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Michel Grabisch, Agnieszka Rusinowska, Xavier Venel. Diffusion in countably infinite networks. 2019. halshs-02340011

HAL Id: halshs-02340011
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2019.17

Maison des Sciences Économiques, 106-112 boulevard de L'Hôpital, 75647 Paris Cedex 13
https://centredeconomiesorbonne.univ-paris1.fr/
ISSN : 1955-611X

Diffusion in countably infinite networks

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Version of July 24, 2019

Abstract

We investigate the phenomenon of diffusion in a countably infinite society of individuals interacting with their neighbors. At a given time, each individual is either active (i.e., has the status or opinion 1) or inactive (i.e., has the status or opinion 0). The configuration of the society describes active and inactive individuals. The diffusion mechanism is based on an aggregation function, which leads to a Markov process with an uncountable set of states, requiring the involvement of σ-fields. We focus on two types of aggregation functions – strict, and Boolean. We determine absorbing, transient, and irreducible sets under strict aggregation functions. We show that segregation of the society cannot happen, and its state evolves towards a mixture of infinitely many active and infinitely many inactive agents. In our analysis, we mainly focus on the network structure. We distinguish networks with a blinker (periodic class of period 2) and those without. ϕ-irreducibility is obtained at the price of a richness assumption of the network, meaning that it should contain infinitely many complex stars and have enough space for storing local configurations. When considering Boolean aggregation functions, the diffusion process becomes deterministic, and the contagion model of Morris (2000) can be seen as a particular case of our framework with aggregation functions. In this case, consensus and non trivial absorbing states as well as cycles can exist.

JEL Classification: C7, D7, D85
Subject Classification OR/MS: Networks/graphs; Probability: diffusion, Markov processes

Keywords: diffusion, countable network, aggregation function, absorbing set, transient set, irreducible set

1 Introduction

Studying diffusion in networks has a very strong and established position in the literature, with a variety of approaches and different ways of modeling diffusion. Adoption of a new technology, product purchasing and marketing, opinion formation, influence, social learning, information cascades and transmission, fashions, contagion and disease infection, financial contagion, among many other related phenomena, they are all relevant to network diffusion; for surveys on these issues see, e.g., two chapters in Jackson (2008) on diffusion through networks, and learning, and several chapters in Bramoullé et al. (2016) on diffusion, learning and contagion (e.g., Lamberson (2016), Golub and Sadler (2016), Cabrales et al. (2016)); see also e.g., Rogers (2003) for overviews of the literature on diffusion and its applications, and Silverman (2001) for "word-of-mouth marketing".

Our point of departure is the work of Morris (2000) on contagion, where contagion means a diffusion phenomenon occurring when one of two actions can spread from a finite set of individuals to the whole (countably infinite) population. The basic mechanism in Morris (2000) is a deterministic process, known under the name of *threshold model* (see, e.g., Granovetter (1978)), where an agent becomes infected (or *active*, as we will use this term throughout the paper) if the number of infected agents in its neighborhood exceeds a given threshold. To introduce some randomness in this mechanism and to make it more realistic, we replace the threshold mechanism by the following: an agent becomes active with some probability, which is a function of the status (active or inactive) of every agent in its neighborhood. The probability is equal to 1 (respectively, 0) if all agents in the neighborhood are active (respectively, inactive), and it is required that the probability monotonically increases with the number of active agents. Such a function determining the probability to be active is called an *aggregation function*. This relates the present paper with previous works of two of the authors that study in depth models of opinion dynamics based on aggregation functions (see Grabisch and Rusinowska (2013); Förster et al. (2013)).

The essential differences with our previous works are the following: 1) We assume that the society of agents is not any more finite but countably infinite, like in Morris (2000); 2) The underlying network is not any more complete but has some structure, defined by a neighborhood relation with very weak requirements, as introduced in Morris (2000). We elaborate on these two points. First, we argue that the countably infinite assumption is a natural approximation of the real world when speaking of contagion or diffusion phenomena. Indeed, diffusion never stops, does not have any border, but continues and continues. Considering, for instance, the spread of a disease in Paris, there is clearly no reason why it will not be transmitted to other French cities, then to its neighboring countries, Belgium, Luxembourg, Germany, Switzerland, Italy, Spain, and so on. Second, the neighborhood relation defined by Morris (2000) amounts to considering as network a connected graph where each agent has a bounded number of neighbors, so that almost any kind of network can be considered via this neighborhood relation.

The general aim of the paper is to study the evolution in the long run of the diffusion process, and to answer some basic questions like: Is a cascade possible, leading to a state where all agents are active, or all inactive? Is segregation possible, i.e., a stable state where a part of the population is active while the other is not? Are other remarkable phenomena, like cycles, possible? Our mechanism of diffusion is Markovian, and we

investigate all its absorbing and transient classes, which permits to give a precise answer to the above questions. However, the assumption on the number of agents (countably infinite) makes the number of possible states of the process to be *uncountably* infinite. This has a strong consequence on the analysis, as it requires the use of σ-fields and much more complex notions than classical ones, e.g., irreducibility.

Our study provides complete results in two cases of aggregation functions: strict, and Boolean. A strict aggregation function means that if any agent in the neighborhood of agent i becomes active, then the probability that i becomes active is strictly higher. A Boolean aggregation function simply means that the result is either 0 or 1. Then the model becomes deterministic and we are back to the initial model of Morris (2000).

We summarize our main findings. In the case of a strict aggregation function, we distinguish two types of networks: those with a blinker and those without. Roughly speaking, a blinker amounts to having a partition of the society in two blocks, one being active and the other inactive on odd time steps, and vice versa on even time steps (i.e., a blinker is a periodic class of period 2). Blinkers exist on many networks with a regular structure and low degree (e.g., the $\mathbb{Z}^2$ plane where each agent has 4 neighbors: north, east, south, west; the hexagonal pavement, regular hierarchies, etc.). Although their probability of occurence is zero, they are of utmost importance in the analysis as they divide the network in two parts, say the *even agents* and the *odd agents*. We show that for networks not having a blinker, there always exist a subnetwork admitting a blinker. As a consequence, the results obtained when a blinker exists carry over to the general case with few changes. Our main result gives the exhaustive list of all possible absorbing and transient ϕ-irreducible sets. Most notably, we find that: 1) The states where every agent is active or every agent is inactive are absorbing, which can result from a cascade effect; 2) Any state with a finite number of odd or even agents being active or inactive is necessarily transient, so that no segregation can occur, and the state of the society evolves towards a mixture of infinitely many active and infinitely many inactive agents, either odd or even. ϕ-irreducibility is proved at the price of an additional hypothesis on the structure of the network (*not* on the aggregation function, interestingly), which we call *richness*: it says that the network should contain infinitely many "complex stars" (roughly speaking, stars with at least 3 branches of length at least 2), and that the network has enough "space" to store a local configuration of active/inactive agents on its neighbors. Most networks satisfy this assumption and we characterize networks having complex stars.

As for Boolean aggregation functions, the results fairly differ. Indeed, while a cascade effect to the limit cases where all agents are active or all are inactive is still possible, there are many possibilities to reach other absorbing states (which is interpreted as segregation), or cycles, depending on the value of the threshold. A more detailed study depends essentially on the topology of the network, as we showed with $\mathbb{Z}^2$ and the neighborhoods of 4 and 8 neighbors.

The remainder of this paper is organized as follows. In Section 2 we introduce the main framework of countable networks and aggregation functions, informal and formal definitions of our diffusion process, and key concepts useful for the analysis of Markov chains with an uncountable set of states. We also present some first results that are independent of specific assumptions on the aggregation functions. Section 3 is dedicated to the study of deterministic configurations and the introduction of the notions of a

fixed point and a blinker. In Section 4 we focus on strict aggregation functions and networks with a blinker. We highlight some ϕ-irreducible classes, countable transient and absorbing sets, and we prove in Section 5 that under some additional assumptions on the network, the latter sets are also ϕ-irreducible. Section 6 is dedicated to the case of strict aggregation functions where the network admits no blinker. Section 7 is devoted to the study of Boolean aggregation functions. In Section 8, we mention briefly the case of nonstrict and nonBoolean aggregation functions, whose more detailed analysis is left for future research. In Section 9, some related literature on diffusion and social learning is recalled.

2 The framework

2.1 Countable networks

Let $\mathcal{X}$ be the set of agents (society) considered to be countably infinite. We consider on $\mathcal{X}$ a *neighborhood relation*: $x \sim y$ (x is neighbor of y) is a binary relation which is irreflexive, symmetric, *bounded* (each x has at most γ neighbors, where γ is a fixed constant), and *connected*. The latter means that for every $x, y \in \mathcal{X}$, there exists a finite path connecting x to y, i.e., there exist $x_1, \dots, x_k \in \mathcal{X}$ such that $x_1 = x$, $x_k = y$ and for each $i = 1, \dots, k-1$, $x_i \sim x_{i+1}$. To each agent x we associate its *neighborhood* $\Gamma^\sim(x)$ as the set of neighbors of x (if no confusion occurs, $\Gamma^\sim$ is simply denoted by Γ). Note that irreflexiveness means that $x \notin \Gamma(x)$. Moreover, we assume that for any x, we have an order on the neighbors in $\Gamma(x)$.

$\mathcal{X}$ together with the neighborhood relation $\sim$ define an undirected graph, denoted by $(\mathcal{X}, \sim)$, whose set of nodes is $\mathcal{X}$, and there is a link between x and y iff $x \sim y$. Such a graph, where $\sim$ is a neighborhood relation, is called a *(countable) network*. We give several examples of networks below, borrowed from Morris (2000).

Example 1 (**2-dimensional grid**)**.** Consider $\mathcal{X} = \mathbb{Z}^2$ and the neighborhood of x is defined as the set containing all agents within a certain distance of x, excluding x. Taking for example the Euclidean distance, the neighborhood of x within distance 1 is formed by the 4 agents at north, south, east and west position, i.e., $(x_1, x_2 + 1)$, $(x_1, x_2 - 1)$, $(x_1 + 1, x_2)$ and $(x_1 - 1, x_2)$ respectively. We call this the *1-neighborhood* of x. With distance $\sqrt{2}$, we get 8 neighbors (4 on the diagonals in addition to the previous ones). We may consider more general distances, not necessarily symmetric on both coordinates. Figure 1 shows the graphs corresponding to the 1-neighborhood and $\sqrt{2}$-neighborhood.

Example 2 (d-**dimensional grid**)**.** An immediate generalization of the previous example is to consider $\mathcal{X} = \mathbb{Z}^d$, with d any positive integer, and $\sim$ again defined by the Euclidean distance.

Example 3 (**hexagonal pavement**)**.** The hexagonal pavement is such that every node has 3 neighbors (see Figure 2).

Example 4 (**hierarchy**)**.** A hierarchy is a graph where each agent x has $m(x) \geq 1$ subordinates, and one superior, except the root which has no superior (see Figure 3). The numbers $m(x)$ may differ for each agent.

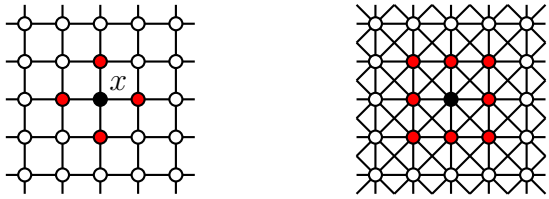


Figure 1: Graphs with $\mathcal{X} = \mathbb{Z}^2$ and $\sim$ corresponding to the Euclidean distance of 1 (left) and $\sqrt{2}$ (right). The neighborhood of x (black node) is the set of red nodes.

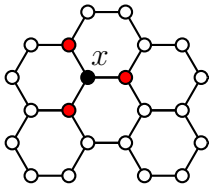


Figure 2: Graph of the hexagonal pavement

We assume that each agent can have one of two statuses 0 and 1, which can be interpreted in various ways (opinion on a given subject, adoption of a new technology, infection by some disease, etc.). We will call this value the *status* of agent x. As we are interested in this paper by the spread over $\mathcal{X}$ of one of the actions, we say that agent x is *active* if his status is 1 and that he is *inactive* if his status is 0.

At a given time, the society is composed of active and inactive agents. The *configuration* of the society describes who is active and who is inactive. Hence, a configuration can be represented either by a 0-1-valued function on $\mathcal{X}$ (0 for inactive, 1 for active), or by the set of active agents. The set of all possible configurations is $\Omega := \{0,1\}^{\mathcal{X}} \equiv 2^{\mathcal{X}}$. In the sequel, we will freely use one or the other representation, whichever is most convenient. We elaborate on both representations below.

In the function representation, a configuration is a function $\omega \in \Omega$, where $\omega(x) = 1$ if x is active, and $\omega(x) = 0$ if x is inactive. For every $X \subset \mathcal{X}$, we denote by π_X the projection from Ω to $\{0,1\}^X$. Every element of $\{0,1\}^X$ will be called a *partial configuration* restricted to X. Given a subset X', a *partial configuration in* X' is a

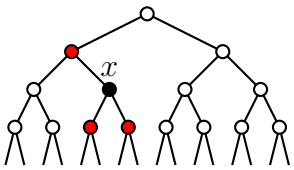


Figure 3: Graph of a hierarchy with $m(x) = 2$ for each agent x

partial configuration for a subset X of X'. A partial configuration $\theta \in \{0,1\}^X$ defines naturally the set $\theta^+ = \{\omega \in \Omega, \pi_X(\omega) = \theta\}$ of all configurations compatible with θ. This set will be called the *cylinder* generated by θ. Two particular configurations are the constant functions $\mathbf{1}, \mathbf{0}$, which correspond respectively to the cases where all agents are active and no agent is active. This representation is particularly convenient for defining formally the dynamical system.

In the set representation, the state of the society is represented by the set of active agents. We have clearly a one-to-one correspondence between the two formulations since any configuration ω is uniquely defined by the set X of active agents, with the correspondence $1_X \equiv \omega$. For example, $\mathbf{1}, \mathbf{0}$ correspond respectively to $\mathcal{X}$ and $\varnothing$. A partial configuration becomes a pair of disjoint sets (X, Y) such that agents in X are active and agents in Y are inactive. It naturally induces the cylinder $(X, Y)^+$ of configurations such that agents in X are active, agents in Y are inactive, and the status of agents outside $X \cup Y$ is not constrained. When X and Y are finite, we will say that the cylinder is *finite*. This second representation is particularly convenient when using the notion of neighborhood.

2.2 Informal definition of the diffusion process

We now describe how the diffusion evolves in the society. Given a natural number n, we denote by $[n]$ the set $\{1, \ldots, n\}$ and for every integer $\gamma \geq 1$, we denote by $[0,1]^{\leq \gamma} = \cup_{l=1}^{\gamma} [0,1]^l$.

We assume that each agent's status is randomly updated as a function of the statuses of its neighbors. An *aggregation function* is a mapping $A : [0,1]^{\leq \gamma} \to [0,1]$ which is nondecreasing w.r.t. each coordinate, and satisfies $A(1, \ldots, 1) = 1$, $A(0, \ldots, 0) = 0$. It is *symmetric* or *anonymous* if $A(z_1, \ldots, z_l) = A(z_{\sigma(1)}, \ldots, z_{\sigma(l)})$ for every $l \in [\gamma]$ and every permutation σ on $[l]$.

