Refinements to the generic existence of equilibrium in incomplete markets

Lionel De Boisdeffre

CES
Centre d'Économie de la Sorbonne
UMR 8174

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2019.11

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Lionel de Boisdeffre,[1]

(June 2019)

Abstract

The paper demonstrates the generic existence of general equilibria in incomplete markets with well behaved price properties. The economy has two periods and an ex ante uncertainty over the state of nature to be revealed at the second period. Securities pay off in cash or commodities at the second period, conditionally on the state of nature to be revealed. They permit financial transfers across periods and states, which are insufficient to span all state contingent claims to value, whatever the spot price to prevail. Under smooth preference and the standard Radner (1972) perfect foresight assumptions, equilibrium is shown to exist, except for a closed set of measure zero of endowments and securities. The proof provides additional arguments and insights to Duffie-Shafer's (1985) on the same subject and refines it in two ways. First, equilibrium is shown to exist generically for any norm values of commodity prices on any spot market, and for any collection of state prices. Second, assets need no longer pay off in commodities, but may in any mix of cash and goods.

Key words: sequential equilibrium, temporary equilibrium, perfect foresight, existence, rational expectations, financial markets, asymmetric information, arbitrage.

[1] University of Paris 1-Panthéon-Sorbonne, 106-112 Bd. de l'Hôpital, 75013 Paris. Email address: lionel.de.boisdeffre@wanadoo.fr

0

1 Introduction

This paper demonstrates the generic existence of equilibrium in incomplete financial markets with differential information. It presents a two-period pure exchange economy, with an ex ante uncertainty over the state of nature to be revealed at the second period.

When assets pay off in goods, equilibrium needs not exist, as shown by Hart (1975). His example is based on the collapse of the span of assets' payoffs, that occurs at clearing prices. Attempts to restore the existence of equilibrium noticed that the above "*bad*" prices could only occur exceptionally, as a consequence of Sard's theorem. These attempts include Mc Manus (1984), Repullo (1984), Magill & Shafer (1984, 1985), for potentially complete markets (i.e., complete for at least one price), and Duffie-Shafer (1985, 1986), for incomplete markets. These papers build on differential topology arguments, and demonstrate the generic existence of equilibrium, namely, existence except for a closed set of measure zero of economies, parametrized by the assets' payoffs and the agents' endowments.

The current model extends Duffie-Shafer's (1985) in two ways. First, its financial structure may cover any mix of nominal and real assets. Second, it normalizes (to arbitrary values) equilibrium prices on every spot market. In Duffie-Shafer (1985), the value of one particular consumer's endowment is normalized across all states of nature. The aim is to prove existence of equilibrium under the perfect foresight assumption. The relevance and the means of inferring correct prices are no issues.

In the current paper, however, normalizing price anticipations in every state of nature to relevant values is an important issue, because it is a step towards dropping the perfect foresight assumption, also called the rational expectation assumption.

1

This standard assumption, privileged by Radner (1972), states that agents know the map between the state of nature and the spot price to obtain. It is seen as unrealistic by most theorists, including Radner (1982) himself, for whom the assumption "*seems to require of the traders a capacity for imagination and computation far beyond what is realistic*".

Yet, to our best knowledge, no definition of a sequential equilibrium (as opposed to the temporary) was given so far, which dropped the assumption. The temporary equilibrium drops the assumption, but allows agents to form erroneous forecasts. When agents have no price model, they typically face endogenous uncertainty over future spot prices and need focus their anticipations on sets of relevant values, so that one of them be self-fulfilling ex post. The current paper is a step in this direction, which we develop in a companion paper.

The current proof uses standard differential topology arguments, introduced by Debreu (1970, 1972) for the study of general equilibrium. Following Duffie-Shafer (1985), we define so-called "*pseudo-equilibria*" and prove their full existence from modulo 2 degree theory. The generic existence of equilibria is then derived from Sard's theorem and Grassmannians' properties. Relative to Duffie-Shafer's, the proof provides additional arguments and insights. It refines the latter by exhibiting well behaved properties of the equilibrium prices.

The paper is organized as follows: Section 2 introduces the model and its concepts of equilibrium, pseudo-equilibrium, Grassmannians and their main properties. Section 3 presents the pseudo-equilibrium manifold and states and proves the existence theorems. An Appendix proves a technical Lemma.

2 The model

Throughout the paper, we consider a pure-exchange economy with two periods, $t \in \{0, 1\}$, and an uncertainty, at $t = 0$, upon which state of nature will randomly prevail, at $t = 1$. Consumers exchange goods, on spots markets, and assets of all kinds, on typically incomplete financial markets. The sets, I, S, L and J, respectively, of consumers, states of nature, consumption goods and assets are all finite. The state of the first period $(t = 0)$ is denoted by $s = 0$ and we let $S' := \{0\} \cup S$. Similarly, $l = 0$ denotes the unit of account and we let $L' := \{0\} \cup L$.

2.1 The commodity and financial markets

Agents consume or exchange the consumption goods, $l \in L$, on both periods' spot markets. Commodity prices on spot markets are restricted to the positive quadrant, $P := \{p := (p_s) \in \mathbb{R}_{++}^{L \times S'} : \|p_s\| = 1, \forall s \in S'\}$. Normalization to one is assumed for convenience but non restrictive. In any state, $s \in S'$, that bound could be replaced by any positive value without changing the model's results.

Consumers may operate transfers across states by exchanging, at $t = 0$, finitely many assets, $j \in J$ (with $\#J \leqslant \#S$), which pay off, at $t = 1$, conditionally on the realization of the anticipated states, $s \in S$. These conditional payoffs may be nominal or real or a mix of both. The generic payoffs of an asset, $j \in J$, in a state, $s \in S$, are thus a bundle, $v_j(s) := (v_j^l(s)) \in \mathbb{R}^{L'}$, of the quantities, $v_j^0(s)$, of cash, and $v_j^l(s)$, of each good $l \in L$, which are delivered if state s prevails.

These payoffs define a $(S \times L') \times J$ real matrix, V, identified to a map, $V : S \times \mathbb{R}_+^L \rightarrow \mathbb{R}^J$, relating the forecasts of a state and its spot price, $\omega := (s, p) \in S \times \mathbb{R}_+^L$, to the row of all assets' payoffs in cash, $V(\omega) \in \mathbb{R}^J$, delivered if both state s and price p obtain.

3

Thus, at asset price $q \in \mathbb{R}^J$, agents may buy or sell portfolios of assets, $z = (z_j) \in \mathbb{R}^J$, for $q \cdot z$ units of account at $t = 0$, against the promise of delivery of a flow, $V(\omega) \cdot z$, of conditional payoffs across forecasts, $\omega \in S \times \mathbb{R}_+^L$. For notational purposes, we let:

- $\mathbb{R}^{\Sigma \times J}$ be the set of all $\Sigma \times J$ matrices for $\Sigma = S \times L'$ or $\Sigma = S$;

- $V'(s) \in \mathbb{R}^J$, for every matrix, $V' \in \mathbb{R}^{S \times J}$, and state, $s \in S$, be the matrix' s^{th} row;

- $V'(\omega) \in \mathbb{R}^J$, for every $V' \in \mathbb{R}^{(S \times L') \times J}$, and forecast, $\omega := (s, p) \in S \times \mathbb{R}_+^L$, be the matrix' row of cash payoffs if ω obtains;

- $V'_p \in \mathbb{R}^{S \times J}$, for every $V' \in \mathbb{R}^{(S \times L') \times J}$ and every $p := (p_s) \in P$, be defined by $V'_p(s) := V'(s, p_s)$, for every $s \in S$;

- $\mathcal{G} := \{ V' \in \mathbb{R}^{S \times J}, \ rank \ V' = \#J \}$ and $\mathcal{G}^* := \{ <V'> : \ V' \in \mathcal{G} \}$, in which $<V'>$ denotes the span in $\mathbb{R}^S$ of the matrix' columns.

