that a coalition being winning in a partition implies that the combined weight of its members

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is maximal over all coalitions in the partition. A plurality game is *decisive* if in every partition there is exactly one winning coalition. We show that decisive plurality games with at most four players, majority games with an arbitrary number of players, and almost symmetric decisive plurality games with an arbitrary number of players are precisely supportive. Complete characterizations of a partition's winning coalitions are provided as well.

JEL Classification: C71, D62, D72

Keywords: plurality game, plurality voting, precise support, simple game in partition function form, winning coalition

1 Introduction

Usually, immediately after parliamentary elections have taken place, political parties, media, voters, etc., discuss who is the winner of the election. In the case of plurality voting systems such type of discussions might become even heated since the winning candidate is required to garner more votes than any other single opponent; he need not, as in the case of majority voting, poll more votes than the combined opposition. In this paper, we frame such debates in the setup of simple cooperative games in partition function form and address the question if it is possible to assign weights to political parties that somehow measure who is the winner of an election. If that is possible, we say that the corresponding simple game supports a plurality voting democracy.

Any simple cooperative game assigns the worth of one to coalitions of parties that are winning, and zero to coalitions that are not winning (i.e., losing). Typically, whether a coalition is winning or losing might depend on the way how players outside the coalition are organized into coalitions. It can be that a coalition that is negotiating to form a government is winning if the other parties are not organized, but if some other parties form a (minority) coalition, it might be attractive for one of the negotiating parties to stop negotiations and start negotiations to form a government with the 'new coalition'.

The issue of identifying winning coalitions in plurality voting is not necessarily reserved to political elections, but can easily find its place in organizational or institutional considerations, like university recruitments, calls for a new management team in a department, calls for group research proposals in an institute, etc.

In the present paper, we model situations with such externalities as plurality games. A plurality game is a special type of *simple game in partition function form,* that is, to each pair of a coalition and a partition containing this coalition, it assigns either a worth of one (if the coalition is winning in the partition) or a worth of zero. We call such games *plurality games* if in every partition, there is at least one coalition which wins. In this case, winning does not necessarily mean that the coalition has a majority and can pass a bill, but simply that it is the strongest in a given coalitional configuration as represented by a partition. So, a party that does not have the majority, but is considered as the winner of the election before any negotiation to form a government has taken place, has worth one in the *discrete partition* (i.e., the partition into singletons). For example, it is common practice that the party that got the most votes in an election takes the initiative to form a government. Although this does not imply that eventually this party will be in the government, it obviously gives the party an advantage as long as no coalitions are formed yet.

Within the model of plurality games, we study the possibility of assigning weights to players (parties) such that a coalition being winning in a partition implies that the sum of its players' weights is maximal over all coalitions in the partition. If this is possible for a given game, then we call the game *precisely supportive.* In that case, we can say that the game supports a plurality voting democracy. An important role is played by *decisive* plurality games, where in each partition there is *exactly* one winning coalition in it. In Section 2, we provide an example of a five-player decisive plurality game which is not precisely supportive. In Section 3, we show that *small decisive plurality games* (that is, games with at most four players) are precisely supportive. A plurality game with more than four players turns out to be precisely supportive when it is a *majority game*, i.e., when a partition's winning coalition is of maximal size in it. If we consider majority games which are symmetric in a specific sense, then only the equal weights for all players make the game precisely supportive (Section 4). Notice that, intuitively, not all players can be symmetric in a decisive plurality game, because for such a game exactly one singleton is winning in the discrete partition. The closest we can get to symmetry in such a game, is to require that all players but one (the winner in the discrete partition) are symmetric. We call such games *almost symmetric* and show in Section 5 that almost symmetric decisive plurality games with an arbitrary number of players are precisely supportive. For this, we define *the power of the winner in the discrete partition* in a specific way, show how it shapes the structure of the possible candidates for a winning coalition in a partition, and explicitly use it in the construction of suitable weights.

The rest of the paper is organized as follows. In Section 2, we include the formal definitions of plurality games and precise support, and provide an example illustrating that even decisive plurality games are not necessarily precisely supportive. Sections 3 and 4 are devoted to small games and majority games, respectively. Section 5 presents our results with respect to almost symmetric decisive plurality games, while Section 6 provides an overview of the related literature. All proofs are collected in Appendix A (proofs from Section 3), Appendix B (proofs from Section 4), and Appendix C (proofs from Section 5).

2 Plurality games

In order to specify our ideas, we introduce the concept of a plurality game, which is a special type of a simple cooperative game in partition function form. We give below the corresponding definitions, after which we introduce the notion of a plurality game being precisely supporting a plurality voting democracy.

All games we consider will be defined on a fixed and finite player set $N = \{1, \ldots, n\}$ with $n \geq 2$, whose non-empty subsets are called *coalitions*. A collection π of coalitions is called a *coalition structure* if π is a partition of N, i.e., if all coalitions in π are non-empty, pair-wise disjoint, and their union is N. We denote by $\mathcal{P}$ the set of all partitions (coalition structures) of N. For $\pi \in \mathcal{P}$ and $i \in N$, the notation $\pi(i)$ stands for the coalition in π containing player i. The partition $\pi^d \in \mathcal{P}$ with $\pi^d(i) = \{i\}$ for each $i \in N$, is called the *discrete partition*. A pair $(S; \pi)$ consisting of a non-empty coalition $S \subseteq N$ and a partition $\pi \in \mathcal{P}$ with $S \in \pi$ is called an *embedded coalition*. The set of all embedded coalitions is $E = \big\{(S; \pi) \in 2^N \times \mathcal{P} \mid S \in \pi\big\}$.

For partition $\pi \in \mathcal{P}$ and set of players $S \subset N$, we denote by $\pi_S = \{T \cap S \mid T \in \pi\}$ the partition on S induced by π. Further, we will often write $\{T_1, \ldots, T_k, \pi_S\}$ for $\{T_1, \ldots, T_k, S_1, \ldots, S_p\}$ if $\pi_S = \{S_1, \ldots, S_p\}$.

Simple games in partition function form *A simple game in partition function form* is a pair (N, v), where the partition function $v : E \to \{0, 1\}$ with $v(N; \{N\}) = 1$ specifies which embedded coalitions are winning. In other words, an embedded coalition $(S; \pi) \in E$ is *winning* in the game (N, v)

if and only if $v(S;\pi) = 1$. We sometimes say that coalition S is winning in partition π when $(S;\pi)$ is a winning embedded coalition. The set of all winning embedded coalitions in the game v is denoted by $E_w(v)$. Notice that this game form allows to model externalities of coalition formation. For instance, it can be that a coalition contained in two partitions π and π' is winning in π but losing in π'. We often write a simple game in partition function form by its partition function v and use the following notion of inclusion, borrowed from Alonso-Meijide et al. (2017): For $(S';\pi'), (S;\pi) \in E$, we say that $(S';\pi')$ is *weakly included* in $(S;\pi)$, denoted by $(S';\pi') \subseteq (S;\pi)$, if $S' \subseteq S$ and for each $T \in \pi \setminus \{S\}$ there exists $T' \in \pi'$ with $T \subseteq T'$. A game v is then defined as *monotonic* if $(S';\pi'), (S;\pi) \in E$ with $(S';\pi') \subseteq (S;\pi)$ implies $v(S';\pi') \leq v(S;\pi)$. Besides a nonnegative effect when a coalition grows, this monotonicity notion reflects an idea of negative externalities when players outside a coalition form larger coalitions. In particular, it implies that when a coalition is winning in a partition, then it is winning in every finer partition that contains this coalition. A reasonable motivation here is that in a finer partition there is 'less resistance' against the winning coalition. Clearly, a winning coalition can become losing in a coarser partition since other players forming coalitions might give a 'stronger resistance' against the winning coalition, or make the latter more likely to 'break down'.

Plurality games We call a simple game in partition function form v a *plurality game* if (i) it is monotonic, and (ii) for each $\pi \in \mathcal{P}$ we have $v(S;\pi) = 1$ for at least one $S \in \pi$. A plurality game v is *decisive*, if for each $\pi \in \mathcal{P}$ we have that $v(S;\pi) = 1$ for exactly one $S \in \pi$. We assume, w.l.o.g., that in a decisive plurality game v, it is player 1 who wins (as singleton) in the discrete partition, i.e., $v(\{i\};\pi^d) = 1$ if and only if $i = 1$.