We distinguish 3 cases of aggregation functions:

1. A is strict: $A(z) = 0$ iff z is a 0 vector and $A(z) = 1$ iff z is a 1 vector.
2. A is 0-1-valued (Boolean aggregation function).
3. None of the above applies: A is nonstrict and nonBoolean, i.e., there exist $l \in [\gamma]$ and $z \in [0,1]^\gamma$ s.t. $0 < A(z) < 1$, and there exists $z' \neq 0$ s.t. $A(z') = 0$ or $z'' \neq 1$ s.t. $A(z'') = 1$.

We assume that the probability for an agent x to be active at time $t+1$, given the configuration ω at time t, is

$$P(x \mid \omega) = A(\pi_{\Gamma(x)}(\omega)), \tag{1}$$

where $A : [0,1]^{\leq \gamma} \to [0,1]$ is an aggregation function. In words, the probability for x to be active at the next stage is obtained by aggregating the vector of statuses of all the neighbors of x in the configuration ω. The more active agents are in the neighborhood,

the higher the probability. Moreover, x is active (respectively, inactive) for sure if all its neighbors (respectively, none of its neighbors) were active. We make the additional assumption that the probability for the agents to be active conditionally on ω is independent across the agents. We will see in the next section how to define rigorously this process.

From (1) and this informal definition, we can already highlight one specific aspect of the dynamic process. Using the set representation, we define for any set $X \subseteq \mathcal{X}$ its *closure* $\mathsf{clo}(X) = \overline{X}$ and *interior* $\mathsf{int}(X) = \mathring{X}$ by

$$\mathsf{clo}(X) = \overline{X} = \{x \in \mathcal{X} : \Gamma(x) \cap X \neq \varnothing\}$$
$$\mathsf{int}(X) = \mathring{X} = \{x \in \mathcal{X} : \Gamma(x) \subseteq X\}.$$

It is sometimes convenient to iterate the operator several times. We will define int^n (respectively, clo^n) to be the nth iteration of the operator int (respectively, clo) Accordingly, we use the same notation for configurations: $\overline{\omega}, \mathring{\omega}, \mathsf{clo}(\omega), \mathsf{int}(\omega)$. Obviously, $\mathring{X} \subseteq \overline{X}$, but it is not true in general that $\mathring{X} \subseteq X \subseteq \overline{X}$ (see Example 5 below). Also, it is easy to see that clo and int, viewed as mappings on $(2^{\mathcal{X}}, \subseteq)$, are monotone:

$$X \subset X' \Rightarrow \mathring{X} \subseteq \mathring{X}' \text{ and } \overline{X} \subseteq \overline{X}'.$$

With this new notation, we obtain:

$$x \in \mathring{X} \Rightarrow P(x \mid X) = 1, \quad x \notin \overline{X} \Rightarrow P(x \mid X) = 0. \tag{2}$$

The above implications become equivalences if and only if A is a strict aggregation function. This shows that, given that X is the set of active agents at time t, the set X' of active agents at time $t+1$ lies in the interval $[\mathring{X}, \overline{X}] := \{Y \in 2^{\mathcal{X}} \mid \mathring{X} \subseteq Y \subseteq \overline{X}\}$ with probability 1. Moreover, if the aggregation function is strict and X is finite, then X' can be *any* set in $[\mathring{X}, \overline{X}]$ with a positive probability.

Example 5. Consider $\mathcal{X} = \mathbb{Z}^2$ and the 1-neighborhood. Let ω be the configuration defined by $\omega(n, m) = 1$ iff $|n| + |m| = 1$, denoted by X in set representation (Figure 4 (left)). $\mathring{X}$ and $\overline{X}$ are given in Figure 4 (middle) and (right), respectively. One can see that $\mathring{X} \not\subseteq X \not\subseteq \overline{X}$, however $\mathring{X} \subseteq \overline{X}$.

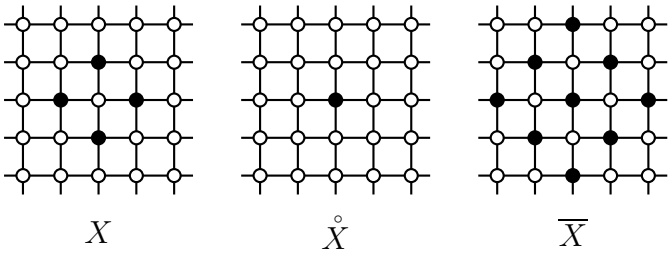


Figure 4: Interior and closure of a set X

2.3 Formal definition of the diffusion process

We define the diffusion process as a Markov process whose set of states is the set Ω of configurations, based on Equation (1) and the independence across agents. As Ω is uncountable, the definition of the Markov process requires to work on σ-fields (see, e.g., Hernández-Lerma and Lasserre (2003)).

We consider on Ω the product topology, i.e., the coarsest topology that makes the projections on each coordinate continuous. A basis of this topology is

$$\left\{\Pi_{i\in I}U_i, \text{ s.t. } U_i \subseteq \{0,1\} \text{ and there is only a finite number of } i \text{ s.t. } U_i \neq \{0,1\}\right\}$$

Reformulating the previous line with the definitions introduced in Subsection 2.1, the topology is generated by the finite cylinders. Since we consider the product of finite sets, the previous basis is countable. We consider on this set the Borelian σ-field denoted by $\mathcal{T}$. It is equal to the product σ-field since $\{0,1\}$ is finite (hence separable, Kallenberg (1997)). The set Ω with the product topology is usually called the Cantor set.

Let us now define a Markov Kernel on Ω, i.e., a mapping K from $\Omega \times \mathcal{T}$ to $[0,1]$ such that:

- For every $\omega \in \Omega$, $K(\omega, \cdot)$ is a probability on $\mathcal{T}$,
- For every $\mathcal{A} \in \mathcal{T}$, $\mathcal{K}(\cdot, \mathcal{A})$ is measurable.

Fix $\omega \in \Omega$. In order to define $K(\omega, \cdot)$, we first construct a family of probability distributions $(\mu_{Y,\omega})_{Y\subset\mathcal{X},Y\text{finite}}$ on partial configuration. Consider $h \in \{0,1\}^Y$, we set

$$\mu_{Y,\omega}(\{h\}) = \Pi_{y\in Y}\left(P(y \mid \omega)h(y) + (1 - P(y \mid \omega))(1 - h(y))\right), \tag{3}$$

where $P(y \mid \omega) = A(\pi_{\Gamma(y)}(\omega))$. Notice that by definition, the distribution over Y depends only on the projection of ω on $\overline{Y}$. For every Y, $\mu_{Y,\omega}$ is a measure on the Borel σ-field of Y (finite set). Moreover, this family of measures satisfies the assumption of Kolmogorov Extension Theorem (Aliprantis and Border, 2006, Corollary 15.27). It follows that there exists $K(\omega, \cdot)$, a probability distribution over Ω and the infinite product σ-field. By construction, it satisfies the first hypothesis of a kernel.

We now check that given $\mathcal{A} \in \mathcal{T}$, the function $K(\cdot, \mathcal{A})$ is measurable. We first consider a set $\mathcal{A}$ in the basis of $\mathcal{T}$. Hence, it is a finite cylinder, i.e., there exist two finite disjoint sets X, Y such that $\mathcal{A} = (X, Y)^+$. Then we know that $K(\omega, (X, Y)^+)$ depends only on the restriction of ω to agents in $\overline{X \cup Y}$, hence on a finite set of agents. By definition of the product topology, the function is continuous and therefore measurable. Let

$$\mathcal{H} = \{\mathcal{A} \in \mathcal{T}, \ \omega \to K(\omega, \mathcal{A}) \text{ is measurable}\} \subset \mathcal{T}$$

be the set of sets in $\mathcal{T}$ such that the mapping is measurable. Then $\mathcal{H}$ is a σ-field since:

- it is nonempty: $\varnothing \in \mathcal{H}$;
- it is stable under complementation since $K(\omega, \overline{\mathcal{A}}) = 1 - K(\omega, \mathcal{A})$;

- it is stable under countable unions: Let $(\mathcal{A}_n)_{n\in\mathbb{N}}$, then

$$K(\omega, \cup_{n\in\mathbb{N}}\mathcal{A}_n) = \sum_{n\in\mathbb{N}} K(\omega, \mathcal{A}_n).$$

The countable sum of measurable mappings is also measurable.

Since $\mathcal{H}$ contains the finite cylinders, it follows that $\mathcal{T} \subset \mathcal{H}$ and we have the equality: for every $\mathcal{A} \in \mathcal{T}$, the function $K(\cdot, \mathcal{A})$ is measurable.

We can now rewrite Equations (2) in terms of the Markov chain. Switching to the set notation, $(X, Y)^+$ is the interval $[X, Y^c]$, and with some abuse of notation we keep the same symbol for the σ-field $\mathcal{T}$. Given that X is the present configuration, the configuration at next time step lies in the interval $[\overset{\circ}{X}, \overline{X}]$ with probability 1:

$$K(X, \mathcal{A}) = 1 \text{ if } [\overset{\circ}{X}, \overline{X}] \subseteq \mathcal{A} \qquad (X \subseteq \mathcal{X}).$$

Observe that if X is finite, $[\overset{\circ}{X}, \overline{X}]$ can easily be seen to be an element of $\mathcal{T}$ since

$$[\overset{\circ}{X}, \overline{X}] = \bigcap_{x\notin\overline{X}} (\overset{\circ}{X}, \{x\})^+.$$

The same conclusion holds with infinite sets as well, so that any interval $[X, Y]$ is an element of $\mathcal{T}$, and so are the singletons $\{X\}$ for any $X \subseteq \mathcal{X}$.

Formula (3) permits to compute the kernel when everything is finite. Specifically, with X, Y finite, we have $K(X, \{Y\}) = \mu_{\overline{X},X}(\{Y\})$. We can also compute the kernel for every element of the basis. Supposing $\mathcal{A} = (X, Y)^+$, we have simply $K(Z, \mathcal{A}) = \mu_{X\cup Y,Z}(\{X\})$. By σ-additivity, we can deduce that $K(X, \mathcal{A}) = 0$ if $|X|$ is infinite and $|\mathcal{A} \cap [\overset{\circ}{X}, \overline{X}]|$ is finite.

2.4 Properties and first results

We first introduce some useful notions for the study of Markov chains with an uncountable set of states and present several results that do not depend on additional assumptions on the aggregation function or the neighborhood.

Definition 1. A set $\mathcal{A} \in \mathcal{T}$ is called *absorbing* if $K(\omega, \mathcal{A}) = 1$ for every $\omega \in \mathcal{A}$.

Proposition 1. $\{\mathbf{1}\}$ (everybody is active) and $\{\mathbf{0}\}$ (nobody is active) are absorbing sets.

Definition 2. A set $\mathcal{A} \in \mathcal{T}$ is called *transient* if for every configuration $\omega \in \mathcal{A}$ there exists $n \in \mathbb{N}$ such that $K^n(\omega, \mathcal{A}) < 1$.

Definition 3. Let ω be a configuration and $\mathcal{A} \in \mathcal{T}$ be a set of configurations. We say that $\mathcal{A}$ is *reachable* from ω if

$$\sum_{n\in\mathbb{N}} K^n(\omega, \mathcal{A}) > 0.$$

In order to prove that a set is reachable, we will mainly rely on the following definition of a trajectory. Indeed, the existence of a trajectory from ω to $\mathcal{A}$ is equivalent to the fact that $\mathcal{A}$ is reachable from ω.

Definition 4. A *trajectory* from ω to $\mathcal{A}$ is a finite sequence of sets of configurations $(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ such that

- $\mathcal{A}_1 = \{\omega\}$,
- $\mathcal{A}_n = \mathcal{A}$,
- for every $1 \leq l \leq n$, $\forall m \in \mathcal{A}_l$, $K(m, \mathcal{A}_{l+1}) > 0$.

We will often apply Definition 4 such that every set $\mathcal{A}_i$ is a cylinder generated by a partial configuration (X_i, Y_i). When doing so, it will be easier to speak directly about a *trajectory of partial configurations*.

Definition 5. Given ϕ a non-trivial σ-finite measure on $\mathcal{T}$, the Markov chain is ϕ-*irreducible* if

$$\sum_{n=1}^{+\infty} K^n(\omega, \mathcal{A}) > 0, \forall \omega \in \Omega \text{ whenever } \phi(\mathcal{A}) > 0.$$

In Proposition 1, we saw that the singletons $\{\mathbf{0}\}$ and $\{\mathbf{1}\}$ are absorbing sets. This implies that the diffusion Markov process K is not irreducible.

Lemma 1. For any non-trivial σ-finite measure ϕ on $\mathcal{T}$, K is not ϕ-irreducible.

Proof. We prove the result by contradiction. Let ϕ be such that the Markov chain is ϕ-irreducible. Since $\{\mathbf{0}\}$ is absorbing, for every $\mathcal{A} \in \mathcal{T}$ such that $\mathbf{0} \notin \mathcal{A}$, we need $\phi(\mathcal{A}) = 0$. Similarly if $\mathbf{1} \notin \mathcal{A}$, we need $\phi(\mathcal{A}) = 0$. ϕ is non-trivial, hence there exists $\mathcal{A}^*$ such that $\phi(\mathcal{A}^*) \neq 0$. It follows that $\mathcal{A}^*$ contains $\mathbf{0}$ and $\mathbf{1}$. We now consider the following partition of $\mathcal{A}^*$: $\{\mathbf{0}\}$, $\{\mathbf{1}\}$ and $\mathcal{A}^* \setminus \{\mathbf{0}, \mathbf{1}\}$. Then $\phi(\mathcal{A}^* \setminus \{\mathbf{0}, \mathbf{1}\}) = 0$ and therefore either $\{\mathbf{0}\}$ or $\{\mathbf{1}\}$ has a strictly positive measure under ϕ, which contradicts the fact that a set without $\mathbf{0}$ or $\mathbf{1}$ has a zero measure. □

Since K is not ϕ-irreducible on $\mathcal{T}$, we may look for sub-fields of $\mathcal{T}$ where K is ϕ-irreducible. We introduce for any $\mathcal{A} \in \mathcal{T}$ the *trace* of $\mathcal{T}$ on $\mathcal{A}$, defined by $\mathcal{T}_{|\mathcal{A}} = \{\mathcal{B} \in \mathcal{T} : \mathcal{B} \subseteq \mathcal{A}\}$, and $\phi_{\mathcal{A}}$ the conditional probability on A.

Definition 6. A set of configurations $\mathcal{A} \in \mathcal{T}$ is a ϕ-*irreducible set* if for every $\omega \in \mathcal{A}$, every $\mathcal{B} \in \mathcal{T}_{|\mathcal{A}}$ such that $\phi_{\mathcal{A}}(\mathcal{B}) > 0$, $K^n(\omega, \mathcal{B}) > 0$ for some n.