- $p \mathbin{\square} x \in \mathbb{R}^S$, for every pair $(p, x) \in P \times \mathbb{R}^{L \times S'}$, be the vector, whose components are the scalar products, $p_s \cdot x_s$, for every $s \in S$.

- S be identified to $\#S$, L to $\#L$, J to $\#J$, I to $\#I$, whenever needed;

- $v^* := J.S.(L + 1) = \dim \mathbb{R}^{(S \times L') \times J}$;

- $v^{**} := (S - J).J$;

- $e^* := I.L.(S + 1)$;

- $l^* := (S + 1).(L - 1) = \dim P$.

2.2 The consumer's behaviour and concept of equilibrum

Each agent, $i \in I$, receives an endowment, $e_i := (e_{is})$, granting the commodity bundles, $e_{i0} \in \mathbb{R}_{++}^L$ at $t = 0$, and $e_{is} \in \mathbb{R}_{++}^L$, in each state, $s \in S$, if it prevails. Given

4

prices, $p := (p_s) \in P$, for commodities, and $q \in \mathbb{R}^J$, for securities, and given the endowment and payoff collections, $(e := (e_i), V) \in \mathbb{R}^{L \times S' \times I} \times \mathbb{R}^{(S \times L') \times J}$, the generic i^{th} agent's consumption set is $X := \mathbb{R}_{++}^{L \times S'}$, and her budget set is:

$$B_i(p, q, e_i, V) := \{ (x, z) \in X \times \mathbb{R}^J : p_0 \cdot (x_0 - e_{i0}) \leqslant -q \cdot z \text{ and } p \,\square\, (x - e_i) \leqslant V_p \, z \};$$

Each consumer, $i \in I$, has preferences represented by a utility function, $u_i : X \to \mathbb{R}$ and optimizes her consumption in the budget set. The above economy is denoted by $\mathcal{E}_{(e,V)} = \{(I, S, L, J), V, e := (e_i)_{i \in I}, (u_i)_{i \in I}\}$. Given $(e', V') \in X^I \times \mathbb{R}^{(S \times L') \times J}$, we define the economy $\mathcal{E}_{(e',V')} = \{(I, S, L, J), V', e', (u_i)_{i \in I}\}$ in the same way as above, and its equilibrium and pseudo-equilibrium concepts as follows:

Definition 1 *Given the endowments, $e' := (e'_i) \in X^I$, and payoff matrix, $V' \in \mathbb{R}^{(S \times L') \times J}$, a collection of prices, $(p, q) \in P \times \mathbb{R}^J$, and strategies, $(x_i, z_i) \in B_i(p, q, e'_i, V')$, for each $i \in I$, is an equilibrium of the economy, $\mathcal{E}_{(e',V')}$, if the following Conditions hold:*

(a) $\forall i \in I$, $x_i \in \arg\max u_i(x)$, for $(x, z) \in B_i(p, q, e'_i, V')$;

(b) $\sum_{i \in I} (x_i - e'_i) = 0$;

(c) $\sum_{i \in I} z_i = 0$.

The state prices, $\lambda = (\lambda_s) \in \mathbb{R}_{++}^S$, are said to support an equilibrium of the economy, $\mathcal{E}_{(e',V')}$, if the equilibrium prices, $(p, q) \in P \times \mathbb{R}^J$, meet the condition: $q = \sum_{s \in S} \lambda_s V'_p(s)$.

The economy is called standard if it meets the following conditions:

Assumption A1 *: $\forall i \in I$, u_i is C^∞;*

Assumption A2 *(Inada Conditions): $\forall (i, s, l, x := (x_s^l)) \in I \times S' \times L \times X$,*

$\partial u_i(x)/\partial x_s^l \in \mathbb{R}_{++}$, $\lim_{x_s^l \to 0} \partial u_i(x)/\partial x_s^l = \infty$ *(where $x_s^l \to 0$ stands for "x_s^l tends to zero while other components of x are fixed"), $\lim_{x_s^l \to \infty} \partial u_i(x)/\partial x_s^l = 0$ (where $x_s^l \to \infty$ stands for "x_s^l tends to infinity at other components of x fixed"), $\lim_{x_s^l \to 0} u_i(x) = 0$;*

Assumption A3 *(strict concavity): $\forall i \in I$, $h^T D^2 u_i(x) h < 0$, $\forall h \neq 0$, $Du_i(x) \cdot h = 0$;*

5

Definition 2 *Let $\lambda := (\lambda_s) \in \mathbb{R}_{++}^S$ be given. The collection of a scalar, $y \in \mathbb{R}_{++}$, prices, $p := (p_s) \in P$, a payoff matrix, $V' \in \mathbb{R}^{(S \times L') \times J}$, a vector space, $G \in \mathcal{G}^*$, consumptions, $x_i := (x_{is}) \in X$, and endowments, $e_i' := (e_{is}') \in X$, for each $i \in I$, is said to be a λ-pseudo-equilibrium of the economy, $\mathcal{E}_{(e', V')}$, if the following conditions hold:*

(a) $x_1 \in \arg\max \ u_1(x),$ *for* $x \in \{ \ x := (x_s) \in X \ : p_0 \cdot (x_0 - e_{10}') + \sum_{s \in S} \ \lambda_s \ p_s \cdot (x_s - e_{1s}') = 0 \ \};$

(b) *for every* $i \in I \backslash \{1\},$ $x_i \in \arg\max \ u_i(x),$

for $x \in \{ \ x := (x_s) \in X \ : \ p_0 \cdot (x_0 - e_{i0}') + \sum_{s \in S} \ \lambda_s \ p_s \cdot (x_s - e_{is}') \ \ and \ \ p \ \square \ (x - e_i') \in G \ \};$

(c) $< V_p' > \subset G;$

(d) $\sum_{i \in I} \ (x_i - e_i') = 0;$

(e) $p_0 \cdot e_{10}' + \sum_{s \in S} \ \lambda_s \ p_s \cdot e_{1s}' = y.$

Given $(e', V') \in X^I \times \mathbb{R}^{(S \times L') \times J}$, *we say that* $(y, p, G) \in \mathbb{R}_{++} \times P \times \mathcal{G}^*$ *is a λ-pseudo-equilibrium, if there exists $x \in X^I$, such that (x, y, p, G, e', V') is a λ-pseudo-equilibrium along Conditions (a) to (e), above. We let $\mathcal{E}_\lambda^*$ be the pseudo-equilibrium manifold, or the set of collections, (y, p, G, e', V'), such that (y, p, G) is a λ-pseudo-equilibrium, given (e', V'). We define a projection map, $\pi^\lambda : (y, p, G, e', V') \in \mathcal{E}_\lambda^* \mapsto (e', V') \in X^I \times \mathbb{R}^{(S \times L') \times J}$.*

Remark 1 We chose to define pseudo-equilibria and equilibria with reference to financial structures mixing both nominal and real assets. This is no restriction. All arguments and results of this paper hold if assets pay off in goods or cash only.