Precise support We say that a plurality game v is *precisely supportive*, if there exists a weight vector $w \in X_+^N := \left\{w \in \mathbb{R}_+^N \mid \Sigma_{i\in N} w_i = 1\right\}$ such that for each $(S;\pi) \in E$, $(S;\pi) \in E_w(v)$ implies $w(S) := \Sigma_{i\in S} w_i \geq \Sigma_{i\in T} w_i := w(T)$ for each $T \in \pi$. We call w a *supporting weight vector* for the plurality game. In words, a plurality game is precisely supportive if there exist nonnegative weights for the players, such that the winning coalitions in a partition can be identified by their weights in the sense that they have maximal weights in the partition.

The five-player plurality game given below is decisive but it is not precisely supportive. We slightly abuse notation and write for instance $\underline{12}, 34, 5$ to denote the partition $\{\{1,2\},\{3,4\},\{5\}\}$ with the coalition $\{1,2\}$ being winning in it, i.e., $\underline{12}, 34, 5$ means that $v(\{1,2\}\,;\{\{1,2\}\,,\{3,4\}\,,\{5\}\})=1$.

$\underline{12345}$	$\underline{234}, 1, 5$	$\underline{12}, 34, 5$	$\underline{15}, 24, 3$
$\underline{1234}, 5$	$\underline{125}, 34$	$\underline{12}, 3, 45$	$\underline{15}, 34, 2$
$\underline{1235}, 4$	$\underline{125}, 3, 4$	$\underline{12}, 35, 4$	$\underline{15}, 2, 3, 4$
$\underline{1245}, 3$	$\underline{135}, 24$	$\underline{12}, 3, 4, 5$	$\underline{25}, 34, 1$
$\underline{1345}, 2$	$\underline{135}, 2, 4$	$\underline{13}, 24, 5$	$\underline{25}, 1, 3, 4$
$\underline{2345}, 1$	$\underline{235}, 14$	$\underline{13}, 25, 4$	$24, \underline{1}, 35$
$\underline{123}, 45$	$\underline{235}, 1, 4$	$\underline{13}, 45, 2$	$24, \underline{1}, 3, 5$
$\underline{123}, 4, 5$	$\underline{145}, 23$	$\underline{13}, 2, 4, 5$	$\underline{23}, 1, 45$
$\underline{124}, 35$	$\underline{145}, 2, 3$	$14, \underline{23}, 5$	$\underline{23}, 1, 4, 5$
$\underline{124}, 3, 5$	$\underline{245}, 13$	$\underline{14}, 25, 3$	$35, \underline{1}, 2, 4$
$\underline{134}, 25$	$\underline{245}, 1, 3$	$\underline{14}, 35, 2$	$\underline{34}, 1, 2, 5$
$\underline{134}, 2, 5$	$\underline{345}, 12$	$\underline{14}, 2, 3, 5$	$\underline{45}, 1, 2, 3$
$\underline{234}, 15$	$\underline{345}, 1, 2$	$\underline{15}, 23, 4$	$\underline{1}, 2, 3, 4, 5$

Suppose that the above game were precisely supportive with $w \in X_+^N$. We should then have $w_4 + w_5 \geq w_1$ (due to $\underline{45}, 1, 2, 3$) and $w_1 \geq w_2 + w_4$ (due to $24, \underline{1}, 3, 5$) and thus, (by adding these two inequalities) $w_5 \geq w_2$. On the other hand, from $w_1 \geq w_2 + w_4$ and $w_2 + w_3 \geq w_1$ (by $\underline{23}, 1, 45$) follows $w_3 \geq w_4$. Finally, from $w_1 \geq w_2 + w_4$ and $w_2 + w_5 \geq w_1$ (by $\underline{25}, 1, 3, 4$) we have $w_5 \geq w_4$.

We also have $w_3 + w_4 \geq w_1$ (by $\underline{34}, 1, 2, 5$), and it follows with $w_1 \geq w_2 + w_4$ (see above) that $w_3 \geq w_2$. We have $w_1 \geq w_3 + w_5$ (by $24, \underline{1}, 35$) and thus, with $w_3 + w_4 \geq w_1$ (by $\underline{34}, 1, 2, 5$), we have $w_4 \geq w_5$. Summarizing, so far we have $w_1 \geq w_3 \geq w_4 = w_5 \geq w_2$.

Further, we have $w_2 + w_5 \geq w_3 + w_4$ (by $\underline{25}, 34, 1$) and thus, with $w_4 = w_5$ and $w_3 \geq w_2$ (see above), this gives $w_2 = w_3$. Hence, $w_1 \geq w_3 = w_4 = w_5 = w_2$ should hold.

Moreover, from $w_2 + w_3 \geq w_1 + w_4$ (by $14, \underline{23}, 5$) and $w_2 = w_4$, follows $w_3 \geq w_1$. We conclude then that all weights should be equal. However, $w_1 \geq w_2 + w_4$ (by $24, \underline{1}, 35$) then implies that $w_1 = w_2 = w_4 = 0$, which results in all weights being equal to zero. Thus, we have a contradiction to $w \in X_+^N$.

In view of the above example, in what follows we first concentrate on decisive plurality games with at most four players (Section 3), and then consider additional restrictions on games with an arbitrary number of players guaranteeing that these games are precisely supportive (Sections 4 and 5).

3 Small games

We start our analysis by showing that every decisive small game (a game with at most four players) is precisely supportive (Theorem 1). This result is in accordance with corresponding results for the case of standard simple games (simple games without externalities). For instance, von Neumann and Morgenstern (1944) show that all simple games with less than four players, all proper or strong simple games with less than five players, and all constant sum games with less than six players have voting representations, i.e., they can be represented by weights assigned to the players. However, there are constant-sum games with six players for which representative weights do not exist. In their study of rough weightedness of small games, Gvozdeva and Slinko (2011) show that all games with at most four players, all strong or proper games with at most five players, and all constant-sum games with at most six players are roughly weighted. Finally, it is worth mentioning that Shapley (1962) also explicitly analyses games with four or less players when discussing several properties of weighted majority games. In particular, he studies homogeneous games where the weights can be assigned in such a way that all minimal winning coalitions have the same weight.

For decisive small games, we describe the type of coalitions that can be candidates for a winning coalition in a partition. Specifically, it turns out that the winning coalition in a partition is either one of maximal size, or the one containing the winner (assumed to be player 1) in the discrete partition.

Proposition 1 *Let v be a decisive plurality game with at most four players. Then for each $(S;\pi) \in E$, $(S;\pi) \in E_w(v)$ implies either $|S| \geq |T|$ for each $T \in \pi$, or $S = \pi(1)$.*

The above conclusion is very helpful for the proof of our main result in this section.

Theorem 1 *Every decisive plurality game with at most four players is precisely supportive.*

One of the messages of Proposition 1 is that, in particular cases, it is exactly the size of a coalition that matters. This fact invites us to focus on majority games.

4 Majority games and symmetric players

A plurality game v is a *majority game* if for all $(S;\pi) \in E$, $(S;\pi) \in E_w(v)$ implies $|S| \geq |T|$ for each $T \in \pi$. It is easy to verify that $w = \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)$ is a supporting weight vector for every majority game.

Theorem 2 *Every majority game is precisely supportive.*

Majority games can be symmetric in at least two senses we describe now. Given a plurality game v, we say that two players $i, j \in N$ are

- *α-symmetric* in v, if for each $\pi \in \mathcal{P}$ with $j \notin \pi(i)$, $(\pi(i);\pi) \in E_w(v)$ implies that $\pi(i)\setminus\{i\}\cup\{j\}$ is winning in the partition $\left\{\pi(i) \setminus \{i\} \cup \{j\}, \pi(j) \setminus \{j\} \cup \{i\}, \pi_{N\setminus(\pi(i)\cup\pi(j))}\right\}$;
- *β-symmetric* in v, if for each $\pi \in \mathcal{P}$ with $S, T \in \pi$ and $i, j \notin S \cup T$, $(S;\pi) \in E_w(v)$ implies that $S \cup \{i\}$ is winning in the partition $\left\{S \cup \{i\}, T \cup \{j\}, \pi(i) \setminus \{i\}, \pi(j) \setminus \{j\}, \pi_{N\setminus(S\cup T\cup\pi(i)\cup\pi(j))}\right\}$.