Remark 1. If $\phi(\mathcal{A}) > 0$, then the condition $\phi_{\mathcal{A}}(\mathcal{B}) > 0$ is equivalent to $\phi(\mathcal{B}) > 0$. But when $\phi(\mathcal{A}) = 0$, the condition is a non-trivial condition whereas $\phi(\mathcal{B}) > 0$ is never true.

Definition 7. A set of configurations $\mathcal{A} \in \mathcal{T}$ is a ϕ-*irreducible class* if it is both absorbing and a ϕ-irreducible set.

Proposition 2. $\{\mathbf{1}\}$ and $\{\mathbf{0}\}$ are finite ϕ-irreducible classes for any ϕ.

3 Deterministic configurations

Before considering the set of all configurations, it is interesting to focus on configurations such that the transition is deterministic, i.e., sets X such that $\mathring{X} = \overline{X}$. We distinguish two cases: $\mathring{X} = \overline{X} = X$ (then X is called a *fixed point*), or $\mathring{X} = \overline{X} \neq X$ (then X is called a *blinker*, for reasons that will become clear later).

Proposition 3. The following holds:

1. Consider X such that $\overline{X} = X$. Then X is a fixed point.
2. There is no other fixed point than $\mathcal{X}$ and $\varnothing$.

Proof. 1. Consider X such that $\overline{X} = X$. Take $x \in X$. We claim that $\Gamma(x) \subseteq X$, proving that $\mathring{X} = X$, and therefore X is a fixed point. Indeed, suppose not: there exists $y \in \Gamma(x)$ with $y \notin X$. Since $\sim$ is symmetric, $\Gamma(y) \ni x$, so that $y \in \overline{X} \setminus X$, a contradiction.

2. Let X be a fixed point and suppose that $X \neq \varnothing, \mathcal{X}$. Then there exists $x \in X$ and $y \notin X$. By connectedness, there exists a finite path $x_1 = x, x_2, \ldots, x_k = y$ connecting x to y. However, as $X = \mathring{X}$, $\Gamma(x_1) \subseteq X$, and consequently $\Gamma(x_2) \subseteq X$, $\Gamma(x_3) \subseteq X$, etc., till $\Gamma(x_{k-1}) \subseteq X$. However, $y = x_k \in \Gamma(x_{k-1})$, a contradiction. □

In particular, we obtain that $\{\mathbf{0}\}$ and $\{\mathbf{1}\}$ are the only absorbing singletons.

Proposition 4. Consider X such that $\mathring{X} = \overline{X} \neq X$. The following holds:

1. X^c has the same property.
2. $\mathring{X} = X^c$

Proof. We first observe that $\mathring{X} = \overline{X} \neq X$ is equivalent to the two conditions:

- Any $x \in \mathcal{X}$ has all its neighbors in X or none of them. $(*)$
- Either there exists $x \notin X$ which has (all) neighbors in X or there exists $x \in X$ with no neighbor in X. $(**)$

1. The two conditions $(*)$ and $(**)$ are invariant to the change $X \to X^c$.
2. Assume by $(**)$ that there exists $x \in X$ with no neighbor in X (the other case works by $(*)$). We claim that $X \cap \overline{X} = \varnothing$. Indeed, suppose not and consider $y \in X \cap \overline{X}$. By connectedness, there exists a finite path $x_1 = x, x_2, \ldots, x_k = y$ connecting x to y. However, $x_2 \in \Gamma(x_1)$, therefore $x_2 \in X^c$. By symmetry of $\sim$, $\Gamma(x_2) \ni x_1$, which by $(*)$ implies that $\Gamma(x_2) \subseteq X$, i.e., $x_2 \in \overline{X}$. Therefore, $x_3 \in X$, but $x_3 \notin \overline{X}$ because by symmetry $X^c \ni x_2 \in \Gamma(x_3)$. Hence, x_ℓ with odd index ℓ belongs to $X \cap (\overline{X})^c$ while those with even indices belong to $X^c \cap \overline{X}$. Therefore, y can never be reached, a contradiction.

   By the same argument, it follows that $X^c \cap (\overline{X})^c = \varnothing$. Consequently, $X^c = \overline{X}$.

□

A direct consequence of Proposition 4 is that if X is a blinker, $\{X, X^c\}$ is a periodic absorbing class of period 2, which explains the name "blinker". Moreover, if there exists a blinker, it is unique up to complementation.

Lemma 2. If a blinker exists, it is unique up to complementation.

The proof below is constructive.

Proof. We construct a set X by the following algorithm.

0. Choose $x \in \mathfrak{X}$ and set $X = X_0 = \{x\}$, $Y = Y_0 = \varnothing$.
1. For any $x \in X_0$, compute $\Gamma(x)$, set $Y_0 = \bigcup_{x \in X_0} \Gamma(x)$ and $Y \leftarrow Y \cup Y_0$.
2. If $Y \cap X \neq \varnothing$, STOP: there is no blinker. Otherwise, for any $y \in Y_0$, compute $\Gamma(y)$, set $X_0 = \bigcup_{y \in Y_0} \Gamma(y)$ and $X \leftarrow X \cup X_0$.
3. If $Y \cap X \neq \varnothing$, STOP: there is no blinker. Otherwise, go to Step 1.

Observe that the construction of Y is in fact the construction of $\overline{X}$. After countably infinitely many steps, X is a blinker and $Y = X^c$. The blinker is unique up to complement because $\sim$ is connected: the starting point has no importance. □

Let us revisit the networks given in Examples 1 to 4 and examine if blinkers exist (the algorithm in the proof of Lemma 2 permits to do this). For $\mathbb{Z}^2$ with the 1-neighborhood, a blinker exists, formed by all nodes (n, m) such that $n + m$ is even (see Figure 5 (left)). However, no blinker exists with the $\sqrt{2}$-neighborhood, nor any other distance. The result can be extended to $\mathbb{Z}^d$ as well, for any positive integer d. A blinker also exists for the hexagonal pavement (1 node out of 2 on each hexagon; see Figure 5 (middle)) and for the hierarchy (all nodes of odd-numbered layers; see Figure 5 (right)).

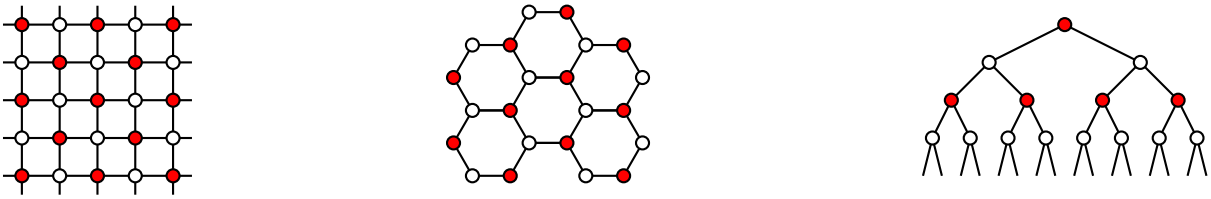


Figure 5: Blinkers for $\mathbb{Z}^2$ endowed with the 1-neighborhood (left), for the hexagonal pavement (middle), and for the hierarchy (right)

An important consequence of Lemma 2 is that if a blinker X exists, we can make a partition of $\mathfrak{X}$ in two parts, corresponding to X and X^c. Let us denote this partition by $\{\mathfrak{X}_e, \mathfrak{X}_o\}$ which will be called respectively *even positions* and *odd positions*, recalling the example of $\mathbb{Z}^d$ and the 1-neighborhood. It is important to keep in mind that any $x \in \mathfrak{X}_e$ (resp., $\mathfrak{X}_o$) has its neighborhood contained in $\mathfrak{X}_o$ (resp., $\mathfrak{X}_e$). In particular, the status at even stages at even positions is independent of the status at odd stages at even positions.

4 Strict aggregation functions and networks with a blinker: Absorbing and transient sets

In this section, we focus on the case of strict aggregation functions and networks $(\mathcal{X}, \sim)$ with a blinker. A key element of the analysis is to define a partition of the set of configurations into blocks, according to the "number" of active/inactive nodes on odd and even positions.

4.1 Definition of a partition of Ω and main results

A block of the partition is characterized by a 4-uple (a, b, c, d), where a, b, c, d are the "numbers" of nodes respectively in $\mathcal{X}_e$ (even positions) with inactive status, in $\mathcal{X}_e$ with active status, in $\mathcal{X}_o$ with inactive status , and in $\mathcal{X}_o$ with active status . The "numbers" of nodes are coded by $0, F, \infty$, where 0 means that there is no node, F means that there are nodes but finitely many, and ∞ means that there are infinitely many nodes.

There are a priori $3^4 = 81$ possible blocks in the partition, but actually only 25 are nonempty. Indeed, if we restrict ourselves to even positions, either 0 or 1 has to appear infinitely often, therefore only 5 possible cases remain: $(0, \infty)$, (F, ∞), $(\infty, 0)$, (∞, F) and (∞, ∞). Since what happens on odd and on even positions are independent, there are 25 nonempty blocks in the partition.

For example, with this notation we have:

- $(\infty, 0, \infty, 0)$ is the singleton $\{\mathbf{0}\} = \{\varnothing\}$,
- $(0, \infty, 0, \infty)$ is the singleton $\{\mathbf{1}\} = \{\mathcal{X}\}$,
- $(0, \infty, \infty, 0)$ is the singleton $\{\mathcal{X}_e\}$ which is the configuration where even agents are active whereas odd agents are inactive,
- $(\infty, 0, 0, \infty)$ is the singleton $\{\mathcal{X}_o\}$ which is the configuration where even agents are inactive whereas odd agents are active.

We now establish the following theorem.

Theorem 1. The following sets of configurations are respectively:

(i) finite ϕ-irreducible classes:

- $(\infty, 0, \infty, 0)$,
- $(0, \infty, 0, \infty)$,
- $(0, \infty, \infty, 0) \cup (\infty, 0, 0, \infty)$.

(ii) countable absorbing sets:

- $(\infty, \infty, \infty, \infty)$,
- $(\infty, 0, \infty, \infty) \cup (\infty, \infty, \infty, 0)$,
- $(0, \infty, \infty, \infty) \cup (\infty, \infty, 0, \infty)$.

(iii) countable transient sets:

- (∞, F, ∞, F),
- (F, ∞, F, ∞),
- $(F, \infty, \infty, F) \cup (\infty, F, F, \infty)$,
- $(\infty, F, \infty, \infty) \cup (\infty, \infty, \infty, F)$,
- $(F, \infty, \infty, \infty) \cup (\infty, \infty, F, \infty)$,
- $(\infty, F, \infty, 0) \cup (\infty, 0, \infty, F)$,
- $(F, \infty, 0, \infty) \cup (0, \infty, F, \infty)$,
- $(0, \infty, \infty, F) \cup (\infty, F, 0, \infty)$,
- $(F, \infty, \infty, 0) \cup (\infty, 0, F, \infty)$.

Theorem 1.(i) reformulates and synthesizes previous results (Propositions 2, 3 and 4) thanks to this new notation. The rest of the section is devoted to the proof of Theorem 1.(ii) and Theorem 1.(iii). In Section 4.2, we focus on the "number" of inactive/active nodes on even/odd positions and prove that the sets containing only a null or infinite number of each element are absorbing. Then, in Section 4.3, we prove that the sets containing finitely many elements are transient.

4.2 Persistence of the "number" of elements

The following proposition states that the "number" of active agents on even (respectively odd) positions yields the same "number" of active agents on odd (respectively even) positions at the next stage. We establish the result for the case of active agents on even positions. The same result holds if active is replaced by inactive and/or $\mathcal{X}_e$ by $\mathcal{X}_o$.

Proposition 5. For every $\omega \in \Omega$, let ω' be a configuration randomly chosen according to the kernel $K(\omega, \cdot)$. Then:

- if $\{x \in \mathcal{X}_e \mid \omega(x) = 1\}$ is empty then ω' has almost surely no active agents on odd positions;
- if $\{x \in \mathcal{X}_e \mid \omega(x) = 1\}$ is finite then ω' has almost surely a finite (maybe zero) number of active agents on odd positions;
- if $\{x \in \mathcal{X}_e \mid \omega(x) = 1\}$ is infinite then ω' has almost surely an infinite number of active agents on odd positions.

Proof. Denote by $\Theta = \{x \in \mathcal{X}_e \mid \omega(x) = 1\}$ the set of active agents on an even position.

First, it is clear that if this set is empty, then every odd position is surrounded by only inactive agents and therefore is inactive with probability 1 at the next stage. Hence, there is almost surely no active node on odd positions in ω'.

Secondly, let x be an odd position. Since the aggregation function is strict, the probability that $\omega'(x)$ is equal to 1 is strictly positive, if and only if, there exists a neighbor x' of x such that $\omega(x') = 1$. The set of neighbors of odd positions is the set of even positions and each agent has at most γ neighbors, therefore almost surely

$$|\{x \in \mathcal{X}_o \mid \omega'(x) = 1\}| \leq \gamma |\{x \in \mathcal{X}_e \mid \omega(x) = 1\}|.$$

Hence, if the right-hand side is finite, so is the left-hand side.

Finally, we assume that $\{x \in \mathcal{X}_e \mid \omega(x) = 1\}$ is infinite. We compute the probability that ω' has a finite number of active nodes on odd states. If this is the case, it implies that there exists an infinite number of elements $x \in \Theta$ such that

$$\forall x' \in \Gamma(x),\ \omega'(x') = 0.$$

But by assumption, x' has at least one neighbor which is active, it follows that the probability that $\omega'(x') = 0$ is strictly smaller than 1. Since the probabilities are independent, it follows that the total probability of the event is 0. □

Theorem 1.(ii) is then an immediate consequence of the above proposition.

4.3 Transient set

We now prove that if there is a strictly positive but finite number of active agents at odd positions, then there is a positive probability to reach a configuration where there are no active agents anymore on odd positions. Similar results hold for inactive and/or for odd positions. This result can be expressed in terms of reachability.

Lemma 3. Let $X \subset \mathcal{X}_o$.

- If X is a finite set, then $\mathsf{int}(\mathsf{int}(X)) \subset X$.
- If X is a co-finite set, then $X \subset \mathsf{clo}(\mathsf{clo}(X)))$.

Proof. We prove the first result. The second one can be deduced immediately by complementation.

Let us prove the result by contradiction. First, we know that $\mathsf{int}(\mathsf{int}(X)) \subset \mathcal{X}_o$. Take $x \in \mathsf{int}(\mathsf{int}(X))$ and assume that $x \notin X$. By definition of $\mathcal{X}_o$ and $\mathcal{X}_e$, we know that $\mathsf{int}(X) \cap \mathcal{X}_o = \varnothing$, hence $\Gamma(x) \not\subseteq X$, i.e., there exists $y \in \Gamma(x)$ s.t. $y \notin X$. Since $x \in \mathsf{int}(\mathsf{int}(X))$, $\Gamma(x) \subseteq \mathsf{int}(X)$ and therefore $y \in \mathsf{int}(X)$. It follows that $\Gamma(y) \subseteq X$. However, by symmetry of $\sim$, we have $x \in \Gamma(y)$, so $x \in X$, a contradiction. This proves that $\mathsf{int}(\mathring{X}) \subseteq X$.

