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Abstract

This paper examines the valuation of American knock-out and knock-in step options. The structures of the immediate exercise regions of the various contracts are identified. Typical properties of American vanilla calls, such as uniqueness of the optimal exercise boundary, up-connectedness of the exercise region or convexity of its t-section, are shown to fail in some cases. Early exercise premium representations of step option prices, involving the Laplace transforms of the joint laws of Brownian motion and its occupation times, are derived. Systems of coupled integral equations for the components of the exercise boundary are deduced. Numerical implementations document the behavior of the price and the hedging policy. The paper is the first to prove that finite maturity exotic American Options written on a single underlying asset can have multiple disconnected exercise regions described by a triplet of coupled boundaries.

Keywords: Risk management; American options; Step options; Occupation time; Multiple exercise boundaries.

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1. Introduction

Step options, introduced by Linetsky (1999), are modified versions of single barrier options.[1] A step option, parametrized by a knock-out (knock-in) rate, is a contract that gradually loses (gains) value depending on the cumulative excursion time of the underlying asset above or below a given barrier[2]. The payoff is the same as that of the plain vanilla counterpart, except that it is discounted (appreciated) by a *knock-out* (*knock-in*) factor. When the adjustment factor is an exponential (linear) function of the occupation time, the contract is a *proportional* (*simple*) step option. A standard knock-out (knock-in) step option is cheaper (more expensive) than its plain vanilla counterpart. Moreover, the gradual adjustment of the option ensures continuity at the barrier, which is important from a risk management perspective. In a way, step options can be viewed as "limited regrets" alternatives to one-touch barrier options. Step options are mainly traded over-the-counter. They can serve as a benchmark for the analysis and the design of certain classes of occupation-time derivatives and related structured products. They can also be used to model and help financial decision-making.

Linetsky (1999) provides valuation formulas and Greeks, such as Delta and Gamma, for European-style step options. For these contracts, the adjustment factor is based on the occupation time from the inception date to the maturity date. Closed-form formulas for European-style double barrier step options are obtained by Davydov & Linetsky (2002). European Step options are also studied under a class of nonlinear volatility diffusions by Campolieti, Makarov, & Wouterloot (2013) and hyper-exponential jump-diffusion processes by Wu & Zhou (2016). Xing & Yang (2013) consider American-style step options, for which the adjustment factor is based on the occupation time from the inception date to the exercise date. They provide various analytical formulas that are useful for pricing perpetual American (up-and-out) step put options only. In the finite maturity case, they resort to a PDE approach and a finite difference procedure to characterize and compute the step option price.[3] Rodosthenous & Zhang (2018) examine a class of American knock-out barrier options where the random expiration date is determined by the time spent below a threshold and the underlying asset price follows a spectrally negative Levy process. In their setting the knock-out time has an exponential distribution independent of the underlying price, so that the problem can be recast as a perpetual American step option pricing problem. They characterize the value of the

[1] Barrier options are popular instruments that have been extensively studied. Standard references include Merton (1973), Rubinstein & Reiner (1991), Geman & Yor (1996) and Schröder (2000). Recent contributions can be found in Lin & Palmer (2013) and Golbabai, Ballestra, & Ahmadian (2014). See also Grant, Vora, & Weeks (1997), Davidov & Linetsky (2001) and Kaishev & Dimitrova (2009), among others, for the family of path-dependent options. The pricing of exotic options such as barrier options is an important topic that is examined in the operations research literature; see e.g., Kou (2007), Feng & Linetsky (2008), Cai, Chen, & Wan (2009), Dingec & Hörmann (2012), Giesecke & Smelov (2013), Jin, Li, Tan, & Wu (2013), Wang & Tan (2013), Sesana, Marazzina, & Fusai (2014), Date & Islyaev (2015), Fusai, Germano, & Marazzina (2016), Phelan, Marazzina, Fusai, & Germano (2018).

[2] Step options belong to the class of occupation time derivatives. Other examples of contracts involving occupation times include quantile options, Parisian options and corridor options (see Dassios (1995), Chesney, Jeanblanc-Picqué, & Yor (1997) and Fusai (2000); see also Section A.2.4 in Broadie & Detemple (2004)).

[3] In their introduction, Xing & Yang (2013) stress that "unlike the perpetual case, the closed form of the pricing formula is impossible." Our paper provides an explicit formula for the price of a finite maturity American step option, parametrized by the relevant optimal exercise boundaries.

contract and show that the optimal exercise region consists of bands, which can be disconnected.

This paper focuses on *American step options* with proportional adjustment factors that can either deflate or inflate the exercise payoff, when the underlying price follows a geometric Brownian motion process. It extends the literature in several directions. First, it provides a comprehensive analysis of the exercise regions of all types of proportional step call option contracts. This analysis reveals the possibility of non-standard features. It shows, in particular, that the exercise region of a step call can fail to be up-connected. That is, immediate exercise, although optimal at a given underlying asset price, can cease to be optimal if the underlying price increases. The exercise region can also be composed of several disconnected sub-regions. Immediate exercise can then be optimal at low and high underlying prices, yet may fail to be the best policy at intermediate prices. In such cases, the exercise region cannot be described by a single exercise boundary, instead multiple boundaries are required.

Second, the article derives analytic valuation formulas that identify the premium associated with early exercise. These Early Exercise Premium (EEP) representations of step option prices rely on the structural properties of the exercise regions uncovered.[4] The formulas depend on the Laplace transforms of the joint laws of Brownian motion and its excursion times. When the exercise region consists of disconnected sub-regions, the EEP has several corresponding components.

Third, it shows that optimal exercise boundaries, satisfy recursive integral equations. This characterization is a by-product of the EEP representation, following from the fact that immediate exercise is optimal on the boundary. In the case of multiple boundaries, a system of coupled integral equations is obtained. Coupling implies that each boundary component is affected by the others.

Fourth, it develops a numerical approach for the resolution of such a system of recursive integral equation. The approach exploits the fact that the value of the step option on any particular exercise boundary, depends on other boundary components only through their future values. This follows from the fact that the underlying asset price can only be at one point at any given time. Other boundary components matter because the underlying price can cross other boundaries at future times. Given this local separation (uncoupling) property, the algorithm solves for the different boundary components sequentially, at any given time, taking future boundary values as given. It then proceeds recursively through time following a standard backward iteration procedure.

The paper relates to several other branches of the literature. First, it complements a literature dealing with unusual features of exercise regions associated with real options or financial derivatives. Battauz, De Donno, & Sbuelz (2012, 2015) and De Donno, Palmowski, & Tumilewicz (2019), for instance, examine economic settings leading to negative discount rates and show that a double continuation region can then arise.[5] Under such circumstances, it is optimal to wait if the underlying asset price is either high or low. If the current price is high, immediate exercise becomes optimal

[4]EEP representation formulas were introduced by Kim (1990), Jacka (1991) and Carr, Jarrow, & Myneni (1992) in order to price American vanilla options in the standard market model with constant coefficients.

[5]Gold loans, i.e., loans collateralized by gold, are popular instruments in India, that fall in this category. Gold loans often entail a prepayment provision for the borrower, hence are American options. The effective discount rate is negative if the borrowing rate exceeds the riskfree rate, as is typically the case. See Battauz et al. (2015) for details.

if the price decreases and hits an upper boundary B^u. If it is low, it is optimal to exercise at the first hitting time of a lower boundary $B^d < B^u$. The immediate exercise region is therefore unique and corresponds to the region in between the two boundaries. A double continuation region can also arise with positive interest rate for capped options: Broadie & Detemple (1995) document this phenomenon for options with growing caps and Detemple & Kitapbayev (2018) for two-level caps. The case of step options studied here differs significantly from the above literature. In particular, it admits, under specific circumstances, a double exercise region that requires the determination of three exercise boundaries. Immediate exercise is then optimal at the first time any of the three boundaries is hit. Rodosthenous & Zhang (2018) also obtain a double exercise region in their infinite horizon version of the problem. One difference with the present study is that their exercise boundaries are constant. Another difference is that the price can jump over the intermediate region. In our setting the boundaries are curves that depend on time and the intermediate region only appears starting at an endogenous time.

Second, the paper may help to shed light on economic phenomena where discounting is affected by the path of the underlying process. The valuation of Executive Stock Options - hereafter ESOs -, which is in the realm of real options, is an example in this vein. Here, the executive may decide to exercise her finitely-lived ESOs prematurely and leave the firm if an interesting opportunity arises. If the likelihood of this event, i.e., the intensity rate, depends on the time spent by the underlying price above a barrier, the executive holds a step option with an automatic exercise provision.[6,7] Contracts of this type are valued by Carr & Linetsky (2000). Another relevant application concerns R&D projects. Here, the likelihood of achieving success before a competitor does can depend on the ability of the firm to invest resources in the discovery process. If performance is poor, for instance if the stock price wanders below a threshold for extensive periods of time, the firm may have to curtail expenses or refocus on core activities, both of which raise the likelihood of failure. Conversely, solid performance may raise the likelihood of success as more resources are devoted to research activities. The evaluation of such a project entails the simultaneous consideration of both types of events. In both of these examples, the payoff depends on the occupation time(s) of some set(s). Optionality of decisions implies that the holder of the claim, i.e., the executive in the first example or the firm in the second one, can decide to optimally exercise or discontinue their operations. Such decision-making problems are similar to the American step option problem examined in this paper. The comprehensive review article by Trigeorgis & Tsekrekos (2018) provides perspective on additional potential applications to decision problems in real options contexts.

The rest of this paper is organized as follows. Section 2 presents the model and the valuation problem. Section 3 describes properties of the exercise region and of the step option price. Section 4 provides valuation formulas and integral equations for the exercise boundary components. Section 5 discusses hedging policies. The numerical procedure is presented and applied in Section 6.

[6]The likelihood of receiving an outside offer increases if the firm performs well, above some threshold, and with the duration of this outperformance.

[7]An alternative motivation for this specification rests on the existence of liquidity needs and the increasing desire to cash in as the time spent above the strike increases.

Conclusions follow. Appendix A states background results pertaining to the density of a Brownian motion killed at some rate below zero. Appendix B presents the proofs of the theorems in Section 3. Appendix C provides technical results about exercise boundaries. Appendix D describes properties of the numerical scheme for implementation.

2. The model

We consider the standard frictionless market with continuous trading. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, $\widetilde{W}$ a standard $\mathbb{P}$-Brownian and $\mathcal{F}_{(\cdot)}$ the $\mathbb{P}$-augmentation of the filtration generated by $\widetilde{W}$. There are two assets, a risky asset and a riskless asset. The riskless asset pays interest at the constant and positive rate r. The risky asset price follows a geometric Brownian motion, $dS_t/S_t = (\mu - \delta)\, dt + \sigma d\widetilde{W}_t$, with constant parameters (μ, δ, σ). The coefficient μ is the asset's expected (total) rate of return, σ its standard deviation and δ its dividend yield, assumed to be non-negative. The market model is denoted by $\mathcal{M}(r, \delta, \sigma)$.

It is well known that there exists an *equivalent martingale (risk neutral) measure* $\mathbb{Q}$ such that,

$$\frac{dS_t}{S_t} = (r - \delta)\, dt + \sigma dW_t$$

where W is a $\mathbb{Q}$-Brownian motion. The filtration generated by W is also $\mathcal{F}_{(\cdot)}$. The price at time t can be written as $S_t = S_0 N_{0,t}$ where $N_{v,s} \equiv e^{\alpha\sigma(s-v)+\sigma(W_s - W_v)}$ and $\alpha\sigma = r - \delta - \sigma^2/2$. The cumulative time spent by S above $(^+)$ or below $(^-)$ a constant barrier H, within a given time period $[s,t]\,, s \leq t$, is,

$$O^{\pm}_{s,t} \equiv O_{s,t}\left(S, A^{\pm}(H)\right) = \int_s^t 1_{\{S_v \in A^{\pm}(H)\}} dv$$

where $A^{\pm}(H) \equiv \{x \in \mathbb{R}_+ : \pm (x - H) \geq 0\}$. The cumulative excursion time $O^{\pm}_{0,t}$ is additive, so that $O^{\pm}_{0,t} = O^{\pm}_{0,v} + O^{\pm}_{v,t}$ for any $v \in [0,t]$. In the sequel, the generic notation $O_{0,t}$, without superscript, is used for statements that apply to both cases or when the intended case is clear from the context. The terminologies "cumulative excursion time"and "occupation time" are used interchangeably.

A *proportional step option* (or step option for short) initiated at date 0 with maturity date T, strike price K and constant excursion barrier H has an exercise payoff equal to,

$$e^{-\rho O_{0,\tau}} (S_\tau - K)^+ \ \text{(call)}, \qquad e^{-\rho O_{0,\tau}} (K - S_\tau)^+ \ \text{(put)}$$

at the exercise date $\tau \in \mathcal{T}$, where $\rho \in \mathbb{R}$ is a depreciation/appreciation rate. When the set of possible exercise dates is $\mathcal{T} = \{T\}$, the step option is European-style. When it is the set of stopping times $\mathcal{T} = \mathcal{S}([0,T])$ of the filtration taking values between 0 and T, the step option is American-style. The exercise payoff of a step option is the same as the payoff of a vanilla option, except that it is deflated by a factor $e^{-\rho O_{0,\tau}}$ when the knock-out rate ρ is positive $(\rho \geq 0)$ or inflated by the same factor when the knock-in rate ρ is negative $(\rho < 0)$. In both cases, the factor depends exponentially on the cumulative excursion time above or below a given barrier during the entire life of the option.

So, when the rate ρ is positive, the deflating factor is similar to a discount factor with stochastic continuously compounded discount rate $\rho O_{0,\tau}$. When ρ is negative, the inflating factor corresponds to an appreciation factor with stochastic continuously compounded appreciation rate $-\rho O_{0,\tau}$.

Standard valuation principles show that the value of a European step call is,

$$esc\left(S_t,t\right)=E_t^*\left[b_{t,T}e^{-\rho O_{0,T}}\left(S_T-K\right)^+\right]=e^{-\rho O_{0,t}}E_t^*\left[b_{t,T}e^{-\rho O_{t,T}}\left(S_T-K\right)^+\right] \tag{1}$$

where $b_{t,T}\equiv e^{-r(T-t)}$ and $E_t^*\left[\cdot\right]=E^*\left[\cdot|\,\mathcal{F}_t\right]$ is the expectation under the risk neutral measure.[8] The second equality above follows from the additivity of cumulative excursion times. European step options are priced by Linetsky (1999). For $t=0$, he shows that,

$$\begin{aligned}esc(S,0)&=E_x\left[b_{0,T}e^{\alpha\left(W_T'-x\right)-\alpha^2T/2-\rho O_{0,T}}\left(He^{\sigma W_T'}-K\right)1_{\left\{W_T'>k'\right\}}\right]\\&=e^{-\gamma T-\alpha x}\left[H\Psi_\rho'\left(\alpha+\sigma;k',x,T\right)-K\Psi_\rho'\left(\alpha;k',x,T\right)\right]\end{aligned} \tag{2}$$

where $k'=\frac{1}{\sigma}\log\frac{K}{H}$, $x=\frac{1}{\sigma}\log\frac{S}{H}$, $\gamma=r+\frac{\alpha^2}{2}$, and W' is a Brownian motion starting at x at time $t=0$. The conditional expectation $E_x\left[\cdot\right]$ is associated with W' and

$$\Psi_\rho'\left(\alpha;k',x,T\right)=E_x\left[e^{\alpha W_T'-\rho O_{0,T}}1_{\left\{W_T'>k'\right\}}\right]=\int_{k'}^{\infty}e^{\alpha s}E_x\left[e^{-\rho O_{0,T}};W_T'\in ds\right]. \tag{3}$$

Performing a change of variables leads to a representation similar to the standard Black-Scholes call option price formula,

$$\begin{aligned}esc(S,0)&=E^*\left[b_{0,T}e^{-\rho O_{0,T}}\left(S_T-K\right)^+\right]\\&=e^{-\gamma T}\left[S\Psi_\rho\left(\alpha+\sigma;k,h,T\right)-K\Psi_\rho\left(\alpha;k,h,T\right)\right]\end{aligned} \tag{4}$$

where

$$\Psi_\rho\left(\alpha;k,h,T\right)=\int_k^\infty e^{\alpha s}E\left[e^{-\rho O_{0,T}^h};W_T\in ds\right] \tag{5}$$

with $k=\frac{1}{\sigma}\log\frac{K}{S}$, $h=\frac{1}{\sigma}\log\frac{H}{S}$, $\gamma=r+\frac{\alpha^2}{2}$, W a standard Brownian motion, $E[\cdot]$ its associated conditional expectation, and $O_{0,T}^h$ its cumulative occupation time below or above h. Formula (5) requires the calculation of $E\left[e^{-\rho O_{0,T}^h};W_T\in ds\right]$, whose Laplace transform can be derived by using results from Feynman-Kac (see Kac (1949, 1951, 1980)). Inverting the Laplace transform gives expressions for Ψ. The function Ψ is given in Proposition 1 (See Appendix A). In order to distinguish between *up-and-out* and *down-and-out* or *up-and-in* and *down-and-in* step calls, it suffices to substitute $O_{0,T}^+,\Psi_\rho^+$ or $O_{0,T}^-,\Psi_\rho^-$ in the relevant places.

Let $SC\left(S,O,t\right)$ be the value at t of an American step call initiated at date 0. Standard results

[8] The right hand side of (1), $E_t^*\left[b_{t,T}e^{-\rho O_{t,T}}\left(S_T-K\right)^+\right]$, is the price at time t of a European step option initiated at t, with maturity T when the underlying asset price is S_t. It is identical to $E^*\left[b_{0,T-t}e^{-\rho O_{0,T-t}}\left(S_{T-t}-K\right)^+\right]$, the price at time 0 of a step option with maturity $T-t$ when the underlying price is $S_0\equiv S_t$.

can be invoked to write (see Karatzas (1988)),

$$SC(S,O,t) = \sup_{\tau \in \mathcal{S}([t,T])} E_t^* \left[e^{-r(\tau - t) - \rho O_{0,\tau}} (S_\tau - K)^+ \right].$$

Equivalently, using $O_{0,\tau} = O_{0,t} + O_{t,\tau} \equiv O + O_{t,\tau}$,

$$SC(S,O,t) = \exp(-\rho O) \times \sup_{\tau \in \mathcal{S}([t,T])} E^* \left[e^{-r(\tau - t) - \rho O_{t,\tau}} (S N_{t,\tau} - K)^+ \right].$$

where the expectation is unconditional because the (non overlapping) increments of a Brownian motion are independent. Let $sc(S,t)$ be the value of an American step call with the same contractual characteristics (H, K, T), but initiated at date t. If the first contract is still alive, the two values are related by,

$$SC(S,O,t) = \exp(-\rho O)\, sc(S,t) \equiv \exp(-\rho O_{0,t})\, sc(S,t)$$

where,

$$\begin{aligned} sc(S,t) &= \sup_{\tau \in \mathcal{S}([t,T])} E_t^* \left[e^{-r(\tau - t) - \rho O_{t,\tau}} (S_\tau - K)^+ \right] \\ &= \sup_{\tau \in \mathcal{S}([t,T])} E^* \left[e^{-r(\tau - t) - \rho O_{t,\tau}} (S N_{t,\tau} - K)^+ \right]. \end{aligned}$$

When $\rho \geq 0$, these options are knock-out step calls. When $\rho < 0$, they are knock-in step calls.

3. Properties of American step options

Knock-out step options are examined first. Properties of immediate exercise regions and price functions are described in Sections 3.1 and 3.2. Knock-in step options are studied in Section 3.3. Proofs of the theorems in this section are in Appendix B.

To set the stage, it is useful to recall a few properties of standard American call options whose prices in the market $\mathcal{M}(r, \delta, \sigma)$ are generically denoted by $C(S,t)$. Let,

$$\mathcal{E}^c \left\{ (S,t) \in \mathbb{R}_+ \times [0,T] : C(S,t) = (S-K)^+ \right\}$$

be the immediate exercise region of a standard American call option. It is then well known that $\mathcal{E}^c$ is non-empty, closed, up-connected and right-connected (see Detemple (2006)). Non-emptiness of $\mathcal{E}^c$ ensures that immediate exercise is optimal at some point $(S,t) \in \mathbb{R}^+ \times [0,T]$. That $\mathcal{E}^c$ is closed means that immediate exercise is optimal at (S,t) if it is optimal at each point of any sequence $\{(S_n, t_n) : n \in \mathbb{N}\}$ belonging to $\mathcal{E}^c$ that converges to (S,t). Up-connectedness guarantees that it remains optimal to exercise when the underlying price increases to $S' \geq S$, if it is optimal at (S,t). Finally, right-connectedness ensures that it remains optimal to exercise if time moves forward, i.e., at (S, t') where $t' > t$, if it is at (S, t). This last property is intuitive because the holder of a shorter maturity option has less opportunities to exercise, hence cannot do better than the holder of the longer maturity option, given the same initial circumstances (S, t).

Consider now a market $\mathcal{M}(r + \rho, \delta + \rho, \sigma)$ with interest rate $r + \rho$ and dividend yield $\delta + \rho$ and denote by $C(S, t; \rho)$ the price of a similar standard American call option. It is worth stressing that the risk neutral dynamics of the asset price in $\mathcal{M}(r + \rho, \delta + \rho, \sigma)$ is the same as in $\mathcal{M}(r, \delta, \sigma)$. However, the discount factor is significantly different. When ρ is positive (respectively negative), the preference for immediacy is greater (respectively lower) in $\mathcal{M}(r + \rho, \delta + \rho, \sigma)$ than it is in $\mathcal{M}(r, \delta, \sigma)$. Now, if $\mathcal{E}^c_\rho$ stands for the immediate exercise region of this American call option, then, when $\rho \geq 0$ (respectively $\rho < 0$), $\mathcal{E}^c \subseteq \mathcal{E}^c_\rho$ (respectively $\mathcal{E}^c_\rho \subseteq \mathcal{E}^c$). Moreover, one has $\mathcal{E}^c = \mathcal{E}^c_0$.

3.1. *Knock-out step options: immediate exercise region*

The immediate exercise region of the knock-out step call ($\rho \geq 0$),

$$\mathcal{E}^{sc} = \left\{(S, O, t) \in \mathbb{R}_+ \times [0, T] \times [0, T] : SC(S, O, t) = e^{-\rho O}(S - K)^+\right\}$$

is the set of points (S, O, t) where immediate exercise is optimal. The continuation region $\mathcal{C}^{sc}$ is the complement of $\mathcal{E}^{sc}$. Our first lemma provides a useful reduction in the dimensionality of the problem. The result follows from the relation $SC(S, O, t) = \exp(-\rho O)\, sc(S, t)$.

Lemma 3.1. *i)* $(S, O, t) \in \mathcal{E}^{sc}$ *if and only if* $(S, t) \in \mathcal{E}^{sc}_o$ *where,*

$$\mathcal{E}^{sc}_o = \left\{(S, t) \in \mathbb{R}_+ \times [0, T] : sc(S, t) = (S - K)^+\right\}.$$

ii) $\mathcal{E}^{sc} = \{(S, O, t) \in \mathbb{R}_+ \times [0, T] \times [0, T] : (S, t) \in \mathcal{E}^{sc}_o\}$.

The continuation region associated with sc, i.e., the complement of $\mathcal{E}^{sc}_o$, is $\mathcal{C}^{sc}_o$. Let $\mathcal{E}^{sc}_o(t)$ be the t-section of $\mathcal{E}^{sc}_o$, i.e., the set of underlying prices such that immediate exercise is optimal at time t. Mathematically, $\mathcal{E}^{sc}_o(t) = \left\{S \in \mathbb{R}_+ : sc(S, t) = (S - K)^+\right\}$ for given $t \in [0, T]$.

Knock-out step calls can be down-and-out or up-and-out, depending on whether the occupation time tallies the time spent below or above the barrier H. Where relevant, superscripts are used to indicate the nature of the contract (e.g., $\mathcal{E}^{dosc}_o, \mathcal{C}^{dosc}_o, \mathcal{E}^{uosc}_o, \mathcal{C}^{uosc}_o$).

To simplify notation, define the ratios $\kappa \equiv r/\delta$, $\kappa_\rho \equiv (r + \rho)/(\delta + \rho)$ and let $x \vee y = \max\{x, y\}$, $x \wedge y = \min\{x, y\}$. Also, for prices S_1, S_2, let $S^\lambda \equiv \lambda S_1 + (1 - \lambda) S_2$. Finally note that $\kappa \geq 1 \Longleftrightarrow \kappa \geq \kappa_\rho \geq 1$ and $\kappa \leq 1 \Longleftrightarrow \kappa \leq \kappa_\rho \leq 1$.

Theorem 3.1. *The exercise region of a knock-out step call option,* $\mathcal{E}^{sc} = [0, T] \times \mathcal{E}^{sc}_o$*, has the following properties*
(i) Non-emptiness: $\mathcal{E}^c \subseteq \mathcal{E}^{sc}_o \subseteq \mathcal{E}^c_\rho$ *where* $\mathcal{E}^c \neq \varnothing$
(ii) Right-connectedness: $(S, t) \in \mathcal{E}^{sc}_o \Longrightarrow (S, v) \in \mathcal{E}^{sc}_o$ *for* $t \leq v \leq T$

(iii) Up-connectedness and subregion connectedness:

(iii.a) $(S,t) \in \mathcal{E}_o^{uosc} \Longrightarrow (\lambda S,t) \in \mathcal{E}_o^{uosc}$ *for* $\lambda \geq 1$

(iii.b) $(S,t) \in \mathcal{E}_o^{dosc} \cap \mathcal{E}^c \Longrightarrow (\lambda S,t) \in \mathcal{E}_o^{dosc}$ *for* $\lambda \geq 1$

(iii.c) $\mathcal{E}_o^{dosc} = \mathcal{E}_o^{dosc,1} \cup \mathcal{E}_o^{dosc,2}$ *for some* $\mathcal{E}_o^{dosc,j}, j = 1,2$ *connected such that* $\mathcal{E}_o^{dosc,1} \cap \mathcal{E}_o^{dosc,2} = \varnothing$

(iv) t-section convexity: for any $t \in [0,T]$*,*

(iv.a) $(S_1,S_2) \in \mathcal{E}_o^{uosc}(t)^2 \Longrightarrow S^\lambda \in \mathcal{E}_o^{uosc}(t)$ *for* $\lambda \in [0,1]$

(iv.b) $(S_1,S_2) \in \left(\mathcal{E}_o^{dosc}(t) \cap \mathcal{E}^c(t)\right)^2 \Longrightarrow S^\lambda \in \mathcal{E}_o^{dosc}(t)$ *for* $\lambda \in [0,1]$

(iv.c) $(S_1,S_2) \in \mathcal{E}_o^{dosc,j}(t)^2 \Longrightarrow S^\lambda \in \mathcal{E}_o^{dosc,j}(t)$ *for* $\lambda \in [0,1]$, $j = 1,2$

(v) Continuation subregions:

(v.a) If $\kappa_\rho K > S > H \vee K$ *or* $H \wedge \kappa K > S > K$*, then* $(S,t) \notin \mathcal{E}_o^{uosc}$

(v.b) If $H \wedge \kappa_\rho K > S > K$ *or* $\kappa K > S > H \vee K$*, then* $(S,t) \notin \mathcal{E}_o^{dosc}$

Intuition for property *(i)* is as follows. When $\rho \geq 0$, the discounted vanilla call payoff in market $\mathcal{M}(r+\rho, \delta+\rho, \sigma)$ is smaller than in market $\mathcal{M}(r,\delta,\sigma)$. Immediate exercise is therefore optimal in $\mathcal{M}(r+\rho,\delta+\rho,\sigma)$ if it is optimal in $\mathcal{M}(r,\delta,\sigma)$. That is, $\mathcal{E}^c \subseteq \mathcal{E}_\rho^c$. The step call initiated at time t, i.e., sc, can be viewed as a vanilla call in market $\mathcal{M}\left(r+\rho 1_{\{S\in A\}}, \delta+\rho 1_{\{S\in A\}}, \sigma\right)$ with stochastic interest $r+\rho 1_{\{S\in A\}}$ and stochastic dividend yield $\delta+\rho 1_{\{S\in A\}}$. The discounted step call payoff therefore lies in between the discounted payoffs of the call options in $\mathcal{M}(r+\rho,\delta+\rho,\sigma)$ and $\mathcal{M}(r,\delta,\sigma)$. It follows that immediate exercise must be optimal for the step call if it is optimal for the vanilla call in $\mathcal{M}(r,\delta,\sigma)$. Likewise, immediate exercise must be optimal for the vanilla call in $\mathcal{M}(r+\rho,\delta+\rho,\sigma)$, if it is optimal for the step call. Figure 2 in Subsection 6.2 illustrates the relationships between the optimal exercise regions.