Each of these symmetry notions expresses an independence idea. Two players being α-symmetric in a game imposes restrictions in situations where one of the two players belongs to a winning coalition in a partition. In such a case, α-symmetry requires the coalition to remain winning in the partition obtained by exchanging the places of these two players. On the other hand, the notion of β-symmetry is based on the idea that the relation between two coalitions, one of which is winning in a partition, should be preserved when the two players are correspondingly added to these coalitions. *A plurality game v is α-symmetric (β-symmetric)* if all players are α-symmetric (β-symmetric) in v.

Our next result provides a characterization of α-symmetric majority games and shows that, on the class of majority games, α-symmetry implies β-symmetry.

Proposition 2 *A majority game is α-symmetric if and only if all coalitions of maximal size in a partition are winning in it. Moreover, any α-symmetric majority game is also β-symmetric.*

It is worth mentioning that not every β-symmetric majority game is also α-symmetric. For instance, let us consider the four-player decisive majority game v defined below.

$\underline{1234}$	$\underline{12}, 34$	$\underline{14}, 2, 3$
$\underline{123}, 4$	$\underline{12}, 3, 4$	$\underline{23}, 1, 4$
$\underline{124}, 3$	$\underline{13}, 24$	$\underline{24}, 1, 3$
$\underline{134}, 2$	$\underline{13}, 2, 4$	$\underline{34}, 1, 2$
$\underline{234}, 1$	$\underline{14}, 23$	$\underline{1}, 2, 3, 4$

Notice first that all players from $N \setminus \{1\}$ are β-symmetric in v. To see this, take, for example, players 2 and 3, and a partition with two coalitions, say S and T, that do not contain these two players ($2, 3 \notin S \cup T$), such that one coalition, say S, is winning, and the other coalition T is losing in the partition. In fact, the only partition of this type is the discrete partition with $\left(\{1\}\, ; \pi^d\right) \in E_w(v)$. Since the coalition $\pi(1)$ containing player 1 is winning in any partition consisting of two two-player coalitions, irrespective of which player joins player 1, players 2 and 3 are β-symmetric in v. A similar argument for player 4 shows that all players in $N \setminus \{1\}$ are β-symmetric in v. To see that player 1 is also β-symmetric with the other players in v, take player 1 and, for example, player 2, and notice that there is no partition with at least two coalitions that do not contain players 1 and 2, and where one of these two coalitions is winning. Therefore, players 1 and 2, and thus all players, are obviously β-symmetric in v. However, from the fact that $\{1\}$ is the unique winning coalition in the discrete partition, it immediately follows that the game is not α-symmetric.

If we require a majority game to be α-symmetric, then the equal weight vector used to prove Theorem 2, is the only one making the game precisely supportive.

Proposition 3 *A majority game is α-symmetric if and only if its supporting weight vector w is given by $w_i = \frac{1}{n}$ for all $i \in N$.*

As one can see, this result is clearly driven by the fact that, in an α-symmetric majority game, all largest coalitions in a partition are winning in it.

5 Almost symmetric games

We have defined decisive plurality games as games where there is exactly one winning coalition in each partition. Since the coalition consisting of player 1 is supposed to be the unique winner in the discrete partition, we have then that no decisive plurality game is α-symmetric. In this section, we restrict the set of players who are α-symmetric (β-symmetric) in a decisive plurality game and study the impact of this restriction on the game's precise support.

More precisely, we call a decisive plurality game v *almost α-symmetric* if all players but one are α-symmetric in v. Clearly then, due to the decisiveness of v, the following result is immediate.

Proposition 4 *Let v be a decisive plurality game with at least three players. If v is almost α-symmetric, then all players in $N \setminus \{1\}$ are α-symmetric in v.*

Recall from Proposition 2, that for the case of majority games, β-symmetry is implied by α-symmetry. Let us now define a decisive plurality game v to be *almost β-symmetric* if all players but player 1 are β-symmetric in v, and show that *almost α-symmetry and almost β-symmetry are independent properties.*

For this, let us first show that almost α-symmetry does not imply almost β-symmetry. To see this, consider the six-player almost α-symmetric decisive game v with player set $N = \{1, i, j, k, \ell, m\}$ and coalition $\pi(1)$ winning in every partition $\pi \in \mathcal{P}$, except in the partitions of type $\{\{1\}, \{i, j, k\}, \{\ell, m\}\}$, $\{\{1\}, \{i, j, k\}, \{\ell\}, \{m\}\}$, $\{\{1\}, \{i, j, k, \ell\}, \{m\}\}$, and $\{\{1\}, \{i, j, k, \ell, m\}\}$ where it is always the largest coalition in the partition that wins. The game is indeed almost α-symmetric since exchanging any two players from $N \setminus \{1\}$ does not change the size of the (corresponding) winning coalition in a partition. Suppose now that the game v is almost β-symmetric and consider for instance players ℓ and m. Then $1, \underline{ijk}, \ell m$ should imply that $1m, \underline{ijk\ell}$ which is in contradiction to the fact that $\{1, m\}$ wins in the partition $\{\{1, m\}, \{i, j, k, \ell\}\}$.

To show that almost β-symmetry does not imply almost α-symmetry, let us take the five-player decisive majority game v where, for every partition $\pi \in \mathcal{P}$ containing exactly two coalitions S and T of maximal size, we have that $1 \in S \cup T$ implies $(\pi(1); \pi) \in E_w(v)$, while $1 \notin S \cup T$ implies $(\pi(2); \pi) \in E_w(v)$. Recall additionally that, by supposition, $\left(\{1\}; \pi^d\right) \in E_w(v)$ holds. Obviously, players 3, 4 and 5 are β-symmetric in v. To show that players 2 and 3 are also β-symmetric in v, notice that the only partitions containing two coalitions that do not contain either of these two players, such that one of these coalitions is winning, are $1, \underline{45}, 2, 3$; $4, \underline{15}, 23$; $4, \underline{15}, 2, 3$; $5, \underline{14}, 23$; $5, \underline{14}, 2, 3$; and the discrete partition. By majority, the winning coalition in these partitions stays winning after either player 2 or player 3 joins, showing that players 2 and 3, and thus all players in $N \setminus \{1\}$ are β-symmetric in v; thus, the game is almost β-symmetric. Notice, however, that the game is not almost α-symmetric since, considering players 2 and 4, we have that $\{2,5\}$ wins in $\{\{2,5\},\{3,4\},\{1\}\}$ but $\{4,5\}$ loses in $\{\{2,3\},\{4,5\},\{1\}\}$.

In what follows, we call a decisive plurality game v *almost symmetric* if all players but player 1 are both α- and β-symmetric in v. Almost symmetric decisive plurality games turn out to be precisely supportive (Theorem 3). We show this by proving important implications (stated in Propositions 4-6) on the winning embedded coalitions in such games. In these results, we also explicitly state the type of almost symmetry (i.e., almost symmetry, almost α-symmetry, or almost β-symmetry) which is used in the corresponding proofs.

We start by showing that in almost α-symmetric games, a winning coalition in a partition either contains the winner (player 1) in the discrete partition, or has strictly more players than any other coalition (not containing player 1) in the partition.

Proposition 5 *Let v be a decisive and almost α-symmetric plurality game. Then $(S;\pi) \in E_w(v)$ implies either $S = \pi(1)$ or $|S| > |T|$ for each $T \in \pi \setminus \{S, \pi(1)\}$.*

This proposition gives as a corollary that in a partition having two or more coalitions of maximal size, the winning coalition should contain the winner in the discrete partition.

Corollary 1 *Let v be a decisive and almost α-symmetric plurality game, and let $\pi \in \mathcal{P}$. If $\pi \setminus \{\pi(1)\}$ contains at least two largest coalitions, then $(\pi(1);\pi) \in E_w(v)$.*

Define $\mathcal{P}_1 = \{\pi \in \mathcal{P} \mid \{1\} \in \pi\}$ as the collection of those partitions that contain singleton $\{1\}$. In what follows, we make use of the *power* $p_1(v)$ *of player* 1 in a decisive plurality game v, and define it as

$$p_1(v) = \max_{\pi \in \mathcal{P}_1} \{t \mid \exists T \in \pi \text{ with } |T| = t \text{ and } (\{1\};\pi) \in E_w(v)\},$$

that is, it is the *size of a largest coalition which loses against* $\{1\}$ in some partition from $\mathcal{P}_1$. We denote by $\mathcal{P}^*$ the set of all partitions in which there is only one largest coalition which does not contain player 1. For $\pi \in \mathcal{P}^*$, S_π stands for the largest coalition in $\pi \setminus \{\pi(1)\}$.