Suppose that for all $x \in X$, $x \in \mathsf{int}(\mathsf{int}(X))$. It follows that for all $x \in X$, $\Gamma(x) \subseteq \mathsf{int}(X)$, hence

$$\forall x \in X,\ \forall y \in \Gamma(x), \forall z \in \Gamma(y), z \in X. \tag{4}$$

Since X is finite, pick some $t \in \mathcal{X}_o \cap X^c$. By connectedness of $\sim$, there should exist a finite path $x_0 = x, x_2, \ldots, x_{2k} = t$ from $x \in X$ to $t \notin X$. However, no such path can exist by Equation (4).

Therefore, there exists $x \in X$ such that $x \notin \mathsf{int}(\mathsf{int}(X))$. This proves that the inclusion is strict. □

We can immediately deduce from Lemma 3 the following proposition.

Proposition 6. For every $\omega \in \Omega$, if $\{x \in \mathcal{X}_o \mid \omega(x) = 1\}$ is finite then the set of configurations with only 0 on odd positions is reachable.

Proof. As $X := \{x \in \mathcal{X}_o \mid \omega(x) = 1\}$ is finite, Lemma 3 implies that the sequence $\mathsf{int}^2(X), \mathsf{int}^4(X), \ldots$ converges to $\varnothing$ in a finite number of steps. Since A is a strict aggregation rule, for every $n \geq 0$, the probability to reach $\mathsf{int}^{2n+2}(X)$ from $\mathsf{int}^{2n}(X)$ is positive, hence we constructed a trajectory between X and $\varnothing$ with even stages: hence a trajectory from a configuration in $(.,.,.,F)$ to a configuration in $(.,.,.,0)$. □

Remark 2. Let us stress out that this lemma does not say anything about the presence of active agents at an even stage on an even position.

We can prove in a similar way the same result for $X \subset \mathcal{X}_e$. This yields the proof of Theorem 1.(iii).

5 Strict aggregation functions and networks with a blinker: ϕ-irreducible sets

The next step is to study in detail ϕ-irreducibility. Contrarily to the previous results which hold without additional assumptions on the network, we need now some particular conditions to establish ϕ-irreducibility. In this section, based on necessary conditions on $(\mathcal{X}, \sim)$, we will prove that the sets highlighted in Theorem 1 are ϕ-irreducible. Hence, we provide a decomposition of the set of configurations into three types: sets in case (i) are finite absorbing ϕ-irreducible sets, those in case (ii) are infinite absorbing ϕ-irreducible sets, and the ones in case (iii) are transient ϕ-irreducible sets.

In the remainder, we may consider on Ω the following measure ϕ such that for every two finite disjoint sets X and Y,

$$\phi((X,Y)^+) = \frac{1}{2^{|X|+|Y|}}.$$

For every X^* and Y^*, we impose the restriction of ϕ conditionally on $(X^*, Y^*)^+$ defined on the trace of $\mathcal{T}$ on $(X^*, Y^*)^+$ such that for every two finite disjoint sets X and Y disjoint with $X^* \cup Y^*$,

$$\phi_{X^*,Y^*}((X^* \cup X, Y^* \cup Y)^+) = \frac{1}{2^{|X|+|Y|}}.$$

In general, the sets presented in the previous section are not ϕ-irreducible for any $(\mathcal{X}, \sim)$. We present two counterexamples in order to highlight two different types of problem.

Example 6. Consider the network defined by $\mathcal{X} = \mathbb{Z}$ and the 1-neighborhood:

$$\forall x \in \mathcal{X}, \Gamma(x) = \{x-1, x+1\}.$$

We want to show that the set of configurations $(\infty, \infty, \infty, 0) \cup (\infty, 0, \infty, \infty)$ is not ϕ-irreducible. Consider the configuration ω in this set defined by

$$\omega(x) = \begin{cases} 0 \text{ if } x \text{ is odd,} \\ 0 \text{ if } x \leq 0 \text{ and } x \text{ is even,} \\ 1 \text{ if } x > 0 \text{ and } x \text{ is even.} \end{cases}$$

and the cylinder $\mathcal{A} = (X, Y)^+$ with $X = \{0, 4\}$ and $Y = \{2\}$. We show that the cylinder $\mathcal{A}$ is not reachable from ω. The reason is that at any stage, the current configuration ω' has the form $\omega'(x) = \omega(x - z)$, for some $z \in \mathbb{Z}$, so that the succession 1-0-1 (when looking only at even positions in an even stage or only at odd positions in an odd stage) is not possible. Let us show this for the first transition from ω. The new configuration ω' is such that $\omega'(2k+1) = 1$ for any $k \geq 1$, $\omega'(1)$ can be either 1 or 0, and $\omega'(x) = 0$ otherwise. If $\omega'(1) = 0$, then the claim is true with $z = 1$, and if $\omega'(1) = 1$, then the claim is true with $z = -1$. As ω' is equal to ω up to a shift, the same reasoning applies at every stage.

Example 7. Consider the plane graph completed by two additional agents linked to $(0, 0)$. Formally, let $\mathcal{X} = \mathbb{Z}^2 \cup \{\alpha, \beta\}$. We assume that the neighborhoods are given by

- $\Gamma(\alpha) = \Gamma(\beta) = \{(0,0)\}$,
- $\Gamma((0,0)) = \{(-1,0),(0,-1),(1,0),(0,1),\alpha,\beta\}$,
- for every $(n,m) \neq (0,0)$, $\Gamma((n,m)) = \{(n-1,m),(n,m-1),(n+1,m),(n,m+1)\}$.

Then, this network admits a blinker with $\mathcal{X}_e = \{(n,m) \text{ s.t. } n+m \text{ is even}\}$ and $\mathcal{X}_o = \{(n,m) \text{ s.t. } n+m \text{ is odd}\} \cup \{\alpha,\beta\}$. Moreover, since α and β have a unique neighbor, we know that for every dynamic of opinion: the status of α (resp. β) at stage t is the status of agent $(0,0)$ at stage $t-1$. In particular, both statuses are equal and it is impossible to reach the cylinder $(\{\alpha\},\{\beta\})^+$ of configurations where α is active whereas β is inactive.

Intuitively, in Example 6, the set of configurations is not ϕ-irreducible because the graph is a line, and there is not enough room to move 0 and 1 without erasing patterns that we want to preserve. In Example 7, the graph is enough large to move patterns, but as the nodes α,β share the same neighbor, there is no possibility to store somewhere a partial configuration where α and β would take different values.

The outline of the rest of the section is the following. First, we describe our sufficient conditions on the graph in order to obtain ϕ-irreducibility. Then, we turn to the description of the algorithm that allows to construct trajectories between different configurations. We then deduce the final result.

5.1 Conditions on graphs

The first definition is related to the problem encountered in Example 6.

Definition 8. A *complex star* is a 7-uple $(s_*, s_1, s_2, s_3, s'_1, s'_2, s'_3) \in \mathcal{X}^7$ such that:

- s_1, s_2, s_3 are 3 distinct nodes;
- $\{s_1, s_2, s_3\} \subseteq \Gamma(s_*)$;
- $s'_1 \in \Gamma(s_1)$, $s'_2 \in \Gamma(s_2)$ and $s'_3 \in \Gamma(s_3)$;
- $s_* \notin \{s'_1, s'_2, s'_3\}$.

Informally, s_* is the center of a star with three branches that have at least a depth of two. Recall that by assumption $(\mathcal{X},\sim)$ admits a blinker, and therefore we know that $\{s'_1, s'_2, s'_3, s_*\} \cap \{s_1, s_2, s_3\} = \varnothing$. Also, note that we do not assume that s'_1, s'_2 and s'_3 are distinct.

Roughly speaking, a complex star is a device permitting to store an inactive status or an active status along time. Its existence prevents the graph to be similar to a line. In the following proposition, we identify a sequence with the set of its values.

Proposition 7. The network $(\mathcal{X},\sim)$ does not contain a complex star if and only if it has one of the two following forms:

- First case:

1. there exists a sequence $(x_m)_{m\in\mathbb{N}}$ in $\mathcal{X}^{\mathbb{N}}$ such that all nodes are different, $\Gamma(x_0) = \{x_1\}$ and for every $n \geq 1$, $\Gamma(x_n) \cap (x_m)_{m\in\mathbb{N}} = \{x_{n-1}, x_{n+1}\}$.
2. for all $x \in \mathcal{X}$, if $x \notin (x_m)_{m\in\mathbb{N}}$ then there exists $n \in \mathbb{N}$ such that $\Gamma(x) = \{x_n\}$.

- Second case:

1. there exists a sequence $(x_m)_{m\in\mathbb{Z}}$ in $\mathcal{X}^{\mathbb{Z}}$ such that all nodes are different and for every $n \in \mathbb{Z}$, $\Gamma(x_n) \cap (x_m)_{m\in\mathbb{Z}} = \{x_{n-1}, x_{n+1}\}$.
2. for all $x \in \mathcal{X}$, if $x \notin (x_m)_{m\in\mathbb{Z}}$ then there exists $n \in \mathbb{Z}$ such that $\Gamma(x) = \{x_n\}$.

Proof. The "if" part is obvious. As for the "only if" part, let us first assume that there exists a double ray in the graph, i.e., there exists a sequence $(x_m)_{m\in\mathbb{Z}}$ in $\mathcal{X}^{\mathbb{Z}}$ such that all nodes are different and for every $n \in \mathbb{Z}$, $\{x_{n-1}, x_{n+1}\} \subseteq \Gamma(x_n)$.

Moreover, since any node on the double ray admits already two branches of size 2, the absence of complex star implies that:

- any additional node linked to the ray has no other neighbor. We will say that it forms an antenna.
- for every distinct $i, j \in \mathbb{Z}$ which are not consecutive there is no link between x_i and x_j, since similarly this would generate a complex star with x_i as a center. Therefore, $\Gamma(x_n) \cap (x_m)_{m\in\mathbb{Z}} = \{x_{n-1}, x_{n+1}\}$.

Hence, we obtain the second case in our characterization of networks without complex stars.

Let us suppose now that there exists no double ray. Since $(\mathcal{X}, \sim)$ is an infinite connected graph with bounded degree, we know by Proposition 8.2.1. from Diestel (2005) that there exists a ray, i.e., there exists $(x_i)_{i\geq 0}$ such that $x_1 \in \Gamma(x_0)$ and

$$\forall i \geq 1, \{x_{i-1}, x_{i+1}\} \subseteq \Gamma(x_i).$$

The preceding reasoning on additional nodes can be applied for every $i \geq 2$ (it is unknown if x_0 or x_1 have more than one neighbors): any node not in $(x_i)_{i\geq 0}$ and connected to $(x_i)_{i\geq 2}$ has only one neighbor.

Denote by $\mathcal{X}_1 = \cup_{i\geq 2}(\{x_i\} \cup \Gamma(x_i))$ and consider the graph with the set of nodes $\mathcal{X}_2 = (\mathcal{X} \setminus \mathcal{X}_1) \cup \{x_1\}$ and the relation $\sim$. This new graph is still connected. Moreover, it is finite, otherwise there would exist a ray which could be connected to the previous ray to generate a double ray. Let us consider a node x' at maximal distance from x_1 in the subgraph $(\mathcal{X}_2, \sim)$. We define the new sequence $(x'_n)_{n\geq 0}$ with $x'_0 = x'$, made from the path from x' to x_1 concatenated with the ray $(x_n)_{n\geq 1}$.

- The reasoning on antennas can now be applied to any node in $\{x'_n, n \geq 2\}$.
- Moreover, every neighbor of x'_1 (except x'_2) has only x'_1 as neighbor, otherwise x' would not be a farthest node from x_1.
- Finally, there cannot be any internal link in the ray $(x'_n)_{n\geq 0}$, since any such link has at least one extremity in $\{x'_n, n \geq 2\}$ and would create a complex star.

□

The existence of a complex star is not sufficient to ensure ϕ-irreducibility, as it is shown by the network of Example 7. As explained above, the failure is due to the fact that the desired partial configuration cannot be stored on the unique neighbor of α, β. This motivates the following definition. A similar definition can be stated on $\mathcal{X}_e$ or on $\mathcal{X}$.

Definition 9. We say that a partial configuration (X, Y) of $\mathcal{X}_o$ can be *stored* if there exists a mapping θ from $\mathcal{X}_o$ to $\mathcal{X}_e$ such that

1. for every $x \in X \cup Y$, $\theta(x) \in \Gamma(x)$,
2. for every $x \in X$ and every $y \in Y$, $\theta(x) \neq \theta(y)$.

θ is called a *storing function*. Observe that $\theta(X \cup Y) \subseteq \overline{X \cup Y}$.

Remark 3. It is easy to check that if a partial configuration (X_1, Y_1) of $\mathcal{X}_o$ can be stored and a partial configuration (X_2, Y_2) of $\mathcal{X}_e$ can be stored then the partial configuration $(X_1 \cup X_2, Y_1 \cup Y_2)$ can also be stored. Moreover, if any partial configuration can be stored, then any configuration can be stored as well.

We now present classes of graphs where a storing function exists. We focus on injective storing functions which permit an easy analysis. Consider a partial configuration (X, Y) in $\mathcal{X}_o$, storable by θ and suppose in addition that θ is an injection from $X \cup Y$ to $\overline{X \cup Y} \subseteq \mathcal{X}_e$. In terms of graph theory, we may rephrase this by saying that the bipartite graph $(X \cup Y, \overline{X \cup Y}, \sim)$ admits a *matching of* $X \cup Y$. A graph is bipartite when the set of nodes can be partitioned into A, B, with no edge between nodes in A and no edge between nodes in B. As $(\mathcal{X}, \sim)$ admits a blinker, for any $X \subseteq \mathcal{X}_o$, $(X, \overline{X}, \sim)$ forms a bipartite graph.

There are classes of graphs $(\mathcal{X}, \sim)$ which admit a matching of $X \cup Y$ for any partial configuration (X, Y) in $\mathcal{X}_o$ and in $\mathcal{X}_e$: these are the k-regular graphs and the hierarchies (see Example 4). A graph is *k-regular* ($k \geq 2$, integer) if each node has exactly k neighbors.

Proposition 8. If $(\mathcal{X}, \sim)$ is k-regular or is a hierarchy, then any configuration can be stored.

Proof. Due to the previous discussion, it suffices to prove that for any partial configuration (X, Y) on $\mathcal{X}_o$ (or $\mathcal{X}_e$), the bipartite graph $(X \cup Y, \overline{X \cup Y}, \sim)$ admits a matching of $X \cup Y$. We consider first the case of a k-regular graph. Observe that by assumption, the degree of each node of $X \cup Y$ in the bipartite graph is k, while the degree of each node of $\overline{X \cup Y}$ in the bipartite graph is at most k (some of these nodes have necessarily neighbors outside $X \cup Y$). Consider $S \subseteq X \cup Y$ and the set of neighbors of S, denoted by $\Gamma(S) := \bigcup_{x \in S} \Gamma(x)$. By the above observation, the number of edges from S to $\Gamma(S)$ is $k|S|$, and the total number of edges arriving at $\Gamma(S)$ is at most $k|\Gamma(S)|$. Therefore $|\Gamma(S)| \geq |S|$. Then, by Hall's Theorem (see, e.g., (Diestel, 2005, Th. 2.1.2)), a matching of $X \cup Y$ exists.