Claim 1 *Let* $\lambda := (\lambda_s) \in \mathbb{R}_{++}^S$ *be given and* $(x := (x_i), y, p, G, e' := (e_i'), V')$ *be a λ-pseudo-equilibrium of a standard economy, $\mathcal{E}_{(e', V')}$, such that $< V_p' > = G$. Then, the economy, $\mathcal{E}_{(e', V')}$, has an equilibrium, $(p, q, [(x_i, z_i)])$, supported by the state prices, λ.*

Proof Let $\lambda := (\lambda_s) \in \mathbb{R}_{++}^S$ be given and let (x, y, p, G, e', V') be a λ-pseudo-equilibrium such that $< V_p' > = G$. From Condition (b) of Definition 2, there exists $z_i \in \mathbb{R}^J$, for each $i \in I \backslash \{1\}$, such that $p \ \square \ (x_i - e_i') = V_p' z_i$. Let $z_1 := -\sum_{i \in I \backslash \{1\}} z_i$. Then, $\sum_{i \in I} z_i = 0$ holds by construction. Moreover, Condition (d) of Definition 2 implies,

6

from above: $p \,\square\, (x_1 - e'_1) = -(\sum_{i \in I \setminus \{1\}} p_s \cdot (x_{is} - e'_{is}))_{s \in S} = -\sum_{i \in I \setminus \{1\}} V'_p z_i = V'_p z_1.$

Let $z := (z_i) \in \mathbb{R}^{J \times I}$, $q := \sum_{s \in S} \lambda_s V'_p(s)$ and $\mathcal{C} := (p, q, x, z)$ be given from above. From Definition 2 and above the relations $(x_i, z_i) \in B_i(p, q, e'_i, V')$ hold, for every $i \in I$, and the collection, $\mathcal{C} := (p, q, x, z)$, meets Conditions (b)-(c) of Definition 1 of equilibrium.

Let $i \in I \setminus \{1\}$ be given. From Assumption $A2$, the budget set $B_i(p, q, e'_i, V')$ can be replaced by $B'_i(p, q, e'_i, V') := \{(x, z) \in X \times \mathbb{R}^J : p_0 \cdot (x_0 \text{-} e'_{i0}) = -q \cdot z \text{ and } p \,\square\, (x\text{-}e'_i) = V'_p z\}$ in Definition 1 at no cost. From the definition of q, the pseudo-equilibrium budget set, $B'_i := \{x \in X \;:\; p_0 \cdot (x_0 - e'_{i0}) + \sum_{s \in S} \lambda_s p_s \cdot (x_s - e'_{is}) = 0 \;\; and \;\; p \,\square\, (x\text{-}e'_i) \in G\}$, coincides with $B^*_i := \{ x \in X : \exists z \in \mathbb{R}^J, (x, z) \in B'_i(p, q, e'_i, V') \}$. Since x_i is optimal in $B'_i = B^*_i$, the strategy (x_i, z_i) is optimal in $B_i(p, q, e'_i, V')$ from above. We show similarly that (x_1, z_1) is optimal in $B_1(p, q, e'_1, V')$. Then, $\mathcal{C}$ also meets Condition (a) of Definition 1, and, from above, defines an equilibrium of the economy $\mathcal{E}_{(e', V')}$. $\qquad\square$

2.3 A characterization of the set $\mathcal{G}^*$

The set, $\mathcal{G}^*$, of $\#J$-dimensional subspaces of $\mathbb{R}^S$, is referred to as a Grassmannian. To present the topological properties of the Grassmannian, we need introduce a distance between its elements. A traditional approach to this problem builds on the concept of principal angles between the elements of $\mathcal{G}^*$, along Definition 3:

Definition 3 *The principal angles, $\theta_j \in [0, \pi/2]$, for $j \in \{1, ..., \#J\}$, between two vector spaces, $(G, G') \in \mathcal{G}^{*2}$, are defined by $\#J$ pairs of vectors, (u_j, v_j), for $j \in \{1, ..., \#J\}$, solving the problems $\cos\theta_j := u_j \cdot v_j = \max_{(u,v) \in G \times G'} u \cdot v$, subject to $\|u\| = \|v\| = 1$, and $u \cdot u_{j'} = v \cdot v_{j'} = 0$ for each $j' \in \{1, ..., j - 1\}$. They define a distance, $d : (G, G') \in \mathcal{G}^{*2} \mapsto d(G, G') = \sqrt{\sum_{j \in J} \sin^2\theta_j}$, and a related topology, τ, on $\mathcal{G}^*$, referred to throughout.*

To check that d is, indeed, a distance, the reader may refer to Shor-Sloane (1998),

7

or Calderbank and alii (1999). A distance had to be defined on $\mathcal{G}^*$, which is no Euclidean space, before characterizing this set, along the following Claim 2.

We define $\mathcal{Z} := \{ W \in \mathbb{R}^{(S-J) \times S} : \textit{the rows of matrix } W \textit{ form an orthonormal set } \}$ and $\mathcal{Z}^* := \{ W \in \mathbb{R}^{S \times J} : \exists W' \in \mathcal{Z}, \textit{ such that } W'.W = 0, \textit{ and the columns of } W \textit{ are orthonormal} \}$. We can characterize the Grassmannian manifold owing to the latter vector spaces:

Claim 2 *Let G be a sub-vector space of $\mathbb{R}^S$. The following Assertions hold:*

(i) $(G \in \mathcal{G}^*) \Leftrightarrow (\exists W \in \mathcal{Z} : G = \{ z \in \mathbb{R}^{\mathcal{S}} : Wz = 0 \});$

(ii) $\mathcal{G}^* = \{ <W> : W \in \mathcal{Z}^* \};$

(iii) $\mathcal{G}^*$ *is compact.*

Proof Assertion *(i)* Let $W \in \mathcal{Z}$ and $G = \{ z \in \mathbb{R}^{\mathcal{S}} : Wz = 0 \}$ be given. Since all rows of W are orthonormal and their number is $S-J$, G is a J-dimensional sub-vector space of $\mathbb{R}^S$. That is, $G := \{ z \in \mathbb{R}^{\mathcal{S}} : Wz = 0 \} \in \mathcal{G}^*$, whenever $W \in \mathcal{Z}$. Conversely, let $G \in \mathcal{G}^*$ be given. Since G is a J-dimensional vector space, we may construct a $(S-J) \times S$ matrix W, whose rows form an orthonormal set and such that $G = \{ z \in \mathbb{R}^{\underline{\mathbf{S}}} : Wz = 0 \}$. $\square$

Assertion *(ii)* stems tautologically from Assertion *(i)*. First, let $W \in \mathcal{Z}^*$ be given. There exists $W' \in \mathcal{Z}$, such that $W'.W = 0$. Then, each column of W belongs to $G := \{ z \in \mathbb{R}^{\mathcal{S}} : W'z = 0 \}$, which implies, from Assertion *(i)*: $<W> \subset G \in \mathcal{G}^*$. Hence, the inclusion $\{ <W> : W \in \mathcal{Z}^* \} \subset \mathcal{G}^*$ holds.