Property *(ii)* reflects the fact that, in the market under consideration, a derivative with a shorter time-to-maturity has a smaller set of possible exercise times. Optimality of immediate exercise for the longer maturity option therefore implies optimality of immediate exercise for the shorter maturity contract.

Property *(iii.a)* ensures that immediate exercise of an up-and-out step call remains optimal for all greater underlying prices. The property holds because 1) the call payoff cannot grow faster than the increase in the price of the underlying asset and 2) the discount associated with future occupation times increases as the underlying asset price increases. That is, $(\lambda S N_{t,\tau} - K)^+ \leq (SN_{t,\tau} - K)^+ + (\lambda - 1) SN_{t,\tau}$ and $e^{-\rho O_{t,\tau}^\lambda} \leq e^{-\rho O_{t,\tau}}$. The combination of these two properties ensures that $e^{-\rho O_{t,\tau}^\lambda}(\lambda S N_{t,\tau} - K)^+ \leq e^{-\rho O_{t,\tau}}(SN_{t,\tau} - K)^+ + (\lambda - 1) SN_{t,\tau}$ for any stopping time $\tau \in \mathcal{S}([t,T])$. The price bound $sc(\lambda S, t) \leq sc(S,t) + (\lambda - 1) S$ follows. Optimality of immediate exercise at (S,t) then gives $sc(\lambda S,t) \leq \lambda S - K$. Feasibility of immediate exercise at $(\lambda S, t)$ ensures the reverse inequality $\lambda S - K \leq sc(\lambda S, t)$. It follows that $sc(\lambda S,t) = \lambda S - K$ so that immediate exercise is optimal at $(\lambda S, t)$. It is interesting to note that this argument does not carry through for down-an-out step options. In this case, the discount factor associated with the occupation time increases with the asset price adding value to any waiting policy. As a result, immediate exercise is no longer assured to be optimal at the higher underlying price. Nevertheless, as stated in *(iii.b)*, up-connectedness holds over the subregions where it is optimal to exercise a vanilla call. Properties *(iv.a)* and *(iv.b)*, which assert the convexity of the t-sections $\mathcal{E}_o^{uosc}(t)$ and $\mathcal{E}_o^{dosc}(t)$ over some range, follow immediately from up-connectedness in *(iii.a)* and *(iii.b)*.

Property *(iii.c)* is striking, because it suggests the possibility of disjoint exercise subregions. Such a configuration can emerge when $\kappa K \geq H > \kappa_\rho K \vee K$. Indeed, in the region $\mathcal{R}_l \equiv \{S : \kappa K \geq H > S > \kappa_\rho K \vee K\}$, the local gain from immediate exercise, which consists of dividends collected net of the interest loss on the strike, is

$$\delta S - rK + \rho (S - K) 1_{\{S \leq H\}},$$

a positive quantity. At times approaching maturity, the likelihood of exiting $\mathcal{R}_l$ vanishes, implying the eventual optimality of immediate exercise. In contrast, if the underlying price belongs to $\mathcal{R}_m \equiv \{S : \kappa K \geq S > H > \kappa_\rho K \vee K\}$, the local gain from immediate exercise $\delta S - rK$ becomes nonpositive (because $\kappa K \geq S$). Immediate exercise is therefore suboptimal throughout $\mathcal{R}_m$. As $\mathcal{E}^c \subseteq \mathcal{E}_o^{sc}$, it becomes again optimal to exercise when the price is sufficiently large. A necessary condition is $S \in \mathcal{R}_u \equiv \{S : S > \kappa K \vee K \vee \kappa_\rho K\}$. Hence, there exist subregions $\mathcal{E}_o^{dosc,1} \subseteq \mathcal{R}_u \times [0, T]$ and $\mathcal{E}_o^{dosc,2} \subseteq \mathcal{R}_l \times [0, T]$ such that $\mathcal{E}_o^{dosc,1} \cap \mathcal{E}_o^{dosc,2} = \varnothing$ and $\mathcal{E}_o^{dosc} = \mathcal{E}_o^{dosc,1} \cup \mathcal{E}_o^{dosc,2}$. Moreover, subregions $\mathcal{E}_o^{dosc,j}, j = 1, 2$ must be connected. If not, the value of delaying exercise at a point inside a disconnected area is negative, i.e., the instantaneous waiting benefit is negative, leading to value losses. Connectedness of the subregions also implies t-section convexity of the subregions, as asserted in *(iv.c)*.

Finally, property *(v)* identifies regions where immediate exercise is suboptimal. In these areas, the local net gain from exercising is,

$$S\left(\delta + \rho 1^{\pm}\right) - K\left(r + \rho 1^{\pm}\right)$$

where $1^+ = 1_{\{S \geq H\}}$ and $1^- = 1_{\{S \leq H\}}$, which is negative. Hence, immediate exercise gives a local loss and is dominated by some waiting policy. Likewise, immediate exercise when $S < K$ is trivially suboptimal.

Figures 1 and 2 illustrate the structure of the exercise region for a down-and-out step call.

3.2. Knock-out step options: price function

Recall that $sc(S,t) = \sup_{\tau \in \mathcal{S}([t,T])} E^* \left[e^{-r(\tau - t) - \rho O_{t,\tau}} (S N_{t,\tau} - K)^+\right]$. The next theorem describes properties of $sc(S,t)$ in the knock-out case.

Theorem 3.2. *The knock-out price function $sc(\cdot,\cdot) : \mathbb{R}_+ \times [0,T] \to \mathbb{R}_+$ has the following properties,*
(i) Continuity: $sc(\cdot,\cdot)$ is continuous on $\mathbb{R}_+ \times [0,T]$
(ii) Time monotonicity: $sc(S,\cdot)$ is nonincreasing on $[0,T]$ for all $S \in \mathbb{R}_+$
(iii) Space monotonicity: $sc^{do}(\cdot,t)$ is nondecreasing on $\mathbb{R}_+$ for all $t \in [0,T]$
(iv) Quasi-convexity: $sc^{do}\left(S^\lambda, t\right) \leq sc^{do}(S_1,t) \vee sc^{do}(S_2,t)$ for $(S_1, S_2, t) \in$

$\mathbb{R}_+^2 \times [0,T]$ *and* $\lambda \in [0,1]$

(v) Bounded slope: $|sc(\lambda S,t) - sc(S,t)| \leq M(\lambda - 1)S$ *for* $\lambda \geq 1$ *and some* $0 < M < \infty$

The continuity of the price function with respect to (S,t) follows from the continuity of the occupation time and of the payoff function. Property *(ii)* is the counterpart of right-connectedness in Theorem 3.1 *(ii)*. The monotonicity property *(iii)* reflects the fact that the down-and-out step option payoff is increasing in S whereas the occupation time, thus the discount rate, is decreasing in S. Note that *(iii)* need not hold for up-and-out step options. In this case the discount rate increases with S which may offset the payoff gain. In fact, for low values of S the American up-and-out step call approaches its European counterpart, which is known to be non-monotonic (Linetsky (1999)).

The quasi-convexity of the down-and-out step call price in *(iv)* is a consequence of *(iii)*. This property is weaker than the typical convexity of the price for a vanilla call. Convexity can fail, for a down-and-out step call, because the knock-out factor is not convex (see Corollary 1 for further details). In the case of an up-and-out step call, even quasi-convexity can fail.

Finally, it is of interest to note that the slope of a knock-out step call is bounded. This property ensures continuity of the call price with respect to the underlying asset price S. It also ensures that it is possible to hedge the claim using replicating policies involving bounded proportions of the underlying asset price.

The next corollary shows that global convexity fails for the down-and-out step call.

Corollary 1. *The down-and-out price function* $sc^{do}(\cdot,\cdot) : \mathbb{R}_+ \times [0,T] \to \mathbb{R}_+$ *can be concave over some range of prices. More specifically, suppose* $\mathcal{E}_o^{dosc} = \mathcal{E}_o^{dosc,1} \cup \mathcal{E}_o^{dosc,2}$ *for some* $\mathcal{E}_o^{dosc,j} \neq \varnothing, j = 1,2$ *such that* $\mathcal{E}_o^{dosc,1} \cap \mathcal{E}_o^{dosc,2} = \varnothing$ *and let* $\left(S^\lambda, t\right) \in \left(\mathcal{E}_o^{dosc,1} \cup \mathcal{E}_o^{dosc,2}\right)^c$ *such that* $S^\lambda = \lambda S_1 + (1-\lambda) S_2$ *where* $(S_j,t) \in \mathcal{E}_o^{dosc,j}$, $j = 1,2$*. Then,* $\lambda sc(S_1,t) + (1-\lambda) sc(S_2,t) < sc\left(S^\lambda, t\right)$*.*

Intuition for the result is straightforward. If the exercise region consists of disconnected subsets, any underlying price located in between these regions is a continuation point. The step call price therefore exceeds the exercise value at that continuation point. But the step call price also equals the exercise value at higher and lower underlying prices in the neighboring exercise subregions. Linearity of the exercise payoff combined with the optimality (suboptimality) of exercise in (between) the subregions ensures that the convex combination of the call prices at points in the subregions is strictly less than the option price at the intermediate continuation point.

Proof of Corollary 1. Assume $\mathcal{E}_o^{dosc,j} \neq \varnothing, j = 1,2$ and let $(S_j,t) \in \mathcal{E}_o^{dosc,j}$, $j = 1,2$. Consider the convex combination $S^\lambda = \lambda S_1 + (1-\lambda) S_2$ and note that $\left(S^\lambda, t\right) \in \left(\mathcal{E}_o^{dosc,1} \cup \mathcal{E}_o^{dosc,2}\right)^c$, i.e., $\left(S^\lambda, t\right)$ is in the continuation region. Then $sc^{do}\left(S^\lambda, t\right) > S^\lambda - K$. But $(S_j,t) \in \mathcal{E}_o^{dosc,j}$ means that $sc^{do}(S_j,t) = S_j - K$, $j = 1,2$. Hence,

$$\lambda sc(S_1,t) + (1-\lambda) sc(S_2,t) = \lambda (S_1 - K) + (1-\lambda)(S_2 - K) = S^\lambda - K < sc\left(S^\lambda, t\right)$$

which demonstrates the claim. □

The relation between American step options and American barrier and vanilla options is clarified next. Let $C^{do}(S,t;H)$ (resp. $C^{uo}(S,t;H)$) be the price of an American down-and-out (resp. up-and-out) call with knock-out barrier H.

Theorem 3.3. *The following limits apply:*
(i) If $\rho \to +\infty$, then $sc^{do}(S,t) \to C^{do}(S,t;H)$ and $sc^{uo}(S,t) \to C^{uo}(S,t;H)$
(ii) If $\rho \to 0$, then $sc(S,t) \to C(S,t)$ and $SC(S,t) \to C(S,t)$

Property *(i)* shows that the American one-touch knock-out call without rebate is the limit of the American step call option as the discount factor ρ becomes very large. In that instance, the discount factor explodes to infinity as soon as the occupation time becomes positive. The option payoff, in that event, becomes null. The step payoff therefore corresponds to the payoff of the one-touch knock-out call. Property *(ii)* examines the polar case where ρ vanishes. The limit step call payoff is simply the vanilla call payoff. The limit prices of the American step calls coincide with the price of the vanilla American call.

3.3. Knock-in step options

The immediate exercise region, for knock-in step calls (i.e., $\rho < 0$), is described next. Theorem 3.4 deals with the case $\delta + \rho \geq 0$, Theorem 3.5 with $\delta + \rho < 0$. We assume $r + \delta \geq 0$ throughout.

Theorem 3.4. *(Knock-in step call) Consider the case $\rho < 0$ and assume $\delta + \rho \geq 0$ and $r + \rho \geq 0$. The exercise region of the knock-in step call, $\mathcal{E}^{sc} = [0,T] \times \mathcal{E}_o^{sc}$, satisfies,*
(i) Non-emptiness: $\mathcal{E}_\rho^c \subseteq \mathcal{E}_o^{sc} \subseteq \mathcal{E}^c$ where $\mathcal{E}^c, \mathcal{E}_\rho^c \neq \varnothing$
(ii) Right-connectedness: $(S,t) \in \mathcal{E}_o^{sc} \Longrightarrow (S,v) \in \mathcal{E}_o^{sc}$ for $t \leq v \leq T$
(iii) Up-connectedness and subregion connectedness:
(iii.a) $(S,t) \in \mathcal{E}_o^{disc} \Longrightarrow (\lambda S,t) \in \mathcal{E}_o^{disc}$ for $\lambda \geq 1$
(iii.b) $(S,t) \in \mathcal{E}_o^{uisc} \cap \mathcal{E}_\rho^c(t) \Longrightarrow (\lambda S,t) \in \mathcal{E}_o^{uisc}$ for $\lambda \geq 1$
(iii.c) $\mathcal{E}_o^{uisc} = \mathcal{E}_o^{uisc,1} \cup \mathcal{E}_o^{uisc,2}$ for some $\mathcal{E}_o^{uisc,j}, j = 1,2$ connected such that $\mathcal{E}_o^{uisc,1} \cap \mathcal{E}_o^{uisc,2} = \varnothing$
(iv) t-section convexity: for any $t \in [0,T]$,
(iv.a) $(S_1,S_2) \in \mathcal{E}_o^{disc}(t)^2 \Longrightarrow S^\lambda \in \mathcal{E}_o^{disc}(t)$ for $\lambda \in [0,1]$
(iv.b) $(S_1,S_2) \in \left(\mathcal{E}_o^{uisc}(t) \cap \mathcal{E}_\rho^c(t)\right)^2 \Longrightarrow S^\lambda \in \mathcal{E}_o^{uisc}(t)$ for $\lambda \in [0,1]$
(iv.c) $(S_1,S_2) \in \mathcal{E}_o^{uisc,j}(t)^2 \Longrightarrow S^\lambda \in \mathcal{E}_o^{uisc,j}(t)$ for $\lambda \in [0,1]$, $j = 1,2$
(v) Continuation subregions:
(v.a) If $\kappa_\rho K > S > H \vee K$ or $H \wedge \kappa K > S > K$, then $(S,t) \notin \mathcal{E}_o^{uisc}$
(v.b) If $H \wedge \kappa_\rho K > S > K$ or $\kappa K > S > H \vee K$, then $(S,t) \notin \mathcal{E}_o^{disc}$.

Properties *(i)*, *(ii)* and *(v)* are similar to those in Theorem 3.1. For Property *(i)*, note that the discounted payoff relations are the opposite of those described in Theorem 3.1. For *(ii)*, it could be stressed that the property is universal across step options. It does not depend on the sign of ρ. For *(v)*, the same intuitions as for knock-out step options apply. In the regions considered immediate exercise is dominated by some waiting policy.

For *(iii.b)*, note that $\mathcal{E}_0^{uisc}(t)$ is up-connected over the subregion $\mathcal{E}_\rho^c(t)$, because $\mathcal{E}_\rho^c \subseteq \mathcal{E}_0^{sc}$. It is also convex over that subregion, as stated in *(iv.b)*. These two properties hold for up-and-in and down-and-in contracts. Moreover, as claimed in *(iii.a)*, $\mathcal{E}_0^{disc}(t)$ is up-connected over the whole region if the net dividend yield is positive, i.e., $\delta + \rho \geq 0$. In this instance, the discounted payoff is bounded above,

$$e^{-\rho O_{t,\tau}^\lambda} (\lambda S N_{t,\tau} - K)^+ \leq e^{-\rho O_{t,\tau}} (S N_{t,\tau} - K)^+ + (\lambda - 1) S e^{-\rho(\tau - t)} N_{t,\tau}.$$

The value of the upper bound is,

$$sc^{di}(S,t) + (\lambda - 1) S E^* \left[e^{-(\delta+\rho)(\tau-t)} \eta_{t,\tau}^\sigma \right] \leq sc^{di}(S,t) + (\lambda - 1) S$$

if $\delta + \rho \geq 0$. If immediate exercise is optimal at (S,t), the right hand side becomes $\lambda S - K$ and $sc^{di}(\lambda S, t) \leq \lambda S - K$. The optimality of immediate exercise at $(\lambda S, t)$ follows. Up-connectedness implies the convexity property *(iv.a)*. Finally, for *(iii.c)*, note that when $\kappa_\rho K > H > \kappa K \vee K$, the exercise region can split as $\mathcal{E}_0^{uisc} = \mathcal{E}_0^{uisc,1} \cup \mathcal{E}_0^{uisc,2}$ for some $\mathcal{E}_0^{uisc,j}, j = 1, 2$ connected such that $\mathcal{E}_0^{uisc,1} \cap \mathcal{E}_0^{uisc,2} = \varnothing$. In the region $\mathcal{R}_m \equiv \{S : \kappa_\rho K \geq S \geq H > \kappa K \vee K\}$, the local gain $\delta S - rK + \rho(S - K) 1^+$ is nonpositive, implying the suboptimality of immediate exercise. In $\mathcal{R}_u \equiv \{S : S \geq \kappa_\rho K \geq H > \kappa K \vee K\}$, it is nonnegative implying the possible optimality of immediate exercise. In $\mathcal{R}_d \equiv \{S : \kappa_\rho K \geq H > S \geq \kappa K \vee K\}$, the local gain $\delta S - rK$ is again nonnegative allowing for the possibility of early exercise. The argument for t-section convexity *(iv.c)* parallels the one for *(iv.c)* in Theorem 3.1.

The next result deals with the case $\delta + \rho < 0$ and $r + \rho \geq 0$. In this instance, $r > \delta$, implying $\kappa > 1 > \kappa_\rho$. Moreover, $\mathcal{E}_\rho^c = \{(S,T) : S \geq K\}$. It is never optimal to exercise the vanilla call prior to maturity in market $\mathcal{M}(r + \rho, \delta + \rho, \sigma)$.

Theorem 3.5. *(Knock-in step call) Consider the case $\rho < 0$ and assume $\delta + \rho < 0$ and $r + \rho \geq 0$. The exercise region of the knock-in step call, $\mathcal{E}^{sc} = [0,T] \times \mathcal{E}_o^{sc}$, satisfies,*

(i) Non-emptiness:

(i.a) If $H \leq \kappa K$, then $\mathcal{E}_o^{uisc} = \mathcal{E}_\rho^c = \{(S,T) : S \geq K\}$ and $\mathcal{E}_o^{disc} \subseteq \mathcal{E}^c$

(i.b) If $H > \kappa K$, then $\mathcal{E}_o^{uisc} \subseteq \mathcal{E}^c \cap \{S \leq H\}$ and $\mathcal{E}_0^{disc} \subseteq \mathcal{E}^c \cap \{S \geq H\}$

(ii) Right-connectedness: $(S,t) \in \mathcal{E}_o^{sc} \Longrightarrow (S,v) \in \mathcal{E}_o^{sc}$ for $t \leq v \leq T$

(iii) Up-connectedness and subregion connectedness:

(iii.a) $(S,t) \in \mathcal{E}_o^{disc} \Longrightarrow (\lambda S, t) \in \mathcal{E}_o^{disc}$ for $\lambda \geq 1$

(iii.b) If $H \leq \kappa K$, then $(S,T) \in \mathcal{E}_o^{uisc} \Longrightarrow (\lambda S, T) \in \mathcal{E}_o^{uisc}$ for $\lambda \geq 1$

(iii.c) If $H > \kappa K$, then $(S,t) \in \mathcal{E}_o^{uisc} \not\Longrightarrow (\lambda S, t) \in \mathcal{E}_o^{uisc}$ for $\lambda \geq 1$

(iv) t-section convexity: for any $t \in [0,T]$

(iv.a) $(S_1, S_2) \in \mathcal{E}_o^{disc}(t)^2 \Longrightarrow S^\lambda \in \mathcal{E}_o^{disc}(t)$ for $\lambda \in [0,1]$

(iv.b) $(S_1, S_2) \in \mathcal{E}_o^{uisc}(t)^2 \Longrightarrow S^\lambda \in \mathcal{E}_o^{uisc}(t)$ *for* $\lambda \in [0,1]$

(v) Continuation subregions:

(v.a) If $H \wedge \kappa K > S$*, then* $(S,t) \notin \mathcal{E}_o^{disc}$*.for* $t < T$

(v.b) If $S > H \vee \kappa K$*, then* $(S,t) \notin \mathcal{E}_o^{uisc}$ *for* $t < T$

When $\delta + \rho < 0$ and $r + \rho \geq 0$, the most significant novelty is that early exercise becomes suboptimal when the occupation time is active. Indeed, the up-and-in local gain $(\delta + \rho 1^+) S - (r + \rho 1^+) K$ is negative for $S \geq H$ and so is the down-and-in local gain $(\delta + \rho 1^-) S - (r + \rho 1^-) K$ for $S < H$. Properties *(i)* and *(v)* follows from this.

Property *(ii)*, *(iii.a)* and *(iv.a)* are as before. For *(iii.a)*, it is useful to note that $\mathcal{E}_o^{disc} \subseteq \mathcal{E}^c \cap \{S \geq H\}$ where $\{S \geq H\}$ is up-connected. The proof actually uses the delayed exercise premium (DEP) representation to show that any waiting policy at the point $(\lambda S, t)$ is dominated by immediate exercise. This follows because immediate exercise is optimal at (S, v) for any $v \in [t,T]$ and because the net benefit of waiting $rK - \delta\lambda S_v$ is non-positive in the region $\left\{\left(\widehat{S}, v\right) : \widehat{S} \geq S \text{ and } v \in [t,T]\right\}$ (see the proof for details). For *(iii.b)*, the exercise region $\mathcal{E}_o^{uisc}$ reduces to the exercise set $\{S \geq K\}$ at maturity T, which is up-connected. For *(iii.c)*, recall that $\mathcal{E}_0^{uisc} \subseteq \mathcal{E}^c \cap \{S \leq H\}$, where $\{S \leq H\}$ is not up-connected.

Properties *(i.b)* and *(iv.b)* show that the exercise region of the up-and-in step call is a convex subset of $\mathcal{E}^c \cap \{S \leq H\}$ when $H > \kappa K$. This subset is proper because the step call price is strictly larger than the standard call price. The exercise region is also right-connected by *(ii)*. It can therefore be described by a pair of boundaries that collide at some point in the interval $[t^*, T]$ where t^* solves $B^c(t^*) = H$.

4. Valuation formulas

In this section we derive analytic formulas for the price of an American step call option. The results of Appendix C enable us to define early exercise boundaries and to provide an EEP representation of the price for each type of step call option, and for different configurations of the parameters r, δ, K and H.

4.1. Up-and-out step call

Define the boundary,

$$B^{uo}(t) = \inf\{S : (S,t) \in \mathcal{E}_o^{uosc}(t)\}$$

for all $t \in [0,T)$ and recall that there are only two possible parameter configurations $\kappa \geq \kappa_\rho \geq 1$ and $\kappa \leq \kappa_\rho \leq 1$.

Theorem 4.1 (EEP representation)**.** *The value of the American-style up-and-out step call has the EEP decomposition* $sc^{uo}(S,t) = esc^{uo}(S,t) + eepsc^{uo}(S,t)$ *where* $esc^{uo}(S,t)$ *is the value of a*

European up-and-out step call and $eepsc^{uo}(S,t)$ *is the early exercise premium,*

$$esc^{uo}(S,t) = E_t^*\left[b_{t,T}e^{-\rho O_{t,T}^+}(S_T - K)^+\right]$$

$$eepsc^{uo}(S,t) = E_t^*\left[\int_t^T b_{t,v}e^{-\rho O_{t,v}^+}\phi_v dv\right]$$

and the local gains from early exercise $\phi_v \equiv \phi(S_v, B^{uo}(v))$ *are,*

$$\phi_v = \left(\delta S_v - rK + \rho(S_v - K)1_{\{S_v \geq H\}}\right)1_{\{S_v \geq B^{uo}(v)\}}.$$

The immediate exercise boundary B^{uo} *solves the recursive integral equation,*

$$B^{uo}(t) - K = sc(B^{uo}(t), t)$$

$$B^{uo}(T_-) = \begin{cases} \kappa_\rho K \vee K & \text{if } (\kappa K \vee K) \wedge \kappa_\rho K \geq H \\ H \vee K & \text{if } (\kappa K \vee K) \geq H \geq \kappa_\rho K \\ \kappa K \vee K & \text{if } H \geq (\kappa K \vee K) \vee \kappa_\rho K. \end{cases}$$

The optimal exercise policy is $\tau^{uo} = \inf\{t \in [0,T] : S_t = B^{uo}(t)\}$ *or* $t = T$ *if no such time exists in* $[0,T]$ *and* $S_T \geq K$*.*

Theorem 4.1 shows that the price of an American-style step option can be decomposed into the sum of its European-style counterpart and an EEP, i.e., a premium for exercising before the maturity date when the local gains are sufficiently positive. The integral equation for the boundary is derived from the fact that on the optimal exercise boundary there is no value for waiting and the option value equals its payoff.

Proof of Theorem 4.1. For an up-and-out step call $\{S \in A\} = \{S \geq H\}$. By the strong Markov property (see Peskir & Shiryaev (2006), Chapter III), the price function $sc^{uo}(\cdot,\cdot) : \mathbb{R}_+ \times [0,T] \to \mathbb{R}_+$ is $C^{2,1}$ in $\mathcal{C}_o^{sc}$. It is also continuously differentiable at the exercise boundary B^{uo} by Theorem C.1 in Appendix C, and in $\mathcal{E}_o^{sc}$ where $sc^{uo}(S,t) = S - K$. Applying Ito's lemma to $b_{t,v}e^{-\rho O_{t,v}^+}sc^{uo}(S_v, v)$ gives

$$\begin{aligned} b_{t,T}e^{-\rho O_{t,T}^+}sc^{uo}(S_T,T) = sc^{uo}(S,t) &+ \int_t^T b_{t,v}e^{-\rho O_{t,v}^+}sc_s^{uo}(S_v,v)1_{S_v \in \mathcal{C}_o^{sc}(v)}dS_v \\ &+ \frac{1}{2}\int_t^T b_{t,v}e^{-\rho O_{t,v}^+}sc_{ss}^{uo}(S_v,v)1_{S_v \in \mathcal{C}_o^{sc}(v)}d[S]_v \\ &+ \int_t^T b_{t,v}e^{-\rho O_{t,v}^+}\left(sc_t^{uo}(S_v,v) - \left(r + \rho 1_{\{S \geq H\}}\right)sc^{uo}(S_v,v)\right)1_{S_v \in \mathcal{C}_o^{sc}(v)}dv \\ &+ \int_t^T b_{t,v}e^{-\rho O_{t,v}^+}1_{S_v \in \mathcal{E}_o^{sc}(v)}\left(dS_v - \left(r + \rho 1_{\{S \geq H\}}\right)(S_v - K)dv\right) \end{aligned}$$

where $sc_s^{uo}(\cdot,\cdot), sc_{ss}^{uo}(\cdot,\cdot), sc_t^{uo}(\cdot,\cdot)$ are the partial derivatives of the price function. Taking the conditional expectation under the risk neutral measure, applying the no-arbitrage condition and rearranging yields

$$sc^{uo}(S,t) = E_t^*\left[b_{t,T}e^{-\rho O_{t,T}^{+}}sc^{uo}(S_T,T)\right] + E_t^*\left[\int_t^T b_{t,v}e^{-\rho O_{t,v}^{+}}\phi_v dv\right]$$

where

$$\phi_v = \left(\left(\delta + \rho 1_{\{S\geq H\}}\right)S_v - \left(r + \rho 1_{\{S\geq H\}}\right)K\right)1_{S_v\in\mathcal{E}_o^{sc}(v)}.$$

With the identification $\mathcal{E}_o^{sc}(v) = \{S_v \geq B^{uo}(v)\}$, the formula stated follows. That the immediate exercise boundary satisfies $B^{uo}(t) - K = sc^{uo}(B^{uo}(t),t)$ is straightforward. The limit $B^{uo}(T_-) = \lim_{t\uparrow T} B^{uo}(t)$ follows from Theorem C.2. □

Explicit formulas for the price components are given next using the function Ψ in Proposition 1.