Suppose now that $(\mathcal{X}, \sim)$ is a hierarchy. We know that $\mathcal{X}_o$ is the set of nodes in the odd layers of the hierarchy. Take $x \in X \cup Y$. It suffices to define $\theta(x)$ as one of the subordinates of x, which lies on the next even layer. This will form a matching of $X \cup Y$ as two distinct nodes in $X \cup Y$ have necessarily disjoint sets of subordinates. □

This leads to the following condition on the network.

Richness Assumption. $(\mathcal{X}, \sim)$ is said to be *rich* if:

1. there exists an infinite number of complex stars,
2. any partial configuration (X, Y) on $\mathcal{X}_o$ and on $\mathcal{X}_e$ can be stored.

Observe that all the networks introduced in Examples 1 to 4 satisfy the richness assumption (except $(\mathbb{Z}^d, \sim)$ with $d = 1$ which has no complex star, since they are all k-regular or a hierarchy, and they contain infinitely many complex stars.

Remark 4. Notice that we do not assume only the existence of a complex star but the existence of an infinite number of complex stars. A challenging issue is to relax this condition. As it will appear in the proof, the question become complicated if there is only a finite number of complex stars since we cannot decompose anymore the procedure into elementary procedures.

Finally, let us present the following statement inspired by Example 7 that will be used later in the paper.

Lemma 4. Suppose that in $(\mathcal{X}, \sim)$ there exist three agents x, y and z with

$$\Gamma(x) = \Gamma(y) = \{z\}.$$

Then $(\mathcal{X}, \sim)$ does not satisfy the richness assumption.

Proof. The configuration where x is active and y is inactive cannot be stored since necessarily $\theta(x) = \theta(y)$. □

5.2 ϕ-reachability result

We can now state our result on ϕ-reachability.

Theorem 2. Assume that $(\mathcal{X}, \sim)$ satisfies the Richness Assumption. Then the following sets are ϕ-irreducible:

- $(\infty, 0, \infty, 0)$,
- $(0, \infty, 0, \infty)$,
- $(0, \infty, \infty, 0) \cup (\infty, 0, 0, \infty)$,
- $(\infty, \infty, \infty, \infty)$,
- $(\infty, 0, \infty, \infty) \cup (\infty, \infty, \infty, 0)$,
- $(0, \infty, \infty, \infty) \cup (\infty, \infty, 0, \infty)$,
- (∞, F, ∞, F),
- (F, ∞, F, ∞),
- $(F, \infty, \infty, F) \cup (\infty, F, F, \infty)$,
- $(\infty, F, \infty, \infty) \cup (\infty, \infty, \infty, F)$,
- $(F, \infty, \infty, \infty) \cup (\infty, \infty, F, \infty)$,
- $(\infty, F, \infty, 0) \cup (\infty, 0, \infty, F)$,
- $(F, \infty, 0, \infty) \cup (0, \infty, F, \infty)$,
- $(0, \infty, \infty, F) \cup (\infty, F, 0, \infty)$,
- $(F, \infty, \infty, 0) \cup (\infty, 0, F, \infty)$.

In order to prove Theorem 2, we establish the following two propositions. The theorem is then an immediate consequence.

Proposition 9. Assume that $(\mathcal{X}, \sim)$ satisfies the Richness Assumption. Let ω be a configuration such that there exist both 0 and 1 on an odd position.

- Let $X \subset \mathcal{X}_e$ and $Y \subset \mathcal{X}_e$ be two finite sets, then the cylinder $(X, Y)^+$ is reachable from ω in an odd number of stages.
- Let $X \subset \mathcal{X}_o$ and $Y \subset \mathcal{X}_o$ be two finite sets, then the cylinder $(X, Y)^+$ is reachable from ω in an even number of stages.

Proposition 10. Assume that $(\mathcal{X}, \sim)$ satisfies the Richness Assumption. Let ω be a configuration such that there exist both 0 and 1 on an even position.

- Let $X \subset \mathcal{X}_e$ and $Y \subset \mathcal{X}_e$ be two finite sets, then the cylinder $(X, Y)^+$ is reachable from ω in an even number of stages.
- Let $X \subset \mathcal{X}_o$ and $Y \subset \mathcal{X}_o$ be two finite sets, then the cylinder $(X, Y)^+$ is reachable from ω in an odd number of stages.

In Propositions 9 and 10 we restricted ourselves to finite cylinders. We can extend these results by using the fact that Ω is a polish set. Let $\mathcal{A}$ be such that $\phi(\mathcal{A}) > 0$. ϕ is a Borel measure and therefore is tight:

$$\forall \mathcal{A} \in \mathcal{T},\ \phi(\mathcal{A}) = \sup_{K \text{ compact } \subset A} \phi(K).$$

It follows that there exists a closed set K such that $\phi(K) > 0$. Any open set can be written as a countable union of finite cylinders, and therefore any closed set can be written as a countable intersection of complements of finite cylinders. However, the complement of a finite cylinder is a union of finite cylinders:

$$((X, Y)^+)^c = \left(\bigcup_{x \in X} (\varnothing, \{x\})^+ \right) \cup \left(\bigcup_{y \in Y} (\{y\}, \varnothing)^+ \right).$$

Hence, a closed set can be written as a countable union of countable intersections of finite cylinders, and therefore one of these intersections has a non-zero measure under ϕ: there exists $(X_n, Y_n)_{n \in \mathbb{N}}$ a sequence of finite cylinders such that

$$\phi \left(\bigcap_{n \in \mathbb{N}} (X_n, Y_n)^+ \right) > 0.$$

This is only possible if $X_\infty = \bigcup_{n \in \mathbb{N}} X_n$ and $Y_\infty = \bigcup_{n \in \mathbb{N}} Y_n$ are finite and disjoint. In this case, we have

$$\bigcap_{n \in \mathbb{N}} (X_n, Y_n)^+ = (X_\infty, Y_\infty)^+.$$

$(X_\infty, Y_\infty)^+$ is reachable from ω and therefore $\mathcal{A}$ is reachable from ω.

In the next section, we will prove Proposition 9. Proposition 10 can be proven in a very similar way. The proof relies on constructing an explicit trajectory between the initial partial configuration ω to any other partial configuration ω' compatible with the condition in the lemma. In order to do so, the intuition is to fix the status of each agent in the network one by one. In doing so, one needs to be careful about not erasing a status that has been fixed before in the procedure.

5.3 Proof of Proposition 9

The exposition of the proof is divided into several parts. We first present two partial mechanisms of transitions. The first one is a storing mechanism of partial configuration conditionally on the existence of a storing function. The second one is the propagation of an active status or the propagation of an inactive status in the network. These two procedures are defined locally, i.e., on partial configurations.

Then, we will combine these partial mechanisms into a global well-defined procedure. Hence, it will induce a trajectory. The key point is to ensure that these elementary mechanisms are not conflicting each other by using the same nodes at the same time.

We restrict ourselves to what happens on $\mathcal{X}_o$. But any of the result can be expressed on $\mathcal{X}_e$. The bipartite structure of $(\mathcal{X}, \sim)$ ensures that one can decompose the dynamic into two parts: the status of odd stages/odd positions and even stages/even positions on one side and the status of even stages/odd positions and odd stages/even positions on the other side.

5.3.1 Storing and propagation

We begin by remarking that one can store the status of a node on one of its neighbor. Let $\omega \in \Omega$ be a configuration such that agent x in $\mathcal{X}_o$ is active, and x' be one of its neighbors. The following sequence of cylinders does the job:

- for every $l \in 2\mathbb{N}$, $\mathcal{A}_l = (\{x\}, \varnothing)^+$,
- for every $l \in 2\mathbb{N}+1$, $\mathcal{A}_l = (\{x'\}, \varnothing)^+$.

The next mechanism for storing partial configurations is merely a generalization of the previous idea.

Lemma 5. If a partial configuration (X, Y) of $\mathcal{X}_o$ can be stored then for every $n \in 2\mathbb{N}$,

$$\forall \omega \in (X,Y)^+,\ K^n(\omega, (X,Y)^+) > 0.$$

The procedure used in the proof will be called "storing a partial configuration in the global procedure".

Proof. We assume that the finite configuration (X, Y) of $\mathcal{X}_o$ can be stored. Hence, there exists θ such that for all $x \in X \cup Y$, $\theta(x) \in \Gamma(x)$ and for all $x \in X$, for all $y \in Y$, $\theta(x) \neq \theta(y)$. Define $\theta(X) = \{\theta(x),\ x \in X\}$ and $\theta(Y) = \{\theta(y),\ y \in Y\}$ We consider the following sequence of cylinders:

- for every $l \in 2\mathbb{N}$, $\mathcal{A}_l = (X, Y)^+$,
- for every $l \in 2\mathbb{N}+1$, $\mathcal{A}_l = (\theta(X), \theta(Y))^+$.

By construction, $\theta(X) \subseteq \overline{X}$ and $\theta(Y) \subseteq \overline{Y}$. Moreover, by symmetry of the neighborhood relation, $X \subseteq \overline{\theta(X)}$ and $Y \subseteq \overline{\theta(Y)}$. By definition of θ we know that $\theta(X) \cap \theta(Y) = \varnothing$. Hence, this sequence is well defined and

$$\forall m \in \mathcal{A}_l,\ K(m, \mathcal{A}_{l+1}) > 0.$$

Moreover, this can be ensured by only imposing restriction on the status of agents in $X \cup Y$ at even stages and in $\overline{X \cup Y}$ at odd stages. □

Several storing procedures can be run at the same time if they operate on disjoint sets of nodes with disjoint neighborhood.

We now turn to the problem of propagating the status of a node in a partial configuration to any node in the network. This procedure will be called "propagating in the global procedure".

Lemma 6. Let (X, Y) be a partial configuration of $\mathcal{X}_o$ such that X is not empty. Then, for any $\omega \in (X, Y)^+$ and for every $x' \in \mathcal{X}$,

- if $x' \in \mathcal{X}_e$ then there exists $n \in 2\mathbb{N}$ such that

$$K^n(\omega, (\{x'\}, \varnothing)^+) > 0,$$

- if $x' \in \mathcal{X}_o$ then there exists $n \in 2\mathbb{N} + 1$ such that

$$K^n(\omega, (\{x'\}, \varnothing)^+) > 0.$$

Proof. Let (X, Y) be a partial configuration of $\mathcal{X}_o$ such that X is not empty. Let $x \in X$ be an active agent in $\mathcal{X}_o$ and x' be another agent. Consider a path $x_0, \dots, x_n$ between $x = x_0$ and $x' = x_n$. Informally, we propagate the status of x to x' along this path. More formally, consider the following sequence of cylinders:

$$\forall l \in \{0, \dots, n\},\ \mathcal{A}_l = (\{x_l\}, \varnothing)^+.$$

This sequence is well defined, since for every $l \in \{0, \dots, n-1\}$, $x_{l+1} \in \Gamma(x_l)$, and we have by construction

$$\forall m \in \mathcal{A}_l,\ K(m, \mathcal{A}_{l+1}) > 0.$$

We obtain that a configuration where x' is active is reachable and moreover, depending if x' is even or odd, we obtain the condition on the number of iterations. □

A similar procedure can be described for inactive agents.

A key point in the previous construction is that the propagation requires a constraint at stage l only on the status of x_l. All other agents can be indifferently active or not. This will allow us to combine the procedure with other procedures without having them interfering with each other.

5.3.2 Global procedure

We now turn to establish the general procedure. Let $X \subset \mathcal{X}_o$ and $Y \subset \mathcal{X}_o$ be two disjoint finite sets. Since the graph satisfies the Richness Assumption, we know that (X, Y) can be stored. Moreover, there exists an infinite number of complex stars, therefore there exists a complex star such that all nodes of the star are disjoint from $X \cup \overline{X} \cup Y \cup \overline{Y}$. Denote by $(s_*, s_1, s_2, s_3, s'_1, s'_2, s'_3)$ the nodes of this complex star, we assume that $s_* \in \mathcal{X}_e$. Hence, $s_1, s_2, s_3 \in \mathcal{X}_o$ and $s'_1, s'_2, s'_3 \in \mathcal{X}_e$. We also define a distance on the graph as the length of a shortest path between a node and s_*:

$$\forall x \in \mathcal{X}, d(x, s_*) = \inf\{n \in \mathbb{N}, \exists (x_l)_{1 \le l \le n} \text{ a path between } x \text{ and } s_*\}.$$

This is a well-defined distance since the graph is connected and there exists at least one path between s_* and x. We define $\mathcal{X}_n = \{x \in \mathcal{X}, d(s_*, x) \leq n\}$ and $\mathcal{X}_{n,e} = \mathcal{X}_n \cap \mathcal{X}_e$ and $\mathcal{X}_{n,o} = \mathcal{X}_n \cap \mathcal{X}_o$.

Let ω be a configuration with at least one active agent and one inactive agent in $\mathcal{X}_o$. Let us prove that there exists a trajectory with an odd number of steps from ω to $(X, Y)^+$.

In order to prove this result, we decompose the construction of the trajectory into several steps. First, we will focus on partial configurations restricted to the star where one agent is active and one agent is inactive. We will see that there exists a trajectory between any such partial configurations. In particular, it is possible to exchange the statuses of two agents. Next, we will prove that there is a trajectory from ω to a configuration where the active and inactive statuses are on the branches of the complex star. Then, we will fix the status of every agent by decreasing distance to the center of the star. The key point is that while we fix a new status, the previous statuses are preserved.

From the complex star to the complex star. Let $\mathcal{I}$ be the set of configurations such that at least one node in $\{s_1, s_2, s_3\}$ has value 0 and at least one node in $\{s_1, s_2, s_3\}$ has value 1. Formally, we have

$$\mathcal{I} = (\{s_1\}, \{s_2\})^+ \cup (\{s_2\}, \{s_1\})^+ \cup (\{s_1\}, \{s_3\})^+ \cup (\{s_3\}, \{s_1\})^+ \cup (\{s_2\}, \{s_3\})^+ \cup (\{s_3\}, \{s_2\})^+.$$

Lemma 7. For every pair of partial configurations (X', Y') and (X'', Y'') in $\mathcal{I}$ there exists a trajectory of even length between (X', Y') and (X'', Y'').

Proof. Consider a restricted configuration on $\{s_1, s_2, s_3\}$. We assume without loss of generality that s_1 is active whereas s_2 is inactive. We want to prove that we can reach any restricted configuration on $\{s_1, s_2, s_3\}$.

First, let us prove that we can generate both the configuration where s_3 is inactive or active without using any node outside the complex star $\{s_*, s_1, s_2, s_3, s'_1, s'_2, s'_3\}$. Let us show that we can reach the case where s_3 is infected. We consider the following sequence of restricted configurations:

- s_1 is active and s_2 is inactive,
- s_* is active and s'_2 is inactive,
- s_1, s_3 are active and s_2 is inactive.