Conversely, let $G \in \mathcal{G}^*$ be given. From Assertion *(i)*, there exists $W' \in \mathcal{Z}$, such that $G = \{ z \in \mathbb{R}^{\mathcal{S}} : W'z = 0 \}$. Since G is J-dimensional, there exists a set, $\{W_j\}$, of J orthonormal vectors of G. Then, the matrix, W, whose columns are the vectors W_j (for $j \in J$) belongs to $\mathcal{Z}^*$ and is such that $<W> = G$. Hence, $\mathcal{G}^* \subset \{ <W> : W \in \mathcal{Z}^* \}$. $\square$

Assetion *(iii)* Let $\{G^k\}_{k \in \mathbb{N}}$ be a given sequence of elements of $\mathcal{G}^*$. From Assertion

(ii), there exists a representing sequence, $\{W^k\}_{k \in \mathbb{N}}$, of elements of $\mathcal{Z}^*$, i.e., such that $< W^k > = G^k$, for all $k \in \mathbb{N}$. Since $\mathcal{Z}^*$ is compact in an Euclidean space, we may assume that $\{W^k\}_{k \in \mathbb{N}}$ converges, say to $W \in \mathcal{Z}^*$, for the Euclidean norm. From Assertion (ii), the relation $G := < W > \in \mathcal{G}^*$ holds, whereas, from Calderbank and al. (1999, p. 130), $d(G^k, G) = \|W^{k.T}W^k - W.^TW\| / \sqrt{2}$ holds for all $k \in \mathbb{N}$. Hence, from above, $\lim_{k \to \infty} d(G^k, G) = 0$, that is, $\mathcal{G}^*$ is compact for the topology τ. $\qquad \square$

2.4 The other main properties of the set $\mathcal{G}^*$

Let Σ be the set of permutations between states, $s \in S$. For every $\sigma \in \Sigma$, we let $P_\sigma \in \mathbb{R}^{S \times S}$ be the corresponding permutation matrix. That is, for every $V' \in \mathbb{R}^{S \times J}$, $P_\sigma.V' \in \mathbb{R}^{S \times J}$ is obtained by permuting the matrix' rows along σ. From the definition of $\mathcal{G}$, for every $V' \in \mathcal{G}$, there exists $\sigma \in \Sigma$, which needs not be unique, such that the last J rows of $P_\sigma.V'$ are linearly independent. Thus, for each $\sigma \in \Sigma$, we let:

$$\mathcal{G}_\sigma := \{V' \in \mathbb{R}^{S \times J} : P_\sigma.V' = \begin{pmatrix} W \\ V^* \end{pmatrix} \in \mathbb{R}^{S \times J}, \text{ with } W \in \mathbb{R}^{(S-J) \times J} \text{ and } V^* \in \mathbb{R}^{J \times J} \text{ invertible}\}$$

and $\mathcal{G}_\sigma^* := \{ < V' > : V' \in \mathcal{G}_\sigma \}$.

Given $\sigma \in \Sigma$, the generic vector space $G \in \mathcal{G}_\sigma^*$, admits, from above, a unique matrix representation of the form $P_\sigma^{-1}.\begin{pmatrix} -\Phi_\sigma(G) \\ I \end{pmatrix}$, where $\Phi_\sigma(G) \in \mathbb{R}^{(S-J) \times J}$ takes arbitrary values when G varies. We define the map, $\Psi_\sigma : G \in \mathcal{G}_\sigma^* \mapsto P_\sigma^{-1}.\begin{pmatrix} -\Phi_\sigma(G) \\ I \end{pmatrix}$, and let:

- $[\, I \mid \Phi_\sigma(G)\,]$ be the $(S-J) \times S$ matrix, whose first $(S-J)$ columns are those of the identiy matrix, $I \in \mathbb{R}^{(S-J) \times (S-J)}$, followed by the columns of $\Phi_\sigma(G) \in \mathbb{R}^{(S-J) \times J}$;

- $K_\sigma : P \times \mathcal{G}_\sigma^* \times \mathbb{R}^{(S \times L') \times J} \to \mathbb{R}^{(S-J) \times J}$ be the map defined by $K_\sigma(p, G, V') := [\, I \mid \Phi_\sigma(G)\,].P_\sigma.V_p'$.

9

Claim 3 *Let $\sigma \in \Sigma$ and $G \in \mathcal{G}_\sigma^*$ be given. The following Assertions hold:*

(i) $\{\mathcal{G}_\sigma\}_{\sigma \in \Sigma}$ *is an open cover of* $\mathcal{G}$;

(ii) $G = \{z \in \mathbb{R}^\mathcal{S} : [\, I \mid \Phi_\sigma(G) \,].P_\sigma\, z = 0\}$;

(iii) Φ_σ *is a homeomorphism;*

(iv) $\{\mathcal{G}_\sigma^*\}_{\sigma \in \Sigma}$ *is an open cover of* $\mathcal{G}^*$;

(v) $\mathcal{G}^*$ *is a manifold without boundary;*

(vi) *the map* $(p, V') \in P \times \mathcal{V} \to K_\sigma(p, G, V') \in \mathbb{R}^{(S-J) \times J}$ *is* C^∞;

(vii) *the sets* $\operatorname{Im} K_\sigma$, $\mathcal{G}_\sigma$, $\mathcal{G}_\sigma^*$ *and* $\mathcal{G}^*$ *are manifolds of dimension* $v^{**} := (S - J).J$; *the derivative* $D_{V'}\, K_\sigma(p, G, V')$ *has full rank,* v^{**}.

Proof We set $\sigma \in \Sigma$ as given and, to simplify, we will assume w.l.o.g. that $\sigma = Id$, unless stated otherwise. Assertion (i) results from the definitions. $\qquad\square$

Assertion (ii) Let $G \in \mathcal{G}_{Id}^*$ be given. The relation $[\, I \mid \Phi_{Id}(G) \,] \begin{pmatrix} -\Phi_{Id}(G) \\ I \end{pmatrix} = 0$ holds from the definition. Let $z \in G$ be given. From above, there exists $z' \in \mathbb{R}^J$, such that $z = \begin{pmatrix} -\Phi_{Id}(G) \\ I \end{pmatrix} z'$. Hence, the relation $[\, I \mid \Phi_{Id}(G) \,]\, z = [\, I \mid \Phi_{Id}(G) \,] \begin{pmatrix} -\Phi_{Id}(G) \\ I \end{pmatrix} z' = 0$ holds and the relation $G \subset \{z \in \mathbb{R}^\mathcal{S} : [\, I \mid \Phi_{Id}(G) \,]\, z = 0\}$ follows.

Conversely, let $z := \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in \mathbb{R}^S$ be such that $[\, I \mid \Phi_{Id}(G) \,]\, z = 0$, where $z_1 \in \mathbb{R}^{S-J}$ and $z_2 \in \mathbb{R}^J$. From the above definitions, the relation $[\, I \mid \Phi_{Id}(G) \,]\, z = 0$ is written $z_1 = -\Phi_{Id}(G)\, z_2$, that is, $z = \begin{pmatrix} -\Phi_{Id}(G) \\ I \end{pmatrix} z_2 \in <\begin{pmatrix} -\Phi_{Id}(G) \\ I \end{pmatrix}> = G$. The converse relation, $\{z \in \mathbb{R}^S : [I|\Phi_{Id}(G)]\, z = 0\} \subset G$, follows, and Assertion (ii) holds fom above. $\square$

Assertion (iii) Assume w.l.o.g. that $\sigma = Id$. From the above definitions, the map $\Psi_{Id} : G \in \mathcal{G}_{Id}^* \mapsto \Psi_{Id}(G) = \begin{pmatrix} -\Phi_{Id}(G) \\ I \end{pmatrix} \in \mathcal{G}_{Id}$ is one-to-one and onto, and so is Φ_{Id}.