Corollary 2. *The EEP components of the American-style up-and-out step call value are,*

$$esc^{uo}(S,t) = e^{-\gamma(T-t)}\left[S_t\Psi_\rho^u(\alpha+\sigma;k,h,T-t) - K\Psi_\rho^u(\alpha;k,h,T-t)\right]$$

$$eepsc^{uo}(S,t) = \int_t^T e^{-\gamma(v-t)}\Phi_v dv$$

where, $\gamma = r + \frac{\alpha^2}{2}$ *and*

$$\begin{aligned}\Phi_v &= \delta S_t\Psi_\rho^u(\alpha+\sigma;b^{uo}(v),h,v-t) - rK\Psi_\rho^u(\alpha;b^{uo}(v),h,v-t)\\ &\quad + \rho\left(S_t\Psi_\rho^u(\alpha+\sigma;b^{uo}(v)\vee h,h,v-t) - K\Psi_\rho^u(\alpha;b^{uo}(v)\vee h,h,v-t)\right),\end{aligned}$$

with $b^{uo}(v) = \frac{1}{\sigma}\log\frac{B^{uo}(v)}{S}$.

Our next corollary gives the step call boundary when H is sufficiently large.

Corollary 3. *Suppose that* $H \geq \kappa K \vee K \vee \kappa_\rho K$*. Let* t^* *be such that* $H = B^c(t^*)$ *if an interior solution exists, or* $t^* = 0$ *if* $H \geq B^c(t)$ *for all* $t \in [0,T]$*. Then* $B^{uo}(t) = B^c(t)$ *for all* $t \in [t^*,T]$.

In the case under consideration, the exercise boundary of the vanilla call falls below the barrier for all $t \in [t^*,T]$. That is $B^c(t) \leq H$ for all $t \in [t^*,T]$. The up-and-out step call holder can therefore implement the exercise policy of the vanilla call and obtain the same exercise payoff. It follows that $esc^{uo}(S,t) = C(S,t)$ and $B^{sc}(t) = B^{uo}(t)$ for $t \in [t^*,T]$.

4.2. Down-and-out step call

Define the boundaries,

$$B^{do}(t) = \inf\left\{S : (S,t) \in \mathcal{E}_o^{dosc}(t) \text{ and } S \geq (\kappa K \vee K) \vee \kappa_\rho K\right\}$$

and, if $\kappa K \geq H \geq \kappa_\rho K > K$,

$$B_{lu}^{do}(t) = \sup\left\{S : (S,t) \in \mathcal{E}_o^{dosc}(t) \text{ and } H \geq S \geq \kappa_\rho K\right\}$$

$$B_{ld}^{do}(t) = \inf\left\{S : (S,t) \in \mathcal{E}_o^{dosc}(t) \text{ and } H \geq S \geq \kappa_\rho K\right\}.$$

The parameter configurations for (κ, κ_ρ) are the same as for the up-and-out step call.

Theorem 4.2 (EEP representation)**.** *The value of the American-style down-and-out step call has the EEP decomposition* $sc^{do}(S,t) = esc^{do}(S,t) + eepsc^{do}(S,t)$ *where* $esc^{do}(S,t)$ *is the value of a European down-and-out step call and* $eepsc^{do}(S,t)$ *is the early exercise premium,*

$$esc^{do}(S,t) = E_t^*\left[b_{t,T} e^{-\rho O_{t,T}^-}(S_T - K)^+\right]$$

$$eepsc^{do}(S,t) = E_t^*\left[\int_t^T b_{t,v} e^{-\rho O_{t,v}^-} \phi_v dv\right]$$

and the local gains from early exercise $\phi_v \equiv \phi\left(S_v, B^{do}(v), B_{lu}^{do}(v), B_{ld}^{do}(v)\right)$ *are,*

$$\phi_v = \begin{cases} \phi_{1v} & \text{if } (\kappa K \vee K) \wedge \kappa_\rho K \geq H \\ \phi_{2v} & \text{if } H \geq (\kappa K \vee K) \vee \kappa_\rho K \\ \phi_{3v} & \text{if } \kappa K \geq H \geq \kappa_\rho K > K. \end{cases}$$

$$\phi_{1v} = (\delta S_v - rK)\, 1_{\left\{S_v \geq B^{do}(v)\right\}}$$

$$\phi_{2v} = \left(\delta S_v - rK + \rho(S_v - K)\, 1_{\{S_v \leq H\}}\right) 1_{\left\{S_v \geq B^{do}(v)\right\}}$$

$$\phi_{3v} = (\delta S_v - rK)\, 1_{\left\{S_v \geq B^{do}(v)\right\}} + (\delta S_v - rK + \rho(S_v - K))\, 1_{\left\{B_{lu}^{do}(v) \geq S_v \geq B_{ld}^{do}(v)\right\}}$$

If $(\kappa K \vee K) \wedge \kappa_\rho K \geq H$ *or* $H \geq (\kappa K \vee K) \vee \kappa_\rho K$*, the immediate exercise boundary* B^{do} *is unique and solves the recursive integral equation,*

$$B^{do}(t) - K = sc\left(B^{do}(t), t; B^{do}(\cdot), B_{lu}^{do}(\cdot), B_{ld}^{do}(\cdot)\right)$$

$$B^{do}(T_-) = \begin{cases} \kappa K \vee K & \text{if } (\kappa K \vee K) \wedge \kappa_\rho K \geq H \\ \kappa_\rho K \vee K & \text{if } H \geq (\kappa K \vee K) \vee \kappa_\rho K. \end{cases}$$

If $\kappa K \geq H \geq \kappa_\rho K > K$*, the boundary of the immediate exercise region can consist of* 3 *pieces,* $B^{do}, B_{lu}^{do}, B_{ld}^{do}$*, that solve the system of coupled recursive integral equations,*

$$\begin{cases} B^{do}(t) - K = sc\left(B^{do}(t), t; B^{do}(\cdot), B_{lu}^{do}(\cdot), B_{ld}^{do}(\cdot)\right); & B^{do}(T_-) = \kappa K \\ B_{lu}^{do}(t) - K = sc\left(B_{lu}^{do}(t), t; B^{do}(\cdot), B_{lu}^{do}(\cdot), B_{ld}^{do}(\cdot)\right); & B_{lu}^{do}(T_-) = H \\ B_{ld}^{do}(t) - K = sc\left(B_{ld}^{do}(t), t; B^{do}(\cdot), B_{lu}^{do}(\cdot), B_{ld}^{do}(\cdot)\right); & B_{ld}^{do}(T_-) = \kappa_\rho K. \end{cases}$$

The optimal exercise policy is,

$$\tau^{do} = \inf\left\{t \in [0,T] : S_t = B^{do}(t) \;\; or\; S_t = B^{do}_{lu}(t) \;\; or\; S_t = B^{do}_{ld}(t)\right\}$$

or $t = T$ if no such time exists in $[0,T]$ and $S_T \geq K$.

Proof of Theorem 4.2. The proof proceeds along the same lines as the proof of Theorem 4.1. □

Corollary 4. *The EEP components of the American down-and-out step call value are*

$$esc^{do}(S,t) = e^{-\gamma(T-t)}\left[S_t \Psi^d_\rho(\alpha+\sigma; k, h, T-t) - K\Psi^d_\rho(\alpha; k, h, T-t)\right]$$

$$eepsc^{do}(S,t) = \int_t^T e^{-\gamma(\upsilon-t)}\Phi_\upsilon d\upsilon$$

where,

$$\Phi_\upsilon = \begin{cases} \Phi^1_\upsilon & \text{if } (\kappa K \vee K) \wedge \kappa_\rho K \geq H \\ \Phi^2_\upsilon & \text{if } H \geq (\kappa K \vee K) \vee \kappa_\rho K \\ \Phi^3_\upsilon & \text{if } \kappa K \geq H \geq \kappa_\rho K > K. \end{cases}$$

and Φ^1_υ, Φ^2_υ, Φ^3_υ are given by,

$$\Phi^1_\upsilon = \delta S_t \Psi^d_\rho\left(\alpha+\sigma; b^{do}(\upsilon), h, \upsilon - t\right) - rK\Psi^d_\rho\left(\alpha; b^{do}(\upsilon), h, \upsilon - t\right),$$

$$\begin{aligned}\Phi^2_\upsilon = \Phi^1_\upsilon &+ \rho S_t\left[\Psi^d_\rho\left(\alpha+\sigma; b^{do}(\upsilon)\wedge h, h, \upsilon - t\right) - \Psi^d_\rho(\alpha+\sigma; h, h, \upsilon - t)\right] \\ &- \rho K\left[\Psi^d_\rho\left(\alpha; b^{do}(\upsilon)\wedge h, h, \upsilon - t\right) - \Psi^d_\rho(\alpha; h, h, \upsilon - t)\right],\end{aligned}$$

$$\begin{aligned}\Phi^3_\upsilon = \Phi^1_\upsilon &+ (\delta+\rho) S_t\left[\Psi^d_\rho\left(\alpha+\sigma; b^{do}_{ld}(\upsilon), h, \upsilon - t\right) - \Psi^d_\rho\left(\alpha+\sigma; b^{do}_{lu}(\upsilon), h, \upsilon - t\right)\right] \\ &- (r+\rho) K\left[\Psi^d_\rho\left(\alpha; b^{do}_{ld}(\upsilon), h, \upsilon - t\right) - \Psi^d_\rho\left(\alpha; b^{do}_{lu}(\upsilon), h, \upsilon - t\right)\right]\end{aligned}$$

with $b^{do}(\upsilon) = \frac{1}{\sigma}\log\frac{B^{do}(\upsilon)}{S}$, $b^{do}_{ld}(\upsilon) = \frac{1}{\sigma}\log\frac{B^{do}_{ld}(\upsilon)}{S}$ and $b^{do}_{lu}(\upsilon) = \frac{1}{\sigma}\log\frac{B^{do}_{lu}(\upsilon)}{S}$.

Remark 4.3. *The proof of Corollary 4 uses,*

$$\begin{aligned}&\int_a^b e^{\alpha s} E\left[e^{-\rho O^{h,-}_{0,T}}; W_T \in ds\right] \\ &= \int_a^\infty e^{\alpha s} E\left[e^{-\rho O^{h,-}_{0,T}}; W_T \in ds\right] - \int_b^\infty e^{\alpha s} E\left[e^{-\rho O^{h,-}_{0,T}}; W_T \in ds\right] \\ &= \Psi^d_\rho(\alpha; a, h, T) - \Psi^d_\rho(\alpha; b, h, T).\end{aligned}$$

4.3. Down-and-in step call

Define the boundary $B^{di}(t) = \inf\left\{S : (S,t) \in \mathcal{E}_o^{disc}(t)\right\}$. The price decomposition for the down-and-in step call is described next.

Theorem 4.4 (EEP representation)**.** *Suppose that $r + \rho \geq 0$. The value of the American-style down-and-in step call has the EEP decomposition $sc^{di}(S,t) = esc^{di}(S,t) + eepsc^{di}(S,t)$ where $esc^{di}(S,t)$ is the value of a European down-and-in step call and $eepsc^{di}(S,t)$ is the early exercise premium,*

$$esc^{di}(S,t) = E_t^*\left[b_{t,T} e^{-\rho O_{t,T}^-}(S_T - K)^+\right]$$

$$eepsc^{di}(S,t) = E_t^*\left[\int_t^T b_{t,v} e^{-\rho O_{t,v}^-}\phi_v dv\right]$$

and the local gains from early exercise $\phi_v \equiv \phi\left(S_v, B^{di}(v)\right)$ are,

$$\phi_v = \begin{cases} \left((\delta S_v - rK) + \rho(S_v - K) 1_{\{S_v \leq H\}}\right) 1_{\left\{S_v \geq B^{di}(v)\right\}} & \text{if } \delta + \rho \geq 0 \\ (\delta S_v - rK) 1_{\left\{S_v \geq B^{di}(v)\right\}} & \text{if } \delta + \rho < 0 \end{cases}.$$

The immediate exercise boundary B^{di} is unique and solves the recursive integral equation,

$$B^{di}(t) - K = sc\left(B^{di}(t), t; B^{di}(\cdot)\right)$$

$$B^{di}(T_-) = \begin{cases} \begin{cases} (H \wedge \kappa_\rho K) \vee \kappa K & \text{if } \kappa_\rho K \geq \kappa K \geq K \\ K & \text{if } K \geq \kappa K \geq \kappa_\rho K \end{cases} & \delta + \rho \geq 0 \\ H \vee \kappa K & \delta + \rho < 0 \end{cases}$$

The optimal exercise policy is $\tau^{di} = \inf\left\{t \in [0,T] : S_t = B^{di}(t)\right\}$ or $t = T$ if no such time exists in $[0,T]$ and $S_T \geq K$.

Proof of Theorem 4.4. The arguments in the proof of Theorem 4.1 can be adapted to prove Theorem 4.4. There are two cases as discussed in Theorems 3.4 and 3.5: $\delta + \rho \geq 0$ and $\delta + \rho < 0$. □

Corollary 5. *The EEP components of the American down-and-in step call value are*

$$esc^{di}(S,t) = e^{-\gamma(T-t)}\left[S_t \Psi_\rho^d(\alpha + \sigma; k, h, T-t) - K\Psi_\rho^d(\alpha; k, h, T-t)\right]$$

$$eepsc^{di}(S,t) = \int_t^T e^{-\gamma(v-t)}\Phi_v dv$$

where,

$$\Phi_v = \begin{cases} \Phi_v^1 & \text{if } \delta + \rho \geq 0 \\ \Phi_v^2 & \text{if } \delta + \rho < 0 \end{cases}$$

and Φ^1_υ, Φ^2_υ *are given by,*

$$\Phi^1_\upsilon = \Phi^2_\upsilon + \rho S_t \left[\Psi^d_\rho\left(\alpha+\sigma; b^{di}(\upsilon)\wedge h, h, \upsilon - t\right) - \Psi^d_\rho(\alpha+\sigma; h, h, \upsilon - t)\right]$$
$$- \rho K \left[\Psi^d_\rho\left(\alpha; b^{di}(\upsilon)\wedge h, h, \upsilon - t\right) - \Psi^d_\rho(\alpha; h, h, \upsilon - t)\right],$$

with

$$\Phi^2_\upsilon = \delta S_t \Psi^d_\rho\left(\alpha+\sigma; b^{di}(\upsilon), h, \upsilon - t\right) - rK\Psi^d_\rho\left(\alpha; b^{di}(\upsilon), h, \upsilon - t\right)$$

and $b^{di}(\upsilon) = \frac{1}{\sigma}\log\frac{B^{di}(\upsilon)}{S}$.

4.4. Up-and-in step call

Define the boundaries,

$$B^{ui}(t) = \inf\left\{S : (S,t) \in \mathcal{E}^{uisc}_o(t) \text{ and } S \geq (\kappa K \vee K) \vee \kappa_\rho K\right\}$$

and, if $\kappa_\rho K \geq H \geq \kappa K > K$,

$$B^{ui}_{lu}(t) = \sup\left\{S : (S,t) \in \mathcal{E}^{uisc}_o(t) \text{ and } H \geq S \geq \kappa K\right\}$$

$$B^{ui}_{ld}(t) = \inf\left\{S : (S,t) \in \mathcal{E}^{uisc}_o(t) \text{ and } H \geq S \geq \kappa K\right\}.$$

Theorem 4.5 (EEP representation)**.** *Suppose that* $r+\rho \geq 0$*. The value of the American-style up-and-in step call has the EEP decomposition* $sc^{ui}(S,t) = esc^{ui}(S,t) + eepsc^{ui}(S,t)$ *where* $esc^{ui}(S,t)$ *is the value of a European up-and-in step call and* $eepsc^{ui}(S,t)$ *is the early exercise premium,*

$$esc^{ui}(S,t) = E^*_t\left[b_{t,T}e^{-\rho O^+_{t,T}}(S_T - K)^+\right]$$

$$eepsc^{ui}(S,t) = E^*_t\left[\int_t^T b_{t,v}e^{-\rho O^+_{t,v}}\phi_v dv\right]$$

and the local gains from early exercise $\phi_v \equiv \phi\left(S_v, B^{ui}(v), B^{ui}_{lu}(v), B^{ui}_{ld}(v)\right)$ *are,*

$$\phi_v = \begin{cases} \begin{cases} \phi_{1v} & \text{if } H \geq \kappa_\rho K \geq \kappa K \geq K \\ \phi_{2v} & \text{if } \kappa_\rho K \geq H \geq \kappa K \geq K \\ \phi_{3v} & \text{if } \kappa_\rho K \geq \kappa K \geq K \text{ and } \kappa K \geq H \\ \phi_{4v} & \text{if } K \geq \kappa K \geq \kappa_\rho K \end{cases} & \delta+\rho \geq 0 \\ (\delta S_v - rK)1_{\left\{B^{ui}_{lu}(v) \geq S_v \geq B^{ui}_{ld}(v)\right\}} & \delta+\rho < 0 \end{cases}$$

$$\phi_{1v} = \left(\delta S_v - rK + \rho(S_v - K)1_{\{S_v \geq H\}}\right)1_{\{S_v \geq B^{ui}(v)\}}$$

$$\phi_{2v} = \left(\delta S_v - rK + \rho(S_v - K)1_{\{S_v \geq H\}}\right)1_{\{S_v \geq B^{ui}(v)\}} + (\delta S_v - rK)1_{\left\{B^{ui}_{lu}(v) \geq S_v \geq B^{ui}_{ld}(v)\right\}}$$

$$\phi_{3v} = \left(\delta S_v - rK + \rho(S_v - K)1_{\{S_v \geq H\}}\right)1_{\{S_v \geq B^{ui}(v)\}}$$

$$\phi_{4\upsilon} = \left(\delta S_\upsilon - rK + \rho\left(S_\upsilon - K\right) 1_{\{S_\upsilon \geq H\}}\right) 1_{\{S_\upsilon \geq B^{ui}(\upsilon)\}}.$$

In the case $\delta + \rho \geq 0$ and if $H \geq \kappa_\rho K \geq \kappa K \geq K$ or $K \geq \kappa K \geq \kappa_\rho K$, the immediate exercise boundary B^{ui} is unique and solves the recursive integral equation,

$$B^{ui}(t) - K = sc\left(B^{ui}(t), t; B^{ui}(\cdot)\right); \qquad B^{ui}(T_-) = \kappa K \vee K.$$

If $\kappa_\rho K \geq \kappa K \geq K$ and $\kappa K \geq H$, the boundary is unique and solves,

$$B^{ui}(t) - K = sc\left(B^{ui}(t), t; B^{ui}(\cdot)\right); \qquad B^{ui}(T_-) = \kappa_\rho K \vee K = \kappa_\rho K.$$

If $\kappa_\rho K \geq H \geq \kappa K \geq K$, the boundary of the immediate exercise region can consist of 3 pieces, $B^{ui}, B^{ui}_{lu}, B^{ui}_{ld}$, that solve the system of coupled recursive integral equations,

$$\begin{cases} B^{ui}(t) - K = sc\left(B^{ui}(t), t; B^{ui}(\cdot), B^{ui}_{lu}(\cdot), B^{ui}_{ld}(\cdot)\right); & B^{ui}(T_-) = \kappa_\rho K \\ B^{ui}_{lu}(t) - K = sc\left(B^{ui}_{lu}(t), t; B^{ui}(\cdot), B^{ui}_{lu}(\cdot), B^{ui}_{ld}(\cdot)\right); & B^{ui}_{lu}(T_-) = H \\ B^{ui}_{ld}(t) - K = sc\left(B^{ui}_{ld}(t), t; B^{ui}(\cdot), B^{ui}_{lu}(\cdot), B^{ui}_{ld}(\cdot)\right); & B^{ui}_{ld}(T_-) = \kappa K. \end{cases}$$

In the case $\delta + \rho < 0$, the boundary of the immediate exercise region can consist of 2 pieces, B^{ui}_{lu}, B^{ui}_{ld}, that solve the system of coupled recursive integral equations,

$$\begin{cases} B^{ui}_{lu}(t) - K = sc\left(B^{ui}_{lu}(t), t; B^{ui}_{lu}(\cdot), B^{ui}_{ld}(\cdot)\right); & B^{ui}_{lu}(T_-) = H \\ B^{ui}_{ld}(t) - K = sc\left(B^{ui}_{ld}(t), t; B^{ui}_{lu}(\cdot), B^{ui}_{ld}(\cdot)\right); & B^{ui}_{ld}(T_-) = \kappa K. \end{cases}$$

When $\mathcal{E}^{uisc}_o$ has 3 pieces, the optimal exercise policy is,

$$\tau^{ui} = \inf\left\{t \in [0,T] : S_t = B^{ui}(t) \text{ or } S_t = B^{ui}_{lu}(t) \text{ or } S_t = B^{ui}_{ld}(t)\right\}$$

or $t = T$ if no such time exists in $[0,T]$ and $S_T \geq K$. Other cases are covered by adjusting the number of boundaries in τ^{ui}.

Proof of Theorem 4.5. See the proof of Theorem 4.1, and consider the two cases $\delta + \rho \geq 0$ and $\delta + \rho < 0$. □

Corollary 6. *The EEP components of the American up-and-in step call value are,*

$$esc^{ui}(S,t) = e^{-\gamma(T-t)}\left[S_t \Psi^u_\rho\left(\alpha + \sigma; k, h, T-t\right) - K\Psi^u_\rho\left(\alpha; k, h, T-t\right)\right]$$

$$eepsc^{ui}(S,t) = \int_t^T e^{-\gamma(\upsilon - t)} \Phi_\upsilon d\upsilon$$

where,

$$\Phi_\upsilon = \begin{cases} \begin{cases} \Phi_\upsilon^1 & \text{if } H \geq \kappa_\rho K \geq \kappa K \geq K \\ \Phi_\upsilon^2 & \text{if } \kappa_\rho K \geq H \geq \kappa K \geq K \\ \Phi_\upsilon^3 & \text{if } \kappa_\rho K \geq \kappa K \geq K \text{ and } \kappa K \geq H \\ \Phi_\upsilon^4 & \text{if } K \geq \kappa K \geq \kappa_\rho K \end{cases} & \delta + \rho \geq 0 \\ \Phi_\upsilon^5 & \delta + \rho < 0 \end{cases}$$

and Φ_υ^1, Φ_υ^2, Φ_υ^3, Φ_υ^4, Φ_υ^5 *are given by,*

$$\begin{aligned} \Phi_\upsilon^1 &= \delta S_t \Psi_\rho^u \left(\alpha + \sigma; b^{ui}(\upsilon), h, \upsilon - t\right) - rK\Psi_\rho^u \left(\alpha; b^{ui}(\upsilon), h, \upsilon - t\right) \\ &+ \rho S_t \Psi_\rho^u \left(\alpha + \sigma; b^{ui}(\upsilon) \vee h, h, \upsilon - t\right) - \rho K\Psi_\rho^u \left(\alpha; b^{ui}(\upsilon) \vee h, h, \upsilon - t\right) \end{aligned}$$

$$\begin{aligned} \Phi_\upsilon^2 &= \Phi_\upsilon^1 + \delta S_t \left[\Psi_\rho^u \left(\alpha + \sigma; b_{ld}^{ui}(\upsilon), h, \upsilon - t\right) - \Psi_\rho^u \left(\alpha + \sigma; b_{lu}^{ui}(\upsilon), h, \upsilon - t\right)\right] \\ &- rK \left[\Psi_\rho^u \left(\alpha; b_{ld}^{ui}(\upsilon), h, \upsilon - t\right) - \Psi_\rho^u \left(\alpha; b_{lu}^{ui}(\upsilon), h, \upsilon - t\right)\right], \end{aligned}$$

$$\Phi_\upsilon^3 = \Phi_\upsilon^4 = \Phi_\upsilon^1,$$

$$\begin{aligned} \Phi_\upsilon^5 &= \delta S_t \left[\Psi_\rho^u \left(\alpha + \sigma; b_{ld}^{ui}(\upsilon), h, \upsilon - t\right) - \Psi_\rho^u \left(\alpha + \sigma; b_{lu}^{ui}(\upsilon), h, \upsilon - t\right)\right] \\ &- rK \left[\Psi_\rho^u \left(\alpha; b_{ld}^{ui}(\upsilon), h, \upsilon - t\right) - \Psi_\rho^u \left(\alpha; b_{lu}^{ui}(\upsilon), h, \upsilon - t\right)\right] \end{aligned}$$

with $b^{ui}(\upsilon) = \frac{1}{\sigma}\log\frac{B^{ui}(\upsilon)}{S}$, $b_{ld}^{ui}(\upsilon) = \frac{1}{\sigma}\log\frac{B_{ld}^{ui}(\upsilon)}{S}$ *and* $b_{lu}^{ui}(\upsilon) = \frac{1}{\sigma}\log\frac{B_{lu}^{ui}(\upsilon)}{S}$.

5. Hedging American step options

Hedge ratios follow from the valuation formulas in the previous section by differentiating with respect to the underlying asset price. Let $\psi_\rho(\alpha; k, h, T) \equiv \partial_S \Psi_\rho^d(\alpha; k, h, T)$ be the first derivative of the function $\Psi_\rho(\alpha; k, h, T)$ with respect to S. Proposition 2 in Appendix A gives the expression for ψ_ρ. The next two theorems describe the delta hedge ratios of down-and-out and down-and-in American step calls.

Theorem 5.1. *Consider an American down step call option and let* $\Delta_{sc}^d = \partial SC^d(S, O, t)/\partial S$ *be its delta hedge ratio. Then* Δ_{sc}^d *is continuous on* $\{\mathbb{R}_+ \backslash H\} \times [0, T)^2$ *for* $0 \leq |\rho| < \infty$ *and given by* $\Delta_{sc}^d = \Delta_{esc}^d + \Delta_{eepsc}^d$ *where* Δ_{esc}^d *is the hedge for the European step call and* Δ_{eepsc}^d *the hedge for the early exercise premium. The terms in this decomposition are,*

$$\Delta_{esc}^d = e^{-\lambda(T-t)} \begin{bmatrix} \Psi_\rho^d(\alpha + \sigma; k, h, T - t) \\ +S_t \psi_\rho^d(\alpha + \sigma; k, h, T - t) - K\psi_\rho^d(\alpha; k, h, T - t) \end{bmatrix}$$

$$\Delta_{eepsc}^d = \int_t^T e^{-\lambda(\upsilon - t)} \sum_{i=1}^n \begin{bmatrix} g_i(S, K)\Psi_\rho^d(\alpha_i; k_i, h_i, \upsilon - t) \\ +G_i(S, K)\psi_\rho^d(\alpha_i; k_i, h_i, \upsilon - t) \end{bmatrix} d\upsilon$$

where $G_i(S,K), i=1,...,n,$ *are the functions in the representations of* $\Phi_\upsilon \equiv \sum_{i=1}^n G_i(S,K)\Psi_\rho^d(\alpha_i;k_i,h_i,\upsilon-t)$ *in Corollaries 4 and 5. Function* $g_i(S,K) \equiv \partial_S G_i(S,K)$ *is the derivative of* $G_i(S,K)$ *with respect to the price* S*. The parameters* (α_i,k_i,h_i) *are drawn from the following sets* $\alpha_i \in \{\alpha,\alpha+\sigma,-\alpha,-\alpha-\sigma\}$, $k_i \in \{b,h,k,-b,-h,-k\}$, *and* $h_i \in \{h,-h\}$, *as appropriate.*

The hedge ratio decomposition parallels the EEP decomposition of the American step call. Note that the formula for the EEP hedge involves the derivative of the function $\Phi_\upsilon = \sum_{i=1}^n G_i(S,K)\Psi_\rho^d(a_i;k_i,h_i,\upsilon-t)$ in the integrand of the premium (see Corollaries 4 and 5). Indeed, straightforward differentiation gives

$$\partial_S\Phi_\upsilon = \sum_{i=1}^n \Big(G_i(S,K)\partial_S\Psi_\rho^d(\alpha_i;k_i,h_i,\upsilon-t) + \partial_S G_i(S,K)\Psi_\rho^d(\alpha_i;k_i,h_i,\upsilon-t)\Big)$$

where $\partial_S G_i(S,K) = g_i(S,K)$ and $\partial_S\Psi_\rho^d(\alpha_i;k_i,h_i,\upsilon-t) = \psi_\rho^d(\alpha_i;k_i,h_i,\upsilon-t)$. It is also important to note that the EEP hedge is parameterized by the exercise boundary. The evaluation of the delta hedge, therefore requires the determination of the immediate exercise boundary. This issue is addressed in the next section.