We constructed a trajectory between our original configuration and the one we wanted. Notice that in fact, we only use as auxiliary state s_* and s'_2, hence there are no constraints on the statuses of s'_1 and s'_3 (if they are different from s'_2). By exchanging the role of s_1 and s_2, we obtain a procedure such that s_3 is inactive.

Once s_3 is fixed, it remains to fix by the same argument the statuses of s_1, s_2: if s_3 is active, by considering s_2 as the inactive, fix the status of s_1 as it should be and then finally fix the status of s_2; if s_3 is inactive, exchange s_1 with s_2. □

From ω to the complex star. We show that there exists a trajectory of odd length from ω to $\mathcal{I}$.

Lemma 8. Let $\omega \in \Omega$. Assume that there exist $x, x' \in \mathcal{X}_o$ such that $\omega(x) = 1$ and $\omega(x') = 0$. Then there exists $n \in 2\mathbb{N}$ such that

$$K^n(\omega, \mathcal{I}) > 0.$$

Proof. By assumption, there exists $x, y \in \mathcal{X}_o$ such that x is active and y is inactive in configuration ω. We will distinguish by which one is closer to the center of the star s_*. Notice that since x and y are both in $\mathcal{X}_o$ (we assumed a blinker), their distance to s_* differ by an even number. We choose a shortest path p from x to s_*.

- If $d(x, s_*) < d(y, s_*)$ then p does not go through y. Consider the shortest path from y to s_*. By definition of a complex star, there exists $i \in \{1, 2, 3\}$ such that s_i is not on the shortest path between y and s_*. Let us assume without loss of generality that it is s_1. We propagate the active status to s_1 through p while storing y on any of its neighbors. By the distance assumption, none of the neighbors of y can be on p or it will contradict the assumption on distances. We distinguish two cases:
  - the shortest path from y to s_* goes through s_2 (resp. s_3): one propagates the inactive status to s_2 (resp. s_3) through the shortest path while storing the status of s_1 in s_*.
  - the shortest path from y to s_* does not go through $\{s_1, s_2, s_3\}$. Therefore, there exists $s_4 \in \Gamma(s_*)$ such that $s_4 \notin \{s_1, s_2, s_3\}$. One can propagate the inactive status to s_4 through the shortest path while storing the status of s_1 in s_*. To conclude, by applying Lemma 7 to the star $\{s_*, s_1, s_1', s_2, s_2', s_4, s_4'\}$, one can place the inactive status in s_2 while preserving the active status at s_1.
- If $d(x, s_*) > d(y, s_*)$ then the proof is similar as the first case by inverting the role of active and inactive agents.
- If $d(x, s_*) = d(y, s_*)$ then we need to distinguish three cases depending on the neighbors of y and x:
  - If y has only one neighbor, not on p: We follow the proof of the first case by storing the status of y on this neighbor.
  - If y has more than one neighbor: Then, one of them has to be outside of p. Indeed, if two of them are on the path p, then it means than one of these neighbors is at distance less than $d(x, s_*) - 3$ of s_* which would put y at a distance less than $d(x, s_*) - 2$, a contradiction. We follow the proof of the first case by storing the status of y on this neighbor.
  - If y has only one neighbor which is on p: Then by assumption on the distances, there exists $z \in \mathcal{X}$ in the neighborhods of x and y. The case $\Gamma(x) = \Gamma(y) = \{z\}$ is ruled out by Lemma 4. Hence, $|\Gamma(x)| > 1$ and we can treat this case by exchanging the role of x and y: first propagate the status of y while storing x on its other neighbor.

□

From the star to the cylinder $(X, Y)^+$. By assumption, the complex star is disjoined from $X \cup \overline{X} \cup Y \cup \overline{Y}$. This condition ensures that the storing algorithm for $(X, Y)^+$ from Lemma 5 can be applied without using the star. Hence, we are free to manipulate the status of nodes in the star.

Lemma 9. Let $\omega \in \mathcal{I}$. Then for every $z \in \mathcal{X}_o$, one can place an active agent at z in $n \in 2\mathbb{N}$ stages. Moreover, this can be done without changing the values for nodes at a distance greater than z from s_*.

Proof. Let us consider ω_* a configuration in $\mathcal{I}$ and $z \in \mathcal{X}_o$ not on the complex star and with no neighbors on the complex star. Let us say without loss of generality that we want to place an active agent at z and that s_1 is active whereas s_2 is inactive.

By assumption, there exists a global storing mechanism for any configuration denoted by θ. Moreover, by the condition on the neighborhoods, the status of any agent z' is stored on one of its neighbors which is not in $\{s_*, s_1, s_2, s_3, s'_1, s'_2, s'_3\}$.

Consider a shortest path between s_* and z. There are two cases:

- there exists a shortest path not going through $\{s_1, s_2, s_3, s'_1, s'_2, s'_3\}$, then the procedure is as follows: propagate the active status from s_1 to s_* and then to z while storing s_2 on s'_2 and s_1 at s_*. On every agent at a distance further away from z, store the status by using the global storing mechanism.

- every shortest path goes through $\{s_1, s_2, s_3, s'_1, s'_2, s'_3\}$. Assume that it goes through s'_i, then there also exists a shortest path such that it goes through s_i, since by construction s'_i is at distance 2 from s_*. We consider this shortest path. We have several cases:

  - if it goes through s_1: then simply propagate s_1 to z storing at s'_1, and store s_2 at s_*. Forget about s_3.
  - if it goes through s_2: first consider the path from the current configuration to the one where s_2 is active and s_1 is inactive by Lemma 7. Store s_1 outside the path that we need. Then, we are back to the previous case.
  - if it goes through s_3: consider the path from the current configuration to the one where s_3 is active and s_1 is inactive by Lemma 7 and finish like in the previous case.

The storing mechanism ensures that anything that happens outside of the path and the stars can be preserved. □

Lemma 10. Let $\omega \in \mathcal{I}$. Then for every $z \in \mathcal{X}_o$, one can place an inactive agent at z in $n \in 2\mathbb{N}$ stages. Moreover, that can be done without changing the values for nodes at a distance greater than z from x_*.

By iterating the previous lemma by decreasing distance, we obtain Proposition 9 conditionally on the fact that the complex star is centered at an even node. Nevertheless, the previous proof still holds when the star is centered in $\mathcal{X}_o$. Indeed, it is sufficient to notice that now the propagation from ω to $\mathcal{I}$ is done in an odd number of stages and the propagation from the star to $\mathcal{X}_o$ (resp. $\mathcal{X}_e$) is done in an odd (resp. even) number of stages, yielding a total of even (resp. odd) number of stages. Hence, Proposition 9 is proven.

6 Strict aggregation functions and symmetric neighborhood

In this section, we focus on strict aggregation functions but we assume that the graph does not admit a blinker. We saw in Sections 4 and 5 that we could separate what was happening at odd positions (respectively even positions) at odd times from what was happening at odd positions (respectively even positions) at even times. This is not possible anymore, so that the results of Section 4 cannot be applied. On the other hand, we will see that any graph admits at least one subgraph with a blinker. Since a path in the subgraph is still a path in the graph, the procedures of Section 5 can still be used.

Example 8. Consider $\mathcal{X} = \mathbb{Z}^2$ and the $\sqrt{2}$-neighborhood (for the Euclidean distance) defined by, for any $(n, m) \in \mathcal{X}$,

$$\Gamma((n,m)) = \{(p,q),\ p \in \{n-1, n, n+1\} \text{ and } q \in \{m-1, m, m+1\}\}.$$

(see Figure 6 (left)). This graph does not admit a blinker. Nevertheless, by focusing on the 1-neighborhood, we can still define

$$\mathcal{X}_e = \{(n,m) \text{ s.t. } n+m \text{ is even}\}$$

and

$$\mathcal{X}_o = \{(n,m) \text{ s.t. } n+m \text{ is odd}\}.$$

This amounts to considering the subgraph where all diagonal links (i.e., links internal to $\mathcal{X}_e$ and $\mathcal{X}_o$) have been deleted. In this subgraph, a blinker exists (see Figure 6 (middle)) and we can apply the procedures of Section 5.

More generally, any network admits a subnetwork with a blinker. Indeed, given $(\mathcal{X}, \sim)$ and $c \in \mathcal{X}$, one can define the distance from any node x to c:

$$\forall x \in \mathcal{X}, d(x,c) = \inf\{n \in \mathbb{N}, \exists (x_l)_{1 \le l \le n} \text{ path between } x \text{ and } c\}.$$

This yields naturally two sets

$$\mathcal{X}_o^c = \{x \in \mathcal{X},\ \text{s.t. } d(x,c) \text{ is odd}\}$$

and

$$\mathcal{X}_e^c = \{x \in \mathcal{X},\ \text{s.t. } d(x,c) \text{ is even}\}.$$

Let us define from $\sim$ the new relation $\approx^c$ on $\mathcal{X}$ as follows: internal edges in $\mathcal{X}_o^c$ and in $\mathcal{X}_e^c$ for $\sim$ are deleted, and only edges between a node in $\mathcal{X}_o^c$ and $\mathcal{X}_e^c$ for $\sim$ remain. The induced subnetwork $(\mathcal{X}, \approx^c)$ admits a blinker exactly given by $\mathcal{X}_o^c$ and $\mathcal{X}_e^c$. Moreover, first this subgraph is connected, because by construction every node is connected to c, and second every node has a finite number of neighbors. Hence, $\approx^c$ is a neighborhood relation. This yields the following proposition.

Proposition 11. Given a network $(\mathcal{X}, \sim)$, there exists a subnetwork $(\mathcal{X}, \approx)$ that admits a blinker.

Remark 5.

1. Suppose $(\mathcal{X}, \sim)$ does not have a blinker and consider a subgraph $(\mathcal{X}, \approx)$ which admits a blinker. Does there always exist a node c such that $\approx$ is equal to $\approx^c$, the relation induced by the distance to c? In other words, is the above mechanism able to generate any subgraph with a blinker? The answer is no, as shown by the above example with $\mathcal{X} = \mathbb{Z}^2$. Figure 6 (right) shows the subgraph induced by the distance to the central node $(0, 0)$ (or to any arbitrary node), which fairly differs from the one on the middle figure.

2. Notice that there may exist several subgraphs admitting a blinker, as shown in Figure 6. The choice of the subgraph is important, since it is possible for some partial configuration to be stored for one subgraph and not for another one, as shown by Example 9.

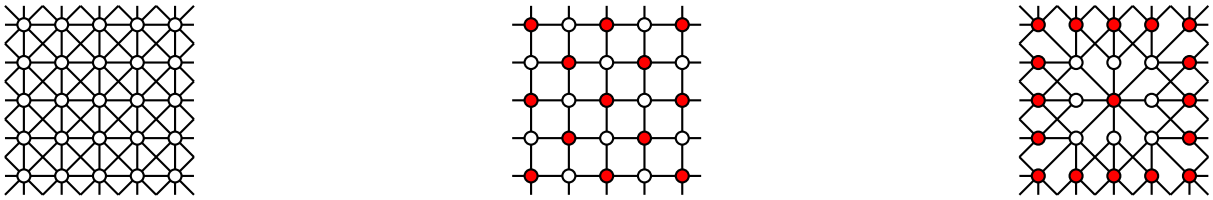


Figure 6: The $\mathbb{Z}^2$ network with the $\sqrt{2}$-neighborhood (left); The $\mathbb{Z}^2$ subnetwork with the 1-neighborhood (middle); The $(\mathbb{Z}^2, \approx^c)$ subnetwork where $c = (0, 0)$ (right). White and red nodes indicate the blinker.

Example 9. We slightly modify the network $(\mathcal{X}, \sim)$ of Example 7 as follows:

- $\Gamma^{\sim}(\alpha) = \{(0, 0), \beta\}$,
- $\Gamma^{\sim}(\beta) = \{(0, 0), \alpha\}$,

i.e., α and β are linked together. This network does not admit a blinker since we have a cycle of length 3. If we consider the graph generated by the distance to $(0, 0)$ as subgraph, we obtain exactly the network of Example 7 where some configurations cannot be stored. On the contrary, let us consider the subgraph $(\mathcal{X}, \approx)$ where the link between $(0, 0)$ and β has been deleted, hence:

- $\Gamma^{\approx}(\alpha) = \{(0, 0), \beta\}$,
- $\Gamma^{\approx}(\beta) = \{\alpha\}$,
- $\Gamma^{\approx}((0, 0)) = \{(-1, 0), (0, -1), (1, 0), (0, 1), \alpha\}$,
- for every $(n, m) \neq (0, 0)$, $\Gamma^{\approx}((n, m)) = \Gamma^{\sim}((n, m))$.

Then, this new graph admits a blinker and every configuration can be stored by storing separately what happens on $\{\alpha, \beta\}$ and $\mathbb{Z}^2$.

In general, the absence of blinker in the original graph $(\mathcal{X}, \sim)$ implies that the graph is not bipartite, from which we deduce the following proposition.

Proposition 12. Let $(\mathcal{X}, \sim)$ be a network without blinker. For every subnetwork $(\mathcal{X}, \approx)$ with a blinker decomposition into $\mathcal{X}_o$ and $\mathcal{X}_e$, at least one of the two following statements holds:

- there exists $x \in \mathcal{X}_o$ such that $\Gamma^{\sim}(x) \cap \mathcal{X}_o \neq \varnothing$;
- there exists $x \in \mathcal{X}_e$ such that $\Gamma^{\sim}(x) \cap \mathcal{X}_e \neq \varnothing$.

Let $(\mathcal{X}, \approx)$ be a subnetwork of $(\mathcal{X}, \sim)$ with a blinker and let $(\mathcal{X}_e, \mathcal{X}_o)$ be the decomposition associated to the blinker. We consider the partition of the set of configurations where we count how many 1 and 0 exist in ω, without separating odd and even positions. Formally, a block of the partition is a 2-uple (a, b) such that $a, b \in \{0, F, \infty\}$ and there are a positions with value 0 and b positions with value 1. Since there is an infinite number of positions, there are only 5 different blocks of the partition which are nonempty.

Theorem 3. Assume that there exists $(\mathcal{X}, \approx)$ a subnetwork of $(\mathcal{X}, \sim)$ with a blinker that satisfies the Richness Assumption. We have the following decomposition:

- $(\infty, 0)$ and $(0, \infty)$ are absorbing and finite ϕ-irreducible classes,
- (∞, ∞) is absorbing and infinite ϕ-irreducible class,
- (∞, F) and (F, ∞) are transient and ϕ-irreducible sets.

Proof. The result concerning $(\infty, 0)$ and $(0, \infty)$ has already been shown in the general case (Proposition 2). We denote by $\mathcal{X}_o$ and $\mathcal{X}_e$ the blinker decomposition associated to the subgraph $(\mathcal{X}, \approx)$.

(∞, ∞) is absorbing:

We have to prove that if $\omega \in (\infty, \infty)$ and ω' is randomly chosen with probability $K(\omega, .)$, then with probability 1, $\omega' \in (\infty, \infty)$. Let us compute the probability that ω' has a finite or null number of 1. Denote by

$$\Theta = \{x \in \mathcal{X},\ \omega(x) = 1\}$$

then $\Gamma(\Theta) = \cup_{x \in \Theta} \Gamma(x)$ is infinite. But by assumption, any $x' \in \Gamma(\Theta)$ has at least one neighbor with value 1, it follows that the probability that $\omega'(x') = 0$ is strictly smaller than $\rho < 1$. Since the probabilities are independent, it follows that the total probability of infinitely many to be equal to 0 is 0. The same reasoning holds for 0.