We show it is bicontinuous for the two topologies. The continuity of Ψ_{Id} is obvious from Definition 3. To show that Ψ_{Id}^{-1} is continous we let $W = \Psi_{Id}(G)$ and $G \in \mathcal{G}_{Id}^*$ be given. As an immediate corollary of Claim 2, there exists a neighbourhood, U, of W in $\Psi_{Id}(\mathcal{G}_{Id}^*)$, a scalar, $K > 0$, and a map, $\overline{W} \in U \mapsto V_{\overline{W}} \in \mathbb{R}^{J \times J}$, such that, $V_{\overline{W}}$ is invertible, $\Lambda(\overline{W}) := \overline{W}.V_{\overline{W}} \in \mathcal{Z}^*$ and $\|V_{\overline{W}}\| < K$, for every $\overline{W} \in U$. It follows that the map, $\Lambda : U \to \mathcal{Z}^*$, is ($K$-Lipschitzian) continuous. Then, the continuity of Ψ_{Id}^{-1} at W stems from the following relations:

$$d(\Psi_{Id}^{-1}(\overline{W}), \Psi_{Id}^{-1}(W)) = \|\Lambda(\overline{W}).^T\Lambda(\overline{W}) - \Lambda(W).^T\Lambda(W)\| / \sqrt{2},$$

which hold, for every $\overline{W} \in U$, from Calderbank and alii (1999, p. 130).

Indeed, let $W^k \in U$, for every $k \in \mathbb{N}$, be such that $\lim_{k \to \infty} \|W - W^k\| = 0$. The Calderbank relations and the continuity of Λ imply: $\lim_{k \to \infty} d(\Psi_{Id}^{-1}(W^k), \Psi_{Id}^{-1}(W)) = 0$. That is Ψ_{Id}^{-1} is continuous at $W \in \mathcal{G}_{Id}$, hence, continuous. □

Assertions (iv) and (v) result from Assertions (i)-(iii) and the definition of $\mathcal{G}^*$. □

Assertion (vi) results from the definition of K_σ. □

Assertion (vii) Let $\sigma \in \Sigma$ be given. From Assertion (iii), $\mathcal{G}_\sigma^*$ is homeomorphic to:

$$\{V' = P_\sigma^{-1}.\begin{pmatrix} W \\ I \end{pmatrix}, W \in \mathbb{R}^{(S-J) \times J}\}, \text{ whose dimension is } v^{**} := (S - J).J.$$

Hence, from the definitions, Assertions $(i) - (iv)$ and above, $\mathcal{G}^*$, $\mathcal{G}_\sigma$, $\mathcal{G}$ and $\operatorname{Im} K_\sigma$ are all manifolds of the same dimension, v^{**}.

Let $\mathcal{J}$ be the set of last J states. To check that $D_{V'}K_{Id}(p, G, V')$ has full rank (hence, also $D_{V'}K_\sigma(p, G, V')$ for $\sigma \in \Sigma$), for every $(p, G, V') \in P \times \mathcal{G}_\sigma^* \times \mathcal{V}$, we write:

$$K_{Id}(p, G, V') := [\, I \mid \Phi_{Id}(G)].V_p' = V_p'(S \backslash \mathcal{J}) + \Phi_{Id}(G).V_p'(\mathcal{J}),$$

11

where $V_p'(S \backslash \mathcal{J})$ is the $(S-J) \times J$ matrix, whose rows are the top $(S-J)$ rows of V_p' and $V_p'(\mathcal{J})$ is the $J \times J$ matrix, whose rows are the last J rows of V_p'.

The derivatives of $K_{Id}(p, G, V')$ with respect to payoffs, for $s \in S \backslash \mathcal{J}$, are of the form of a $(S-J) \times (S-J)$ block diagonal matrix, P, of diagonal elements:

the $J \times J(L+1)$ matrices $P(s) = \begin{pmatrix} (1, p_s) & 0 & 0 & .. & 0 \\ 0 & (1, p_s) & 0 & .. & 0 \\ : & : & & & : \\ 0 & 0 & 0 & .. & (1, p_s) \end{pmatrix}$, for every $s \in S \backslash \mathcal{J}$.

The matrix P, therefore, has rank $(S-J).J$. It follows from above that the derivative $D_{V'} K_{Id}(p, G, V')$ has maximal rank, $v^{**} = (S-J)J$. $\qquad\square$

3 The pseudo-equilibrium manifold and existence theorems

This Section defines demand correspondences, characterizes the pseudo-equilibrium manifold, $\mathcal{E}_\lambda^*$, given $\lambda \in \mathbb{R}_{++}^S$, and states and proves the existence Theorems.

3.1 The demand and excess demand correspondences

We recall the notations of sub-Section 2.1 and let $\lambda = (\lambda_s) \in \mathbb{R}_{++}^S$ be given, throughout. For agent $i = 1$, we define her demand, $D_1^\lambda : \mathbb{R}_{++} \times P \to X$, by $D_1^\lambda(y, p) :=$ arg max $u_1(x)$, for $x \in \{ x \in X : p_0 \cdot e_{10}' + \sum_{s \in S} \lambda_s \, p_s \cdot e_{1s}' = y \}$. In the latter problem, $y > 0$ is taken as given. As classical results, in a standard economy, D_1^λ is a C^∞ map, such that, given y, $\lim_{p \to \bar{p}} \|D_1^\lambda(y, p)\| = +\infty$ whenever $\bar{p} \in \partial P \backslash \{0\}$.

The demand correspondence, $D_i^\lambda : P \times \mathcal{G}^* \times X \to X$, defined by $D_i^\lambda(p, G, e_i') :=$ arg max $u_i(x)$, for $x \in \{ x \in X_i : p_0 \cdot (x_0 - e_{i0}') + \sum_{s \in S} \lambda_s p_s \cdot (x_s - e_{is}') = 0 \ and \ p \,\square\, (x - e_i') \in G \}$, for each $i \in I \backslash \{1\}$, is also a C^∞ map, as a standard result.

Using Walras' law, we pick up one good, say $l = 1$. We recall that $\dim P = l^* :=$ $(S+1)(L-1)$. For every $i \in I$, and every consumption $x_i \in X$, we denote by $x_i^* :=$ $(x_{is}^*) \in \mathbb{R}_{++}^{l^*}$, the extracted vector, which drops consumptions in all goods $l = 1$. We denote similarly (with stars) the extracted demands in $\mathbb{R}^{l^*}$. Given $(y, p, G, e' := (e_i')) \in$ $\mathbb{R}_{++}^{l^*+1} \times \mathcal{G}^* \times \mathbb{R}_{++}^{e^*}$, the excess demand (in $\mathbb{R}^{l^*}$) is then:

$$Z^\lambda(y, p, G, e') := D_1^{\lambda*}(y, p) + \sum_{i \in I \setminus \{1\}} D_i^{\lambda*}(p, G, e_i') - \sum_{i \in I} e_i'^*.$$

It defines a demand correspondence, $Z^\lambda : \mathbb{R}_{++}^{l^*+1} \times \mathcal{G}^* \times \mathbb{R}_{++}^{e^*} \to \mathbb{R}^{l^*}$. It follows from above that Z^λ is a C^∞ map, whose (partial) derivative satisfies $D_{e_1^*} Z^\lambda(y, p, W, (e_i')) = -I$, where I is the $l^* \times l^*$ identity matrix. We notice from the limit property of D_1^λ that $\lim_{(y,p,G,e') \to (\overline{y}, \overline{p}, \overline{G}, \overline{e}')} \|Z^\lambda(y, p, G, e')\| = +\infty$ whenever $(\overline{y}, \overline{p}, \overline{G}, \overline{e}') \in \mathbb{R}_{++} \times \partial(\mathbb{R}_{++}^{l^*}) \setminus \{0\} \times \mathcal{G}^* \times \mathbb{R}_{++}^{e^*}$.