Our next result explains the crucial role of $p_1(v)$ in determining the winning coalitions in the partitions contained in $\mathcal{P}^*$.

Proposition 6 *Let v be a decisive and almost symmetric plurality game with at least five players, and let $\pi \in \mathcal{P}^*$. Then,*
(1) $|S_\pi| \geq p_1(v) + |\pi(1)|$ implies $(S_\pi; \pi) \in E_w(v)$;
(2) $|S_\pi| < p_1(v) + |\pi(1)|$ implies $(\pi(1); \pi) \in E_w(v)$.

Corollary 1 and Proposition 6 allow for a complete characterization of the winning embedded coalitions in almost symmetric games. For $\pi \in \mathcal{P}$, the coalition $\pi(1)$ is winning in π when either $\pi \setminus \{\pi(1)\}$ contains at least two largest coalitions, or the unique largest coalition in π is smaller than the threshold level $p_1(v) + |\pi(1)|$. In the remaining case, it is the unique largest coalition which wins in the partition π. Notice that, in the two cases where $\pi(1)$ is winning in π, a coalition $S \in \pi$ with $|S| > |\pi(1)|$ may exist; so, it is not necessarily that a largest coalition wins in a partition.

The value $p_1(v)$ is also very important for the proof of our main result in this section (Theorem 3). We show there that the weight vector w defined by $w_1 = \frac{p_1(v)}{p_1(v)+|N|-1}$ and $w_i = \frac{1}{p_1(v)+|N|-1}$ for each $i \in N \setminus \{1\}$ makes every decisive and almost symmetric game precisely supportive.

Theorem 3 *Every decisive and almost symmetric plurality game is precisely supportive.*

In order to shed light on how the results in this section lead to precise support, let us consider a decisive and almost symmetric game v with five players. Notice first that it is impossible for the power of player 1 in the game v to be $p_1(v) = 1$. The reason here is that $p_1(v) = 1$ would imply that in the partition $\{\{1\}, \{2,3\}, \{4,5\}\}$ either $\{2,3\}$ or $\{4,5\}$ should be winning, which is a contradiction to Corollary 1. So, $p_1(v) \in \{2,3,4\}$ should be the case.

If $p_1(v) = 4$, then $(\{1\}; \{\{1\}, \{2,3,4,5\}\}) \in E_w(v)$ and thus, by monotonicity, the winning coalition in any partition contains player 1. Clearly then, the weight vector $w = \left(\frac{1}{2}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}\right)$ makes v precisely supportive.

Suppose next that $p_1(v) = 3$ holds. Since $p_1(v) + |\pi(1)| \geq 4$, we have, by Corollary 1 and Proposition 6, that $\pi(1)$ is winning in all partitions except in $\{\{1\}, \{2,3,4,5\}\}$ (where $\{2,3,4,5\}$ is winning). It can be checked that the weight vector $w = \left(\frac{3}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}\right)$ makes the game v precisely supportive.

Finally, consider the case of $p_1(v) = 2$. Due to $p_1(v) + |\pi(1)| \geq 3$, we have, by Corollary 1 and Proposition 6, that $\pi(1)$ is winning in all partitions except in $\{\{1\}, \{i,j,k\}, \{\ell\}\}$ (where $\{i,j,k\}$ is winning) and $\{\{1\}, \{i,j,k,\ell\}\}$ (where $\{i,j,k,\ell\}$ is winning). Clearly then, the weight vector $w = \left(\frac{1}{3}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right)$ provides precise support for the game v.

Let us finally remark that decisive plurality games are always precisely supportive in case there are no externalities. We say that a plurality game v is *externality-free* if $v(S;\pi) = v(S;\pi')$ holds for all $S \subseteq N$ and for all partitions $\pi, \pi' \in \mathcal{P}$ such that $S \in \pi \cap \pi'$.

Proposition 7 *Every externality-free, decisive plurality game is precisely supportive.*

This proposition makes clear that problems with decisive plurality games being not precisely supportive arise from the existence of (negative) externalities.

6 Related literature

In our study of the question about assigning weights to players in simple partition function form games, we have singled out classes of games which do allow for a weighted representation. Moreover, the precise support of the corresponding games was shown to crucially shape the set of possible winning coalitions in a partition and thus, to shed light on which coalitions are most powerful in the presence of (negative) externalities. This naturally places our work within the strands of literature devoted to the numerical representation of standard simple games as well as to the study of general partition function form games.

The first strand of literature is mainly concerned with the question whether it is (always) possible to represent a standard simple game as a weighted majority game, that is, to find non-negative weights and a positive real number (quota) such that a coalition is winning in the simple game if and only if the combined weights of its members weakly exceeds the quota. As shown by von Neumann and Morgenstern (1944), not all simple games do allow for such weighted majority representation. The question of finding properties that characterize weighted games within the class of simple games was then naturally posted by Isbell (1956, 1958), and answered by Elgot (1961) and Taylor and Zwicker (1992). More precisely, Taylor and Zwicker (1992) characterize weighted voting in terms of the ways in which coalitions can gain or lose by trading among themselves, while Hammet et al. (1981) and Einy and Lehrer (1989) answer the above question by using results about separating convex sets. Peleg (1968) and Sudhölter (1996) show for the case of (constant-sum) weighted majority games that, correspondingly, the nucleolus and the modified nucleolus induce a representation of the game.

If a standard simple game does not allow for a weighted representation, then one might consider rough weights (cf. Taylor and Zwicker 1999). A simple game is roughly weighted if there exist weights and a threshold such that every coalition with the sum of its players' weights being above (respectively below) the threshold is winning (respectively losing). Again, not all standard simple games turn out to be roughly weighted. Gvozdeva and Slinko (2011) give necessary and sufficient conditions for a simple game to have rough weights. Related to the issue of non-weightedness of games, Carreras and Freixas (1996) and Freixas and Molinero (2009) investigate complete simple games that are simple games behaving in some respects as weighted simple games.

All the papers cited above deal with standard simple games, while the focus of our work is on the weighted representation of plurality games which we defined as special type of simple games in partition function form. To the best of our knowledge, we are the first to study and provide conditions assuring weighted representations of such games.

The second strand of related literature deals with general partition function form games as initiated by the seminal works of Thrall (1962) and Thrall and Lucas (1963), and recently surveyed by Kóczy (2018). Besides the investigation of general properties of such games (e.g., Lucas and Marcelli 1978, Maskin 2003, Hafalir 2007, Navarro 2007), there are two main issues of simultaneous interest in the corresponding works: which coalitions will form (cf. Ray 2007), and how the coalitional worths will be allocated to their members. For instance, de Clippel and Serrano (2008) separate the intrinsic payoffs from those due to the externalities of coalition formation. However, the main focus in that literature has been on extending the Shapley value for games with externalities (e.g., Myerson 1977, Gilboa and Lehrer 1991, Albizuri et al. 2005, Macho-Stadler et al. 2007, de Clippel and Serrano 2008, McQuillin 2009, Dutta et al. 2010, Grabisch and Funaki 2012) and on extending different power indices to the class of simple games with externalities (e.g., Bolger 1986, Alonso-Meijide et al. 2017, Alvarez Moros et al. 2017).

Although the study of power indices for simple partition function form games was out of the scope of this paper, we nevertheless used a notion of power (for the winner in the discrete partition) in order to derive our results on precise support for almost symmetric plurality games. In follow-up research, we intend to generalize this notion as to apply to each player, to investigate in detail its properties, and to axiomatically characterize it.

A Appendix: Proofs from Section 3

Proof of Proposition 1. Notice first that for $|N| \leq 3$ the assertion follows by $\{1\}$ winning in the discrete partition and the monotonicity of v. Suppose now that $|N| = 4$, i.e., $N = \{i, j, k, \ell\}$, and let there exist $(S; \pi) \in E$ with $(S; \pi) \in E_w(v)$, $1 \notin S$, and $|S| < |T|$ for some $T \in \pi$. We show that this leads to a contradiction. Since there are only four players, we have $|S| = 1$. Assume, w.l.o.g., that $(S; \pi) = (\{i\}; \{\{i\}, \{j\}, \{k, \ell\}\})$ with $i \neq 1$ holds. But then, by monotonicity of v, $(\{i\}; \{\{i\}, \{j\}, \{k\}, \{\ell\}\}) \in E_w(v)$ should hold as well, giving a contradiction to v being decisive and $\{i\} \neq \{1\}$. By a similar argument, it can be shown that $(S; \pi) \in E_w(v)$ cannot be of the form $\{\{i\}; \{\{i\}, \{j, k, \ell\}\}\}$ with $i \neq 1$. ■

Proof of Theorem 1. Let v be a decisive plurality game defined on the player set N and recall that, by assumption, player 1 wins in the discrete partition. For $|N| = 2$, the weight vector $w = (\frac{1}{2}, \frac{1}{2})$ makes v precisely supportive.