(∞, F) and (F, ∞) are transient:
Let us consider $(\mathcal{X}, \approx)$ and the partition defined in Section 4. Then, if $\omega \in (\infty, F)$ then we know that there are only a finite number of active agents and an infinite number of inactive agents. We have two possibilities:

- $\omega \in (\infty, 0, \infty, F) \cup (\infty, F, \infty, 0)$,
- $\omega \in (\infty, F, \infty, F)$.

We showed in Section 4 that both sets are transient (for the subgraph). It follows that (∞, F) is also transient for the subgraph and therefore for the original graph.

(∞,∞), (∞, F) and (F,∞) are ϕ-irreducible:
Let ω be a configuration with one active agent denoted by x and one inactive agent denoted by y. We consider a subgraph $(\mathcal{X}, \approx)$ with blinker $\mathcal{X}_o$ and $\mathcal{X}_e$. Moreover, by Proposition 12, there exists $c \in \mathcal{X}$ with a neighbor in the same block of the blinker. We will assume in the following that $c \in \mathcal{X}_o$ and there exists $d \in \Gamma(c) \cap \mathcal{X}_o$. Finally, let $(s_*, s_1, s_2, s_3, s'_1, s'_2, s'_3)$ be a complex star in the graph $(\mathcal{X}, \approx)$ centered in $\mathcal{X}_e$ disjoint from $\{c, d\}$.

Our aim is to construct a trajectory to a partial configuration ω' such that there exist both an active agent and an inactive agent on some even positions, and also on some odd positions. We distinguish several cases:

- If $x \in \mathcal{X}_o$ and $y \in \mathcal{X}_o$, then one can apply Lemma 8 to construct a trajectory between ω and $\mathcal{I}$, i.e., with a 1 on an odd position (say, s_1) and a 0 on an odd position (say, s_2). The next step is to propagate the active status from $\mathcal{I}$ to c through the subgraph (even number of stages), from c to d through their direct link (one stage), and then back to s'_1 through the subgraph (odd number of stages), while storing s_1 at s'_1 and s_2 at s_*. The total number of stages being even, we obtain at even stages the value 1 at $s_1 \in \mathcal{X}_o$ and at $s'_1 \in \mathcal{X}_e$. As these two nodes are neighbors, the value 1 can be kept at odd stages as well. We proceed differently for the inactive agent s_2, reaching c and then d as before in an odd number of stages. As in odd stages, 0 is stored at s_*, we have at odd stages a 0 at $s_* \in \mathcal{X}_e$ and at $d \in \mathcal{X}_o$, as desired.
- If $x \in \mathcal{X}_e$ and $y \in \mathcal{X}_e$, then the proof is similar.
- If $x \in \mathcal{X}_o$ and $y \in \mathcal{X}_e$, propagate x to s_1, so that s_1 has value 1 on even stages. Then propagate y to c in an odd number of stages through the subgraph, then propagate to d by the direct link between c and d. Doing so, d has value 0 at even stages. We are back to the previous case with $y' = d \in \mathcal{X}_o$ and $x' = s_1 \in \mathcal{X}_o$.

Let X and Y be two finite sets. We want to prove that $(X, Y)^+$ is reachable from ω'. X can be decomposed into $X_o = X \cap \mathcal{X}_o$ and $X_e = X \cap \mathcal{X}_e$ and Y can be decomposed into $Y_o = Y \cap \mathcal{X}_o$ and $Y_e = Y \cap \mathcal{X}_e$. We can now restrict ourselves to the subgraph $(\mathcal{X}, \approx)$ and apply the results of Section 5. □

7 Boolean aggregation functions

The aggregation function being Boolean, the diffusion model (1) becomes deterministic. In the previous sections, we used heavily the fact that the aggregation function is strict. Since it is not true anymore, we obtain very different results. Let us restrict our attention to anonymous functions. Then there exists a threshold $0 \leq \widetilde{q} \leq \gamma$ such that

$$A(1_S) = \begin{cases} 1 & \text{if } |S| \geq \widetilde{q} \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

This yields the contagion model of Morris (2000) which is also the threshold model introduced, e.g., by Granovetter (1978). In this model, the rule of contagion with threshold $0 \leq q \leq 1$ is the following. Given a configuration $X(t)$ at time t, next configuration $X(t+1)$ is the set of agents having a proportion of neighbors in $X(t)$ at least equal to q:

$$X(t+1) = \left\{ x \in \mathcal{X} \,:\, \frac{|\Gamma(x) \cap X(t)|}{|\Gamma(x)|} \geq q \right\}. \tag{6}$$

Comparing (6) with (1) and (5) yields the equivalence between the two models. The *contagion threshold* ξ is the largest q such that infection spreads over $\mathcal{X}$ from some finite group $X(0)$. Morris (2000) has shown that for any network (in the sense of Section 2.1), $\xi \leq \frac{1}{2}$.

By definition, if q is below the contagion threshold, then the absorbing states are the trivial states $\varnothing$ and $\mathcal{X}$. Otherwise, other nontrivial absorbing classes may occur, as will be shown.

Let us try to find first absorbing states. As we noted earlier, X is an absorbing state if it is a fixed point, i.e., $X(t+1) = X(t)$. We introduce the following convenient notation. The *frontier* of X is the set

$$\partial X = \overline{X} \setminus \mathring{X} = \{x \in \mathcal{X} \,:\, \Gamma(x) \cap X^c \neq \varnothing \text{ and } \Gamma(x) \cap X \neq \varnothing\}.$$

We also introduce $\partial^+ X = \partial X \cap X$ the *inner frontier* of X, and $\partial^- X = \partial X \cap X^c$ the *outer frontier* of X. The following results are immediate.

Proposition 13. Suppose A is an anonymous Boolean aggregation function. X is an absorbing state if and only if:

1. Each inner frontier node $x \in \partial^+ X$ has at least $\lceil \gamma q \rceil$ neighbors in X, and
2. Each outer frontier node $x \in \partial^- X$ has at least $\lfloor \gamma(1-q) \rfloor + 1$ neighbors in X^c.

Proof. X will not lose an element if each element of X has at least γq neighbors in X, and X will not attract a new element if each element of $\mathcal{X} \setminus X$ has strictly less than γq neighbors in X, hence at least $\lfloor \gamma(1-q) \rfloor + 1$ neighbors in $\mathcal{X} \setminus X$. □

We examine two particular networks from Example 1 for illustration.

The case $\mathbb{Z}^2$ with the 1-neighborhood Let us consider various values of q. The following table gives the minimal number of neighbors in X for frontier nodes, so that X is an absorbing state.

q	1/4	1/2	3/4	1
minimal nb of interior neighbors for nodes in X	1	2	3	4
minimal nb of interior neighbors for nodes outside X	4	3	2	1

Let us describe the shape of the frontier of a connected set X with $|X| > 1$ satisfying the above requirements. To this end, we consider a frontier node x of X and the number of its interior neighbors $\eta(x) := |\Gamma(x) \cap X|$.

1. $\eta(x) \geq 1$ for every frontier node x. This is true for any X such that $|X| \geq 2$.

2. $\eta(x) \geq 2$ for every frontier node x. Only the situation $\eta(x) = 1$ is forbidden, which corresponds to "antennas" (see Figure 7, left). Therefore, X is any shape without antennas, with $|X| \geq 3$.

3. $\eta(x) \geq 3$ for every frontier node x. Then additional forbidden situations are "convex corners" and "isthms" (see Figure 7). Then X is any shape without antennas, convex corners and isthms, and $|X| \geq 4$. Note that since X may be infinite, one can have shapes with concave corners (which are allowed) but without convex corners (see Figure 8).

4. Evidently, $\eta(x) \geq 4$ is impossible by definition of frontier nodes.

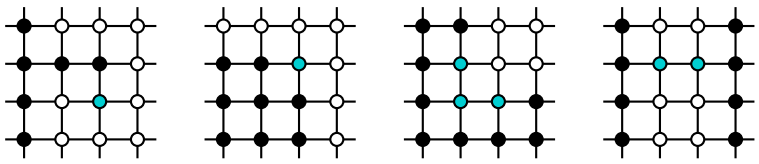


Figure 7: 4×4 portion of $\mathbb{Z}^2$, where X is in black or blue. By convention, the color of nodes extends infinitely in any direction. From left to right: antenna (x such that $\eta(x) = 1$ in blue), convex corner (x such that $\eta(x) = 2$ in blue), concave corner in blue, isthm (x such that $\eta(x) = 2$ in blue)

From the previous analysis, it follows that only $q = \frac{1}{2}$ or $\frac{3}{4}$ may lead to nontrivial absorbing classes. Also, these two cases are exact complements of each other, in the sense that X is a possible absorbing state for $q = \frac{1}{2}$ if and only if $\mathcal{X} \setminus X$ is a possible absorbing state for $q = \frac{3}{4}$. Taking for example the latter, each connected component of X should be of size at least 4, and should have no convex corner, no antenna and no isthm, while each connected component of the complement set should be of size at least 3 and have no antennas. Obvious examples are half-planes, concave corners extending infinitely in both directions, and infinite shapes with squared "holes" of width greater than 1 (see the example in Figure 8).

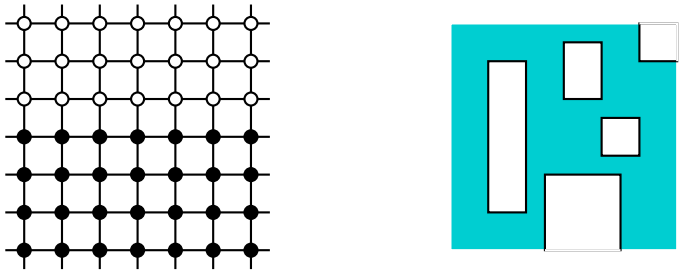


Figure 8: Examples of absorbing states with $q = \frac{3}{4}$. Left: horizontal half-plane. Right: shape with squared holes (in blue; the grid is not represented)

The case of $\mathbb{Z}^2$ with the $\sqrt{2}$-neighborhood We obtain the following table.

q	1/8	1/4	3/8	1/2	5/8	3/4	7/8
minimal nb of interior neighbors for nodes in X	1	2	3	4	5	6	7
minimal nb of interior neighbors for nodes outside X	8	7	6	5	4	3	2

The analysis of possible shapes in this topology reveals to be much more complex and we will not provide a full analysis as in the previous example. Obviously, $q = \frac{1}{8}$ is ruled out. If $\eta(x) = 7$, then the only possibility is that $\mathcal{X} \setminus X$ is a singleton. Since then $\mathcal{X} \setminus X$ cannot satisfy the condition $\eta(x) \geq 2$, this rules out $q = \frac{1}{4}$ and $q = \frac{7}{8}$. If $\eta(x) \geq 3$, as before, antennas and isthms of length greater than 2 are forbidden, but not convex corners. For $\eta(x) \geq 4$, squared convex corners are no more allowed (but 45° corners are possible). For $\eta(x) \geq 5$, no corners are allowed except concave ones. For $\eta(x) \geq 6$, a remarkable situation is where $\mathcal{X} \setminus X$ is a square with edges of length 1. Therefore, as possible absorbing states (we do not pretend to be exhaustive) we find:

1. $q = 3/8$: a connected component of X is a square with edge of length 1 (see Figure 9, left)

2. $q = 1/2$ or $q = 5/8$: the two cases are complements of each other. Taking the latter, a connected component of X is any shape of size at least 6 without convex corners, antennas nor isthms, and each connected component of the complement set should be of size at least 5 and have no antennas, isthms and squared convex corners. For example, half-planes are allowed, as well as infinite shapes with squared holes whose corners are "rounded" (see Figure 9, right).

3. $q = 3/4$: a connected component of $\mathcal{X} \setminus S$ is a square with edge of length 1.

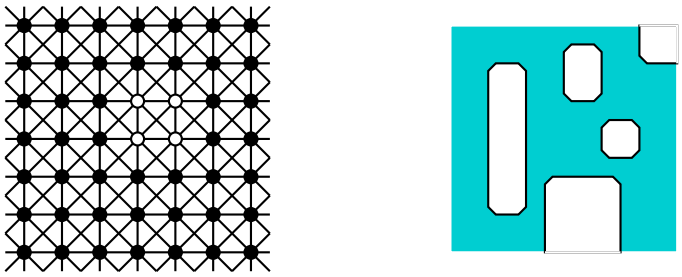


Figure 9: Left: example of absorbing state with $q = \frac{3}{4}$. Right: example with $q = \frac{5}{8}$ (in blue; the grid is not represented)

Other absorbing classes Let us reconsider the notion of trajectory. As the process is deterministic, a (deterministic) trajectory is merely a sequence of states $X_1, X_2, X_3, \ldots$. We show that cycles (periodic trajectories) and infinite (aperiodic) trajectories may exist. Figure 10 presents an example of a cycle, and Figure 11 an example of an infinite trajectory.

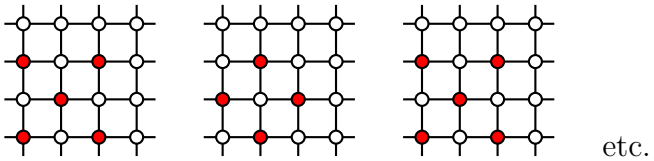


Figure 10: Example of a cycle with $\mathbb{Z}^2$ and the 1-neighborhood ($q = \frac{1}{2}$)

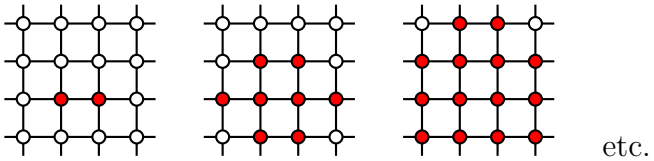


Figure 11: Example of an infinite trajectory with $\mathbb{Z}^2$ and the 1-neighborhood ($q = \frac{1}{4}$). The initial X constantly grows.

Proposition 14. Suppose A is an anonymous and Boolean aggregation function. Then absorbing classes are either singletons $\{X\}$, where $X \in 2^{\mathcal{X}}$, or cycles (periodic trajectories) of nonempty sets $\{X_1, \ldots, X_k\}$ with the condition that all sets are pairwise incomparable by inclusion.

Proof. Every transition being with probability 1, the evolution is a deterministic trajectory, hence a sequence of configurations $X_0, X_1, \ldots, X_k, \ldots$. Note that if a repetition occurs, then the trajectory is periodic. The following cases are exhaustive and exclusive:

1. Fixed points, $X_{k+1} = X_k$ for $k \geq K$ exist, and are absorbing states.
2. Cycles (periodic trajectories) exist, and are periodic absorbing classes.
3. Otherwise, the trajectory does not have any repetition and is infinite. But this cannot be a class, because there would be no transition from state X_{k+1} to X_k.