Proof of Theorem 5.1. Differentiating the price function with respect to S leads to the expressions announced. Continuity follows from the continuity of the functions Ψ_ρ and ψ_ρ. □

The behavior of the hedge at the barrier is important for issuers/sellers of the contract.

Theorem 5.2. *The hedge ratio for the American down step call has the following properties at the barrier, for all* $k \in \mathbb{R}$ *and* $h \in \mathbb{R}$ *and* $0 \le |\rho| < \infty$*:*
(i) Δ_{sc}^d *is continuous,*
(ii) $\Gamma_{sc}^d = \partial\Delta_{sc}^d/\partial S$ *has a finite jump.*

The reason for continuity, Property *(i)*, is because the down-and-out step option does not lose all value when the barrier H is hit. What changes is the rate at which the payoff depreciates if the underlying price wanders below H. This stands in contrast with the classic down-and-out option that has a price discontinuity at the barrier. A related intuition applies for the down-and-in case. Property *(ii)* nevertheless indicates that the rate of change of the hedge jumps at the barrier. This jump in the derivative of the hedge reflects the fact that the adjustment factor in the option payoff becomes active/inactive when the barrier is crossed.

Proof of Theorem 5.2. Consider $f_\rho(z,h,T)\,dz = E\left[e^{-\rho O_{0,T}^{h,-}}; W_T \in dz\right]$, the transition probability, and its double Laplace transform,

$$\int_0^\infty e^{-sT} E\left[g(W_T)\,e^{-\rho O_{0,T}^{h,-}}\right] dT = \int_{-\infty}^\infty g(h)\,f_\rho(z,h,s)\,dh$$

where $g : \mathbb{R} \longrightarrow \mathbb{R}_+$ is a Borel function, $s > 0$ and $\rho > 0$. From Kac (1949), it follows that the function $f(h) = f_\rho(z, h, s)$ is the unique solution to the Sturm-Liouville equation,

$$\frac{1}{2}\frac{\partial^2 f(h)}{\partial h^2} = \left(s + \rho 1_{\{h \le 0\}}\right) f(h), \quad \text{for } h \neq 0$$

subject to the requirements that $\frac{\partial}{\partial h} f_\rho(z, h, s)$ exists and is locally bounded for all $h \neq 0$, that,

$$\lim_{h \to \infty} f_\rho(z, h, s) = 0, \qquad \lim_{h \to -\infty} f_\rho(z, h, s) = 0$$

and at the barrier,

$$\lim_{\epsilon \to 0_+} f_\rho(z, \epsilon, s) - f_\rho(z, -\epsilon, s) = 0, \tag{6}$$

$$\lim_{\epsilon \to 0_+} \frac{\partial}{\partial h} f_\rho(z, \epsilon, s) - \frac{\partial}{\partial h} f_\rho(z, -\epsilon, s) = 0. \tag{7}$$

Given (6)-(7), the functions $f_\rho(z, h, s)$ and $\frac{\partial}{\partial h} f_\rho(z, h, s)$ are continuous at $h = 0$ for $0 \le \rho < \infty$. The continuity of Δ_{sc}^d at the barrier follows.

The proof of *(ii)* follows immediately from *(i)*. □

6. Numerical procedure and implementation

6.1. Computational algorithm

The following algorithm is implemented to compute the immediate exercise boundary. Details about the convergence of the algorithm are presented in Appendix D. For concreteness, consider the case of a down-and-out step call when $\kappa K \ge H \ge \kappa_\rho K > K$. As shown above, 3 boundary components $B^{do}, B_{lu}^{do}, B_{ld}^{do}$ are required to describe the immediate exercise region.

Discretize the time interval as $\{t_n : n = 0, ..., N\}$ where $t_N = T$ and $\Delta t_n = t_n - t_{n-1} = h$ for all $n = 1, ..., N$. To simplify notation, write n for t_n (hence $B^{do}(t_n) = B^{do}(n)$, etc.,...).

1. For $n = N$: set $B^{do}(N) = \kappa K \vee K$, $B_{lu}^{do}(N) = H$, $B_{ld}^{do}(N) = \kappa_\rho K \vee K$
2. For $n = N - 1 : ... : 0$

   (a) Approximate the EEP and the option price by:

   $$eepsc^{do}(S, n) = h \sum_{j=0}^{n} w_j^{(n)} e^{-\lambda(j-n)h} \Phi_j$$

   $$sc^{do}(S, n) = esc^{do}(S, n) + eepsc^{do}(S, n)$$

   (b) Sequentially calculate the boundary components:

      i. $B^{do}(n) - K = sc\left(B^{do}(n), n; B^{do}(\cdot), B_{lu}^{do}(\cdot), B_{ld}^{do}(\cdot)\right)$
      ii. $B_{lu}^{do}(n) - K = sc\left(B_{lu}^{do}(n), n; B^{do}(\cdot), B_{lu}^{do}(\cdot), B_{ld}^{do}(\cdot)\right)$

iii. $B_{ld}^{do}(n) - K = sc\left(B_{ld}^{do}(n), n; B^{do}(\cdot), B_{lu}^{do}(\cdot), B_{ld}^{do}(\cdot)\right)$

In Step 2.a., the outer integral in the EEP can be calculated using a variety of quadrature methods with bounded weights (e.g., trapezoidal rule, barycentric rational quadrature rule, etc.,...). To speed up our computations we have chosen the hybrid quadrature scheme presented in Laminou Abdou & Moraux (2016). In Step 2.b., the determination of the exercise boundary components is a fixed point problem that can be solved by using a root-finding algorithm (e.g., the bisection method).

Note also that the order in which the boundary components are calculated in Step 2.b. does not matter. The option price evaluated at a given boundary point does not depend on the contemporaneous values of the other boundaries. For instance, the value $sc\left(B_{lu}^{do}(n), n; B^{do}(\cdot), B_{lu}^{do}(\cdot), B_{ld}^{do}(\cdot)\right)$ of the contract at $S(n) = B_{lu}^{do}(n)$ does not depend on $B^{do}(n), B_{ld}^{do}(n)$. But it does depend on the future values of the boundary components $\left\{B^{do}(j), B_{lu}^{do}(j), B_{ld}^{do}(j) : j > n\right\}$, which are already known from computations at previous discretization points.

The algorithm described above produces an approximation of the exercise boundary,

$$\left\{B^{do}(n), B_{lu}^{do}(n), B_{ld}^{do}(n) : n = 0, ..., N\right\}$$

along the discretization selected. As $h \to 0$, the approximation converges to the true boundary $\left\{B^{do}(t), B_{lu}^{do}(t), B_{ld}^{do}(t) : t \in [0, T]\right\}$.

6.2. *Numerical Examples*

This section presents numerical results and graphical illustrations to highlight structural and numerical properties of the price, the hedge ratio, the exercise region and the boundaries. For brevity, we focus on the case of down-and-out call options. Parameter values are chosen so as to provide clear illustrations of relevant features.

The outer integral of the EEP is computed numerically by dividing the time interval $[0, T]$ into n subintervals. The inner integrals in the functions Ψ are computed with the MATLAB built-in numerical integration function. For higher values of H and/or ρ, that built-in function can take longer to achieve the precision goal for the computation of Ψ and this affects the global computation time.[9] For $n = 100$, on a computer equipped with an Intel $i7$ core processor and using MATLAB 2017b, the mean computation time of the single boundary (when $H = 95$) is $1mn53sec$. The mean computation time of the three coupled boundaries (when $H = 120$) increases to $1h20mn$. For graphical illustrations, we set $n = 300$; for numerical values of option prices and hedges, we take $n = 1,000$.

Figure 1 shows the structure of the down-and-out exercise region, in particular that it can fail to be up-connected. This scenario occurs in the configuration $\kappa K \geq H \geq \kappa_\rho > K$. The gray areas

[9] As we use the bisection method to determine the roots of the different recursive equations, the algorithm has to repeatedly compute the functions Ψ appearing in the recursive equations in order to determine the boundaries. Here, the tolerance level is set to 10^{-4}.

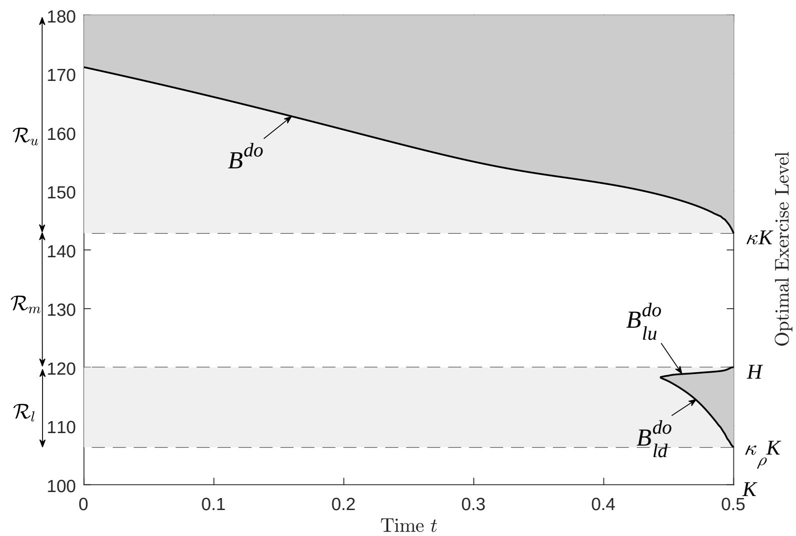

Fig. 1. The early exercise boundaries of a down-and-out step call option in the configuration $\kappa K \geq H \geq \kappa_\rho K > K$. Parameters values: $K = 100$, $H = 120$, $\rho = 0.2$, $r = 0.05$, $\delta = 0.035$, $\sigma = 0.3$, $n = 300$, $tol = 0.0001$, $T = 0.5$.

$\mathcal{R}_u$ and $\mathcal{R}_l$ represent the regions where the local gain from exercising the option early is positive. The darker areas delimited by B^{do},B^{do}_{lu} and B^{do}_{ld} are the regions where early exercise is optimal.

Figure 2 illustrates the relationships between the optimal exercise regions as stated in Property *(i)* of Theorem 3.1. The optimal exercise region of the vanilla call option in the market $\mathcal{M}\,(r+\rho, \delta+\rho)$, denoted by $\mathcal{E}^c_\rho$, is the union of all gray areas. It contains the optimal exercise region of the down-and-out step call which is $\mathcal{E}^{dosc}_o = \mathcal{E}^{dosc,1}_o \cup \mathcal{E}^{dosc,2}_o$. The darkest area is the optimal exercise region of the vanilla American call option in $\mathcal{M}\,(r, \delta)$. Near maturity, the boundaries of some of these regions are very close as they converge to the boundaries of deterministic problems.

Figures 3 and 4 show the early exercise boundary (EEB) for different values of the killing rate ρ and the excursion boundary H. These figures illustrate the fact that an increase in ρ or H, tends to decrease the EEBs. Hence, it is optimal to exercise earlier for options with higher values of ρ or H. When ρ is large (e.g., $\rho = 20$) and the underlying asset price is in the excursion region, waiting can lead to a significant reduction in the value of the gain. This explains why the immediate exercise boundary is mostly flat for $H = 120$ (see Figure 3).

Figure 5 shows the Delta hedges for a European and an American-style down-and-out step call options, as functions of the underlying asset price. As expected, the hedges are continuous functions, even at the barrier. But activation of the occupation time when the barrier is crossed implies that the Deltas are non-differentiable at $S = H$, inducing a finite jump in the Gammas. The hedges decrease rapidly below the barrier, when the underlying price decreases.

Figure 6 illustrates the convergence of our numerical method in the case of a down-and-out step call option. It plots the approximated value (dashed line) and the "true value"(solid line) of the

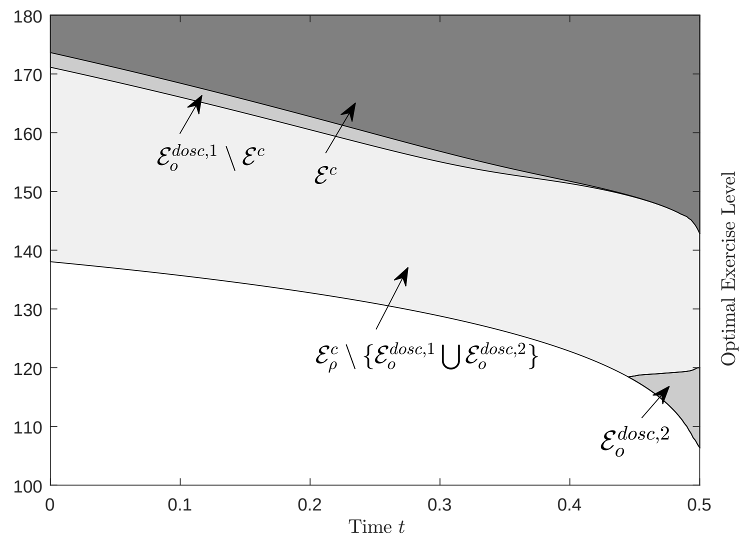

Fig. 2. Optimal exercise regions $\mathcal{E}^c \subseteq \mathcal{E}_o^{dosc} \subseteq \mathcal{E}_\rho^c$. $\mathcal{E}^c$ is the exercise region of a vanilla call option in the market $\mathcal{M}(r, \delta)$. $\mathcal{E}_o^{dosc} = \mathcal{E}_o^{dosc,1} \cup \mathcal{E}_o^{dosc,2}$ is the exercise region of a down-and-out step call option in the configuration $\kappa K \geq H \geq \kappa_\rho K > K$. $\mathcal{E}_\rho^c$ is the exercise region of a vanilla call in the market $\mathcal{M}(r + \rho, \delta + \rho)$. Parameters values: $K = 100$, $H = 120$, $\rho = 0.2$, $r = 0.05$, $\delta = 0.035$, $\sigma = 0.3$, $n = 300$, $tol = 0.0001$, $T = 0.5$.

EEB at the point $t = 0$. The true value is computed by increasing n so as obtain at least six stable decimal digits. For $n \geq 250$ the approximated value has at least a precision of 10^{-6}.

Table 1 provides numerical values for the two hedges. When the underlying price is below the barrier, the two Deltas are close, nevertheless distinct. The size of the difference becomes significant above the barrier, reflecting the need to hedge fluctuations in the EEP.

Stock price	Delta of European DO Step	Delta of American DO Step
80	0.0212	0.0215
90	0.2427	0.2460
100	0.7520	0.7637
110	0.7834	0.7997
120	0.8203	0.8437
130	0.8546	0.8871
140	0.8828	0.9262

Table 1: The hedging ratio ($\partial SC/\partial S$) for a down-and-out step call option. Parameters values: $K = 100$, $H = 95$, $\rho = 20$, $r = 0.05$, $\delta = 0.05$, $\sigma = 0.3$, $n = 1000$, $tol = 0.0001$, $T = 1$.

Table 2 compares the prices of vanilla, step and knockout (down-and-out) calls for different asset prices. It shows that the EEP comprises a substantial part of the American step option price even if the option is deep out of the money. As the underlying price increases, the American option prices become close to each other because the likelihood that the barrier becomes active decreases.

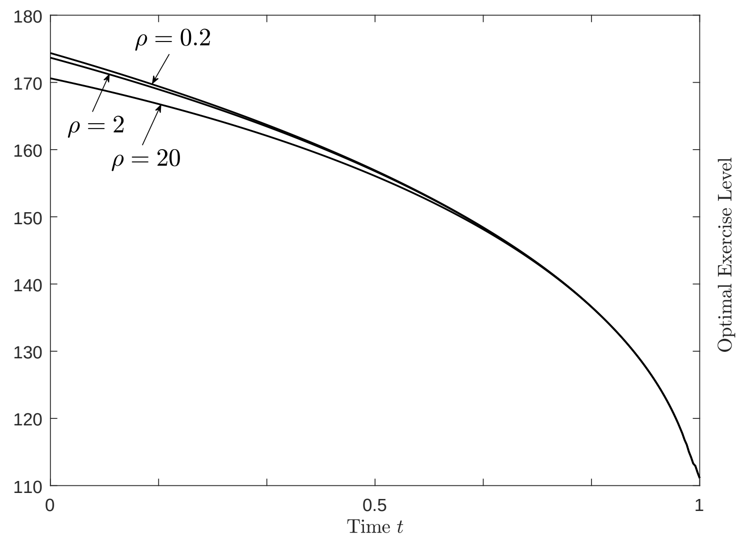

Fig. 3. The early exercise boundaries of a down-and-out step call option for different values of ρ. Parameters values: $K = 100$, $H = 95$, $r = 0.05$, $\delta = 0.035$, $\sigma = 0.3$, $n = 300$, $tol = 0.0001$, $T = 1$.

Table 3 provides robustness results. It shows the impact of the discount rate rho, volatility sigma and dividend yield q on the step call price and its components. The EEP contribution to the step call price (ratio) is most sensitive to the dividend yield and volatility. It is positively related to q and rho, but negatively to sigma.

7. Conclusion

This paper analyzes American-style step options in the standard model with constant coefficients. Several unusual features of the contract were documented. A striking result is that the exercise region of a step call need not be up-connected, implying that immediate exercise, if optimal at a given price, may no longer be optimal at a higher price. Another surprising property is that immediate exercise can be suboptimal at an intermediate point while being optimal at higher and lower underlying prices. When these non-standard features apply, the exercise region can no longer be characterized by a single curve, even though the contract is written on a single underlying asset price. Multiple boundary components are indeed necessary to properly describe the optimal exercise decision and to calculate the value of the contract. Boundary components were shown to satisfy a system of coupled recursive integral equations. EEP decompositions for the price and the hedge ratio were also provided.

Step options are occupation time derivatives, that is derivative contracts whose values depend on the occupation time of some set. As such, the value of the step option slowly dissipates or appreciates. As documented in this study, this feature facilitates hedging and risk management activities. Indeed, even for an American-style step option, hedging is straightforward once the

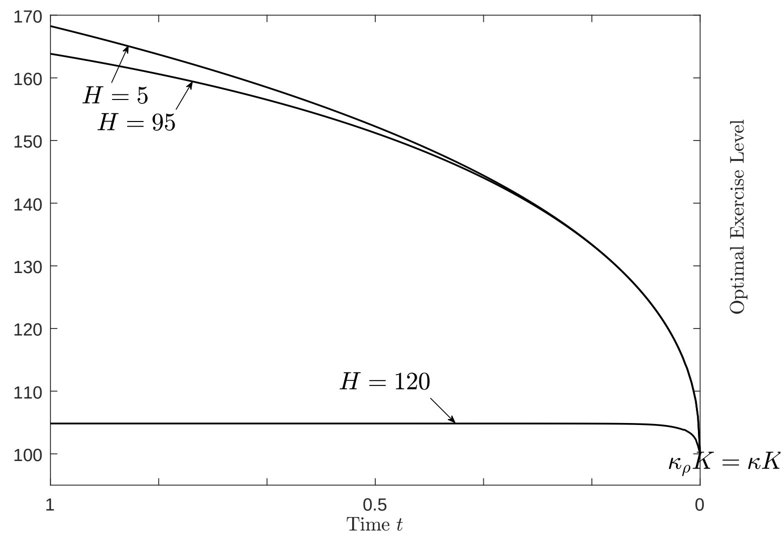

Fig. 4. The early exercise boundaries of a down-and-out step call option for different values of H. Parameters values: $K = 100$, $\rho = 20$, $r = 0.05$, $\delta = 0.05$, $\sigma = 0.3$, $n = 300$, $tol = 0.0001$, $T = 1$.

exercise boundary components have been identified. Of paramount importance for users is the fact that the hedge ratio does not vary drastically when the barrier is crossed.

The principles and methods employed in this study may prove useful to approach the valuation of other American-style contracts. Even though it is possible to write alternative characterizations of the price under minimal assumptions, they are often difficult to exploit unless preliminary knowledge about the structure of the exercise region is obtained. The procedure followed in this paper identifies properties of the exercise region based on dominance arguments. The arguments constructed enable us to prove non-standard properties. The knowledge acquired is then used to derive a decomposition of the price and integral equations for the exercise boundary components. Last, a numerical scheme is developed in order to compute the solution of the system of integral equations obtained. This scheme exploits properties of the equations involved and of the optimal exercise decision.

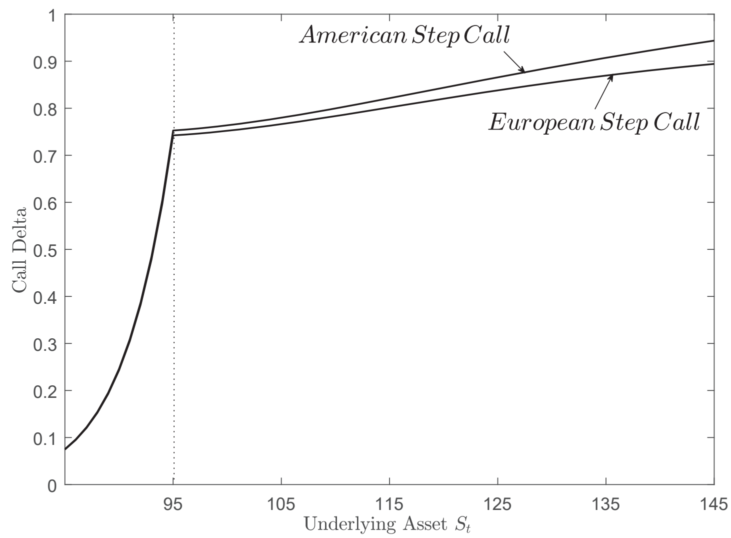

Fig. 5. The delta hedging ratio ($\partial SC/\partial S$) for a down-and-out step call option. Parameters values: $K = 100$, $H = 95$, $\rho = 20$, $r = 0.05$, $\delta = 0.05$, $\sigma = 0.3$, $n = 300$, $tol = 0.0001$, $T = 1$.

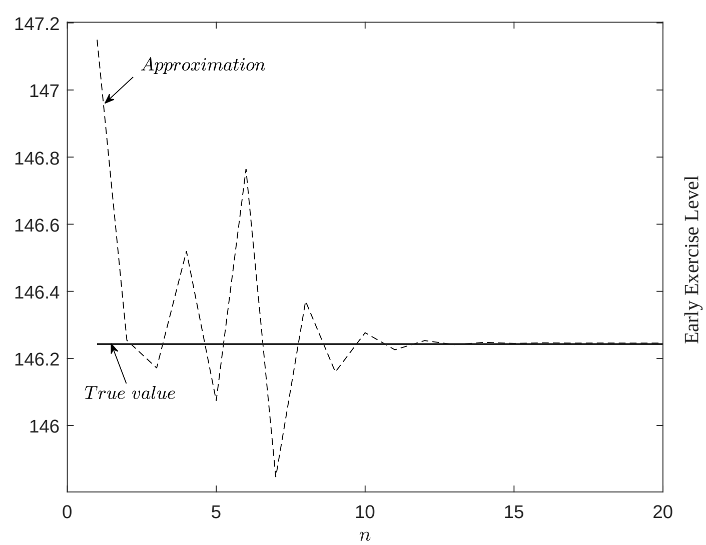

Fig. 6. Approximation vs. "True value"of the EEB of a down-and-out step call option at the point $t = 0$. Parameters values: $K = 100$, $H = 95$, $\rho = 26.34$, $r = 0.05$, $\delta = 0.07$, $\sigma = 0.3$, $n = 300$, $tol = 0.0001$, $T = 1$.

	Standard Call Option				Step Call Option				Standard Barrier Call option			
S	Eur	Am	EEP	Ratio	Eur	Am	EEP	Ratio	Eur	Am	EEP	Ratio
80	0.9120	0.9334	0.0214	2.29%	0.0024	0.0025	0.0001	4.00%	0	0	0	-
85	1.6989	1.7455	0.0466	2.67%	0.0232	0.0243	0.0011	4.53%	0	0	0	-
90	2.8769	2.9692	0.0923	3.11%	0.1970	0.2063	0.0093	4.51%	0	0	0	-
95	4.5018	4.6704	0.1686	3.61%	1.4875	1.5612	0.0737	4.72%	0	0	0	-
100	6.5976	6.8851	0.2875	4.18%	4.5106	4.7457	0.2351	4.95%	3.3321	3.5291	0.1970	5.58%
105	9.1562	9.6192	0.4630	4.81%	7.7467	8.1875	0.4408	5.38%	6.8461	7.2803	0.4342	5.96%
110	12.1431	12.8534	0.7103	5.53%	11.2121	11.9258	0.7137	5.98%	10.5459	11.2859	0.7400	6.56%
115	15.5067	16.5519	1.0452	6.31%	14.9038	15.9798	1.0760	6.73%	14.4248	15.5674	1.1426	7.34%
120	19.1870	20.6712	1.4842	7.18%	18.8033	20.3535	1.5502	7.62%	18.4676	20.1374	1.6698	8.29%
125	23.1235	25.1671	2.0436	8.12%	22.883	25.0418	2.1588	8.62%	22.6529	24.9990	2.3461	9.38%
130	27.2602	29.9994	2.7392	9.13%	27.1115	29.9987	2.8872	9.62%	26.9568	29.9968	3.0400	10.13%
135	31.5489	35.0000	3.4511	9.86%	31.4581	34.9986	3.5405	10.12%	31.3559	34.9959	3.6400	10.40%
140	35.9501	40.0000	4.0499	10.12%	35.8952	39.9985	4.1033	10.26%	35.8287	39.9952	4.1665	10.42%

Table 2: Vanilla, Step, and standard down-and-out call options prices, EEPs and their contribution to the American call prices (ratio). Parameters values: $K = 100$, $H = 95$, $\rho = 26.34$, $r = 0.05$, $\delta = 0.07$, $\sigma = 0.2$, $n = 600$, $tol = 0.0001$, $T = 1$.

σ	δ	European Step Option	American Step Option	EEP	Ratio
$\rho = 0$					
0.2	0.03	8.6525	8.6528	0.0003	0.00%
	0.05	7.5771	7.6626	0.0855	1.12%
	0.1	5.3017	5.9283	0.6266	10.57%
0.4	0.03	16.2107	16.2293	0.0186	0.11%
	0.05	15.0788	15.2503	0.1715	1.12%
	0.1	12.5048	13.2548	0.75	5.66%
$\rho = 5$					
0.2	0.03	6.2126	6.2128	0.0002	0.00%
	0.05	5.3799	5.4513	0.0714	1.31%
	0.1	3.661	4.1871	0.5261	12.56%
0.4	0.03	10.7112	10.727	0.0158	0.15%
	0.05	9.8995	10.0354	0.1359	1.35%
	0.1	8.0791	8.6743	0.5952	6.86%
$\rho = 10$					
0.2	0.03	5.3439	5.3441	0.0002	0.00%
	0.05	4.6096	4.6746	0.065	1.39%
	0.1	3.1067	3.5949	0.4882	13.58%
0.4	0.03	8.8338	8.8482	0.0144	0.16%
	0.05	8.1464	8.2666	0.1202	1.45%
	0.1	6.6121	7.1467	0.5346	7.48%
$\rho = 26.34$					
0.2	0.03	4.2937	4.2939	0.0002	0.00%
	0.05	3.6877	3.7437	0.056	1.50%
	0.1	2.4589	2.9001	0.4412	15.21%
0.4	0.03	6.6088	6.6208	0.012	0.18%
	0.05	6.0798	6.1784	0.0986	1.60%
	0.1	4.9051	5.3635	0.4584	8.55%
$\rho = 50$					
0.2	0.03	3.7522	3.7524	0.0002	0.01%
	0.05	3.2159	3.2669	0.051	1.56%
	0.1	2.1333	2.5488	0.4155	16.30%
0.4	0.03	5.4761	5.4866	0.0105	0.19%
	0.05	5.0319	5.1185	0.0866	1.69%
	0.1	4.0481	4.4649	0.4168	9.34%

Table 3: Sensitivity of value components to down-and-out call options parameters. Parameters values: $K = 100$, $H = 98$, $r = 0.05$, $n = 600$, $tol = 0.0001$, $T = 1$.