If $|N| = 3$, let us take $N = \{1, j, k\}$. By monotonicity of v, $(\{1, j\}; \{\{1, j\}, \{k\}\}) \in E_w(v)$ and $(\{1, k\}; \{\{1, k\}, \{j\}\}) \in E_w(v)$ holds. Take the weight vector w with $w_1 = \frac{1}{2}$ and $w_j = w_k = \frac{1}{4}$. Notice that it makes v precisely supportive independently of the fact whether coalition $\{1\}$ or coalition $\{j, k\}$ is winning in the partition $\{\{1\}, \{j, k\}\}$.

Finally, if $|N| = 4$ let us take $N = \{1, j, k, \ell\}$. Since player 1 is winning in the discrete partition, we have that if supportive weights w do exist, it must hold that $w_1 \geq w_j, w_k, w_\ell$. We distinguish the following cases with respect to bipartitions containing coalitions of size 2.

Case 1 (Player 1 is in no winning coalition in a bipartition of this type). Let v be such that $(\{k, \ell\}; \{\{1, j\}, \{k, \ell\}\}) \in E_w(v)$, $(\{j, \ell\}; \{\{1, k\}, \{j, \ell\}\}) \in E_w(v)$, and $(\{j, k\}; \{\{1, \ell\}, \{j, k\}\}) \in E_w(v)$. Take the vector $w = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$. Notice that it makes v precisely supportive for the winning coalitions in the above partitions. Further, by monotonicity, $(\{k, \ell\}; \{\{1\}, \{j\}, \{k, \ell\}\}) \in E_w(v)$, $(\{j, \ell\}; \{\{1\}, \{k\}, \{j, \ell\}\}) \in E_w(v)$, and $(\{j, k\}; \{\{1\}, \{\ell\}, \{j, k\}\}) \in E_w(v)$. Clearly, w is then a suitable weight vector for these embedded coalitions as well. Again by monotonicity, $(\{j, k, \ell\}; \{\{1\}, \{j, k, \ell\}\}) \in E_w(v)$ and weight vector w still works here. We conclude that w indeed makes v precisely supportive.

Case 2 (Player 1 is in the winning coalition of one bipartition of this type). Let v be such that $(\{1, j\}; \{\{1, j\}, \{k, \ell\}\}) \in E_w(v)$, $(\{j, \ell\}; \{\{1, k\}, \{j, \ell\}\}) \in$

$E_w(v)$, and $(\{j,k\}\,;\{\{1,\ell\}\,,\{j,k\}\}) \in E_w(v)$. The vector w with $w_1 = w_j = \frac{1}{3} > \frac{1}{6} = w_k = w_\ell$ works for these winning embedded coalitions. It can be checked that w is a suitable weight vector also in the embedded coalitions $(S;\{\{k,\ell\},\{1\}\,,\{j\}\})$, irrespective of the winning coalition $S \in \{\{k,\ell\},\{1\}\,,\{j\}\}$. The winning coalitions in the other partitions are determined by monotonicity of the game as displayed in Table 1. In this table, the single underlined coalitions are winning in the corresponding partitions by assumption and the double underlined coalitions are winning by monotonicity. A partition with no underlined coalition displays the fact that the assumptions do not exactly determine which coalition is winning.

$\underline{\underline{1jk\ell}}$	$\underline{1j}, k\ell$	$1, \ell, \underline{\underline{jk}}$
$\underline{\underline{1k\ell}}, j$	$\underline{\underline{1j}}, k, \ell$	$\underline{\underline{1k}}, j, \ell$
$\underline{\underline{1j\ell}}, k$	$1k, \underline{j\ell}$	$\underline{\underline{1\ell}}, j, k$
$\underline{\underline{1jk}}, \ell$	$1, k, \underline{\underline{j\ell}}$	$1, j, k\ell$
$\underline{\underline{jk\ell}}, 1$	$1\ell, \underline{jk}$	$\underline{1}, j, k, \ell$

Table 1: Proof of Theorem 1, Case 2.

Case 3 (Player 1 is in the winning coalition of two bipartitions of this type). Let v be such that $(\{1,j\}\,;\{\{1,j\}\,,\{k,\ell\}\}) \in E_w(v)$, $(\{1,k\}\,;\{\{1,k\}\,,\{j,\ell\}\}) \in E_w(v)$, and $(\{j,k\}\,;\{\{1,\ell\}\,,\{j,k\}\}) \in E_w(v)$. Take the weight vector w with $w_1 = w_j = w_k = \frac{1}{3} > 0 = w_\ell$. It can be checked that w is a suitable weight vector, also irrespective of the winning coalitions in the partitions $\{\{j,\ell\},\{1\}\,,\{k\}\}$ and $\{\{k,\ell\},\{1\},\{j\}\}$. The winning coalitions in the other partitions are determined by monotonicity of the game, see Table 2.

$\underline{\underline{1jk\ell}}$	$\underline{1j}, k\ell$	$1, \ell, \underline{\underline{jk}}$
$\underline{\underline{1k\ell}}, j$	$\underline{\underline{1j}}, k, \ell$	$\underline{\underline{1k}}, j, \ell$
$\underline{\underline{1j\ell}}, k$	$\underline{1k}, j\ell$	$\underline{\underline{1\ell}}, j, k$
$\underline{\underline{1jk}}, \ell$	$1, k, j\ell$	$1, j, k\ell$
$\underline{\underline{jk\ell}}, 1$	$1\ell, \underline{jk}$	$\underline{1}, j, k, \ell$

Table 2: Proof of Theorem 1, Case 3.

Case 4 (Player 1 is in the winning coalition of all (three) bipartitions of this type). Let v be such that $(\{1,j\}\,;\{\{1,j\}\,,\{k,\ell\}\}) \in E_w(v)$, $(\{1,k\}\,;\{\{1,k\}\,,\{j,\ell\}\}) \in E_w(v)$, and $(\{1,\ell\}\,;\{\{1,\ell\}\,,\{j,k\}\}) \in E_w(v)$. Two sub-cases have to be considered:

(4.1) $(\{j,k,\ell\};\{\{1\},\{j,k,\ell\}\}) \in E_w(v)$. Take the vector w with $w_1 = \frac{2}{5}$ and $w_j = w_k = w_\ell = \frac{1}{5}$. It can be checked that w is a suitable weight vector, also in the embedded coalitions with partitions $\{\{j,k\},\{1\},\{\ell\}\}$, $\{\{j,\ell\},\{1\},\{k\}\}$, and $\{\{k,\ell\},\{1\},\{j\}\}$, irrespective of the winning coalitions in these partitions; notice that, due to Proposition 1, there is no winning singleton in these partitions which differs from $\{1\}$. The winners in the other partitions are determined by monotonicity of the game, see Table 3.

$\underline{\underline{1jk\ell}}$	$\underline{1j}, k\ell$	$1, \ell, jk$
$\underline{\underline{1k\ell}}, j$	$\underline{\underline{1j}}, k, \ell$	$\underline{\underline{1k}}, j, \ell$
$\underline{\underline{1j\ell}}, k$	$\underline{1k}, j\ell$	$\underline{\underline{1\ell}}, j, k$
$\underline{\underline{1jk}}, \ell$	$1, k, j\ell$	$1, j, k\ell$
$\underline{jk\ell}, 1$	$\underline{1\ell}, jk$	$\underline{1}, j, k, \ell$

Table 3: Proof of Theorem 1, Case 4.1.