3.2 The pseudo-equilibrium manifold's characterization and properties

We consider the following mappings, for each $\sigma \in \Sigma$:

- $h^\lambda : (y, p, e_1') \in \mathbb{R}_{++}^{l^*+1} \times X_1 \mapsto h^\lambda(y, p, e_1') := (p_0 \cdot e_{10}' + \sum_{s \in S} \lambda_s \ p_s \cdot e_{1s}' - y) \in \mathbb{R}$;

- $K_\sigma : (p, G, V') \in P \times \mathcal{G}^* \times \mathbb{R}^{(S \times L') \times J} \mapsto [\ I \mid \Phi_\sigma(G)\ ].P_\sigma.V_p' \in \mathbb{R}^{v^{**}}$ (as defined above);

- $H_\sigma^\lambda : (y, p, G, e', V') \in \mathbb{R}_{++}^{l^*+1} \times \mathcal{G}^* \times \mathbb{R}_{++}^{e^*} \times \mathbb{R}^{(S \times L') \times J} \mapsto (h^\lambda(y, p, e_1'), Z^\lambda(y, p, G, e'), K_\sigma(p, G, V')) \in \mathbb{R}^{l^*+1+v^{**}}$.

From the definition and Claim 3, the pseudo-equilibrium manifold, $\mathcal{E}_\lambda^*$, is $\cup_{\sigma \in \Sigma} H_\sigma^\lambda(0)^{-1}$. The manifold's properties stem from those of H_σ^λ (for $\sigma \in \Sigma$), which, following Duffie-Shafer (1985), are summarized hereafter.

Claim 4 *Given $\sigma \in \Sigma$, the image 0 is a regular value of the map H_σ^λ, which is C^∞ with respect to the (y, p, e', V') derivatives.*

Proof Let $\sigma \in \Sigma$ be given. The fact that H_σ^λ is C^∞ is standard (see Duffie-Shafer, 1985, pp. 292-293). To show that 0 is regular, consider the derivative of H_σ^λ with respect to y, $e_1'^*$ and V':

$$
D_{(y,e_1'^*,V')} \, H_\sigma^\lambda(y,p,G,e',V') := \begin{pmatrix} D_y h^\lambda(y,p,e_1') = -1 & D_y Z^\lambda(y,p,G,e') & 0 \\ \\ 0 & D_{e_1^*} Z^\lambda(y,p,G,e') = -I & 0 \\ \\ 0 & 0 & D_{V'} K_\sigma(p,G,V') \end{pmatrix}.
$$

This matrix has full rank, $1 + l^* + v^{**}$, from the above Claim 3. Claim 4 follows. □

Claim 5 $\mathcal{E}_\lambda^*$ *is a submanifold of* $\mathbb{R}_{++}^{l^*+1} \times \mathcal{G}^* \times \mathbb{R}_{++}^{e^*} \times \mathbb{R}^{v^*}$ *without boundary of dimension* $e^* + v^*$. *Hence,* π^λ *is a map between manifolds of the same dimension.*

Proof From Claims 3 and 4 and the pre-image theorem, the λ-pseudo-equilibrium set, $\mathcal{E}_\lambda^* = \cup_{\sigma \in \Sigma} H_\sigma^\lambda(0)^{-1}$, is a submanifold (of $\mathbb{R}_{++}^{l^*+1} \times \mathcal{G}^* \times \mathbb{R}_{++}^{e^*} \times \mathbb{R}^{(S \times L') \times J}$) without boundary of dimension $(l^* + 1 + v^{**} + e^* + v^*) - (1 + l^* + v^{**}) = e^* + v^*$. □

Claim 6 *The map* $\pi^\lambda : \mathcal{E}_\lambda^* \to \mathbb{R}_{++}^{e^*} \times \mathbb{R}^{(S \times L') \times J}$ *is proper, that is, the inverse image by* π^λ *of a compact set is compact. Moreover, the set,* $\mathcal{R}_\lambda$, *of regular values of* π^λ *is open and of zero Lebesgue measure complement.*

Proof Let Y be a compact subset of $\mathbb{R}_{++}^{e^*} \times \mathbb{R}^{(S \times L') \times J}$ and let a sequence of elements of $\pi^\lambda(Y)^{-1}$, $\{\mathcal{C}^k := (y^k, p^k, G^k, (e_i^k), V^k)\}_{k \in \mathbb{N}}$, be given. Since Y is compact, we may assume that the sequence $\{(e_i^k), V^k)\}$ converges and denote $((e_i'), V') \in Y$ its limit. From the limit relation on Z^λ and the relation $\mathcal{E}_\lambda^* = \cup_{\sigma \in \Sigma} H_\sigma^\lambda(0)^{-1}$, we may assume that $\{(y^k, p^k)\}$ converges, say to $(y, p) \in \mathbb{R}_{++} \times P$. From Claim 2, the sequence $\{G^k\}$ may be assumed to converge, say to $G \in \mathcal{G}^*$.

Let $\mathcal{C} := (y, p, G, (e_i'), V') := (\lim_{k \to \infty} \mathcal{C}^k)$ be given. From Claims 3 and 4 and Definition 2, there exists $\sigma \in \Sigma$, such that the relations $H_\sigma^\lambda(\mathcal{C}^k) = 0$ hold, for $k \in \mathbb{N}$ big enough, and pass to the limit, which yields: $H_\sigma^\lambda(\mathcal{C}) = 0$. Hence, the sequence $\{\mathcal{C}^k\}$

14

converges to $\mathcal{C} \in \mathcal{E}^*$, which makes $\pi^\lambda(Y)^{-1}$ compact. Thus, π^λ is proper. Since π^λ is proper, its set of singular values, $\mathcal{R}_\lambda^c$, is closed, that is, $\mathcal{R}_\lambda$ is open. From Claim 5 and Sard's theorem (see Milnor, 1997, p. 10), $\mathcal{R}_\lambda^c$ is of zero Lebesgue measure. □

Lemma 1 *There exists a regular value, (e^*, V^*), of π^λ, such that $\#\pi^\lambda(e^*, V^*)^{-1} = 1$.*

Proof See the Appendix. □

3.3 The existence Theorems

We now state and prove the main existence results.