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Appendix A. The function Ψ

This appendix provides formulas for $\Psi_\rho^d(\alpha; k, h, T) = \int_k^\infty e^{\alpha z} f_\rho(z, h, T)\, dz$ where $f_\rho(z, h, T)\, dz = E\left[e^{-\rho O_{0,T}^{h,-}}; W_T \in dz\right]$ is the transition probability density for a Brownian motion started at 0 and killed at rate ρ below h. The expressions correspond to those in Linetsky (1999) modulo a change of variables.

Proposition 1. *The function $\Psi_\rho^d(\alpha; k, h, T)$ is continuous for all $k \in \mathbb{R}$ and $h \in \mathbb{R}$ and*
(i) Region I. $k \geq h$ $(K \geq H)$ and $h \leq 0$ $(S \geq H)$:

$$\begin{aligned}\Psi_\rho^I(\alpha; k, h, T) &= e^{\alpha^2 T/2} N(d_1(T)) - e^{2\alpha h + \alpha^2 T/2} N(d_2(T)) \\ &\quad + \int_0^T \frac{\left(1 - e^{-\rho(T-t)}\right) e^{2\alpha h + \alpha^2 t/2}}{\sqrt{2\pi}\rho\,(T-t)^{3/2}} \left[\alpha N(d_2(t)) + t^{-1/2} N'(d_2(t))\right] dt\end{aligned}$$

(ii) Region II. $k \geq h$ $(K \geq H)$ and $h \geq 0$ $(S \leq H)$:

$$\Psi_\rho^{II}(\alpha; k, h, T) = \int_0^T \frac{\left(1 - e^{-\rho(T-t)}\right) e^{\alpha h + \alpha^2 t/2}}{\sqrt{2\pi}\rho\,(T-t)^{3/2}} \left[\alpha C_1 N(d_3(t)) + C_2 N'(d_3(t))\right] e^{-h^2/[2(T-t)]} dt$$

(iii) Region III. $k \leq h$ $(K \leq H)$ and $h \leq 0$ $(S \geq H)$:

$$\begin{aligned}\Psi_\rho^{III}(\alpha; k, h, T) &= \Psi_\rho^I(\alpha; h, h, T) \\ &\quad + e^{-\rho T}\left[\Psi_{-\rho}^{II}(-\alpha; -h, -h, T) - \Psi_{-\rho}^{II}(-\alpha; -k, -h, T)\right]\end{aligned}$$

(iv) Region IV. $k \leq h$ $(K \leq H)$ and $h \geq 0$ $(S \leq H)$:

$$\begin{aligned}\Psi_\rho^{IV}(\alpha; k, h, T) &= \Psi_\rho^{II}(\alpha; h, h, T) \\ &\quad + e^{-\rho T}\left[\Psi_{-\rho}^{I}(-\alpha; -h, -h, T) - \Psi_{-\rho}^{I}(-\alpha; -k, -h, T)\right]\end{aligned}$$

where

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-z^2/2} dz, \quad N'(x) = \frac{dN(x)}{dx} = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

$$d_1(t) = \frac{-k + \alpha t}{\sqrt{t}}, \quad d_2(t) = \frac{-k + 2h + \alpha t}{\sqrt{t}}, \quad d_3(t) = \frac{-k + h + \alpha t}{\sqrt{t}}$$

$$C_1 = 1 - \frac{h^2}{T-t} + \alpha h, \qquad C_2 = t^{-1/2} C_1 + t^{-3/2} h\,(k-h)\,.$$

Proof of Proposition 1. The function Ψ_ρ can be derived as in Linetsky (1999) (see Borodin & Salminen (2012), Equation (1.5.7)). Alternatively, it can be derived from Ψ'_ρ in (3) using the change of variable $k' = k + x$ and $x = -h$. We have: $\Psi_\rho(\alpha; k, h, T) = e^{\alpha h} \Psi'_\rho(\alpha; k-h, -h, T)$. □

Although the expressions above are simplified, one must resort to numerical integration methods for computations.

Properties of the excursion times of the underlying asset enable us to determine the function $\Psi_\rho^u(\alpha; k, h, T)$. From $t = 0$ to T, the sum of the excursion times above and below the barrier H is the length of the period T, $O_{0,T}^{+} + O_{0,T}^{-} = T$. It follows that,

$$\Psi_\rho^u(\alpha; k, h, T) = \int_k^\infty e^{\alpha s} E\left[e^{-\rho\rho O_{0,T}^{h,+}}; W_T \in ds\right]$$
$$= e^{-\rho T} \int_k^\infty e^{\alpha s} E\left[e^{\rho O_{0,T}^{h,-}}; W_T \in ds\right] = e^{-\rho T} \Psi_{-\rho}^d(\alpha; k, h, T).$$

The next proposition is useful for the derivation of the Delta hedge. It gives the expression of $\psi_\rho(\alpha; k, h, T) \equiv \partial_S \Psi_\rho(\alpha; k, h, T)$ for $K \geq H$.

Proposition 2. *The first derivative of the function $\Psi_\rho(\alpha; k, h, T)$ with respect to S, denoted by $\psi_\rho(\alpha; k, h, T)$, is continuous for all $S \geq 0$. In particular, for $K \geq H$, it is given by,*

- *For $S \geq H$:*

$$\psi_\rho^I(\alpha; k, h, T) = \frac{e^{\alpha^2 T/2} N'(d_1(T))}{\sigma S \sqrt{T}} + \frac{e^{\alpha h + \alpha^2 T/2}}{\sigma S}\left[\alpha N(d_2(T)) + \frac{N'(d_2(T))}{\sqrt{T}}\right]$$
$$+ \frac{e^{2\alpha h}}{\sigma S} \int_0^T \frac{\left(1 - e^{-\rho(T-t)}\right) e^{\alpha^2 t/2}}{\sqrt{2\pi}\rho (T-t)^{3/2}} A_1 dt$$

- *For $S \leq H$:*

$$\psi_\rho^{II}(\alpha; k, h, T) = \int_0^T \frac{\left(1 - e^{-\rho(T-t)}\right) e^{\alpha^2 t/2}}{\sqrt{2\pi}\rho (T-t)^{3/2}} \left[A_2 N(d_3(t)) + A_3 N'(d_3(t)) + A_4\right] e^{\alpha h - \frac{h^2}{2(T-t)}} dt$$

where,

$$A_1 = \frac{d_2(t)}{t} - \frac{\alpha(2+\alpha)}{\sqrt{t}} N'(d_2(t)) - 2\alpha^2 N(d_2(t))$$

$$A_2 = \frac{\alpha e^{\alpha h - h^2/[2(T-t)]}}{\sigma S}\left(\frac{2h}{T-t} - \alpha\right)$$

$$A_3 = \frac{e^{\alpha h - h^2/[2(T-t)]}}{\sigma S}\left[t^{-1/2}\left(\frac{2h}{T-t} - \alpha\right) + t^{-3/2}(h-k)\right]$$

$$A_4 = \frac{e^{\alpha h - h^2/[2(T-t)]}}{\sigma S}\left(\frac{h}{T-t} - \alpha\right)\left[\alpha C_1 N(d_3(t)) + C_2 N'(d_3(t))\right]$$

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-z^2/2} dz, \quad N'(x) = \frac{dN(x)}{dx} = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

$$d_1(t) = \frac{-k + \alpha t}{\sqrt{t}}, \quad d_2(t) = \frac{-k + 2h + \alpha t}{\sqrt{t}}, \quad d_3(t) = \frac{-k + h + \alpha t}{\sqrt{t}}$$

$$C_1 = 1 - \frac{h^2}{T-t} + \alpha h, \quad C_2 = t^{-1/2} C_1 + t^{-3/2} h (k-h).$$

Proof of Proposition 2. Differentiating Ψ_ρ in Proposition 1 with respect to S gives the formulas stated. Note that ψ_ρ is a linear combination of continuous functions. Also, d_1, d_2, d_3 are continuous functions with respect to S and $N(x)$, $N'(x)$ are continuous with respect to x. Continuity of A_1, A_2, A_3, A_4, and ψ_ρ follows. □

Remark A.1. *The formulas for $\psi_\rho(\alpha; k, h, T)$ involve simple mathematical operations and are therefore easy to compute numerically. The computation of hedge ratios is straightforward once the exercise boundary components are known.*

Appendix B. Proofs for Section 3

Proof of Theorem 3.1. Recall that $\rho \geq 0$. Property *(i)* follows from the relation,

$$\begin{aligned} C(S,t;\rho) &\equiv \sup_{\tau \in \mathcal{S}([t,T])} E^*\left[e^{-(r+\rho)(\tau-t)}\left(SN_{t,\tau}-K\right)^+\right] \\ &\leq sc(S,t) = \sup_{\tau \in \mathcal{S}([t,T])} E^*\left[e^{-r(\tau-t)-\rho O_{t,\tau}}\left(SN_{t,\tau}-K\right)^+\right] \\ &\leq \sup_{\tau \in \mathcal{S}([t,T])} E^*\left[e^{-r(\tau-t)}\left(SN_{t,\tau}-K\right)^+\right] = C(S,t). \end{aligned}$$

Thus, $S-K \leq C(S,t;\rho) \leq sc(S,t) \leq C(S,t)$ where the left hand inequality follows from the feasibility of immediate exercise in market $\mathcal{M}(r+\rho, \delta+\rho, \sigma)$. If $(S,t) \in \mathcal{E}^c$ then $C(S,t) = S-K$ and $sc(S,t) \leq S-K$. Feasibility of immediate exercise at (S,t) for the step call implies $sc(S,t) \geq S-K$. Hence $sc(S,t) = S-K$, i.e., $(S,t) \in \mathcal{E}^{sc}_o$. If $(S,t) \in \mathcal{E}^{sc}_o$, then $sc(S,t) = S-K$ and $C(S,t;\rho) \leq S-K$. Feasibility of immediate exercise at (S,t) for the discounted call implies $C(S,t;\rho) \geq S-K$. Thus, $C(S,t;\rho) = S-K$ and $(S,t) \in \mathcal{E}^c_\rho$. Combining these results shows $\mathcal{E}^c \subseteq \mathcal{E}^{sc}_o \subseteq \mathcal{E}^c_\rho$.

To prove property *(ii)* note that,

$$E^*\left[e^{-r(\tau-v)-\rho O_{v,\tau}}\left(SN_{v,\tau}-K\right)^+\right] = E^*\left[e^{-r(\tau-t)-\rho O_{t,\tau}}\left(SN_{t,\tau}-K\right)^+\right]$$

for $\tau \in \mathcal{S}([v,T]) = \mathcal{S}([t, T-(v-t)])$ and $T \geq v \geq t$. It follows immediately that,

$$\begin{aligned} sc(S,v) &= \sup_{\tau \in \mathcal{S}([v,T])} E^*\left[e^{-r(\tau-v)-\rho O_{v,\tau}}\left(SN_{v,\tau}-K\right)^+\right] \\ &= \sup_{\tau \in \mathcal{S}([t,T-(v-t)])} E^*\left[e^{-r(\tau-t)-\rho O_{t,\tau}}\left(SN_{t,\tau}-K\right)^+\right] \\ &\leq \sup_{\tau \in \mathcal{S}([t,T])} E^*\left[e^{-r(\tau-t)-\rho O_{t,\tau}}\left(SN_{t,\tau}-K\right)^+\right] = sc(S,t). \end{aligned}$$

If $(S,t) \in \mathcal{E}^{sc}_o$, then $sc(S,t) = (S-K)^+$. Thus, $sc(S,v) \leq sc(S,t) = (S-K)^+$. As immediate exercise is feasible at time v, i.e., $\tau = v \in \mathcal{S}([v,T])$, the reverse inequality $sc(S,v) \geq (S-K)^+$ also holds. Property *(ii)* follows.

To establish *(iii.a)*, let $\lambda \geq 1$ and define $O^{\lambda}_{t,\tau} \equiv \int_t^{\tau} 1_{\{\lambda SN_{t,v} \geq H\}} dv$. Assume optimality of exercise at (S,t) and not at $(\lambda S, t)$, that is, $(S,t) \in \mathcal{E}^{uosc}_o$ and $(\lambda S, t) \notin \mathcal{E}^{uosc}_o$ for $\lambda \geq 1$. Denote by $\tau > t$ the (future) optimal stopping time (i.e., viewed from t) of the American call option considered at $(\lambda S, t)$. For an up-an-out step call, the following sequence of relations holds,

$$
\begin{aligned}
& sc(\lambda S, t) \\
&= E^* \left[e^{-r(\tau - t) - \rho O^{\lambda}_{t,\tau}} (\lambda S N_{t,\tau} - K)^+ \right] \\
&= E^* \left[e^{-r(\tau - t) - \rho O^{\lambda}_{t,\tau}} (S N_{t,\tau} - K + (\lambda - 1) S N_{t,\tau})^+ \right] \\
&\leq E^* \left[e^{-r(\tau - t) - \rho O^{\lambda}_{t,\tau}} (S N_{t,\tau} - K)^+ \right] + (\lambda - 1) S E^* \left[e^{-r(\tau - t) - \rho O^{\lambda}_{t,\tau}} N_{t,\tau} \right] \\
&\leq E^* \left[e^{-r(\tau - t) - \rho O_{t,\tau}} (S N_{t,\tau} - K)^+ \right] + (\lambda - 1) S E^* \left[e^{-r(\tau - t)} N_{t,\tau} \right] \\
&\leq sc(S,t) + (\lambda - 1) S E^* \left[e^{-\delta(\tau - t)} \eta^{\sigma}_{t,\tau} \right] \\
&\leq sc(S,t) + (\lambda - 1) S E^* \left[\eta^{\sigma}_{t,\tau} \right] = sc(S,t) + (\lambda - 1) S
\end{aligned}
$$

where $\eta^{\sigma}_{t,\tau} = e^{-(r-\delta)(\tau - t)} N_{t,\tau}$ has expectation equal to one. The first inequality follows from the subadditivity of the function $f(x) = x^+$. The second inequality reflects the fact that, for an up-and-out step call, $O^{\lambda}_{t,\tau} \geq O_{t,\tau}$ for $\lambda \geq 1$ and $O^{\lambda}_{t,\tau} \geq 0$. The third inequality uses the definition of the price $sc(S,t)$ and the relation $b_{t,\tau} N_{t,\tau} = e^{-\delta(\tau - t)} \eta^{\sigma}_{t,\tau}$. The fourth inequality follows from $e^{-\delta(\tau - t)} \eta^{\sigma}_{t,\tau} \leq \eta^{\sigma}_{t,\tau}$. The final expression is obtained because the density $\eta^{\sigma}_{t,\tau}$ integrates to 1. Optimality of exercise at (S,t) then implies $sc(\lambda S, t) \leq \lambda S - K$. Feasibility of immediate exercise at $(\lambda S, t)$ gives $sc(\lambda S, t) \geq \lambda S - K$. Thus, $sc(\lambda S, t) = \lambda S - K$.

Property *(iii.b)* is straightforward. Indeed, by *(i)* and the properties of $\mathcal{E}^c$, the exercise region $\mathcal{E}^{dosc}_o = \mathcal{E}^c \cup \left(\mathcal{E}^{dosc}_o \backslash \mathcal{E}^c \right)$ is up-connected over the subregion $\mathcal{E}^c$.

To show *(iii.c)*, note first that immediate exercise can only be optimal in regions where $\delta S_v - rK + \rho (S_v - K) 1_{\{S_v \leq H\}}$ is nonnegative. This implies $\mathcal{E}^{dosc}_o = \mathcal{E}^{dosc,1}_o \cup \mathcal{E}^{dosc,2}_o$ where $\mathcal{E}^{dosc,1}_o \subseteq \{S : S > \kappa K \vee K \vee \kappa_{\rho} K\}$ and $\mathcal{E}^{dosc,2}_o \subseteq \{S : \kappa K \geq H > S > \kappa_{\rho} K\}$. Suppose now that $\mathcal{E}^{dosc,j}_o$ surrounds a set $\mathcal{H}$, where immediate exercise is suboptimal, i.e., $\mathcal{H}$ is a hole in $\mathcal{E}^{dosc,j}_o$. By definition of $\mathcal{H}$, at $(S,t) \in \mathcal{H}$, there exists a waiting policy τ_w, which exercises at the first hitting time of $\mathcal{E}^{dosc,j}_o$ and dominates immediate exercise. By the delayed exercise premium representation (DEP), the value of this policy is,

$$
V(S, t; \tau_w) = S - K + E^*_t \left[\int_t^{\tau_w} b_{t,v} e^{-\rho O_{t,v}} \left(rK - \delta S_v + \rho (K - S_v) 1_{\{S_v \leq H\}} \right) dv \right].
$$

But if $S > \kappa K \vee K \vee \kappa_{\rho} K$, then $rK - \delta S_v + \rho (K - S_v) 1_{\{S_v \leq H\}} < 0$ and $V(S, t; \tau_w) < S - K$, a contradiction. The same argument applies if $\kappa K \geq H > S > \kappa_{\rho} K$. Thus, $sc(\lambda S, t) = S - K$ and $\mathcal{H} = \varnothing$.

Property *(iv)* is a consequence of *(iii)*. Up-connectedness implies t-section convexity. For a down-an-out step option, up-connectedness of the subregion $\mathcal{E}^c$ implies t-section convexity over $\mathcal{E}^c$.

Connectedness of the subregion $\mathcal{E}_o^{dosc,j}$ implies t-section convexity.

For property *(v)*, consider the case of an up-and-out call first. Assume that $\kappa_\rho K > S > H \vee K$, where $\kappa_\rho \equiv \frac{r+\rho}{\delta+\rho}$. Define the stopping times,

$$\tau_{\lambda^+} = \inf\{v \in [t,T] : S_v = \kappa_\rho K\}, \quad \tau_{H\vee K} = \inf\{v \in [t,T] : S_v = H \vee K\}$$

or $\tau_{\lambda^+} = \infty$, $\tau_{H\vee K} = \infty$, if no such times exist in $[t,T]$. Let $\tau_o = \tau_{\lambda^+} \wedge \tau_{H\vee K} \wedge T$. Let $1_v^+ = 1_{\{S_v \geq H\}}$ and consider the dynamic consumption-portfolio policy with initial investment $X_t = S - K$, liquidated at time τ_o and defined by,

$$(c_v, \pi_v) = e^{-\rho O_{t,v}} \left(S_v \left(\delta + \rho 1_v^+ \right) - K \left(r + \rho 1_v^+ \right), S_v \right)$$

for $v \in [t, \tau_o)$. Here, π_v is the dollar amount invested in the dividend-paying asset at time v, $X_v - \pi_v$ is the amount invested in the interest-paying riskless asset and c_v represents the dollar value of consumption. Note that $c_v \leq 0$ in $v \in [t, \tau_o]$, i.e., the policy requires an injection of funds. The value of this policy evolves according to,

$$\begin{aligned} dX_v &= \pi_v \left[\frac{dS_v}{S_v} + \delta dv \right] + (X_v - \pi_v) r dv - c_v dv \\ &= X_v r dv + \pi_v \left(\frac{dS_v}{S_v} - (r - \delta) dv \right) \\ &- e^{-\rho O_{t,v}} \left(S_v \left(\delta + \rho 1_v^+ \right) - K \left(r + \rho 1_v^+ \right) \right) dv \\ &= \left(X_v r - e^{-\rho O_{t,v}} \left(S_v \left(\delta + \rho 1_v^+ \right) - K \left(r + \rho 1_v^+ \right) \right) \right) dv + e^{-\rho O_{t,v}} S_v \sigma dW_v^* \end{aligned}$$

subject to the initial condition $X_t = S - K$. Let $Y_v \equiv e^{-\rho O_{t,v}^\lambda} (S_v - K)$. By Ito's calculus,

$$dY_v = -Y_v \rho 1_v^+ dv + e^{-\rho O_{t,v}^\lambda} S_v \left((r - \delta) dv + \sigma dW_v^* \right); \qquad Y_t = S - K.$$

Inspection of these two dynamics shows that $X_v = Y_v$ for $v \in [t, \tau_o]$, i.e., the value of the dynamic funding policy (c, π, X_t) is $X_v = Y_v$ for $v \in [t, \tau_o]$. Indeed, one has,

$$\begin{aligned} dY_v &= -e^{-\rho O_{t,v}} (S_v - K) \rho 1_v^+ dv + e^{-\rho O_{t,v}} S_v (r - \delta) dv + e^{-\rho O_{t,v}} S_v \sigma dW_v^* \\ &\underbrace{+ e^{-\rho O_{t,v}} K r dv - e^{-\rho O_{t,v}} K r dv}_{=0} \\ &= \left(e^{-\rho O_{t,v}} (S_v - K) r - e^{-\rho O_{t,v}} \left(S_v \left(\delta + \rho 1_v^+ \right) - K \left(r + \rho 1_v^+ \right) \right) \right) dv \\ &+ e^{-\rho O_{t,v}} S_v \sigma dW_v^* \\ &= \left(Y_v r - e^{-\rho O_{t,v}} \left(S_v \left(\delta + \rho 1_v^+ \right) - K \left(r + \rho 1_v^+ \right) \right) \right) dv + e^{-\rho O_{t,v}} S_v \sigma dW_v^*. \end{aligned}$$

Suppose now that immediate exercise of the step option is optimal at (S, t) where $\kappa_\rho K > S > H \vee K$. Buying the step option and selling (shorting) the policy (c, π, X_t) generates an immediate cash flow equal to $X_t - Y_t = 0$. At liquidation τ_o, the cash flow is $X_{\tau_o} - Y_{\tau_o} = 0$. At intermediate dates

$v \in [t, \tau_o]$, the net cash flow generated is $-c_v = e^{-\rho O^{\lambda}_{t,v}} \left(K \left(r + \rho 1^+_v \right) - S_v \left(\delta + \rho 1^+_v \right) \right) \geq 0$ (with strict inequality for $v < \tau_o$), because $\kappa_\rho K > S_v > H \vee K$. The strategy constructed is therefore an arbitrage. The absence of arbitrage opportunities ensures that immediate exercise of the step option is suboptimal at the point considered. The proofs for the region $H \wedge \kappa_\rho K > S > K$ and for the down-and-out call in *(v.b)* are similar. □

Proof of Theorem 3.2. The continuity of the price function relies on an adaptation of the arguments in Jaillet, Lamberton, & Lapeyre (1990). The result follows from the continuity of the occupation time with respect to the initial conditions (S, t), the continuity of the function $(e^y - K)^+$ with respect to y and the continuity of the underlying price process with respect to (S, t).

The proof of *(ii)* follows from the proof of *(ii)* in Theorem 3.1.

To show *(iii)* note that the random variable $(SN_{t,\tau} - K)^+$ is nondecreasing in S whereas $O_{t,\tau} = \int_t^\tau 1_{\{SN_{t,v} \leq H\}} dv$ is nonincreasing in S. The property stated follows. Moreover, *(iii)* implies quasi-convexity as stated in *(iv)*.

To prove *(v)*, consider an arbitrary stopping time $\tau \in \mathcal{S}_{([t,T])}$ and define,

$$G(\tau) \equiv e^{-r(\tau - t)} \left(e^{-\rho O^{\lambda}_{t,\tau}} \left(\lambda SN_{t,\tau} - K \right)^+ - e^{-\rho O_{t,\tau}} \left(SN_{t,\tau} - K \right)^+ \right).$$

Clearly,

$$\begin{aligned} G(\tau) &= e^{-r(\tau-t) - \rho O^{\lambda}_{t,\tau}} \left(\left(\lambda SN_{t,\tau} - K \right)^+ - \left(SN_{t,\tau} - K \right)^+ \right) \\ &\quad + e^{-r(\tau-t) - \rho O^{\lambda}_{t,\tau}} \Psi \left(O_{t,\tau} - O^{\lambda}_{t,\tau} \right) \left(SN_{t,\tau} - K \right)^+ \end{aligned}$$

$$|G(\tau)| \leq \left| \left(\lambda SN_{t,\tau} - K \right)^+ - \left(SN_{t,\tau} - K \right)^+ \right| + \left| \Psi \left(O_{t,\tau} - O^{\lambda}_{t,\tau} \right) \right| SN_{t,\tau}$$

where $\Psi(h) = 1 - e^{-\rho h}$. Thus, if $\tau \in \mathcal{S}_{t,T}$ is optimal at $(\lambda S, t)$, then $0 \leq sc^{do}(\lambda S, t) - sc^{do}(S, t) \leq E^* \left[G(\tau) \right]$ and,

$$\begin{aligned} \left| sc^{do}(\lambda S, t) - sc^{do}(S, t) \right| &\leq E^* \left[|H(\tau)| \right] \\ &\leq E^* \left[\left| \left(\lambda SN_{t,\tau} - K \right)^+ - \left(SN_{t,\tau} - K \right)^+ \right| \right] \\ &\quad + E^* \left[\left| \Psi \left(O_{t,\tau} - O^{\lambda}_{t,\tau} \right) \right| \left(SN_{t,\tau} - K \right)^+ \right]. \end{aligned}$$

For a down-and-out call, $O_{t,\tau} - O^{\lambda}_{t,\tau} \geq 0$, so that,

$$\begin{aligned}
&\left| sc^{do}(\lambda S, t) - sc^{do}(S, t) \right| \\
&\leq E^*\left[\left| (\lambda S N_{t,\tau} - K)^+ - (S N_{t,\tau} - K)^+ \right| \right] + \rho E^*\left[\left| O_{t,\tau} - O^{\lambda}_{t,\tau} \right| (S N_{t,\tau} - K)^+ \right] \\
&\leq (\lambda - 1) S E^*\left[N_{t,\tau} \right] + \rho S E^*\left[\left(O_{t,\tau} - O^{\lambda}_{t,\tau} \right) N_{t,\tau} \right] \\
&\leq \left((\lambda - 1) + \rho E^*\left[\left(O_{t,\tau} - O^{\lambda}_{t,\tau} \right) \eta_{t,\tau} \right] \right) Z S
\end{aligned}$$

where the first line uses $\Psi(h) \equiv 1 - e^{-\rho h} \leq \rho h$ for $h \geq 0$, the second line $|a^+ - b^+| \leq |a - b|$ and the third line $N_{t,\tau} \leq Z \eta_{t,\tau}$ for some $0 < Z < \infty$.