$\underline{\underline{1jk\ell}}$	$\underline{1j}, k\ell$	$\underline{\underline{1}}, \ell, jk$
$\underline{\underline{1k\ell}}, j$	$\underline{\underline{1j}}, k, \ell$	$\underline{\underline{1k}}, j, \ell$
$\underline{\underline{1j\ell}}, k$	$\underline{1k}, j\ell$	$\underline{\underline{1\ell}}, j, k$
$\underline{\underline{1jk}}, \ell$	$\underline{\underline{1}}, k, j\ell$	$\underline{\underline{1}}, j, k\ell$
$jk\ell, \underline{1}$	$\underline{1\ell}, jk$	$\underline{1}, j, k, \ell$

Table 4: Proof of Theorem 1, Case 4.2.

(4.2) $(\{1\};\{\{1\},\{j,k,\ell\}\}) \in E_w(v)$. Then, by monotonicity, in every partition the winning coalition is the one containing player 1. Take the weight vector w with $w_1 = 1 > 0 = w_j = w_k = w_\ell$. It can be checked that w is a suitable weight vector by just applying the monotonicity of the game, see Table 4. ■

B Appendix: Proofs from Section 4

Proof of Proposition 2. We first show that if in a majority game v all coalitions of maximal size in a partition are winning in it, then the game is α-symmetric. Suppose that, on the contrary, there were two players, i and

j, who are not α-symmetric in v. In such a case, there should be $S \subset N$ with $i \in S$ and $j \notin S$ such that $(S;\pi) \in E_w(v)$ but $S \setminus \{i\} \cup \{j\}$ is not winning in the partition $\left\{S \setminus \{i\} \cup \{j\}, \pi(j) \setminus \{j\} \cup \{i\}, \pi_{N\setminus(S\cup\pi(j))}\right\}$. Since S and $S \setminus \{i\} \cup \{j\}$ are of the same size, by v being a majority game, S is a largest coalition in π, and thus $S \setminus \{i\} \cup \{j\}$ is a largest coalition in $\left\{S \setminus \{i\} \cup \{j\}, \pi(j) \setminus \{j\} \cup \{i\}, \pi_{N\setminus(S\cup\pi(j))}\right\}$. By supposition, $(S \setminus \{i\}) \cup \{j\}$ should be winning in that partition, which gives a contradiction.

Let us show next that if a majority game v is α-symmetric, then all coalitions of maximal size in a partition are winning in it. Notice first that, by the definition of a majority game, a winning coalition in a partition should be of maximal size. We are left to show that all coalitions of maximal size are winning. For this, take $(S;\pi) \in E_w(v)$ and suppose that there is $T \in \pi \setminus \{S\}$ with $|T| = |S|$. Since the game is α-symmetric, we can (repeatedly) replace all players from $S \setminus T$ by those from $T \setminus S$ and conclude that $(T;\pi) \in E_w(v)$ should hold.

Finally, consider an α-symmetric majority game v and let us show that the game is also β-symmetric. Take $i, j \in N$, and let $\pi \in \mathcal{P}$ with $S, T \in \pi$ be such that $(S;\pi) \in E_w(v)$ and $i, j \notin S \cup T$. Consider then the partition $\pi' = \left\{S \cup \{i\}, T \cup \{j\}, \pi_{N\setminus(S\cup T\cup\{i,j\})}\right\}$. If S is the unique largest coalition in π, then $S \cup \{i\}$ will be the unique largest coalition also in π'. By the definition of a majority games, $(S \cup \{i\}; \pi') \in E_w(v)$ follows. Suppose now that S and T are two largest coalitions in π. In such a case, $S \cup \{i\}$ and $T \cup \{j\}$ will be two largest coalitions also in π'. Since the majority game v is α-symmetric, it follows from the above characterization of such games that all largest coalitions in π' are winning in it, i.e., $(S \cup \{i\}; \pi') \in E_w(v)$ again follows. We conclude that the game v is β-symmetric as well. ■

Proof of Proposition 3. By Theorem 2, we know that every majority game is precisely supportive, and obviously the game is α-symmetric if it is supported by a weight vector where all weights are equal. This shows the 'if' part.

To show the 'only if' part, suppose that an α-symmetric majority game v is precisely supportive by a weight vector w. Consider the discrete partition π^d and notice that, by Proposition 2, $(\{i\}; \pi^d) \in E_w(v)$ holds for each $i \in N$. Since the game v is precisely supportive, we have that the inequalities $w_k \geq w_\ell$ and $w_\ell \geq w_k$ hold for all $k, \ell \in N$. We conclude then that all weights should be equal, i.e., the weight vector is $w = \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)$. ■

C Appendix: Proofs from Section 5

Proof of Proposition 4. Since $\{1\}$ is the unique winning coalition in the discrete partition, player 1 cannot be α-symmetric in v with any other player due to the decisiveness of the game. By definition of almost α-symmetry of v, all players in $N \setminus \{1\}$ are α-symmetric in v. ■

Proof of Proposition 5. Take $(S;\pi) \in E_w(v)$ and suppose that $S \neq \pi(1)$. Notice that $|S| > 1$ should hold; otherwise, by monotonicity, the single player in S should be winning in the discrete partition, which is in contradiction to the decisiveness of v and $\{1\}$ being the winner in the discrete partition. We now have to show that $|T| \geq |S|$ for some $T \in \pi \setminus \{S, \pi(1)\}$, leads to a contradiction.

Suppose that such a coalition T exists. Take $T' \subseteq T$ with $|T'| = |S|$. Since the game is almost α-symmetric (and all players in $S \cup T$ are α-symmetric in v since both coalitions do not contain player 1), we can (repeatedly) replace all players from S by those from T' and conclude that $\left(T'; \left\{T', S \cup (T \setminus T'), \pi_{N \setminus (S \cup T)}\right\}\right) \in E_w(v)$. By monotonicity, $\left(T; \left\{T, S, \pi_{N \setminus (S \cup T)}\right\}\right) = (T;\pi) \in E_w(v)$ should hold as well, a contradiction to $(S;\pi) \in E_w(v)$ and the decisiveness of the game. ■

The following two lemmas will be used in the proofs of Proposition 6 and Theorem 3.

Lemma 1 *Let v be a decisive and almost α-symmetric plurality game with at least five players. Then $p_v(1) \in \left\{\frac{|N|-1}{2}, \ldots, |N|-1\right\}$ if $|N|$ is odd and $p_v(1) \in \left\{\frac{|N|-2}{2}, \ldots, |N|-1\right\}$ if $|N|$ is even.*

Proof of Lemma 1. Clearly, the largest coalition that could lose against $\{1\}$ in a partition is $N \setminus \{1\}$. Suppose first that $|N|$ is odd and thus, there exists a partition $\pi = \{\{1\}, S, T\}$ with $|S| = |T| = \frac{|N|-1}{2}$. By Corollary 1, $(\{1\};\pi) \in E_w(v)$ follows, and thus, $p_v(1) \geq \frac{|N|-1}{2}$ should hold. If $|N|$ is even, one can take the partition $\pi' = \{\{1\}, S', T', \{i\}\}$ with $|S'| = |T'| = \frac{|N|-2}{2}$ as to conclude again from Corollary 1 that $p_v(1) \geq \frac{|N|-2}{2}$ must be the case. ■

Lemma 2 *Let v be a decisive and almost α-symmetric plurality game with at least five players, and let $\pi \in \mathcal{P}^*$. Then $|S_\pi| \geq p_1(v)$ implies $|S| \leq p_1(v)$ for each $S \in \pi \setminus \{\pi(1), S_\pi\}$.*

Proof of Lemma 2. Suppose not, i.e., $|S_\pi| \geq p_1(v)$ and there exists $S \in$

$\pi \setminus \{\pi(1), S_\pi\}$ such that $|S| > p_1(v)$. If $|N|$ is odd, then $|S_\pi| + |S| > 2p_1(v) \geq |N| - 1$ where the second inequality follows from Lemma 1. Since $1 \notin S_\pi \cup S$, we have a contradiction. If $|N|$ is even, then $|S_\pi| + |S| > 2p_1(v) \geq |N| - 2$ where the second inequality again follows from Lemma 1. Since $1 \notin S_\pi \cup S$, $|S_\pi| + |S| > 2p_1(v)$ implies $p_1(v) = \frac{|N|-2}{2}$. Notice then that $|S_\pi| > |S| \geq p_1(v) + 1 = \frac{|N|}{2}$ holds with the first inequality following from $\pi \in \mathcal{P}^*$ and the second from $|S| > p_1(v)$. We have then $|S_\pi| + |S| > |N|$, a contradiction. ■

Proof of Proposition 6. We consider the two assertions separately.