□

8 Nonstrict nonBoolean aggregation functions

Let us assume for simplicity that the aggregation function A is anonymous. A being nonstrict and nonBoolean implies that there exist $\ell, r \in \{0, \ldots, \gamma - 1\}$ such that

$$A(1_S) = \begin{cases} 0 & \text{iff } |S| \leq \ell \\ 1 & \text{iff } |S| \geq \gamma - r, \end{cases} \tag{7}$$

with $\ell \vee r > 0$ and $\ell + r < \gamma$. This case is more complex than the two previous ones. Indeed, it has the aspect both of the Boolean case and of the strict aggregation case.

In order to highlight this duality, let us consider the following example. Take $\mathbb{Z}^2$ with the 1-neighborhood and the following anonymous aggregation function

$$A(1_S) = \begin{cases} 0 & \text{iff } |S| = 0 \\ 1/2 & \text{iff } |S| = 1, \\ 1 & \text{iff } |S| \geq 2. \end{cases} \tag{8}$$

We see an asymmetry between the role of 0 and 1.

Proposition 5 that focuses on the persistence of the number of elements is only partially true here: the two first bullet points concerning the case with no active agents or the case with a finite number of active agents are still valid. On the contrary, the third one concerning the case with an infinite number of agents is not true anymore, as shown by the following configuration. Let ω be such that

$$\forall (n,m) \in \mathbb{Z}^2,\ \omega(n,m) = 0 \text{ if and only if } n+m \in 10\mathbb{Z}.$$

The transition is deterministic and leads to $\mathbf{1}$ since the inactive agents are all isolated. Proposition 6 that proves the possibility for a finite number of active agents to disappear is not true anymore, since a square of active agents will never disappear. On the contrary, with positive probability a finite set of inactive agents may disappear. Lemma 6 that studies the possibility for a trajectory of diffusion to exist between two configurations seems to be true for active agents but not true for inactive agents.

This makes the study of nonstrict nonBoolean aggregation function pretty tricky, and we leave it for further research. It will need to disentangle the condition for each of the lemmas developed for strict aggregation functions and boolean aggregation functions.

9 Related literature on diffusion and social learning

One of the early and widely used models of diffusion is the Bass model (Bass (1969)) which does not refer to any explicit network structure but incorporates some aspects of imitation. It uses two parameters: one capturing the rate of innovation or spontaneous adoption, and another one representing the rate of imitation or adoption due to others. Since the spread of ideas, products or rumors can be sometimes similar to that of infectious disease, many frameworks of diffusion in networks are inspired by the standard epidemiological models such as the SI, SIS, and SIR models. In the SIR model, any individual of a large finite population is in one of three possible states (susceptible, infected, or recovered), susceptible individuals can become infected, but after recovering they cannot become infected again. Contrary to that, in the SIS model, recovered individuals can become infected again, and in the SI framework once susceptible individuals become infected, they remain infected forever; for more details see e.g., Bailey (1975), Keeling and Rohani (2008), Newman (2010). Different variants of these models are proposed, in particular, in the literature on network diffusion in economics (e.g., Jackson and Rogers (2007), López-Pintado (2008)). Shah and Zaman (2016) consider the problem of detecting the source of a rumor which has spread in a network using only observations about the set of infected nodes, and no information about when the nodes became infected. They

establish the effectiveness of rumor centrality for source detection for generic random trees and the SI model with a generic spreading time distribution.

Other branches of the literature on diffusion in networks concern information cascades, social learning, as well as threshold models in which an individual adopts an action (e.g., a new technology or an opinion on a given subject) when it is adopted by a threshold number or fraction of the individual's neighbors; see e.g., Granovetter (1978), Schelling (1978), Watts (2002), Jackson and Yariv (2005), Jackson and Yariv (2007), Campbell (2013), Lim et al. (2016), Lamberson (2010). Some works consider diffusion models with heterogeneous agents with types possibly correlated with their neighbors (e.g., Galeotti and Rogers (2013), Jackson and López-Pintado (2013)). Numerous works are devoted to Markovian diffusion. Berkhout and Heidergott (2019) are interested in measuring connectivity of the weighted graph associated with a Markov chain. They introduce a new decomposition algorithm for Markov chains and discuss some applications of their decomposition algorithm, e.g., to influence ranking in social networks. The issues of network diffusion are also related to percolation theory from the physics and mathematics literature (e.g., Grimmett (1999), Stauffer and Aharony (1994)) whose basic question whether a liquid on one side of some substance will penetrate to another side can be seen as the question of existence of a path from one side of the network to another one. A different research question but somewhat related to diffusion concerns search and navigation on networks; see e.g., Adamic et al. (2003) for various navigation techniques.

There exists a vast literature on opinion formation and influence; see e.g., Acemoglu and Ozdaglar (2011) for an overview of two methods of modeling social learning through networks: Bayesian learning, where individuals use Bayes' rule to assess the state of the world (e.g., Acemoglu et al. (2011), Bala and Goyal (1998), Banerjee (1992), Banerjee and Fudenberg (2004), Bikhchandani et al. (1992), Ellison and Fudenberg (1993), Gale and Kariv (2003)) and non-Bayesian approach such as imitation models, where individuals consider a weighted average of their neighbors' opinions or actions in a previous period (e.g., DeGroot (1974), Golub and Jackson (2010)). A branch of works in the literature on Bayesian updating and information transmission studies the phenomenon of rumours, see e.g., Banerjee (1993), Bloch et al. (2018). Obviously, the present paper is fairly related to this branch of literature.

Besides Morris (2000) many other studies concern local interaction games, contagion and spreading of some behavior due to local interaction, and the contagion threshold; see e.g., Blume (1995), Ellison (1993), Ellison (2000), Lee and Valentinyi (2000). Numerous works on contagion are devoted to financial networks (e.g., Allen and Gale (2000), Acemoglu et al. (2015), Cabrales et al. (2017), Elliott et al. (2014), Demange (2018); see also Cabrales et al. (2016) for an overview). Another stream of the related literature concerns the theory of conflict (e.g., Baccara and Bar-Isaac (2008)), including studies of optimal network design and defence in a setting with strategic conflict and contagion dynamics (see e.g., Goyal and Vigier (2014)). Also the electrical engineering and computer sciences literature deals with the problem of network design and defence (see, e.g., Alpcan and Basar (2011) and Anderson (2008) for an overview). Klausen et al. (2018) study the issue of threats posed by online extremists in social networks, by using Twitter accounts related to the Islamic State in Iraq and Syria. They present a model for optimal searching the social network to find suspended users new accounts.

Acknowledgment

M. Grabisch and A. Rusinowska acknowledge the financial support by the National Agency for Research, Project DynaMITE (ANR-13-BSH1-0010-01). X. Venel acknowledges the financial support by the National Agency for Research, Project CIGNE (ANR-15-CE38-0007-01). The authors thank Matthew Jackson and Ron Peretz for fruitful discussions.

References

D. Acemoglu and A. Ozdaglar. Opinion dynamics and learning in social networks. *Dynamic Games and Applications*, 1:3–49, 2011.

D. Acemoglu, M. Dahleh, I. Lobel, and A. Ozdaglar. Bayesian learning in social networks. *Review of Economic Studies*, 78:1201–1236, 2011.

D. Acemoglu, A. Ozdaglar, and A. Tahbaz-Salehi. Systemic risk and stability in financial networks. *American Economic Review*, 105(2):564–608, 2015.

L. A. Adamic, R. M. Lukose, and B. A. Huberman. Local search in unstructured networks. In S. Bornholdt and H. G. Schuster, editors, *Handbook of Graphs and Networks: From the Genome to the Internet*. Berlin: Wiley VCH, 2003.

C. Aliprantis and K. Border. *Infinite Dimensional Analysis*. Springer, 2006. 3rd edition.

F. Allen and D. Gale. Financial contagion. *Journal of Political Economy*, 108:1–33, 2000.

T. Alpcan and T. Basar. *Network Security: A Decision and Game Theoretic Approach*. Cambridge, Cambridge University Press, 2011.

R. Anderson. *Security Engineering*. Wiley, 2008. 2nd edition.

M. Baccara and H. Bar-Isaac. How to organize crime? *Review of Economic Studies*, 75: 1039–1067, 2008.

N. Bailey. *The Mathematical Theory of Infectious Diseases and its Applications*. London: Charles Griffin and Company Ltd, 1975.

V. Bala and S. Goyal. Learning from neighbours. *Review of Economic Studies*, 65(3): 595–621, 1998.

A. V. Banerjee. A simple model of herd behavior. *Quarterly Journal of Economics*, 107(3):797–817, 1992.

A. V. Banerjee. The economics of rumours. *Review of Economic Studies*, 60:309–327, 1993.

A. V. Banerjee and D. Fudenberg. Word-of-mouth learning. *Games and Economic Behavior*, 46:1–22, 2004.

F. M. Bass. A new product growth model for consumer durables. *Management Science*, 15:215–227, 1969.

J. Berkhout and B. F. Heidergott. Analysis of Markov influence graphs. *Operations Research*, 67(3):892–904, 2019.

S. Bikhchandani, D. Hirshleifer, and I. Welch. A theory of fads, fashion, custom, and cultural change as informational cascades. *Journal of Political Economy*, 100:992–1026, 1992.

F. Bloch, G. Demange, and R. Kranton. Rumors and social networks. *International Economic Review*, 59:421–448, 2018.

L. E. Blume. The statistical mechanics of best-response strategy revision. *Games and Economic Behavior*, 11:111–145, 1995.

Y. Bramoullé, A. Galeotti, and B. W. Rogers, editors. *The Oxford Handbook of the Economics of Networks*. Oxford University Press, 2016.

A. Cabrales, D. Gale, and P. Gottardi. Financial contagion in networks. In Y. Bramoullé, A. Galeotti, and B. W. Rogers, editors, *The Oxford Handbook of the Economics of Networks*, pages 543–568. Oxford University Press, 2016.

A. Cabrales, P. Gottardi, and F. Vega-Redondo. Risk-sharing and contagion in networks. *Review of Financial Studies*, 30(9):3086–3127, 2017.

A. Campbell. Word-of-mouth communication and percolation in social networks. *American Economic Review*, 103(6):2466–2498, 2013.

M. H. DeGroot. Reaching a consensus. *Journal of the American Statistical Association*, 69:118–121, 1974.

G. Demange. Contagion in financial networks: a threat index. *Management Science*, 64(2):955–970, 2018.

R. Diestel. *Graph Theory (Graduate Texts in Mathematics)*. Springer, 2005.

M. Elliott, B. Golub, and M. O. Jackson. Financial networks and contagion. *American Economic Review*, 104(10):3115–3153, 2014.

G. Ellison. Learning, local interaction, and coordination. *Econometrica*, 61:1047–1072, 1993.

G. Ellison. Basins of attraction, long run equilibria, and the speed of step-by-step evolution. *Review of Economic Studies*, 67:17–45, 2000.

G. Ellison and D. Fudenberg. Rules of thumb for social learning. *Journal of Political Economy*, 101(4):612–643, 1993.

M. Förster, M. Grabisch, and A. Rusinowska. Anonymous social influence. *Games and Economic Behavior*, 82:621–635, 2013.

D. Gale and S. Kariv. Bayesian learning in social networks. *Games and Economic Behavior*, 45(2):329–346, 2003.

A. Galeotti and B. W. Rogers. Strategic immunization and group structure. *American Economic Journal: Microeconomics*, 5(2):1–32, 2013.

B. Golub and M. O. Jackson. Naïve learning in social networks and the wisdom of crowds. *American Economic Journal: Microeconomics*, 2(1):112–149, 2010.

B. Golub and E. Sadler. Learning in social networks. In Y. Bramoullé, A. Galeotti, and B. W. Rogers, editors, *The Oxford Handbook of the Economics of Networks*, chapter 19, pages 504–542. Oxford University Press, 2016.

S. Goyal and A. Vigier. Attack, defence, and contagion in networks. *Review of Economic Studies*, 81(4):1518–1542, 2014.

M. Grabisch and A. Rusinowska. A model of influence based on aggregation functions. *Mathematical Social Sciences*, 66:316–330, 2013.

M. Granovetter. Threshold models of collective behavior. *American Journal of Sociology*, 83:1420–1443, 1978.

G. Grimmett. *Percolation*. Heidelberg: Springer-Verlag, 1999.

O. Hernández-Lerma and J. B. Lasserre. *Markov Chains and Invariant Probabilities*. Springer-Verlag, Basel, 2003.

M. O. Jackson. *Social and Economic Networks*. Princeton University Press, 2008.

M. O. Jackson and D. López-Pintado. Diffusion and contagion in networks with heterogenous agents and homophily. *Network Science*, 1(1):49–67, 2013.

M. O. Jackson and B. W. Rogers. Relating network structure to diffusion properties through stochastic dominance. *The B.E. Journal of Theoretical Economics (Advances)*, 7(1):1–13, 2007.

M. O. Jackson and L. Yariv. Diffusion on social networks. *Economie Publique*, 16:69–82, 2005.

M. O. Jackson and L. Yariv. Diffusion of behavior and equilibrium properties in network games. *American Economic Review*, 97(2):92–98, 2007.

O. Kallenberg. *Foundations of Modern Probability*. Springer-Verlag, New York, 1997.

M. Keeling and P. Rohani. *Modeling Infectious Diseases in Humans and Animals*. Princeton University Press, 2008.

J. Klausen, C. E. Marks, and T. Zaman. Finding extremists in online social networks. *Operations Research*, 66(4):957–976, 2018.

P. J. Lamberson. Social learning in social networks. *The B.E. Journal of Theoretical Economics*, 20(1):36, 2010.

P. J. Lamberson. Diffusion in networks. In Y. Bramoullé, A. Galeotti, and B. W. Rogers, editors, *The Oxford Handbook of the Economics of Networks*, pages 479–503. Oxford University Press, 2016.

I. Lee and A. Valentinyi. Noisy contagion without mutation. *Review of Economic Studies*, 67:47–56, 2000.

Y. Lim, A. Ozdaglar, and A. Teytelboym. A simple model of cascades in networks. Working Paper, 2016.

D. López-Pintado. Diffusion in complex social networks. *Games and Economic Behavior*, 62(2):573–590, 2008.

S. Morris. Contagion. *Review of Economic Studies*, 67:57–78, 2000.

M. E. J. Newman. *Networks: An Introduction*. Oxford University Press, 2010.

E. M. Rogers. *Diffusion of Innovations*. Simon and Schuster, 5th edition, 2003.

T. Schelling. *Micromotives and Macrobehavior*. Norton, 1978.

D. Shah and T. Zaman. Finding rumors sources on random trees. *Operations Research*, 64(3):736–755, 2016.

G. Silverman. *The Secrets of Word-of-Mouth Marketing: How to Trigger Exponential Sales through Runaway Word of Mouth*. New York: Amacom Books, 2001.

D. Stauffer and A. Aharony. *Introduction to Percolation Theory*. CRC Press (online), 1994. 2nd edition.

D. Watts. A simple model of global cascades on random networks. *Proceedings of the National Academy of Sciences*, 99(9):5766–5771, 2002.