As $O_{t,\tau} - O^{\lambda}_{t,\tau} = \int_t^{\tau} 1_{\{H/S \geq N_{t,s} \geq H/S\lambda\}} ds$, Ito's lemma gives,

$$\left(O_{t,\tau} - O^{\lambda}_{t,\tau} \right) \eta_{t,\tau} = \int_t^{\tau} 1_{\{H/S \geq N_{t,s} \geq H/S\lambda\}} \eta_{t,s} ds + \int_t^{\tau} \left(O_{t,s} - O^{\lambda}_{t,s} \right) d\eta_{t,s}.$$

The expectation of the stochastic integral $\int_t^{\tau} \left(O_{t,s} - O^{\lambda}_{t,s} \right) d\eta_{t,s}$ is equal to zero because $O_{t,s} - O^{\lambda}_{t,s}$ is bounded and adapted to the filtration generated by W. Thus,

$$\begin{aligned}
&E^*\left[\left(O_{t,\tau} - O^{\lambda}_{t,\tau} \right) \eta_{t,\tau} \right] \\
&= E^*\left[\int_t^{\tau} 1_{\{H/S \geq N_{t,s} \geq H/S\lambda\}} \eta_{t,s} ds \right] \leq \int_t^{T} E^*\left[1_{\{H/S \geq N_{t,s} \geq H/S\lambda\}} \eta_{t,s} \right] ds \\
&= \int_t^{T} E^*\left[1_{\{H/S \geq N_{t,s} \geq H/S\lambda\}} \frac{N_{t,s}}{e^{(r-\delta)(s-t)}} \right] ds \\
&\leq L \int_t^{T} E^*\left[1_{\{H/S \geq N_{t,s} \geq H/S\lambda\}} N_{t,s} \right] ds \leq L \frac{H}{S} \int_t^{T} E^*\left[1_{\{H/S \geq N_{t,s} \geq H/S\lambda\}} \right] ds
\end{aligned}$$

where L is a upper bound for $e^{-(r-\delta)(s-t)}$ for $s \in [t, T]$, and the last line uses the upper bound $H/S \geq N_{t,s}$ on the event $\{H/S \geq N_{t,s} \geq H/S\lambda\}$. But,

$$E^*\left[1_{\{H/S \geq N_{t,s} \geq H/S\lambda\}} \right] = N\left(d\left(\frac{H}{S}, s - t \right) \right) - N\left(d\left(\frac{H}{S\lambda}, s - t \right) \right)$$

$$d(x, h) = \frac{\log(x) - \left(r - \delta - \frac{1}{2}\sigma^2 \right) h}{\sigma \sqrt{h}}$$

where $N(\cdot)$ is the cumulative standard normal distribution function. As,

$$N(x) - N(x - \Delta) = \int_{x-\Delta}^{x} n(z) dz \leq \frac{1}{\sqrt{2\pi}} \Delta$$

it follows that,

$$E^*\left[1_{\{H/S \geq N_{t,s} \geq H/S\lambda\}} \right] \leq \frac{d(H/S, s - t) - d(H/S\lambda, s - t)}{\sqrt{2\pi}} \leq \frac{\log(\lambda)}{\sigma \sqrt{2\pi (s - t)}}$$

Consequently

$$E^*\left[\left(O_{t,\tau}-O_{t,\tau}^{\lambda}\right)\eta_{t,\tau}\right]\leq L\frac{H}{S}\frac{\log\left(\lambda\right)}{\sigma\sqrt{2\pi}}\int_t^T\frac{1}{\sqrt{s-t}}ds=L\frac{H}{S}\frac{\sqrt{2\left(T-t\right)}}{\sigma\sqrt{\pi}}\log\left(\lambda\right).$$

Finally,

$$\left|sc^{do}\left(\lambda S,t\right)-sc^{do}\left(S,t\right)\right|\leq\left(\left(\lambda-1\right)+\rho L\frac{H}{S}\frac{\sqrt{2\left(T-t\right)}}{\sigma\sqrt{\pi}}\log\left(\lambda\right)\right)ZS.$$

By concavity, $\log\left(\lambda\right)\leq\lambda-1$, implying,

$$\left|sc^{do}\left(\lambda S,t\right)-sc^{do}\left(S,t\right)\right|\leq\left(\lambda-1\right)SM,\ \text{ with }\ M=\left(1+\rho L\frac{H}{S}\frac{\sqrt{2\left(T-t\right)}}{\sigma\sqrt{\pi}}\right)Z.$$

For an up-and-out call, $O_{t,\tau}-O_{t,\tau}^{\lambda}\leq 0$, so that the function $G\left(\tau\right)$ defined above satisfies,

$$\begin{aligned}&e^{-r(\tau-t)}\left(e^{-\rho O_{t,\tau}^{\lambda}}-e^{-\rho O_{t,\tau}}\right)\left(SN_{t,\tau}-K\right)^+\\&\leq G\left(\tau\right)\leq e^{-r(\tau-t)}e^{-\rho O_{t,\tau}}\left(\left(\lambda SN_{t,\tau}-K\right)^+-\left(SN_{t,\tau}-K\right)^+\right).\end{aligned}$$

Bounding terms on the left and right hand sides leads to,

$$-\left(1-e^{-\rho\left(O_{t,\tau}^{\lambda}-O_{t,\tau}\right)}\right)SN_{t,\tau}\leq G\left(\tau\right)\leq\left(\lambda-1\right)SN_{t,\tau}$$

$$\left|G\left(\tau\right)\right|\leq\left(\left(\lambda-1\right)\vee\left(1-e^{-\rho\left(O_{t,\tau}^{\lambda}-O_{t,\tau}\right)}\right)\right)SN_{t,\tau}.$$

Proceeding as in the case of a down-and-out call gives,

$$\left|sc^{uo}\left(\lambda S,t\right)-sc^{uo}\left(S,t\right)\right|\leq E^*\left[\left|G\left(\tau\right)\right|\right]\leq\left(\lambda-1\right)\left(1\vee\rho\right)SE^*\left[N_{t,\tau}\right]\leq\left(\lambda-1\right)SM.$$

for some positive constant M. This completes the proof of Theorem 3.2. □

Proof of Theorem 3.3. *(i)* If $\rho\to+\infty$, $\rho O\to+\infty$ for all stopping times $\tau^{\varepsilon}=\inf\left\{v\in\left[t,T\right]:O_{t,v}\geq\varepsilon\right\}$, where $\varepsilon>0$. As $\varepsilon\downarrow 0$, $\tau^{\varepsilon}\downarrow\tau^{H}=\inf\left\{v\in\left[t,T\right]:SN_{t,v}=H\right\}$.
(ii) If $\rho\to 0$, $e^{-\rho O_{t,\tau}}\left(SN_{t,\tau}-K\right)^+\to\left(SN_{t,\tau}-K\right)^+$ for any $\tau\in\mathcal{S}\left(\left[t,T\right]\right)$. □

Proof of Theorem 3.4. Property *(i)* of Theorem 3.4 is reversed as compared to Property *(i)* in Theorem 3.1, because $C\left(S,t\right)\leq sc\left(S,t\right)\leq C\left(S,t;\rho\right)$. The proof of *(ii)* is standard.

For property *(iii.a)*, in the case of a down-and-in call,

$$
\begin{aligned}
& sc^{di}(\lambda S,t) = E^*\left[e^{-r(\tau-t)-\rho O^{\lambda}_{t,\tau}}(\lambda S N_{t,\tau}-K)^+\right] \\
& = E^*\left[e^{-r(\tau-t)-\rho O^{\lambda}_{t,\tau}}(S N_{t,\tau}-K+(\lambda-1)S N_{t,\tau})^+\right] \\
& \leq E^*\left[e^{-r(\tau-t)-\rho O^{\lambda}_{t,\tau}}(S N_{t,\tau}-K)^+\right] + (\lambda-1) S E^*\left[e^{-r(\tau-t)-\rho O^{\lambda}_{t,\tau}} N_{t,\tau}\right] \\
& \leq E^*\left[e^{-r(\tau-t)-\rho O_{t,\tau}}(S N_{t,\tau}-K)^+\right] + (\lambda-1) S E^*\left[e^{-\delta(\tau-t)-\rho O^{\lambda}_{t,\tau}} \eta^{\sigma}_{t,\tau}\right] \\
& \leq sc^{di}(S,t) + (\lambda-1) S E^*\left[e^{-(\delta+\rho)(\tau-t)} \eta^{\sigma}_{t,\tau}\right] \\
& \leq sc^{di}(S,t) + (\lambda-1) S E^*\left[\eta^{\sigma}_{t,\tau}\right] = sc^{di}(S,t) + (\lambda-1) S
\end{aligned}
$$

where the last line follows from $\delta+\rho \geq 0$. Up-connectedness implies convexity *(iv.a)*.

For *(iii.b)* and *(iv.b)*, note that $\mathcal{E}^{sc}_0(t)$ is up-connected over the region $\mathcal{E}^c_\rho(t)$, because $\mathcal{E}^c_\rho \subseteq \mathcal{E}^{sc}_0$ and $\mathcal{E}^c_\rho(t)$ is up-connected and convex. These properties hold for up-and-in and down-and-in calls.

For *(iii.c)*, consider $\kappa_\rho K > H > \kappa K \vee K$ and note that $\mathcal{E}^{uisc}_0 = \mathcal{E}^{uisc,1}_0 \cup \mathcal{E}^{uisc,2}_0$ for some $\mathcal{E}^{uisc,j}_0$, $j=1,2$ connected such that $\mathcal{E}^{uisc,1}_0 \cap \mathcal{E}^{uisc,2}_0 = \varnothing$. Consider the sets,

$$
\mathcal{R}_m \equiv \{S : \kappa_\rho K > S \geq H > \kappa K \vee K\}, \quad \mathcal{R}_u \equiv \{S : S \geq \kappa_\rho K > H > \kappa K \vee K\}
$$

$$
\mathcal{R}_d \equiv \{S : \kappa_\rho K \geq H > S \geq \kappa K \vee K\}.
$$

In $\mathcal{R}_m$, the local gain $\delta S - rK + \rho(S-K)1^+$ is negative, implying the suboptimality of immediate exercise. In $\mathcal{R}_u$, it is nonnegative implying the possible optimality of immediate exercise. In $\mathcal{R}_d$, the local gain $\delta S - rK$ is again nonnegative allowing for the possible optimality of early exercise. Thus, $\mathcal{E}^{uisc,1}_o \subseteq \mathcal{R}_u \times [0,T]$ and $\mathcal{E}^{uisc,2}_o \subseteq \mathcal{R}_d \times [0,T]$ with $\mathcal{E}^{uisc,1}_o \cap \mathcal{E}^{uisc,2}_o = \varnothing$ and $\mathcal{E}^{uisc}_o = \mathcal{E}^{uisc,1}_o \cup \mathcal{E}^{uisc,2}_o$. If $H \geq \kappa_\rho K > \kappa K \vee K$ or $\kappa_\rho K > \kappa K \vee K \geq H$, the immediate exercise region consists of a single connected set.

Property *(iv.c)* is proved as in Theorem 3.1; *(v)* also hold by similar no-arbitrage arguments. □

Proof of Theorem 3.5. Assume $\rho < 0$. The local gains from exercising are,

$$
(\delta+\rho 1^+) S - (r+\rho 1^+) K = \begin{cases} (\delta+\rho 1^+) S - (r+\rho 1^+) K < 0 & \text{for } S \geq H \\ \delta S - rK & \text{for } S < H \end{cases}
$$

$$
(\delta+\rho 1^-) S - (r+\rho 1^-) K = \begin{cases} \delta S - rK & \text{for } S \geq H \\ (\delta+\rho 1^-) S - (r+\rho 1^-) K < 0 & \text{for } S < H \end{cases}.
$$

It follows immediately that early exercise above (resp. below) H is suboptimal for an up-and-in (resp. down-and-in) step call. Thus, $\mathcal{E}^{uisc}_o = \mathcal{E}^c_\rho = \{(S,T) : S \geq K\}$ and $\mathcal{E}^{disc}_o \subseteq \mathcal{E}^c$ for $H \leq \kappa K$. Also, $\mathcal{E}^{uisc}_o \subseteq \mathcal{E}^c \cap \{S \leq H\}$ and $\mathcal{E}^{disc}_o \subseteq \mathcal{E}^c \cap \{S \geq H\}$ for $H > \kappa K$. This proves *(i)*, and also *(v)*.

Property *(ii)* is standard. To show *(iii.a)*, first recall that $\mathcal{E}_o^{disc} \subseteq \mathcal{E}^c \cap \{S \geq H\}$. Second, by *(ii)*, $(S,t) \in \mathcal{E}_o^{disc} \Longrightarrow (S,v) \in \mathcal{E}_o^{disc}$ for $t \leq v \leq T$. Hence, $S - K = sc^{di}(S,t) = sc^{di}(S,v)$ for $t \leq v \leq T$. Consider the point $(\lambda S, t)$ with $\lambda > 1$ and let $\tau_\lambda = \inf\{v \in [t,T] : \lambda S_v = S\}$ or $\tau_\lambda = T$ if no such time exists in $[t,T]$. Note that immediate exercise is optimal at τ_λ because $O_{t,\tau_\lambda}^\lambda = O_{t,t} = 0$ $((S,t) \in \mathcal{E}_o^{disc} \Longrightarrow \{S \geq H\} \cap \{S \geq \kappa K \geq K\})$. Consider any waiting policy $\tau_w \leq \tau_\lambda$. By the DEP representation, the value of this policy is

$$V(\lambda S, t; \tau_w) = \lambda S - K + E_t^* \left[\int_t^{\tau_w} b_{t,v} e^{-\rho O_{t,v}^\lambda} (rK - \delta \lambda S_v) \, dv \right]$$

where $O_{t,v}^\lambda = 0$ and $rK - \delta\lambda S_v < 0$ for all $v \leq \tau_w$. The latter follows because $\mathcal{E}_0^{disc} \subseteq \mathcal{E}^c$, hence $S \geq \kappa K$ and $\lambda S_v > \kappa K$ for all $v \leq \tau_w$. Thus, $V(\lambda S, t; \tau_w) \leq \lambda S - K$. As immediate exercise is a feasible policy at t with payoff $\lambda S - K$ we get $sc^{di}(\lambda S, t) = \lambda S - K$ and therefore $(\lambda S, t) \in \mathcal{E}_o^{disc}$.

For *(iii.b)*, recall that immediate exercise above H is suboptimal prior to T, for the up-and-in call, if $H \leq \kappa K$. Thus, if $H \leq \kappa K$, then $\mathcal{E}_0^{uisc} = \mathcal{E}_\rho^c = \{(S,T) : S \geq K\}$. Up-connectedness follows. For *(iii.c)*, if $H > \kappa K$, then $\mathcal{E}_0^{uisc} \subseteq \mathcal{E}^c \cap \{S \leq H\}$, which cannot be up-connected.

Property *(iv.a)* follows from *(iii.a)*. For *(iv.b)*, note that the subcase $H \leq \kappa K$ is trivial. In the complementary subcase $H > \kappa K$, $\mathcal{E}_0^{uisc} \subseteq \mathcal{E}^c \cap \{S \leq H\}$. Consider two points $\left(S^1, t\right), \left(S^2, t\right) \in \mathcal{E}_0^{uisc}$, the convex combination $\left(S^\lambda, t\right)$ with $\lambda \in (0,1)$ and the stopping time

$$\tau_\lambda = \inf\left\{v \in [t,T] : \lambda S_v = S^1 \text{ or } \lambda S_v = S^2\right\}$$

or $\tau_\lambda = T$ if no such time exists in $[t,T]$. The DEP representation gives

$$V\left(S^\lambda, t; \tau_w\right) = S^\lambda - K + E_t^* \left[\int_t^{\tau_w} b_{t,v} e^{-\rho O_{t,v}^\lambda} \left(rK - \delta S_v^\lambda\right) dv \right]$$

for any waiting policy $\tau_w \leq \tau_\lambda$. At τ_λ, immediate exercise is optimal. For $v \leq \tau_\lambda$, $rK - \delta S_v^\lambda \leq 0$. The usual argument can then be invoked to conclude that $\left(S^\lambda, t\right) \in \mathcal{E}_0^{uisc}$. □

Appendix C. The early exercise boundaries

The results in this section enable us to define the early exercise boundaries and derive the EEP representation of the price of an American-style step call option. We use the generic superscript $(^{sc})$ for reference to the step call option. Thus $B^{sc}(t)$ can be any optimal exercise boundary for an American step call with single or multiple exercise boundaries.

Theorem C.1. *The price function* $sc(\cdot,\cdot) : \mathbb{R}_+ \times [0,T] \to \mathbb{R}_+$ *of an American-style step call option is continuous at* $B^{sc}(t)$ *for all* t *in* $[0,T)$*. Moreover,* $\partial sc(S,t)/\partial S = 1$ *for* $S = B^{sc}(t)$*.*

Proof of Theorem C.1. a) Upper and lower bounds at the exercise boundaries:

There are three cases. (i) If there is a unique exercise boundary B^{sc}, consider a point (t, S^*)

such that $S^* = B^{sc}(t)$. We know $S^* > K$ (because $B^{sc}(t) > K$ for $t \in [0, T)$) and for all $\varepsilon > 0$

$$sc(S^*, t) - sc(S^* - \varepsilon, t) \leq S^* - K - (S^* - \varepsilon - K)^+ \leq \varepsilon.$$

Taking the limit as $\varepsilon \to 0$, gives

$$\frac{\partial sc\left(S^*_-, t\right)}{\partial S} \equiv \lim_{\varepsilon \downarrow 0} \frac{sc(S^*, t) - sc(S^* - \varepsilon, t)}{\varepsilon} \leq 1.$$

(ii) If there are three boundaries $(B^{sc}, B^{sc}_{lu}, B^{sc}_{ld})$, as in the cases of down-and-out and up-and-in step calls, the upper bound for the slope derived above also applies to points on the upper and lower boundaries (B^{sc}, B^{sc}_{ld}). For the middle boundary B^{sc}_{lu}, which separates an exercise region below it from a continuation region above it, we have the lower bound

$$sc(S^* + \varepsilon, t) - sc(S^*, t) \geq S^* + \varepsilon - K - (S^* - K) = \varepsilon$$

for $S^* = B^{sc}_{lu}$ and any $\varepsilon > 0$, implying

$$\frac{\partial sc\left(S^*_+, t\right)}{\partial S} \equiv \lim_{\varepsilon \downarrow 0} \frac{sc(S^* + \varepsilon, t) - sc(S^*, t)}{\varepsilon} \geq 1.$$

(iii) If there are two boundaries $(B^{sc}_{lu}, B^{sc}_{ld})$, as in the case of an up-an-in step call for specific parameter configurations, then the same arguments show

$$\frac{\partial sc\left(S^*_+, t\right)}{\partial S}_{|S^* = B^{sc}_{lu}} \geq 1 \quad \text{and} \quad \frac{\partial sc\left(S^*_-, t\right)}{\partial S}_{|S^* = B^{sc}_{ld}} \leq 1.$$

b) Reverse inequalities:

To prove the reverse inequalities, fix $\varepsilon > 0$ and consider an optimal stopping time $\tau_\epsilon \in \mathcal{S}([0, T - t])$ for $sc(S^* - \varepsilon, t)$ or for $sc(S^* + \varepsilon, t)$, depending on the case considered. Let $O^{S^* \pm \varepsilon}_{0, \tau_\varepsilon}$ be the occupation time of the relevant set starting from $S^* \pm \varepsilon$. There are two cases.

(i) If $\rho\left(O_{0,\tau_\varepsilon}^{S^*-\varepsilon}-O_{0,\tau_\varepsilon}^{S^*}\right)\geq 0$, corresponding to a down-and-out or an up-and-in step call, then

$$\begin{aligned}
& sc\left(S^*,t\right)-sc\left(S^*-\varepsilon,t\right)\\
& \geq E^*\left[e^{-r\tau_\varepsilon-\rho O_{0,\tau_\varepsilon}^{S^*}}\left(S^*N_{0,\tau_\varepsilon}-K\right)^+\right]-E^*\left[e^{-r\tau_\varepsilon-\rho O_{0,\tau_\varepsilon}^{S^*-\varepsilon}}\left(\left(S^*-\varepsilon\right)N_{0,\tau_\varepsilon}-K\right)^+\right]\\
& =E^*\left[e^{-r\tau_\varepsilon-\rho O_{0,\tau_\varepsilon}^{S^*}}\left(\left(S^*N_{0,\tau_\varepsilon}-K\right)^+-e^{-\rho\left(O_{0,\tau_\varepsilon}^{S^*-\varepsilon}-O_{0,\tau_\varepsilon}^{S^*}\right)}\left(\left(S^*-\varepsilon\right)N_{0,\tau_\varepsilon}-K\right)^+\right)\right]\\
& \geq E^*\left[e^{-r\tau_\varepsilon-\rho O_{0,\tau_\varepsilon}^{S^*}}\left(S^*N_{0,\tau_\varepsilon}-K-e^{-\rho\left(O_{0,\tau_\varepsilon}^{S^*-\varepsilon}-O_{0,\tau_\varepsilon}^{S^*}\right)}\left(\left(S^*-\varepsilon\right)N_{0,\tau_\varepsilon}-K\right)\right)\right]\\
& =E^*\left[e^{-r\tau_\varepsilon-\rho O_{0,\tau_\varepsilon}^{S^*}}\left(S^*N_{0,\tau_\varepsilon}-K\right)\left(1-e^{-\rho\left(O_{0,\tau_\varepsilon}^{S^*-\varepsilon}-O_{0,\tau_\varepsilon}^{S^*}\right)}\right)\right]\\
& +\varepsilon E^*\left[e^{-r\tau_\varepsilon-\rho O_{0,\tau_\varepsilon}^{S^*-\varepsilon}}N_{0,\tau_\varepsilon}\right].
\end{aligned}$$

Under the condition on the occupation time, the first term of the last expression is non-negative. Hence

$$sc\left(S^*,t\right)-sc\left(S^*-\varepsilon,t\right)\geq\varepsilon E^*\left[e^{-r\tau_\varepsilon-\rho O_{0,\tau_\varepsilon}^{S^*-\varepsilon}}N_{0,\tau_\varepsilon}\right]$$

for $\varepsilon>0$. When $\varepsilon\downarrow 0$, the stopping time $\tau_\varepsilon\to 0$, $N_{0,\tau_\varepsilon}\to 1$ and the occupation time $O_{0,\tau_\varepsilon}^{S^*-\varepsilon}\to 0$. By dominated convergence, one finds

$$\frac{\partial sc\left(S_-^*,t\right)}{\partial S}=\lim_{\varepsilon\downarrow 0}\frac{sc\left(S^*,t\right)-sc\left(S^*-\varepsilon,t\right)}{\varepsilon}\geq 1.$$

This result provides lower bounds for the left derivative of the (down-and-out and up-and-in) step call premium at the boundary points B^{sc} and B_{ld}^{sc}. For B_{lu}^{sc} we are looking for an upper bound. Let $S^*=B_{lu}^{sc}\left(t\right)$. Using the same type of arguments as above, along with the condition $\rho\left(O_{0,\tau_\varepsilon}^{S^*-\varepsilon}-O_{0,\tau_\varepsilon}^{S^*}\right)\geq 0$, implies $\rho\left(O_{0,\tau_\varepsilon}^{S^*}-O_{0,\tau_\varepsilon}^{S^*+\varepsilon}\right)\geq 0$ and

$$\begin{aligned}
& sc\left(S^*+\varepsilon,t\right)-sc\left(S^*,t\right)\\
& \leq E^*\left[e^{-r\tau_\varepsilon}\left(e^{-\rho O_{0,\tau_\varepsilon}^{S^*+\varepsilon}}\left(\left(S^*+\varepsilon\right)N_{0,\tau_\varepsilon}-K\right)^+-e^{-\rho O_{0,\tau_\varepsilon}^{S^*}}\left(S^*N_{0,\tau_\varepsilon}-K\right)^+\right)\right]\\
& \leq E^*\left[e^{-r\tau_\varepsilon-\rho O_{0,\tau_\varepsilon}^{S^*+\varepsilon}}\left(\left(\left(S^*+\varepsilon\right)N_{0,\tau_\varepsilon}-K\right)^+-\left(S^*N_{0,\tau_\varepsilon}-K\right)^+\right)\right]\\
& +E^*\left[e^{-r\tau_\varepsilon-\rho O_{0,\tau_\varepsilon}^{S^*+\varepsilon}}\left(1-e^{-\rho\left(O_{0,\tau_\varepsilon}^{S^*}-O_{0,\tau_\varepsilon}^{S^*+\varepsilon}\right)}\right)\left(S^*N_{0,\tau_\varepsilon}-K\right)^+\right]\\
& \leq\varepsilon A_1\left(\varepsilon\right)+A_2\left(\varepsilon\right)
\end{aligned}$$

where

$$A_1\left(\varepsilon\right)\equiv E^*\left[e^{-r\tau_\varepsilon-\rho O_{0,\tau_\varepsilon}^{S^*+\varepsilon}}N_{0,\tau_\varepsilon}\right]$$

$$A_2\left(\varepsilon\right)\equiv e^{|\rho|(T-t)}E^*\left[\left(1-e^{-\rho\left(O_{0,\tau_\varepsilon}^{S^*}-O_{0,\tau_\varepsilon}^{S^*+\varepsilon}\right)}\right)\left(S^*N_{0,\tau_\varepsilon}-K\right)^+\right].$$

The first inequality above is because $sc(S^*,t) \geq E^*\left[e^{-r\tau_\varepsilon-\rho O^{S^*}_{0,\tau_\varepsilon}}(S^* N_{0,\tau_\varepsilon}-K)^+\right]$. The term $\varepsilon A_1(\varepsilon)$ in the last inequality comes from recognizing an asymmetric call spread in the first part of the second inequality. By dominated convergence $\lim_{\varepsilon\downarrow 0} A_1(\varepsilon)\to 1$.[10] The term $A_2(\varepsilon)$ has the upper bound

$$A_2(\varepsilon) \leq S^* e^{|\rho|(T-t)} E^*\left[\Psi\left(O^{S^*}_{0,\tau_\varepsilon}-O^{S^*+\varepsilon}_{0,\tau_\varepsilon}\right)N_{0,\tau_\varepsilon}\right]$$

where $\Psi(h) = 1-e^{-\rho h}$. Straightforward calculations show that

$$\begin{aligned}\Psi\left(O^{S^*}_{0,\tau_\varepsilon}-O^{S^*+\varepsilon}_{0,\tau_\varepsilon}\right) &\leq \rho\left(O^{S^*}_{0,\tau_\varepsilon}-O^{S^*+\varepsilon}_{0,\tau_\varepsilon}\right)\\ &= |\rho|\int_{\tau_H}^{\tau_\varepsilon} 1_{\{H/S^*\geq N_{0,v}>H/(S^*+\varepsilon)\}}1_{\{\tau_H\leq\tau_\varepsilon\}}dv.\end{aligned}$$

where we used the fact that $O^{S^*}_{0,\tau_\varepsilon}-O^{S^*+\varepsilon}_{0,\tau_\varepsilon}=0$ on the event $\{\tau_H>\tau_\varepsilon\}$. Taking expectations and using the Cauchy-Schwartz inequality gives

$$\begin{aligned}A_2(\varepsilon) &\leq e^{|\rho|(T-t)}S^*E^*\left[\Psi\left(O^{S^*}_{0,\tau_\varepsilon}-O^{S^*+\varepsilon}_{0,\tau_\varepsilon}\right)N_{0,\tau_\varepsilon}\right]\\
&\leq e^{|\rho|(T-t)}S^*|\rho|E^*\left[\int_{\tau_H}^{\tau_\varepsilon}1_{\{H/S^*\geq N_{0,v}>H/(S^*+\varepsilon)\}}1_{\{\tau_H\leq\tau_\varepsilon\}}dv N_{0,\tau_\varepsilon}\right]\\
&\leq e^{|\rho|(T-t)}S^*|\rho|\int_0^{T-t}E^*\left[1_{\{H/S^*\geq N_{0,v}>H/(S^*+\varepsilon)\}}1_{\{\tau_H\leq\tau_\varepsilon\}}N_{0,\tau_\varepsilon}\right]dv\\
&\leq |\rho|S^*b_{0,T-t}^{-|r-\delta|-|\rho|}\int_0^{T-t}E^\eta\left[1_{\{H/S^*\geq N_{0,v}>H/(S^*+\varepsilon)\}}1_{\{\tau_H\leq\tau_\varepsilon\}}\right]dv\\
&\leq |\rho|S^*b_{0,T-t}^{-|r-\delta|-|\rho|}\int_0^{T-t}E^\eta\left[1_{\{H/S^*\geq N_{0,v}>H/(S^*+\varepsilon)\}}\right]^{1/2}E^\eta\left[1_{\{\tau_H\leq\tau_\varepsilon\}}\right]^{1/2}dv\\
&= |\rho|S^*b_{0,T-t}^{-|r-\delta|-|\rho|}\left(\int_0^{T-t}Q^\eta\left(\frac{H}{S^*}\geq N_{0,v}>\frac{H}{S^*+\varepsilon}\right)^{1/2}dv\right)Q^\eta(\tau_H\leq\tau_\varepsilon)^{1/2}\end{aligned}$$

where, under the new equivalent martingale measure and using the mean value theorem, the first probability satisfies

$$\begin{aligned}Q^\eta\left(\frac{H}{S^*}\geq N_{0,v}>\frac{H}{S^*+\varepsilon}\right) &= N\left(d^\eta\left(\frac{H}{S^*},v\right)\right)-N\left(d^\eta\left(\frac{H}{S^*+\varepsilon},v\right)\right)\\
&\leq \frac{n(d^\eta(s_\varepsilon,v))}{s_\varepsilon\sigma\sqrt{v}}\left(\frac{H}{S^*}-\frac{H}{S^*+\varepsilon}\right)=\frac{n(d^\eta(s_\varepsilon,v))}{s_\varepsilon\sigma\sqrt{v}}\frac{\varepsilon H}{S^*(S^*+\varepsilon)}\end{aligned}$$

for some intermediate value $s_\varepsilon\in\left[\frac{H}{S^*+\varepsilon},\frac{H}{S^*}\right]$ and, by Lemma C.2 below, the second probability