(1) $|S_\pi| \geq p_1(v) + |\pi(1)|$ implies $(S_\pi; \pi) \in E_w(v)$.

Suppose that $|S_\pi| \geq p_1(v) + |\pi(1)|$. Let $S'_\pi \subseteq S_\pi$ contain exactly $p_1(v) + 1$ players. Consider the partition $\pi' = \left\{\{1\}, S'_\pi, \pi_{N \setminus (\pi(1) \cup S_\pi)}\right\} \cup \left\{\{r\}\right\}_{r \in (\pi(1) \setminus \{1\}) \cup (S_\pi \setminus S'_\pi)}$, and notice that $(\pi'(1); \pi') = (\{1\}; \pi') \in E_w(v)$ would imply, by decisiveness of v, that S'_π loses against $\{1\}$ in π'. By $|S'_\pi| = p_1(v) + 1 > p_1(v)$, we have a contradiction to the definition of $p_1(v)$. So, $(\{1\}; \pi') \notin E_w(v)$. By Lemma 2, each coalition in $\pi_{N \setminus (\pi(1) \cup S_\pi)}$ is of size at most $p_1(v)$, implying that S'_π is the unique largest coalition in π'. We then have, by Proposition 5, that $(S'_\pi, \pi') \in E_w(v)$ holds.

Let S''_π contain S'_π and $|\pi(1)| - 1$ other members of S_π. Having in mind that $|S_\pi \setminus S'_\pi| = |S_\pi| - p_1(v) - 1 \geq |\pi(1)| - 1$ is satisfied, it follows from applying β-symmetry $(|\pi(1)| - 1)$-times and monotonicity that S''_π is winning in the partition $\pi'' = \left\{S''_\pi, \pi(1), \pi_{N \setminus (\pi(1) \cup S_\pi)}\right\} \cup \left\{\{r\}\right\}_{r \in S_\pi \setminus S''_\pi}$. Further, by monotonicity, $(S_\pi; \pi) \in E_w(v)$ showing that assertion (1) holds.

(2) $|S_\pi| < p_1(v) + |\pi(1)|$ implies $(\pi(1); \pi) \in E_w(v)$.

Suppose that $|S_\pi| < p_1(v) + |\pi(1)|$. We split the proof in showing that $(\pi(1); \pi) \in E_w(v)$ follows when either $|S_\pi| \leq p_1(v)$ or $p_1(v) < |S_\pi| < p_1(v) + |\pi(1)|$ holds.

(2.1) ($|S_\pi| \leq p_1(v)$ implies $(\pi(1); \pi) \in E_w(v)$). Suppose that $|S_\pi| \leq p_1(v)$. In view of Proposition 5, it is sufficient to show that $(S_\pi; \pi) \in E_w(v)$ leads to a contradiction. For this, notice that, by the monotonicity of v, S_π would be winning in the partition $\pi^* = \{S_\pi\} \cup \{\{i\}\}_{i \in N \setminus S_\pi}$. Let π' be a partition with respect to which $p_v(1)$ was calculated, that is, there exists $T \in \pi'$ with $|T| = p_v(1)$ and $(\{1\}; \pi') \in E_w(v)$. Take $S \subseteq T$ with $|S| = |S_\pi|$ (such a coalition S does exist due to $|T| = p_v(1) \geq |S_\pi|$), and let $\pi'' = \{S\} \cup \{\{i\}\}_{i \in N \setminus S}$ be the partition containing S with all other players being single. By monotonicity, $(\{1\}; \pi'') \in E_w(v)$ since $\{1\}$ is winning in π'. Due to $1 \notin T \supseteq S$ and $1 \notin S_\pi$, the players in $S_\pi \cup S$ are α-symmetric in v, and we can exchange the players from S_π by those from S (in the partition π^*) as to conclude that S should be the winning coalition in π''. This gives a contradiction to $(\{1\};\pi'') \in E_w(v)$ and the decisiveness of v. Therefore, $(S_\pi;\pi) \notin E_w(v)$, and by Proposition 5, $(\pi(1);\pi) \in E_w(v)$.

(2.2) ($p_1(v) < |S_\pi| < p_1(v) + |\pi(1)|$ implies $(\pi(1),\pi) \in E_w(v)$). Suppose that $p_1(v) < |S_\pi| < p_1(v)+|\pi(1)|$. Let $S'_\pi \subset S_\pi$ contain exactly $p_1(v)$ players. Consider the partition $\pi' = \left\{\{1\}, S'_\pi, \pi_{N\setminus(\pi(1)\cup S_\pi)}\right\} \cup \{\{r\}\}_{r\in(\pi(1)\setminus\{1\})\cup(S_\pi\setminus S'_\pi)}$, and notice that, by Lemma 2, each coalition in $\pi_{N\setminus(\pi(1)\cup S_\pi)}$ is of size at most $p_1(v)$. The latter fact implies that the winning coalition in π' is either $\{1\}$ (if there is a coalition in $\pi_{N\setminus(\pi(1)\cup S_\pi)}$ of size $p_1(v)$ and by Corollary 1) or S'_π (if the size of each coalition in $\pi_{N\setminus(\pi(1)\cup S_\pi)}$ is less than $p_1(v)$ and by Proposition 5). Let us now show that $(S'_\pi;\pi') \in E_w(v)$ leads to a contradiction and thus, $(\{1\};\pi') \in E_w(v)$ should follow.

To get to a contradiction, suppose that $(S'_\pi;\pi') \in E_w(v)$. Consider the partition $\pi'' = \{\{1\}, S'_\pi\} \cup \{\{r\}\}_{r\in N\setminus(\{1\}\cup S'_\pi)}$ and notice that, by the monotonicity of v, $(S'_\pi;\pi'') \in E_w(v)$. Let π''' be a partition with respect to which $p_v(1)$ was calculated, that is, there exists $T \in \pi'''$ with $|T| = p_v(1)$ and $(\{1\};\pi''') \in E_w(v)$. By monotonicity, $(\{1\};\pi^{iv}) \in E_w(v)$, where π^{iv} is the partition containing T with everyone else being single. Since $|S'_\pi| = |T| = p_v(1)$, and all players in $S'_\pi \cup T$ are α-symmetric in v, in the partition π'', we can replace all players from S'_π by those from T as to conclude that T should be winning in the partition containing it with everyone else being single. Thus, we have a contradiction to $(\{1\};\pi^{iv}) \in E_w(v)$ and hence $(S'_\pi;\pi') \notin E_w(v)$, implying that $(\{1\};\pi') \in E_w(v)$.

Having in mind that $|S_\pi| < p_1(v)+|\pi(1)|$ implies $|S_\pi| \leq p_1(v)+|\pi(1)|-1$ and thus, $|\pi(1)|-1 \geq |S_\pi|-p_1(v) = |S_\pi \setminus S'_\pi| > 0$, it follows from monotonicity and applying β-symmetry $|S_\pi \setminus S'_\pi|$-times that Q^* is winning in the partition $\pi^* = \left\{Q^*, S_\pi, \pi_{N\setminus(\pi(1)\cup S_\pi)}\right\} \cup \{\{r\}\}_{r\in\pi(1)\setminus Q^*}$, where Q^* contains player 1 and $|S_\pi \setminus S'_\pi|$ other members of $\pi(1)$. By monotonicity, $(\pi(1);\pi) \in E_w(v)$ then holds. ■

Proof of Theorem 3. Let v be a decisive and almost symmetric plurality game defined on the player set N, and let us take the weight vector $w \in X^N_+$ defined by $w_1 = \frac{p_1(v)}{p_1(v)+|N|-1}$ and $w_i = \frac{1}{p_1(v)+|N|-1}$ for each $i \in N\setminus\{1\}$. In order to show that w is precisely supporting the game v, we explicitly consider the two cases in which N contains either at most four players (Case A)[1] or at

[1]Recall that, by Theorem 1, every decisive small game is precisely supportive. What we show in Case A is that every decisive *and almost symmetric* small game is precisely

least five players (Case B).

Case A ($|N| \leq 4$): For $|N| = 2$, take $N = \{1, 2\}$ and notice that v is then defined by $v(\{1\}; \{\{1\}, \{2\}\}) = 1$ and $v(\{1, 2\}; \{\{1, 2\}\}) = 1$. We have $p_1(v) = 1$ and, by using the above construction of the weight vector, $w = \left(\frac{1}{2}, \frac{1}{2}\right)$. Clearly, w makes the game precisely supportive.