Theorem 1 *For every collection of endowments and payoffs, $(e', V') \in \mathbb{R}_{++}^{e^*} \times \mathbb{R}^{(S \times L') \times J}$, and every $\lambda \in \mathbb{R}_{++}^S$, a standard economy, $\mathcal{E}_{(e', V')}$, admits a λ-pseudo-equilibrium.*

Proof From Lemma 1, there exists at least one λ-pseudo-equilibrium, so the map π^λ is well defined. As standard from modulo 2 degree theory, if $f : X \to Y$ is a smooth proper map between two boundaryless manifolds of same dimension, with Y being connected, the number, $\#f^{-1}(y)$, of elements $x \in X$, such that $y = f(x)$, is the same, modulo 2, for every regular value $y \in Y$. In particular, if one regular value, y, of f, is such that $\#f^{-1}(y) = 1$, then, $f^{-1}(y)$ is non-empty for every $y \in Y$. Indeed, $y \in Y$ is regular by definition if $f^{-1}(y) = \varnothing$. From Claims 2 to 6 and Lemma 1, the map, π^λ, meets all above conditions for $X := \mathcal{E}_\lambda^*$ and $Y := \mathbb{R}_{++}^{e^*} \times \mathbb{R}^{(S \times L') \times J}$. Hence, for every $(e', V') \in \mathbb{R}_{++}^{e^*} \times \mathbb{R}^{(S \times L') \times J}$, a standard economy, $\mathcal{E}_{(e', V')}$, has a λ-pseudo-equilibrium. □

Theorem 2 *For every $\lambda = (\lambda_s) \in \mathbb{R}_{++}^S$, there exists an open set of null comple-ment, $\Omega \subset \mathcal{R}_\lambda$ (along Claim 6), such that, for every $(e', V') \in \Omega$, the state prices λ support an equilibrium of the standard economy, $\mathcal{E}_{(e', V')}$.*

Proof Let $\lambda = (\lambda_s) \in \mathbb{R}_{++}^S$ and $(e^*, V^*) \in \mathcal{R}_\lambda$ be given, a non-empty set from Lemma 1. From Claims 3 and 4, Theorem 1, the implicit function theorem and the definition of a regular value, there exists a pseudo-equilibrium, $\mathcal{C}^* := (y^*, p^*, G^*, e^*, V^*) \in \pi^\lambda(e^*, V^*)^{-1}$, and two open sets, $W \subset \mathcal{E}_\lambda^*$ and $U \subset \mathcal{R}_\lambda$, containing $\mathcal{C}^*$ and (e^*, V^*), respectively, which are mapped homeomorphically by $\pi_{/W}^\lambda$.

We assume w.l.o.g. that $G' \in \mathcal{G}_{Id}^*$ whenever $\mathcal{C}' := (y', p', G', e', V') \in W$. The price map, $(e', V') \in U \mapsto f_1(e', V') \in P$, and the map, $(e', V') \in U \mapsto f_2(e', V') \in \mathcal{G}_{Id}^*$, defined by $(f_1(e', V')\cdot e_1', f_1(e', V'), f_2(e', V'), e', V') \in W$, for $(e', V') \in U$, are continuous from above.

The map $\theta : (y', p', G', e', V') \in W \mapsto (p', \Phi_{Id}(G'), e', V')$ is a homeomorphism from Claim 3 and we let $W^\circ := \theta(W)$ be its image. Following Duffie-Shafer (1985), we let:

$$H^\lambda : \mathcal{C}' := (p', E', e', V') \in W^\circ \mapsto (Z^\lambda(p_0'\cdot e_{10}' + \sum_{s \in S} \lambda_s\, p_s'\cdot e_{1s}', p',\ \Phi_{Id}^{-1}(E'),\ e'),\ K_{Id}(p', \Phi_{Id}^{-1}(E'), V')).$$

Differentiating the relation $H^\lambda(\mathcal{C}') = 0$, which holds from the definition for every $\mathcal{C}' \in W^\circ$, yields: $[D_{(p', E')} H^\lambda(\mathcal{C}')]\, [D\,(f_1, f_2)(e', V')] + D_{(e', V')} H^\lambda(\mathcal{C}') = 0$, for every $\mathcal{C}' \in W^\circ$.

The latter equation states that the rows of $D_{(e', V')} H^\lambda(\mathcal{C}')$ are linear combinations of those of $D\,(f_1, f_2)(e', V')$. From the proof of Claim 4, $D_{(e', V')} H^\lambda(\mathcal{C}')$ has full rank, $l^* + v^{**}$, and so does, from above, $D\,(f_1, f_2)(e', V')$. In particular, $rank\, Df_1(e', V') = l^*$ holds, for every $(e', V') \in U$. We now define the following maps and set:

$\Psi : (e', V') \in U \mapsto (f_1(e', V'), V') \in P \times \mathbb{R}^{(S \times L') \times J}$;

$\Theta : (p', V') \in P \times \mathbb{R}^{(S \times L') \times J} \mapsto V_{p'}' \in \mathbb{R}^{S \times J}$;

$Q := \Theta^{-1}(\mathcal{G})$.

The set $\mathcal{G}$ is (relatively) open, and of null complement from Sard's theorem. The derivatives $D\,\Psi$ and $D_{V'}\,\Theta$ clearly have maximal rank, respectively, $l^* + v^*$ and $S.J$,

so, the maps Ψ and Θ are submersions. Since Θ is a submersion and $\mathcal{G}$ is open and of null complement, so is $Q := \Theta^{-1}(\mathcal{G})$ in $P \times \mathbb{R}^{(S \times L') \times J}$. Let $\Psi(U)$ be the image set of U by Ψ. Then, $Q' := Q \cap \Psi(U)$ is open and of null complement in $\Psi(U)$, which is open.

By the same token, $\Omega_U := \Psi^{-1}(Q')$ is open and of null complement in U. From the above definitions, Theorem 1 and Claim 1, for all $(e', V') \in \Omega_U$, a standard economy, $\mathcal{E}_{(e', V')}$, admits a λ-pseudo-equilibrium, which yields an equilibrium supported by λ.

Applying a classical local to global argument, there exists an open subset, Ω, of $\mathcal{R}_\lambda$, with null complement in $\mathcal{R}_\lambda$, hence, in $\mathbb{R}_{++}^{e^*} \times \mathbb{R}^{(S \times L') \times J}$, such that, for every $(e', V') \in \Omega$, a standard economy, $\mathcal{E}_{(e', V')}$, admits an equilibrium supported by λ. $\square$

Moreover, from Lemma 1, for all $(e', V') \in \Omega$, the number of equilibria in $\pi^\lambda(e', V')^{-1}$ is odd, and each of these equilibria is, from above, a continuous function of (e', V').$\square$

Appendix

Lemma 1 *There exists a regular value, (e^*, V^*), of π^λ, such that $\#\pi^\lambda(e^*, V^*)^{-1} = 1$.*

Proof   We may choose a price, $p^* := (p_s^*) \in P$, and a matrix, $V^* \in \mathbb{R}^{(S \times L') \times J}$, such that the last J rows of $V_{p^*}^*$ form the identity matrix, $I \in \mathbb{R}^{J \times J}$. We let $G^* := < V_{p^*}^* >$. From Assumption $A2$, we may choose endowments such that $\nabla u_i(e_i^*) = p^*$ holds for each $i \in I$. Then, in a standard economy, $(p^*, G^*, (e_i^*), V^*)$ defines a tradeless pure spot market equilibrium by construction, whose allocation, (e_i^*), is Pareto optimal. For any $\lambda \in \mathbb{R}_{++}^S$, this equilibrium also defines a λ-pseudo-equilibrium.