[10] Note that $A_1(\varepsilon)=E^*\left[e^{-\delta\tau_\varepsilon-\rho O^{S^*+\varepsilon}_{0,\tau_\varepsilon}}\eta^\sigma_{0,\tau_\varepsilon}\right]$. It follows that

$$e^{-|\delta+|\rho||T}\leq E^*\left[e^{-|\delta+|\rho||\tau_\varepsilon}\eta^\sigma_{0,\tau_\varepsilon}\right]\leq A_1(\varepsilon)\leq E^*\left[e^{|\delta+|\rho||\tau_\varepsilon}\eta^\sigma_{0,\tau_\varepsilon}\right]\leq e^{|\delta+|\rho||T}.$$

A dominated convergence argument then shows that $\lim_{\varepsilon\downarrow 0}A_1(\varepsilon)\to 1$.

satisfies

$$Q^{\eta}\left(\tau_H \leq \tau_\varepsilon\right) \leq C\left(\varepsilon\right)\varepsilon$$

for sufficiently small ε where $\lim_{\varepsilon\downarrow 0} C\left(\varepsilon\right) = 0$. Then

$$\frac{A_2\left(\varepsilon\right)}{\varepsilon} \leq \left|\rho\right| S^* b_{0,T-t}^{-\left|r-\delta\right|-\left|\rho\right|}\sqrt{\frac{C\left(\varepsilon\right)H}{S^*\left(S^*+\varepsilon\right)}}\int_0^{T-t}\left(\frac{n\left(d^{\eta}\left(s_\varepsilon, v\right)\right)}{s_\varepsilon \sigma\sqrt{v}}\right)^{1/2} dv$$

and

$$\frac{sc\left(S^*+\varepsilon, t\right) - sc\left(S^*, t\right)}{\varepsilon} \leq A_1\left(\varepsilon\right) + \frac{A_2\left(\varepsilon\right)}{\varepsilon}$$

Taking the limit when $\varepsilon \downarrow 0$, using $\lim_{\varepsilon\downarrow 0} A_1\left(\varepsilon\right) \to 1$ and $\lim_{\varepsilon\downarrow 0} C\left(\varepsilon\right) \to 0$, finally leads to

$$\frac{\partial sc\left(S_+^*, t\right)}{\partial S} \leq 1.$$

(ii) If $\rho\left(O_{0,\tau_\varepsilon}^{S^*-\varepsilon} - O_{0,\tau_\varepsilon}^{S^*}\right) \leq 0$, i.e., the case of an up-and-out or a down-and-in step call, there is a single boundary B^{sc}. Set $S^* = B^{sc}\left(t\right)$. We seek to show that the left derivative is bounded below by 1. Straightforward calculations show that

$$\begin{aligned}
& sc\left(S^*, t\right) - sc\left(S^*-\varepsilon, t\right) \\
& \geq E^*\left[e^{-r\tau_\varepsilon - \rho O_{0,\tau_\varepsilon}^{S^*}}\left(S^* N_{0,\tau_\varepsilon} - K\right)^+\right] - E^*\left[e^{-r\tau_\varepsilon - \rho O_{0,\tau_\varepsilon}^{S^*-\varepsilon}}\left(\left(S^*-\varepsilon\right)N_{0,\tau_\varepsilon} - K\right)^+\right] \\
& = E^*\left[e^{-r\tau_\varepsilon - \rho O_{0,\tau_\varepsilon}^{S^*}}\left(\left(S^* N_{0,\tau_\varepsilon} - K\right)^+ - e^{-\rho\left(O_{0,\tau_\varepsilon}^{S^*-\varepsilon} - O_{0,\tau_\varepsilon}^{S^*}\right)}\left(\left(S^*-\varepsilon\right)N_{0,\tau_\varepsilon} - K\right)^+\right)\right] \\
& = E^*\left[e^{-r\tau_\varepsilon - \rho O_{0,\tau_\varepsilon}^{S^*}}\left(S^* N_{0,\tau_\varepsilon} - K - e^{-\rho\left(O_{0,\tau_\varepsilon}^{S^*-\varepsilon} - O_{0,\tau_\varepsilon}^{S^*}\right)}\left(\left(S^*-\varepsilon\right)N_{0,\tau_\varepsilon} - K\right)\right)\right] \\
& = E^*\left[e^{-r\tau_\varepsilon - \rho O_{0,\tau_\varepsilon}^{S^*}}\left(S^* N_{0,\tau_\varepsilon} - K - \left(\left(S^*-\varepsilon\right)N_{0,\tau_\varepsilon} - K\right)\right)\right] \\
& + E^*\left[e^{-r\tau_\varepsilon - \rho O_{0,\tau_\varepsilon}^{S^*}}\left(\left(S^*-\varepsilon\right)N_{0,\tau_\varepsilon} - K\right)\left(1 - e^{-\rho\left(O_{0,\tau_\varepsilon}^{S^*-\varepsilon} - O_{0,\tau_\varepsilon}^{S^*}\right)}\right)\right] \\
& \geq E^*\left[e^{-r\tau_\varepsilon - \rho O_{0,\tau_\varepsilon}^{S^*}}\varepsilon N_{0,\tau_\varepsilon}\right] + S^* E^*\left[e^{-r\tau_\varepsilon - \rho O_{0,\tau_\varepsilon}^{S^*}} N_{0,\tau_\varepsilon}\left(1 - e^{-\rho\left(O_{0,\tau_\varepsilon}^{S^*-\varepsilon} - O_{0,\tau_\varepsilon}^{S^*}\right)}\right)\right] \\
& \geq \varepsilon E^*\left[e^{-\delta\tau_\varepsilon - \rho O_{0,\tau_\varepsilon}^{S^*}}\eta_{0,\tau_\varepsilon}^{\sigma}\right] + S^* e^{\left|\rho\right|\left(T-t\right)} E^*\left[e^{-\delta\tau_\varepsilon}\eta_{0,\tau_\varepsilon}^{\sigma}\left(1 - e^{-\rho\left(O_{0,\tau_\varepsilon}^{S^*-\varepsilon} - O_{0,\tau_\varepsilon}^{S^*}\right)}\right)\right] \\
& \geq \varepsilon E^{\eta}\left[e^{-\delta\tau_\varepsilon - \rho O_{0,\tau_\varepsilon}^{S^*}}\right] + S^* e^{\left(\left|\rho\right| + \left|\delta\right|\right)\left(T-t\right)} E^{\eta}\left[1 - e^{-\rho\left(O_{0,\tau_\varepsilon}^{S^*-\varepsilon} - O_{0,\tau_\varepsilon}^{S^*}\right)}\right] \\
& \equiv \varepsilon A_1\left(\varepsilon\right) + A_2\left(\varepsilon\right)
\end{aligned}$$

where $A_1\left(\varepsilon\right) = E^{\eta}\left[e^{-\delta\tau_\varepsilon - \rho O_{0,\tau_\varepsilon}^{S^*}}\right]$ and the second inequality relies on the condition on the occupation time.

For an up-and-out step call ($\rho > 0$):
If $S^* = B^{sc}\left(t\right) > H$, define $\tau_H = \inf\left\{v \in \left[t, T\right] : \left(S^*-\varepsilon\right)N_{0,v} = H\right\}$ and note that $O_{0,\tau_\varepsilon}^{S^*-\varepsilon} = O_{0,\tau_\varepsilon}^{S^*}$

for $\tau_\varepsilon \leq \tau_H$. It follows that

$$1 - e^{-\rho\left(O^{S^*-\varepsilon}_{0,\tau_\varepsilon} - O^{S^*}_{0,\tau_\varepsilon}\right)} = \left(1 - e^{-\rho\left(O^{S^*-\varepsilon}_{0,\tau_\varepsilon} - O^{S^*}_{0,\tau_\varepsilon}\right)}\right) 1_{\{\tau_H < \tau_\varepsilon\}}$$
$$= \left(1 - e^{-\rho\left(O^{S^*-\varepsilon}_{\tau_H,\tau_\varepsilon} - O^{S^*}_{\tau_H,\tau_\varepsilon}\right)}\right) 1_{\{\tau_H < \tau_\varepsilon\}}.$$

Now, for $h \geq 0$, the continuous function $g(h) = 1 - e^h$ is negative, decreasing and concave, with $g(0) = 0$. It therefore has the lower bound $g(h) \geq g'\left(\overline{h}\right) h$ for $0 \leq \overline{h} \leq h$. Denoting by $-c_\varepsilon$ the relevant derivative in our context then, on the event $\{\tau_H < \tau_\varepsilon\}$, one has

$$1 - e^{-\rho\left(O^{S^*-\varepsilon}_{\tau_H,\tau_\varepsilon} - O^{S^*}_{\tau_H,\tau_\varepsilon}\right)} \geq -c_\varepsilon \left|-\rho\right| \left|O^{S^*-\varepsilon}_{\tau_H,\tau_\varepsilon} - O^{S^*}_{\tau_H,\tau_\varepsilon}\right|$$
$$= -c_\varepsilon \left|\rho\right| \int_{\tau_H}^{\tau_\varepsilon} 1_{\{H/(S^*-\varepsilon) \geq N_{0,v} > H/S^*\}} dv$$
$$\geq -e^{|\rho|(T-t)} \left|\rho\right| \int_0^{T-t} 1_{\{H/(S^*-\varepsilon) \geq N_{0,v} > H/S^*\}} dv.$$

Defining $c \equiv e^{|\rho|(T-t)} \left|\rho\right|$, taking the expectation under the new equivalent martingale measure and using Cauchy-Schwartz give

$$E^\eta \left[1 - e^{-\rho\left(O^{S^*-\varepsilon}_{0,\tau_\varepsilon} - O^{S^*}_{0,\tau_\varepsilon}\right)}\right]$$
$$\geq -cE^\eta \left[\left(\int_0^{T-t} 1_{\{H/(S^*-\varepsilon) \geq N_{0,v} > H/S^*\}} dv\right) 1_{\{\tau_H < \tau_\varepsilon\}}\right]$$
$$\geq -c \int_0^{T-t} E^\eta \left[1_{\{H/(S^*-\varepsilon) \geq N_{0,v} > H/S^*\}} 1_{\{\tau_H < \tau_\varepsilon\}}\right] dv$$
$$= -c \int_0^{T-t} Q^\eta \left(\frac{H}{S^*-\varepsilon} \geq N_{0,v} > \frac{H}{S^*}\right)^{1/2} \left(E^\eta \left[1_{\{\tau_H < \tau_\varepsilon\}}\right]\right)^{1/2} dv$$
$$= -c \int_0^{T-t} Q^\eta \left(\frac{H}{S^*-\varepsilon} \geq N_{0,v} > \frac{H}{S^*}\right)^{1/2} dv Q^\eta \left(\tau_H < \tau_\varepsilon\right)^{1/2}$$

where, using the mean value theorem, the first probability satisfies

$$Q^\eta \left(\frac{H}{S^*-\varepsilon} \geq N_{0,v} > \frac{H}{S^*}\right) = N\left(d^\eta\left(\frac{H}{S^*-\varepsilon}, v\right)\right) - N\left(d^\eta\left(\frac{H}{S^*}, v\right)\right)$$
$$\leq \frac{n\left(d^\eta\left(s_\varepsilon, v\right)\right)}{s_\varepsilon \sigma \sqrt{v}} \left(\frac{H}{S^*-\varepsilon} - \frac{H}{S^*}\right) = \frac{n\left(d^\eta\left(s_\varepsilon, v\right)\right)}{s_\varepsilon \sigma \sqrt{v}} \frac{\varepsilon H}{S^*\left(S^*-\varepsilon\right)}$$

for $s_\varepsilon \in \left[\frac{H}{S^*}, \frac{H}{S^*-\varepsilon}\right]$ and, by Lemma C.1 below, the second probability satisfies $Q^\eta\left(\tau_H < \tau_\varepsilon\right) \leq$

$C(\varepsilon)\,\varepsilon$. Combining these elements gives

$$\begin{aligned}
A_2(\varepsilon) &= S^* e^{(|\rho|+|\delta|)(T-t)} E^\eta \left[1 - e^{-\rho\left(O^{S^*-\varepsilon}_{0,\tau_\varepsilon} - O^{S^*}_{0,\tau_\varepsilon}\right)}\right] \\
&\geq -S^* e^{(|\rho|+|\delta|)(T-t)} c \int_0^{T-t} Q^\eta \left(\frac{H}{S^*-\varepsilon} \geq N_{0,v} > \frac{H}{S^*}\right)^{1/2} dv Q^\eta (\tau_H < \tau_\varepsilon)^{1/2} \\
&\geq -\varepsilon \sqrt{\frac{C(\varepsilon) H}{S^*(S^*-\varepsilon)}} S^* e^{(|\rho|+|\delta|)(T-t)} |\rho| \int_0^{T-t} \sqrt{\frac{n(d^\eta(s_\varepsilon, v))}{s_\varepsilon \sigma \sqrt{v}}} dv \\
&\equiv \varepsilon \widetilde{A}_2(\varepsilon)
\end{aligned}$$

and shows that

$$\frac{sc(S^*,t) - sc(S^*-\varepsilon,t)}{\varepsilon} \geq A_1(\varepsilon) + \widetilde{A}_2(\varepsilon)$$

Taking the limit as $\varepsilon \downarrow 0$, using $\lim_{\varepsilon\downarrow 0} A_1(\varepsilon) \longrightarrow 1$ and $\lim_{\varepsilon\downarrow 0} \widetilde{A}_2(\varepsilon) \longrightarrow 0$ because $\lim_{\varepsilon\downarrow 0} C(\varepsilon) \to 0$, yields the lower bound

$$\frac{\partial sc\left(S^*_-, t\right)}{\partial S} \geq 1.$$

If $H \geq S^* = B^{sc}(t)$, then either $H \geq B^{sc}(t) = B^c(t)$, where $B^c(t)$ is the boundary of a standard American call, or $B^{sc}(t) \leq H < B^c(t)$. This follows because $sc(S,t) \leq c(S,t)$ implying that $B^{sc}(t) \leq B^c(t)$ with equality if $B^c(t) \leq H$. If $H \geq B^{sc}(t) = B^c(t)$, then $sc(S,t) = c(S,t)$ for all $H \geq S$ and $\partial sc\left(S^*_-, t\right)/\partial S = \partial c\left(S^*_-, t\right)/\partial S = 1$. If $B^{sc}(t) \leq H < B^c(t)$, then there exists a standard American call with shorter maturity $T_o < T$ and boundary $B^c(t; T_o)$ such that $sc(S,t) = c(S,t;T_o)$ for all $H \geq S$ and $\partial sc\left(S^*_-, t\right)/\partial S = \partial c\left(S^*_-, t; T_o\right)/\partial S = 1$.

For a down-and-in step call ($\rho < 0$):
If $S^* = B^{sc}(t) > H$, then $O^{S^*-\varepsilon}_{0,\tau_\varepsilon} = O^{S^*}_{0,\tau_\varepsilon} = 0$ for $\tau_\varepsilon \leq \tau_H$. In other respects, the same steps as above apply and establish that $\partial sc\left(S^*_-, t\right)/\partial S \geq 1$.
If $H \geq S^* = B^{sc}(t)$, then it must be noticed that $\delta + \rho > 0$ as otherwise early exercise below H is suboptimal. It follows that $B^{sc}(s) = B(s; r+\rho, \delta+\rho)$ for $s \in [t,T]$, where $B(\cdot; r+\rho, \delta+\rho)$ is the boundary of a standard American call in the market $\mathcal{M}(r+\rho, \delta+\rho, \sigma)$. Hence $sc(S,t) = c(S,t;r+\rho,\delta+\rho)$ for all $H \geq S$ and $\partial sc\left(S^*_-, t\right)/\partial S = \partial c\left(S^*_-, t; r+\rho, \delta+\rho\right)/\partial S = 1$.

c) Combining the bounds:

Combining the results above shows that, for case (i), i.e., $\rho\left(O^{S^*-\varepsilon}_{0,\tau_\varepsilon} - O^{S^*}_{0,\tau_\varepsilon}\right) \geq 0$, comprising down-and-out and up-and-in step calls, we have the smooth pasting conditions

$$\frac{\partial sc\left(S^*_-, t\right)}{\partial S} = 1 \text{ at } S^* = B^{sc}, B^{sc}_{ld}$$

$$\frac{\partial sc\left(S^*_+, t\right)}{\partial S} = 1 \text{ at } S^* = B^{sc}_{lu}.$$

For case (ii), i.e., $\rho\left(O^{S^*-\varepsilon}_{0,\tau_\varepsilon} - O^{S^*}_{0,\tau_\varepsilon}\right) \leq 0$, comprising up-and-out and down-and-in step calls, there is a unique boundary B^{sc} and immediate exercise is optimal for $S \geq B^{sc}$. From the arguments above

$$\frac{\partial sc\left(S_{-}^{*},t\right)}{\partial S}_{|S^{*}=B^{sc}}=1.$$

□

The following auxiliary lemmas are critical for the proof of smooth pasting.[11]

Lemma C.1. *Let $T<\infty$ and consider a point $S=B\left(0\right)>H$. For $\varepsilon\geq 0$ such that $S>S-\varepsilon>H$, let τ_{ε} be the optimal exercise policy at $\left(S-\varepsilon,0\right)$ and $\tau_{H}=\inf\left\{v\in\left[0,T\right]:\left(S-\varepsilon\right)N_{0,v}=H\right\}$. We have $0\leq Q^{\eta}\left(\tau_{H}\leq\tau_{\varepsilon}\right)\leq C\left(\varepsilon\right)\varepsilon$ and*

$$\lim_{\varepsilon\downarrow 0}\frac{Q^{\eta}\left(\tau_{H}\leq\tau_{\varepsilon}\right)}{\varepsilon}=\lim_{\varepsilon\downarrow 0}C\left(\varepsilon\right)=0.$$

Proof of Lemma C.1. Let $F_{v}=S-kv$ and $\tau_{F}=\inf\left\{v\in\left[0,T\right]:\left(S-\varepsilon\right)N_{0,v}=F_{v}\right\}$ be the corresponding first hitting time. Assuming that the boundary $B\left(\cdot\right)$ is strictly decreasing at 0, there exists $k^{o}>0$ such that $F_{v}^{o}=S-k^{o}v\geq B\left(v\right)$ for $v\in\left[0,T\right]$ and $F_{0}^{o}=S=B\left(0\right)$.[12] Let $\tau_{F^{o}}=\inf\left\{v\in\left[0,T\right]:\left(S-\varepsilon\right)N_{0,v}=F_{v}^{o}\right\}$. Then

$$Q^{\eta}\left(\tau_{H}\leq\tau_{\varepsilon}\right)\leq Q^{\eta}\left(\tau_{H}\leq\tau_{F^{o}}\right).$$

Now select $t_{\varepsilon}=\left(k^{o}\right)^{-1}\left(\varepsilon-\frac{1}{2}\varepsilon^{2}\right)$ and consider the fixed threshold $F^{\varepsilon}=S-k^{o}t_{\varepsilon}=S-\varepsilon+\frac{1}{2}\varepsilon^{2}$. Note that $S=B\left(0\right)>F^{\varepsilon}>S-\varepsilon>H$ and define the stopping time $\tau_{F^{\varepsilon}}=\inf\left\{v\in\left[0,T\right]:\left(S-\varepsilon\right)N_{0,v}=F^{\varepsilon}\right\}$. Then

$$\begin{aligned}Q^{\eta}\left(\tau_{H}\leq\tau_{F^{o}}\right)&=Q^{\eta}\left(\tau_{H}\leq\tau_{F^{o}},\tau_{H}>t_{\varepsilon}\right)+Q^{\eta}\left(\tau_{H}\leq\tau_{F^{o}},\tau_{H}\leq t_{\varepsilon}\right)\\&\leq Q^{\eta}\left(\tau_{H}\leq\tau_{F^{\varepsilon}}\right)+Q^{\eta}\left(\tau_{H}\leq t_{\varepsilon}\right)\equiv Q_{1}^{\eta}\left(\varepsilon\right)+Q_{2}^{\eta}\left(\varepsilon\right).\end{aligned}$$

For $Q_{1}^{\eta}\left(\varepsilon\right)$, from the Kolmogorov backward equation under the new equivalent martingale measure and using the mean value theorem, one obtains

$$\begin{aligned}Q_{1}^{\eta}\left(\varepsilon\right)&=\frac{\left(F^{\varepsilon}\right)^{\widetilde{\gamma}}-\left(S-\varepsilon\right)^{\widetilde{\gamma}}}{\left(F^{\varepsilon}\right)^{\widetilde{\gamma}}-H^{\widetilde{\gamma}}}=\frac{\left(S-\varepsilon+\frac{1}{2}\varepsilon^{2}\right)^{\widetilde{\gamma}}-\left(S-\varepsilon\right)^{\widetilde{\gamma}}}{\left(S-\varepsilon+\frac{1}{2}\varepsilon^{2}\right)^{\widetilde{\gamma}}-H^{\widetilde{\gamma}}}\\&\leq\frac{\widetilde{\gamma}\left(s_{\varepsilon}\right)^{\widetilde{\gamma}-1}}{\left(S-\varepsilon+\frac{1}{2}\varepsilon^{2}\right)^{\widetilde{\gamma}}-H^{\widetilde{\gamma}}}\frac{1}{2}\varepsilon^{2}\equiv C_{1}\left(\varepsilon\right)\varepsilon^{2}\end{aligned}$$

where $\widetilde{\gamma}=-1+2\left(\delta-r\right)/\sigma^{2}$ and $s_{\varepsilon}\in\left[S-\varepsilon,S-\varepsilon+\frac{1}{2}\varepsilon^{2}\right]$. As

$$\lim_{\varepsilon\downarrow 0}C_{1}\left(\varepsilon\right)=\frac{1}{2}\lim_{\varepsilon\downarrow 0}\frac{\widetilde{\gamma}\left(s_{\varepsilon}\right)^{\widetilde{\gamma}-1}}{\left(S-\varepsilon+\frac{1}{2}\varepsilon^{2}\right)^{\widetilde{\gamma}}-H^{\widetilde{\gamma}}}=\frac{\widetilde{\gamma}}{2}\frac{S^{\widetilde{\gamma}-1}}{S^{\widetilde{\gamma}}-H^{\widetilde{\gamma}}}=C_{1}\left(0\right),$$

[11]We thank Scott Robertson for suggesting the arguments in the proofs of Lemmas C.1-C.2.

[12]The proof extends to the case of a boundary with a flat section. The line F^{o} is then chosen to have the properties stated at the largest time t_{e} such that $H=B\left(t_{e}\right)$.

we conclude

$$\lim_{\varepsilon\downarrow 0}\frac{Q_1^{\eta}(\varepsilon)}{\varepsilon}\leq\lim_{\varepsilon\downarrow 0}C_1(\varepsilon)\,\varepsilon=0.$$

For $Q_2^{\eta}(\varepsilon)$, the probability of hitting the lower boundary H is under the new equivalent martingale measure

$$\begin{aligned}Q^{\eta}(\tau_H\leq t_\varepsilon)&=N\left(-d^{+,\eta}\left(\frac{S-\varepsilon}{H}\right)\right)+\left(\frac{S-\varepsilon}{H}\right)^{\widetilde{\gamma}}N\left(d^{-,\eta}\left(\frac{S-\varepsilon}{H}\right)\right)\\&=N\left(-\frac{\ln\left((S-\varepsilon)/H\right)+\widetilde{\mu}t_\varepsilon}{\sigma\sqrt{t_\varepsilon}}\right)+\left(\frac{S-\varepsilon}{H}\right)^{\widetilde{\gamma}}N\left(\frac{-\ln\left((S-\varepsilon)/H\right)+\widetilde{\mu}t_\varepsilon}{\sigma\sqrt{t_\varepsilon}}\right)\end{aligned}$$

where $\widetilde{\mu}=\alpha\sigma+\sigma^2=r-\delta+\frac{\sigma^2}{2}$ and $\widetilde{\gamma}=-\frac{2\widetilde{\mu}}{\sigma^2}$. Now applying Lemma C.3 to the two terms of this expression[13] leads to an inequality

$$\begin{aligned}Q^{\eta}(\tau_H\leq t_\varepsilon)\leq&\frac{\sigma\sqrt{2t_\varepsilon}}{\left|-\left[\ln\left((S-\varepsilon)/H\right)+\widetilde{\mu}t_\varepsilon\right]\right|}\exp\left(-\frac{\left(-\left[\ln\left((S-\varepsilon)/H\right)+\widetilde{\mu}t_\varepsilon\right]\right)^2}{2\sigma^2t_\varepsilon}\right)\\&+\left(\frac{S-\varepsilon}{H}\right)^{-\frac{2\widetilde{\mu}}{\sigma^2}}\frac{\sigma\sqrt{2t_\varepsilon}}{\left|-\ln\left((S-\varepsilon)/H\right)+\widetilde{\mu}t_\varepsilon\right|}\exp\left(-\frac{\left(-\ln\left((S-\varepsilon)/H\right)+\widetilde{\mu}t_\varepsilon\right)^2}{2\sigma^2t_\varepsilon}\right)\end{aligned}$$

which gives (because $e^{-x^2}<\frac{1}{1+x^2}$ for all x and after rearrangement)

$$\begin{aligned}Q^{\eta}(\tau_H\leq t_\varepsilon)\leq&\frac{\sigma\sqrt{2t_\varepsilon}}{\left|-\left[\ln\left((S-\varepsilon)/H\right)+\widetilde{\mu}t_\varepsilon\right]\right|}\frac{2\sigma^2t_\varepsilon}{2\sigma^2t_\varepsilon+\left(-\left[\ln\left((S-\varepsilon)/H\right)+\widetilde{\mu}t_\varepsilon\right]\right)^2}\\&+\left(\frac{S-\varepsilon}{H}\right)^{-\frac{2\widetilde{\mu}}{\sigma^2}}\frac{\sigma\sqrt{2t_\varepsilon}}{\left|-\ln\left((S-\varepsilon)/H\right)+\widetilde{\mu}t_\varepsilon\right|}\frac{2\sigma^2t_\varepsilon}{2\sigma^2t_\varepsilon+\left(\left[-\ln\left((S-\varepsilon)/H\right)+\widetilde{\mu}t_\varepsilon\right]\right)^2}\\&=\varepsilon\times\left(\frac{\sigma\sqrt{2t_\varepsilon}}{\left|-\ln\left((S-\varepsilon)/H\right)-\widetilde{\mu}t_\varepsilon\right|}\frac{2\sigma^2t_\varepsilon/\varepsilon}{2\sigma^2t_\varepsilon+\left(-\ln\left((S-\varepsilon)/H\right)-\widetilde{\mu}t_\varepsilon\right)^2}+\right.\\&\left.\left(\frac{S-\varepsilon}{H}\right)^{-\frac{2\widetilde{\mu}}{\sigma^2}}\frac{\sigma\sqrt{2t_\varepsilon}}{\left|-\ln\left((S-\varepsilon)/H\right)+\widetilde{\mu}t_\varepsilon\right|}\frac{2\sigma^2t_\varepsilon/\varepsilon}{2\sigma^2t_\varepsilon+\left(-\ln\left((S-\varepsilon)/H\right)+\widetilde{\mu}t_\varepsilon\right)^2}\right)\\&=\varepsilon\times C_2(\varepsilon)\end{aligned}$$

where $\lim_{\varepsilon\longrightarrow 0}C_2(\varepsilon)=0$ so that

$$\lim_{\varepsilon\downarrow 0}\frac{Q_2^{\eta}(\varepsilon)}{\varepsilon}\leq\lim_{\varepsilon\downarrow 0}C_2(\varepsilon)=0.$$

To prove the claim in Lemma C.1, set $C(\varepsilon)\equiv C_1(\varepsilon)\,\varepsilon+C_2(\varepsilon)$ and take the limit. □

Lemma C.2. *Let $T<\infty$ and consider a point $S=B_{lu}^{sc}(0)<H$. For $\varepsilon\geq 0$ such that $H>S+\varepsilon>S$, let τ_ε be the optimal exercise policy at $(S+\varepsilon,0)$ and $\tau_H=\inf\{v\in[0,T]:(S+\varepsilon)\,N_{0,v}=H\}$.*

[13] For all $\widetilde{\mu}\in\mathbb{R}$, as ε goes to 0, both $\left[-\frac{\ln((S-\varepsilon)/H)+\widetilde{\mu}t_\varepsilon}{\sigma\sqrt{t_\varepsilon}}\right]$ and $\left[\frac{-\ln((S-\varepsilon)/H)+\widetilde{\mu}t_\varepsilon}{\sigma\sqrt{t_\varepsilon}}\right]$ tend to minus infinity because $S-\varepsilon>H$ (so that Lemma C.3 applies).