If $|N| = 3$, take $N = \{1, 2, 3\}$. Recall that $\{1\}$ is winning in the discrete partition and, due to monotonicity, $\pi(1)$ is winning in all partitions $\pi \in \mathcal{P}$ with $|\pi(1)| \geq 2$. If $\{1\}$ additionally wins also in the partition $\{\{1\}, \{2, 3\}\}$, then $p_1(v) = 2$ and we get $w = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right)$. In case that $\{2, 3\}$ wins in $\{\{1\}, \{2, 3\}\}$, then $p_1(v) = 1$ and $w = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$. It can be checked that, in either case, the constructed weights assure precise support of the corresponding game.

Suppose finally that $|N| = 4$ and let us take $N = \{1, j, k, \ell\}$. By construction, $\{1\}$ is winning in the discrete partition. Further, by monotonicity and almost (β-)symmetry, $\pi(1)$ is winning in all partitions $\pi \in \mathcal{P}$ with $|\pi(1)| \geq 2$. Consider now partitions of the type $\{\{1\}, \{j, k\}, \{\ell\}\}$ and notice that, by decisiveness, monotonicity, and $\{1\}$ being wining in the discrete partition, it is impossible that $\{\ell\}$ wins in $\{\{1\}, \{j, k\}, \{\ell\}\}$. By almost ($\alpha$-)symmetry, if $\{j, k\}$ wins in that partition, it is also winning in all partitions of the same type. In such a case, monotonicity requires $\{j, k, \ell\}$ to win in $\{\{1\}, \{j, k, \ell\}\}$. We get then $p_1(v) = 1$ and $w = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$. If it is player 1 who wins in all partitions of type $\{\{1\}, \{j, k\}, \{\ell\}\}$, then we get $p_1(v) = 3$ (if $\{1\}$ wins in $\{\{1\}, \{j, k, \ell\}\}$) or $p_1(v) = 2$ (if $\{j, k, \ell\}$ wins in $\{\{1\}, \{j, k, \ell\}\}$). The corresponding weight vectors making the games precisely supportive are then $\left(\frac{3}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right)$ and $\left(\frac{2}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}\right)$.

Case B ($|N| \geq 5$): Take $\pi \in \mathcal{P}$ and notice that in view of Proposition 5 and by the fact that each player in $N \setminus \{1\}$ has the same weight according to w, it suffices to compare the weights of the coalitions $\pi(1)$ and a largest coalition in $\pi \setminus \{\pi(1)\}$. There are only two possible cases for the partition π: either $\pi \in \mathcal{P}^*$ or $\pi \in \mathcal{P} \setminus \mathcal{P}^*$.

Case B.1 ($\pi \in \mathcal{P}^*$): If $p_1(v) + |\pi(1)| > |S_\pi|$, then $(\pi(1); \pi) \in E_w(v)$ follows from Proposition 6. We have then

$$w(\pi(1)) = \frac{p_1(v) + |\pi(1)| - 1}{p_1(v) + |N| - 1} \geq \frac{|S_\pi|}{p_1(v) + |N| - 1} = w(S_\pi).$$

supported by the weight vector w *specified above.*

On the other hand, if $|S_\pi| \geq p_1(v) + |\pi(1)|$, then we have by Proposition 6 that $(S_\pi; \pi) \in E_w(v)$ and thus,

$$w(S_\pi) = \frac{|S_\pi|}{p_1(v) + |N| - 1} > \frac{p_1(v) + |\pi(1)| - 1}{p_1(v) + |N| - 1} = w(\pi(1)).$$

In either case, v is precisely supported by w.

Case B.2 ($\pi \in \mathcal{P} \setminus \mathcal{P}^*$): By Corollary 1, $(\pi(1); \pi) \in E_w(v)$ holds. Let S be a largest coalition in $\pi \setminus \{\pi(1)\}$ and consider the following possibilities.

(B2.1) ($|N|$ and $|N \setminus \pi(1)|$ are even numbers): We have in this case $|S| \leq \frac{|N|-|\pi(1)|}{2}$ and thus, $w(S) \leq \frac{|N|-|\pi(1)|}{2(p_1(v)+|N|-1)}$. On the other hand,

$$\begin{aligned} w(\pi(1)) &= \frac{p_1(v) + |\pi(1)| - 1}{p_1(v) + |N| - 1} \geq \frac{\frac{|N|-2}{2} + |\pi(1)| - 1}{p_1(v) + |N| - 1} \\ &= \frac{|N| - 4 + 2\,|\pi(1)|}{2\,(p_1(v) + |N| - 1)}, \end{aligned}$$

where the inequality follows due to Lemma 1. Notice further that $|N| - |\pi(1)| > |N| - 4 + 2\,|\pi(1)|$ would be fulfilled only if $|\pi(1)| = 1$, which is in contradiction to both $|N|$ and $|N \setminus \pi(1)|$ being even numbers. We conclude that $|N| - 4 + 2\,|\pi(1)| \geq |N| - |\pi(1)|$, and thus, $w(\pi(1)) \geq w(S)$.

(B2.2) ($|N|$ and $|N \setminus \pi(1)|$ are odd numbers): We have in this case $|\pi(1)| \geq 2$ and $|S| \leq \frac{|N|-|\pi(1)|-1}{2}$, and thus, $w(S) \leq \frac{|N|-|\pi(1)|-1}{2(p_1(v)+|N|-1)}$. On the other hand,

$$\begin{aligned} w(\pi(1)) &= \frac{p_1(v) + |\pi(1)| - 1}{p_1(v) + |N| - 1} \geq \frac{\frac{|N|-1}{2} + |\pi(1)| - 1}{p_1(v) + |N| - 1} \\ &= \frac{|N| - 3 + 2\,|\pi(1)|}{2\,(p_1(v) + |N| - 1)}, \end{aligned}$$

where the inequality follows due to Lemma 1. In this case, $|N| - 3 + 2\,|\pi(1)| > |N| - |\pi(1)| - 1$ always holds and thus, $w(\pi(1)) > w(S)$.

(B2.3) ($|N|$ is even and $|N \setminus \pi(1)|$ is odd): We have in this case $|S| \leq \frac{|N|-|\pi(1)|-1}{2}$ and thus, $w(S) \leq \frac{|N|-|\pi(1)|-1}{2(p_1(v)+|N|-1)}$. On the other hand,

$$\begin{aligned} w(\pi(1)) &= \frac{p_1(v) + |\pi(1)| - 1}{p_1(v) + |N| - 1} \geq \frac{\frac{|N|-2}{2} + |\pi(1)| - 1}{p_1(v) + |N| - 1} \\ &= \frac{|N| - 4 + 2\,|\pi(1)|}{2\,(p_1(v) + |N| - 1)}, \end{aligned}$$

where the inequality holds due to Lemma 1. From $|N| - 4 + 2|\pi(1)| \geq |N| - |\pi(1)| - 1$, we conclude that $w(\pi(1)) \geq w(S)$.

(B2.4) ($|N|$ is odd and $|N \setminus \pi(1)|$ is even): We have in this case $|S| \leq \frac{|N|-|\pi(1)|}{2}$ and thus, $w(S) \leq \frac{|N|-|\pi(1)|}{2(p_1(v)+|N|-1)}$. On the other hand,

$$\begin{aligned} w(\pi(1)) &= \frac{p_1(v) + |\pi(1)| - 1}{p_1(v) + |N| - 1} \geq \frac{\frac{|N|-1}{2} + |\pi(1)| - 1}{p_1(v) + |N| - 1} \\ &= \frac{|N| - 3 + 2|\pi(1)|}{2(p_1(v) + |N| - 1)}, \end{aligned}$$

where the inequality holds due to Lemma 1. In this case, $|N| - 3 + 2|\pi(1)| \geq |N| - |\pi(1)|$ holds, and thus, $w(\pi(1)) \geq w(S)$. ■

Proof of Proposition 7. Consider an externality-free decisive plurality game v with player 1 being the unique winner in the discrete partition. Since there are no externalities, we have that $\{1\}$ is the unique winning coalition in any partition containing $\{1\}$. But then, by monotonicity, $\pi(1)$ is winning in every partition $\pi \in \mathcal{P}$. Taking weights $w_1 = 1$, and $w_i = 0$ for all $i \in N \setminus \{1\}$, then shows that v is precisely supportive. ■

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