Let (x_i) be a λ-pseudo-equilibrium allocation. From the definition, the relation $u_i(x_i) \geqslant u_i(e_i^*)$ holds for every $i \in I$. Assume, by contraposition, that $(x_i) \neq (e_i^*)$. Then,

17

from Assumption $A3$ and above, for $n \in \mathbb{N}$ large enough, $([\frac{x_i}{n} + \frac{(n-1)e_i^*}{n}])$ is attainable and Pareto dominates (e_i^*), in contradiction with above. Hence, $\#\pi^\lambda((e_i^*), V^*)^{-1} = 1.\square$

The last part of the proof of Lemma 1 is to show that $((e_i^*), V^*)$ is a regular value of π^λ. This is equivalent to showing that the derivative, $D_{(y,p,E)} H_{Id}^\lambda(y^*, p^*, \Phi_{Id}^{-1}(E^*), (e_i^*), V^*)$, has full rank, $l^* + 1 + v^{**}$, where - following Duffie-Shafer (1985, p. 296) - we let $E^* := \Phi_{Id}(G^*)$ and $E := \Phi_{Id}(G)$, for every $G \in \mathcal{G}_{Id}^*$, and consider the maps:

$h^*(y, p) := p_0 \cdot e_{10}^* + \sum_{s \in S} \lambda_s \, p_s \cdot e_{1s}^* - y;$

$Z^*(p, E) = Z^\lambda(y^*, p, \Phi_{Id}^{-1}(E), e^*);$

$K^*(p, E) = K_{Id}^\lambda(y^*, p, \Phi_{Id}^{-1}(E), V^*);$

$H^*(y, p, E) = (\, h^*(y, p), \,\, Z^*(p, E), \,\, K^*(p, E) \,);$

$$
D\, H^*(y^*, p^*, E^*) := \begin{pmatrix} D_y \, h^*(y^*, p^*) = -1 & D_p \, h^*(y^*, p^*) & 0 \\ \\ 0 & D_p \, Z^*(p^*, E^*) & D_E \, Z^*(p^*, E^*) \\ \\ 0 & D_p \, K^*(p^*, E^*) & D_E \, K^*(p^*, E^*) \end{pmatrix}.
$$

We show $D\, H^*(y^*, p^*, E^*)$, hence, $D_{(y,p,E)} H_{Id}^\lambda(y^*, p^*, \Phi_{Id}^{-1}(E^*), (e_i^*), V^*)$ have full rank. The arguments are similar to Duffie-Shafer's (1985) and recalled for completeness.

We show, first, that $rank \, D_E \, K^*(p^*, E^*) = v^{**}$. Denoting by $\mathcal{J}$ the set of last J states, we recall that $V_{p^*}^*(\mathcal{J}) = I \in \mathbb{R}^{J \times J}$ is the identity matrix. The derivative of K^* with respect to E at the $S - J$ rows of E^* is the $(S-J) \times (S-J)$ block diagonal matrix, P, of common block diagonal element $I \in \mathbb{R}^{J \times J}$. The rank of P is, therefore, $(S-J)J$. Then, from above, the matrix $D_E \, K^*(p^*, E^*)$ has maximal rank, $v^{**} = (S - J)J$. $\square$

Second, we show that $D_E \, Z^*(p^*, E^*) = 0$. Let $E \in \mathbb{R}^{(S-J) \times J}$ be given. By construction, the gradient's condition, $\nabla u_i(e_i^*) = p^* \in \mathbb{R}_{++}^{L \times S}$, holds for each $i \in I$, making (e_i^*) Pareto optimal. Since affordable to any agent and optimal, (e_i^*) is the demand allocation. Hence, $Z^\lambda(y^*, p^*, \Phi_{Id}^{-1}(E), (e_i^*)) = 0$, for all $E \in \mathbb{R}^{(S-J) \times J}$, and $D_E \, Z^*(p^*, E^*) = 0.\square$

18

The proof that $D_p Z^*(p^*, E^*)$ is non-singular, as a last step, relies on Lemmata 1:

Lemmata 1 *Let $(p, G, i) \in P \times \mathcal{G}^* \times I\backslash\{1\}$ be given. The following Assertions hold:*

(i) $p^T D_p(D_1(y^*, p)) = -D_1(y^*, p)^T$;

(ii) $h^T D_p(D_1(y^*, p)) \, h < 0, \; \forall h \in \mathbb{R}^{L \times S'}\backslash\{0\}$, *such that* $h \cdot D_1(y^*, p) = 0$;

(iii) $p^{*T} D_p(D_i(p^*, G, e_i^*)) = 0$;

(iv) $D_p \left(D_i(p^*, G, e_i^*)\right)$ *is negative semi-definite.*

Proof of Lemmata 1 Assertion *(i)* results from differentiating $p \cdot D_1(y^*, p) = y^*$. □

Assertion *(ii)* From first order conditions, the first agent's gradient and price $p \in P$ are colinear at $D_1(y^*, p)$. Then, Assertion *(ii)* is standard from Assumption *A3*. □

Assertion *(iii)* The satiated budget constraint at the ith agent's demand is written: $p \cdot D_i(p, G, e_i^*) = p \cdot e_i^* = p \cdot D_i(p^*, G, e_i^*)$. Differentiating this at $p = p^*$ yields Assertion *(iii)*. □

Assertion *(iv)* Given $(p, G) \in P \times \mathcal{G}^*$, the relation $D_i(p^*, G, e_i^*) = e_i^*$ holds from above. The relations $p^* \cdot D_i(p, G, e_i^*) \geqslant p^* \cdot e_i^*$ and $p \cdot D_i(p, G, e_i^*) = p \cdot e_i^*$ hold from the definitions. Then, $(p - p^*) \cdot (D_i(p, G, e_i^*) - D_i(p^*, G, e_i^*)) \leqslant 0$ holds for every $(p, G) \in P \times \mathcal{G}^*$. So, the map $D_i(p, G, e_i^*)$ is non-increasing in $p \in P$ and C^∞ and Assertion *(iv)* holds. □

We can now complete the proof of Lemma 1. Assume, by contraposition, that $D_p Z^*(p^*, G^*) \, h = 0$ holds for some $h \neq 0$. The vector h belongs to $\mathbb{R}^{l^*} := \mathbb{R}^{(L-1)S'}$, from the definition of the model's excess demand, Z^λ, hence, of Z^*. We let h° be the vector of $\mathbb{R}^{LS'}$, which coincides with h on $\mathbb{R}^{l^*}$ and whose other components (in good $l = 1$) are all zeros. Then, it follows from the definition of Z^* and above that:

$$0 = D_p Z^*(p^*, G^*) \, h = D_p(D_1^*(y^*, p^*)) \, h + \sum_{i \in I\backslash\{1\}} D_p(D_i^*(p^*, G^*, e_i^*)) \, h \; (\text{for } h \text{ set above}).$$

And, from Lemmata 1 and above, that:

$$0 = p^{*T} D_p(D_1(y^*, p^*)) \ h^\circ + \sum_{i \in I \setminus \{1\}} p^{*T} D_p(D_i(p^*, G^*, e_i^*)) \ h^\circ = -D_1(y^*, p^*)^T \ h^\circ.$$

The above relation, $D_1(y^*, p^*)^T \ h^\circ = 0$ (for $h^\circ \neq 0$), implies, from Lemmata 1-(ii): $h^{\circ T} D_p(D_1(y^*, p)) \ h^\circ < 0$. Then, the relation $h^T D_p \ Z^*(p^*, G^*) \ h < 0$ holds from Lemmata 1-(iv) and above and contradicts the fact that $D_p \ Z^*(p^*, G^*) \ h = 0$, assumed above. This contradiction proves that $D_p Z^*(p^*, G^*)$ and, from above, $DH^*(y^*, p^*, G^*)$ have full rank, i.e., $((e_i^*), V^*)$ is a regular value of π^λ. This completes the proof of Lemma 1.□

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