We have $0 \leq Q^{\eta}(\tau_H \leq \tau_\varepsilon) \leq C(\varepsilon)\varepsilon$ *and*

$$\lim_{\varepsilon \downarrow 0} \frac{Q^{\eta}(\tau_H \leq \tau_\varepsilon)}{\varepsilon} = \lim_{\varepsilon \downarrow 0} C(\varepsilon) = 0.$$

Proof of Lemma C.2. Let $F_v = S + kv$ and $\tau_F = \inf\{v \in [0,T] : (S+\varepsilon) N_{0,v} = F_v\}$ be the corresponding first hitting time. Assuming the boundary $B(\cdot)$ is increasing over $[0,T]$, there exists $k^o > 0$ such that $F_v^o = S + k^o v \leq B(v)$ for $v \in [0,T]$ and $F_0^o = S = B(0)$.[14] Let $\tau_{F^o} = \inf\{v \in [0,T] : (S+\varepsilon) N_{0,v} = F_v^o\}$. Then

$$Q^{\eta}(\tau_H \leq \tau_\varepsilon) \leq Q^{\eta}(\tau_H \leq \tau_{F^o}).$$

Now select $t_\varepsilon = (k^o)^{-1}\left(\varepsilon - \frac{1}{2}\varepsilon^2\right)$ and consider the fixed threshold $F^\varepsilon = S + k^o t_\varepsilon = S + \varepsilon - \frac{1}{2}\varepsilon^2$. Note that $H > S+\varepsilon > F^\varepsilon > S = B(0)$ and define the stopping time $\tau_{F^\varepsilon} = \inf\{v \in [0,T] : (S+\varepsilon) N_{0,v} = F^\varepsilon\}$. Then

$$\begin{aligned} Q^{\eta}(\tau_H \leq \tau_{F^o}) &= Q^{\eta}(\tau_H \leq \tau_{F^o}, \tau_H > t_\varepsilon) + Q^{\eta}(\tau_H \leq \tau_{F^o}, \tau_H \leq t_\varepsilon) \\ &\leq Q^{\eta}(\tau_H \leq \tau_{F^\varepsilon}) + Q^{\eta}(\tau_H \leq t_\varepsilon) \equiv Q_1^{\eta}(\varepsilon) + Q_2^{\eta}(\varepsilon). \end{aligned}$$

For $Q_1^{\eta}(\varepsilon)$, the same type of arguments and similar notations as in Lemma C.1 establish that

$$\begin{aligned} Q_1^{\eta}(\varepsilon) = Q^{\eta}(\tau_H \leq \tau_{F^\varepsilon}) &= \frac{(S+\varepsilon)^{\widetilde{\gamma}} - (F^\varepsilon)^{\widetilde{\gamma}}}{H^{\widetilde{\gamma}} - (F^\varepsilon)^{\widetilde{\gamma}}} = \frac{(S+\varepsilon)^{\widetilde{\gamma}} - \left(S+\varepsilon-\frac{1}{2}\varepsilon^2\right)^{\widetilde{\gamma}}}{H^{\widetilde{\gamma}} - \left(S+\varepsilon-\frac{1}{2}\varepsilon^2\right)^{\widetilde{\gamma}}} \\ &\leq \frac{\widetilde{\gamma}(s_\varepsilon)^{\widetilde{\gamma}-1}}{H^{\widetilde{\gamma}} - \left(S+\varepsilon-\frac{1}{2}\varepsilon^2\right)^{\widetilde{\gamma}}} \frac{1}{2}\varepsilon^2 \equiv C_1(\varepsilon)\varepsilon^2 \end{aligned}$$

for $s_\varepsilon \in \left[S+\varepsilon-\frac{1}{2}\varepsilon^2, S+\varepsilon\right]$. As

$$\lim_{\varepsilon \downarrow 0} C_1(\varepsilon) = \frac{1}{2}\lim_{\varepsilon \downarrow 0} \frac{\widetilde{\gamma}(s_\varepsilon)^{\widetilde{\gamma}-1}}{H^{\widetilde{\gamma}} - \left(S+\varepsilon-\frac{1}{2}\varepsilon^2\right)^{\widetilde{\gamma}}} = \frac{1}{2}\frac{\widetilde{\gamma}(S)^{\widetilde{\gamma}-1}}{H^{\widetilde{\gamma}} - S^{\widetilde{\gamma}}} = C_1(0)$$

we conclude that

$$\lim_{\varepsilon \downarrow 0} \frac{Q_1^{\eta}(\varepsilon)}{\varepsilon} \leq \lim_{\varepsilon \downarrow 0} C_1(\varepsilon)\varepsilon = 0.$$

For $Q_2^{\eta}(\varepsilon)$, the probability of hitting the upper boundary H is, under the new equivalent martingale measure $Q^{\eta}(\tau_H \leq t_\varepsilon)$,

$$\begin{aligned} & N\left(d^{+,\eta}\left(\frac{S+\varepsilon}{H}\right)\right) + \left(\frac{S+\varepsilon}{H}\right)^{\widetilde{\gamma}} N\left(-d^{-,\eta}\left(\frac{S+\varepsilon}{H}\right)\right) \\ &= N\left(\frac{\ln((S+\varepsilon)/H) + \widetilde{\mu}t_\varepsilon}{\sigma\sqrt{t_\varepsilon}}\right) + \left(\frac{S+\varepsilon}{H}\right)^{\widetilde{\gamma}} N\left(-\frac{-\ln((S+\varepsilon)/H) + \widetilde{\mu}t_\varepsilon}{\sigma\sqrt{t_\varepsilon}}\right). \end{aligned}$$

[14] The proof extends to the case where the boundary has a flat section.

where $\widetilde{\mu} = \alpha\sigma + \sigma^2 = r - \delta + \frac{\sigma^2}{2}$ and $\widetilde{\gamma} = -\frac{2\widetilde{\mu}}{\sigma^2}$. Now applying Lemma C.3 to the two terms of this expression[15] leads to an inequality

$$\leq \frac{\sigma\sqrt{2t_\varepsilon}}{\left|\ln\left(\left(S+\varepsilon\right)/H\right)+\widetilde{\mu}t_\varepsilon\right|}\exp\left(-\frac{\left[\ln\left(\left(S+\varepsilon\right)/H\right)+\widetilde{\mu}t_\varepsilon\right]^2}{2\sigma^2 t_\varepsilon}\right)$$
$$+\left(\frac{S+\varepsilon}{H}\right)^{-\frac{2\widetilde{\mu}}{\sigma^2}}\frac{\sigma\sqrt{2t_\varepsilon}}{\left|-\left[-\ln\left(\left(S+\varepsilon\right)/H\right)+\widetilde{\mu}t_\varepsilon\right]\right|}\exp\left(-\frac{\left[-\ln\left(\left(S+\varepsilon\right)/H\right)+\widetilde{\mu}t_\varepsilon\right]^2}{2\sigma^2 t_\varepsilon}\right)$$

which gives (because $e^{-x^2} < \frac{1}{1+x^2}$ for all x and after rearrangement)

$$Q^\eta\left(\tau_H \leq t_\varepsilon\right) \leq \frac{\sigma\sqrt{2t_\varepsilon}}{\left|\ln\left(\left(S+\varepsilon\right)/H\right)+\widetilde{\mu}t_\varepsilon\right|}\frac{2\sigma^2 t_\varepsilon}{2\sigma^2 t_\varepsilon+\left[\ln\left(\left(S+\varepsilon\right)/H\right)+\widetilde{\mu}t_\varepsilon\right]^2}$$
$$+\left(\frac{S+\varepsilon}{H}\right)^{-\frac{2\widetilde{\mu}}{\sigma^2}}\frac{\sigma\sqrt{2t_\varepsilon}}{\left|\ln\left(\left(S+\varepsilon\right)/H\right)-\widetilde{\mu}t_\varepsilon\right|}\frac{2\sigma^2 t_\varepsilon}{2\sigma^2 t_\varepsilon+\left[-\ln\left(\left(S+\varepsilon\right)/H\right)+\widetilde{\mu}t_\varepsilon\right]^2}.$$

Consequently, as in Lemma C.1, one may identify $C_2\left(\varepsilon\right)$ such that i) $Q^\eta\left(\tau_H \leq t_\varepsilon\right) \leq \varepsilon \times C_2\left(\varepsilon\right)$, ii) $\lim_{\varepsilon\longrightarrow 0} C_2\left(\varepsilon\right) = 0$ and finally iii)

$$\lim_{\varepsilon\downarrow 0}\frac{Q_2^\eta\left(\varepsilon\right)}{\varepsilon} \leq \lim_{\varepsilon\downarrow 0} C_2\left(\varepsilon\right) = 0.$$

The claim of Lemma C.2 is then proved by setting $C\left(\varepsilon\right) \equiv C_1\left(\varepsilon\right)\varepsilon + C_2\left(\varepsilon\right)$ and taking the limit. □

Lemma C.3. *For $x < 0$, the following inequality holds*

$$N\left(x\right) < \frac{\sqrt{2}}{|x|}\exp\left(-\frac{x^2}{2}\right).$$

Proof of Lemma C.3. The proof relies on equality (7.1.13) in Abramowitz & Stegun (1965) stating that

$$\frac{1}{x+\sqrt{x^2+2}} < e^{x^2}\int_x^\infty e^{-t^2}dt < \frac{1}{x+\sqrt{x^2+\frac{4}{\pi}}},\ \text{for } x \geq 0.$$

Consider the upper bound. Changing the variable to $t = \frac{u}{\sqrt{2}}$ gives

$$\int_{\sqrt{2}x}^\infty e^{-\frac{u^2}{2}}\frac{du}{\sqrt{2}} < \frac{e^{-x^2}}{x+\sqrt{x^2+\frac{4}{\pi}}}.$$

Dividing both sides by $\sqrt{\pi}$ gives an upper bound for the complement of the normal cdf also called

[15] For all $\widetilde{\mu} \in \mathbb{R}$, as ε goes to 0, both $\left[\frac{\ln((S+\varepsilon)/H)+\widetilde{\mu}t_\varepsilon}{\sigma\sqrt{t_\varepsilon}}\right]$ and $\left[-\frac{-\ln((S+\varepsilon)/H)+\widetilde{\mu}t_\varepsilon}{\sigma\sqrt{t_\varepsilon}}\right]$ tend to minus infinity because $S+\varepsilon < H$ (so that Lemma C.3 applies).

the tail distribution function of the normal distribution

$$N^c\left(\sqrt{2}x\right) = \frac{1}{\sqrt{2\pi}}\int_{\sqrt{2}x}^{\infty} e^{-\frac{u^2}{2}}du < \frac{1}{\sqrt{\pi}}\frac{e^{-x^2}}{x+\sqrt{x^2+4/\pi}} < \frac{e^{-x^2}}{x+\sqrt{x^2+4/\pi}}, \text{ for } x \geq 0.$$

Now, changing the variable from $\sqrt{2}x$ to x, considering the symmetry of the normal distribution ($N^c(x) = N(-x)$) and changing the variable from $-x$ to x, one can conclude that

$$N(x) < \frac{\sqrt{2}e^{-\frac{x^2}{2}}}{-x+\sqrt{x^2+8/\pi}} < \frac{\sqrt{2}e^{-\frac{x^2}{2}}}{\sqrt{x^2+8/\pi}} < \frac{\sqrt{2}}{|x|}e^{-\frac{x^2}{2}}$$

is satisfied for all $x \leq 0$. □

Theorem C.2. *Let B^{sc} be an optimal exercise boundary of an American step call option and define the region $\mathcal{R}_g \equiv \left\{S : \delta S - rK + \rho(S-K)1_{\{S\in A\}} > 0\right\}$ where A is the region where the excursion time counter becomes active. Denote by $\mathcal{R}_g^{sc}$ a subregion of $\mathcal{R}_g$ such that $\mathcal{R}_g^{sc}$ is up-connected and right-connected and $B^{sc} \subset \mathcal{R}_g^{sc}$. We have the following properties*

(i) B^{sc} is monotonic and continuous on $[0,T)$.

(ii) The limit $\lim_{t\uparrow T} B^{sc}(t) = B^{sc}(T_-)$ exists and

(ii.a) if B^{sc} is nonincreasing then $B^{sc}(T_-) = \inf\left\{x : x \in \mathcal{R}_g^{sc}\right\}$.

(ii.b) if B^{sc} is nondecreasing then $B^{sc}(T_-) = \sup\left\{x : x \in \mathcal{R}_g^{sc}\right\}$.

(ii.c) at maturity $B^{sc}(T) = K$.

Proof of Theorem C.2. (i) The monotonicity of B^{sc} is implied by the monotonicity of the price function of the step call option. If B^{sc} is nonincreasing we have $B^{sc}(t_-) \geq B^{sc}(t) \geq B^{sc}(t_+)$ where $B^{sc}(t_-)$ and $B^{sc}(t_+)$ are respectively the left and right limits of the boundary at t.

To show that B^{sc} is right continuous on $[0,T)$, suppose that $(B^{sc}(\nu), \nu) \in \mathcal{E}_o^{sc}$ for all $\nu > t$. As $\mathcal{E}_o^{sc}$ is right-connected (see Theorem 3.1, 3.4 and 3.5), the limit $B^{sc}(t_-)$ exists. Given that $(B^{sc}(\nu), \nu) \in \mathcal{E}_o^{sc}$ and that $\mathcal{E}_o^{sc}$ is a closed set we have $\lim_{\nu\downarrow t}(B^{sc}(\nu), \nu) = (B^{sc}(t_-), t) \in \mathcal{E}_o^{sc}$. That is, $B^{sc}(t_-) \geq B^{sc}(t)$. For a nonincreasing boundary we know that $B^{sc}(t_-) \leq B^{sc}(t)$. Hence, we have $B^{sc}(t_-) = B^{sc}(t)$. A similar argument applies for a non-decreasing boundary. The inequalities are just reversed.

The left continuity can be shown by adapting the arguments in Jacka (1991) to the step call option case. Suppose that we have a jump at some points in $[0,T)$, i.e., $B^{sc}(t_-) > B^{sc}(t)$. This implies that there is an open set $(B^{sc}(t), B^{sc}(t_-)) \equiv \mathcal{R}$ such that $\mathcal{R} \subset \mathcal{E}_o^{sc}$ and for all $x \in \mathcal{R}$ there exists a sequence $\{x_n\} \in \mathcal{C}^{sc}$ with $x = \lim_{n\longrightarrow\infty} x_n$. The strong Markov property ensures that the price function $sc(S,t)$ is $C^{2,1}$ in the continuation region $\mathcal{C}^{sc}$ and satisfies the fundamental PDE

$$sc_t(S,t) + sc_s(S,t)S(r-\delta) + \frac{1}{2}sc_{ss}(S,t)S^2\sigma^2 - \left(r+\rho 1_{\{S\in A\}}\right)sc(S,t) = 0$$

for all $(S,t) \in \mathcal{C}^{sc}$, where $sc_t(S,t)$, $sc_s(S,t)$ are respectively the first derivatives with respect to t and S, and $sc_{ss}(S,t)$ is the second derivative with respect to S. At the limit x, it can be shown that

$$\begin{aligned}\frac{1}{2}sc_{ss}(x,t)\,x^2\sigma^2 &= \left(r+\rho 1_{\{S\in A\}}\right)sc(x,t)-sc_t(x,t)-sc_s(x,t)\,x\,(r-\delta)\\ &= \left(\delta+\rho 1_{\{S\in A\}}\right)x-\left(r+\rho 1_{\{S\in A\}}\right)K.\end{aligned}$$

As $B^{sc}(t) \geq \left(r+\rho 1_{\{S\in A\}}\right)K/\left(\delta+\rho 1_{\{S\in A\}}\right)$ we can deduce that $sc_{ss}(x,t)\geq 0$ for all $x\in\mathcal{R}$ and $sc_{ss}(x,t)\geq\varepsilon>0$ in a subset of $\mathcal{R}$ with positive Lebesgue measure. Using integration by part we get

$$\Delta sc\left(B^{sc}(t),t\right)-\Delta B^{sc}(t)=\int_{B^{sc}(t)}^{B^{sc}(t_-)}\int_{B^{sc}(t)}^{u}sc_{ss}(v,t)\,dvdu.$$

for all (x,t) in $\mathcal{R}$ where $\Delta sc\left(B^{sc}(t),t\right)=sc\left(B^{sc}(t_-),t\right)-sc\left(B^{sc}(t),t\right)$ and $\Delta B^{sc}(t)=\left(B^{sc}(t_-)-K\right)-\left(B^{sc}(t)-K\right)$. By definition of the exercise region $\mathcal{E}_o^{sc}$, the left side of the equation $\Delta sc\left(B^{sc}(t),t\right)-\Delta B^{sc}(t)=0$ which contradicts the fact that $sc_{ss}(x,t)>0$. Thus, the set $\mathcal{R}$ must have null Lebesgue measure, i.e., $B^{sc}(t_-)=B^{sc}(t)$.

(ii) From Properties *(v)* in theorems 3.1, 3.4 and 3.5 we know that $\inf\left\{x : x\in\mathcal{R}_g^{sc}\right\}\leq B^{sc}(T_-)\leq \sup\left\{x : x\in\mathcal{R}_g^{sc}\right\}$. We also know that the uncertainty vanishes as $t\uparrow T$. That is, the optimal exercise boundaries converge to those of a deterministic problem. For a deterministic problem it is optimal to exercise when $\delta S-rK+\rho(S-K)\,1_{\{S\in A\}}\geq 0$. $B^{sc}(T_-)$ converges naturally to the boundaries of $\mathcal{R}_g^{sc}$. When B^{sc} is nonincreasing (resp. nondecreasing) with respect to time it converges to $\inf\left\{x : x\in\mathcal{R}_g^{sc}\right\}$ (resp. $\sup\left\{x : x\in\mathcal{R}_g^{sc}\right\}$). Because B^{sc} has no jump in $[0,T)$, we conclude that $\lim_{t\uparrow T}B^{sc}(t)=B^{sc}(T_-)=\inf\left\{x : x\in\mathcal{R}_g^{sc}\right\}$ (resp. $\lim_{t\uparrow T}B^{sc}(t)=B^{sc}(T_-)=\sup\left\{x : x\in\mathcal{R}_g^{sc}\right\}$) when B^{sc} is nonincreasing (resp. nondecreasing). This completes the proof of *(ii.a)* and *(ii.b)*.

Property *(ii.c)* is obvious. At maturity it pays to exercise the option when $S>K$. □

Appendix D. Convergence of the algorithm in Subsection 6.1

The recursive integral equations for the immediate exercise boundaries all have the form

$$B(t)=f(t,B(t))+\int_a^t g(t,s,B(t),B(s))ds \tag{8}$$

known as a *non-linear Volterra integral equation of the second kind.* The unknown function $\{B(t) : t\in[0,T]\}$ is the exact solution of the equation on $[0,T]$. The functions f and g are defined on $[0,T]\times[0,\infty)$ and on $[0,T]\times[0,T]\times[0,\infty)\times[0,\infty)$, respectively. We assume that $f(t,x)$ and $g(t,s,y,z)$ satisfy Lipschitz conditions

$$|f(t,x)-f(t,y)|\leq L_f\,|x-y| \tag{9}$$

$$|g(t,s,x,z) - g(t,s,y,z)| \leq L_1 |x-y| \tag{10a}$$

$$|g(t,s,z,x) - g(t,s,z,y)| \leq L_2 |x-y| \tag{10b}$$

for constants $0 \leq L_f \leq 1$ and $L_1 > 0$, $L_2 > 0$.

We approximate the unknown function B at a finite number of points in $[0,T]$. Discretizing the time interval as $\{t_n : n = 0, ..., N\}$ where $t_N = T$ and $\Delta t_n = t_n - t_{n-1} = h$ for all $n = 1, ..., N$, we have

$$B_n = f(t_n, B_n) + h \sum_{i=0}^{n} w_i^{(n)} g(t_n, t_i, B_n, B_i) ds, \quad n = 0, ..., N \tag{11}$$

where B_n is the approximated value of B at t_n, and $w_i^{(n)}$, $i = 0, \ldots, n$ are the weights of the quadrature rule used to approximate the integral in (8).

For a call, the exact solution of the recursive equation is bounded: $K \leq B(t) \leq B^* < \infty$ for $t \in [0,T]$ where K is the strike and B^* the constant optimal exercise boundary of the corresponding standard perpetual call option. Define a new approximation function B_n^{K,B^*} such that

$$B_n^{K,B^*} = \begin{cases} B^*, & \text{if } B_n > B^* \\ B_n, & \text{if } K \leq B_n \leq B^* \\ K, & \text{if } B_n < K \end{cases} \tag{12}$$

This new approximation is bounded, with values in $[K, B^*]$. Hereafter we will only consider B_n^{K,B^*} and for notational simplicity we set $B_n \equiv B_n^{K,B^*}$. We will see later that B_n has to be bounded away from zero for the kernel $g(s,t,x,y)$ to be locally Lipshitzian with respect to the third and the fourth arguments.

Theorem D.1. *Consider the approximate solution of (8) by (11)-(12) and assume that*

(i) the functions $f(t,x)$ and $g(t,s,x,y)$ satisfy the Lipshitz conditions (9) and (10a-10b),
(ii) the quadrature rule satisfies the stability condition

$$\sup \left\{ \sum_{i=0}^{n} \left| w_i^{(n)} \right|, \ n \in \mathbb{N} \right\} < \infty.$$

Then $\lim_{h \downarrow 0} |B(t_n) - B_n| = 0$.

The above theorem ensures the convergence of our numerical approximation to the true value of the boundary. In our recursive integral equations, f and g are linear combinations of the function Ψ_ρ. If the latter satisfies the required Lipschitz conditions then both f and g satisfy (i). Assumption (ii) is linked to the choice of the quadrature rule.

Proof of Theorem D.1. Recall that

$$\begin{aligned}\Psi_\rho(\alpha; b, h, T) &= E\left[e^{\alpha W_T - \rho O_{0,T}} 1_{\{W_T > b\}}\right] \\ &= \int_b^\infty e^{\alpha z} E\left[e^{-\rho O_{0,T}}; W_T \in dz\right] \\ &= \int_b^\infty e^{\alpha z} f_\rho(z, h, T) dz\end{aligned}$$

where $f_\rho(z, h, T)dz$ is the transition probability density of a Brownian motion killed at the rate ρ. The continuity of $\Psi_\rho(\alpha; b, h, T)$ with respect to T is obvious and is implied by the continuity of the process $(W_t)_{t\geq 0}$. For $b_1 \leq b_2$, we have

$$\begin{aligned}|\Psi_\rho(\alpha, b_1, h, T) - \Psi_\rho(\alpha, b_2, h, T)| &= \left|\int_{b_1}^{b_2} e^{\alpha z} f_\rho(z, h, T) dz\right| \\ &\leq \int_{b_1}^{b_2} |e^{\alpha z} f_\rho(z, h, T)| \, dz \\ &\leq L \, |b_1 - b_2|\end{aligned}$$

where L is constant. For the third line, it can be shown using the expression for $f_\rho(z, h, T)$, that there exist a constant L such that $\sup_{s\in[0,t], z\in\mathbb{R}} |e^{\alpha z} f_\rho(z, h, t-s)| \leq L < \infty$.[16] It is straightforward to extend this result and show that the function

$$g(t, s, x, y) \equiv \Psi_\rho(\alpha, \log(y/x)/\sigma, h, t-s)$$

satisfies Lipschitz conditions with respect to its third and fourth arguments for $x \in [K, B^*]$, $y \in [K, B^*]$. The result also holds when the kernel is a linear combination of functions Ψ_ρ. Note that the $\log(x)$ function is not Lipshitzian in $\mathbb{R}^+$. However, because both the approximation and the true value of $B(t_n)$ are in $[K, B^*]$, there exist constants a and b such that $0 < a < y/x < b < \infty$, and $\log(x)$ is Lipshitzian in $[a, b]$. We conclude that the functions are Lipschitzian. Now we can write the approximation error $\epsilon_n = B(t_n) - B_n$ as

$$\epsilon_n = f(t, B(t_n)) - f(t, B_n) + h \sum_{i=0}^{n} w_i^{(n)} \left[g(t_n, t_i, B(t_n), B(t_i)) - g(t_n, t_i, B_n, B_i)\right] - I(n; h)$$

where

$$I(n; h) = h \sum_{i=0}^{n} w_i^{(n)} g(t_n, t_i, B(t_n), B(t_i)) - \int_a^{t_n} g(t_n, s, B(t_n), B(s)) ds$$

is the numerical integration error. If the numerical integration method is consistent of order $p > 1$,

[16]The constant L exists for any value of $s \in [0, t]$ and $z \in \mathbb{R}$. See the formulas for $f_\rho(z, h, T)$ in Appendix A.

then there exists a constant c such that

$$\max_{0\le n\le N} |I(n;h)| \le ch^p. \tag{13}$$

Using the Lipschitz conditions, the absolute value of the approximation at t_n is bounded and

$$|\epsilon_n| \le L_f |\epsilon_n| + h \sum_{i=0}^{n} \left|w_i^{(n)}\right| (L_1 |\epsilon_n| + L_2 |\epsilon_i|) + |I(n;h)|.$$

Reorganizing the terms of the inequality yields

$$|\epsilon_n| \le \frac{1}{1 - L_f - hW(L_1 + L_2)} \left[hWL_2 \sum_{i=0}^{n-1} |\epsilon_i| + I(n;h) \right] \tag{14}$$

where the positive constant W is independent of n and such that $\sup_{n,i} \left|w_i^{(n)}\right| \le \sum_{i=0}^{n} \left|w_i^{(n)}\right| \le W$. We know that the first derivative of the European-style option with respect to the underlying asset is bounded above by 1. Therefore $L_f < 1$. For sufficiently small h the coefficient $D(h) = (1 - L_f - hW(L_1 + L_2))^{-1} > 0$, and for some constant $\bar{D}$ such that $D(h) \le \bar{D} < \infty$, (14) becomes

$$\begin{aligned}
|\epsilon_n| &\le \bar{D} \times I(n;h) + C_h \sum_{i=0}^{n-1} |\epsilon_i| \\
&= \bar{D} \times I(n;h) + C_h \left(|\epsilon_{n-1}| + \sum_{i=0}^{n-2} |\epsilon_i| \right) \\
&\le \bar{D} \times I(n;h) + C_h \left(\bar{D} \times I(n-1;h) + \underbrace{C_h \sum_{i=0}^{n-2} |\epsilon_i| + \sum_{i=0}^{n-2} |\epsilon_i|}_{(1+C_h)\sum_{i=0}^{n-2}|\epsilon_i|} \right) \\
&\le \bar{D} \left[I(n;h) + C_h \times I(n-1;h) + \cdots + \left(C_h + \ldots C_h^k\right) \times I(n-k;h) + \cdots + \left(C_h + \ldots C_h^{n-1}\right) \times I(1;h) \right] \\
&= \bar{D} \sum_{k=0}^{n-1} H_k \times I(n-k;h)
\end{aligned}$$

where

$$C_h = hWL_2 / (1 - L_f - hW(L_1 + L_2))$$

$$H_k = C_h + (k-1)C_h^2 + \cdots + (k-1)C_h^{k-1} + C_h^k = \sum_{j=0}^{k} \binom{j}{k} C_h^j \quad \text{with} \quad H_0 = 1.$$

For a $p > 1$ order consistent numerical integration, $I(n-k;h) \to 0$ for $k = 0, \ldots, n-1$ as $h \to 0$. Therefore $\lim_{h\downarrow 0} |B(t_n) - B_n| = 0$. □