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JEL codes:

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Henri Loubergé*, Yannick Malevergne[†] and Béatrice Rey[‡]

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1 Introduction

The work of Eeckhoudt and Schlesinger (2006) provided a refreshing perspective for the study of decisions under risk. Until the publication of their seminal article, and since Friedman and Savage (1948) at least, specific properties of the von Neumann-Morgenstern utility function were associated to specific traits of attitudes towards risk. A negative second derivative of the utility function reflected risk aversion (Friedman and Savage, 1948; Pratt, 1964). A positive third derivative reflected prudence (Kimball, 1990) and was associated to precautionary behavior. A negative fourth derivative was defined as temperance and invoked to explain the demand for risky assets in the presence of background risks (Kimball, 1992). In the same vein, a positive sign of the fifth derivative was more recently associated to edginess to explain the effects of background risk on precautionary saving (Lajeri-Chaherli, 2004). Similarly, a S-shaped utility function or a state-dependent utility function was used to rationalize simultaneous purchasing of insurance and lottery tickets, etc.

Instead of focusing *ab initio* on properties of the utility function, the two authors started from individual choices in simple equal-probability lotteries in order to define risk aversion, prudence, temperance, and so on, and they coined the term “risk apportionment” to describe how these behavioral traits were reflected in individual choices. For example, temperance reflects the preference for not associating an additional zero-mean risk ε_2 to a situation where the decision maker is already exposed to a prevailing zero-mean risk ε_1 . The 50-50 lottery $[\varepsilon_1, \varepsilon_2]$ is preferred to the 50-50 lottery $[0, \varepsilon_1 + \varepsilon_2]$. Preference for risk apportionment means preference for disaggregation of harms, given risk aversion. The beauty of this more primitive approach to attitudes towards risk is that it does not need familiarity with utility theory to be understood, although it is perfectly consistent with the traditional approach based on specific properties of the utility function. Preference for risk apportionment (or harms disaggregation) in successive increasingly complex lotteries translates into alternating signs of successive derivatives of the utility function (mixed risk aversion, as defined by Caballé and Pomansky, 1996).

The approach proposed by Eeckhoudt and Schlesinger (2006) using zero-mean risks received support in experimental work (Deck and Schlesinger, 2010, 2014, 2018; Attema et al., 2019). It was also generalized by Eeckhoudt et al. (2009b) to any couple of risks linked together by properties of increases in risk or stochastic dominance at any order.¹ Indeed, considering four mutually independent risks X_1, Y_1, X_2 and Y_2 such that Y_i dominates X_i by s_i^{th} -order stochastic dominance for $i = 1, 2$, an expected utility maximizer with a mixed risk averse (MRA) utility function up to order $s = s_1 + s_2$

¹Increases in risk were introduced by Rothshild and Stiglitz (1970) and generalized to any order by Ekern (1980). Stochastic dominance was introduced by Hadar and Russel (1969) and Hanoch and Levy (1969) and extended to any order by Ingersoll (1987). The generalization has shown that there is a correspondence between n^{th} degree stochastic dominance and the preference for a non dominated risk by an expected utility maximizer with a mixed risk averse utility function up to order n . An equivalent but less stronger relationship holds for increases in risk of order n (see Ekern, 1980).

prefers the 50-50 lottery $[X_1 + Y_2; Y_1 + X_2]$ to the 50-50 lottery $[X_1 + X_2; Y_1 + Y_2]$. In the former lottery risk apportionment holds. There is disaggregation of harms. In the latter lottery this is not the case. Instead of combining “good with bad” in the two possible lottery outcomes, the lottery yields “bad with bad” in the first outcome and “good with good” in the second outcome.

Eeckhoudt and Schlesinger (2006), as well as Eeckhoudt et al. (2009b), consider additive risks. In the above lotteries, the final outcomes are either $X_1 + Y_2$ and $Y_1 + X_2$ on one hand, or $X_1 + X_2$ and $Y_1 + Y_2$ on the other hand. Subsequent research addressed risk apportionment for multiplicative risks. The analysis is thus restricted to non-negative random variables. Multiplicative risks are observed in various circumstances in economic and social life. For example, investing in an asset denominated in foreign currency exposes the domestic investor to two multiplicative risks, the risk of the asset itself and the risk of variations in the domestic currency value of the foreign currency. Similarly, taking a job with a variable income in a firm exposed to bankruptcy results for the wage earner in a range of final outcomes where the two risks interact multiplicatively. Wang and Li (2010) addressed risk apportionment with multiplicative risks specifically. Building on results obtained by Eeckhoudt et al. (2009a) in a related context – see also Eeckhoudt and Schlesinger (2008) – they reach the conclusion that there is a direct link between multiplicative risk apportionment at order $n + 1$ and the value of n^{th} degree relative risk aversion.² A similar result was obtained by Chiu et al. (2012) using a model combining two additive risks with a multiplicative effect on the first risk, a n -degree shift of stochastic dominance on this risk, and a first-degree shift of stochastic dominance on the second risk. Again, the value of n^{th} degree relative risk aversion is critical to conclude whether risk apportionment in the sense of Eeckhoudt and Schlesinger (2006) is obtained or not.

This literature was soon superseded by work addressing risk apportionment in a bivariate context, a context where the decision-maker’s preferences are driven by the joint effects of two independent risks, for instance risks affecting wealth and health. This work is based on a paper by Denuit et al. (1999) defining mixed risk aversion for bivariate functions. In particular, Jokung (2011) and Denuit and Rey (2013) provided a new look on risk apportionment with additive or multiplicative risks by analyzing these cases as specific cases of risk apportionment in a bivariate context. Indeed, when the two attributes defined in the bivariate context are of the same nature (financial risks, for instance), they can be combined (additively or multiplicatively) to yield a single attribute of utility. This allows the application to the univariate context of the more general results derived in a bivariate context.

In this paper, we start with additive risks and we obtain a simple result for risk apportionment preferences of one decision-maker (DM) in different situations. When the DM displays a preference for risk apportionment when risk X_1 is dominated by

²Their definition of relative risk aversion at order $n + 1$ corresponds to what is generally defined now as relative risk aversion at order n (see Section 3 below).

risk Y_1 at order s_1 and risk X_2 dominated by risk Y_2 at order s_2 , then the same DM necessarily displays a preference for risk apportionment when two other couples of risks (X'_1, Y'_1) and (X'_2, Y'_2) are related by stochastic dominance orderings at orders s'_1 and s'_2 , $s_i \neq s'_i$ for $i = 1, 2$, if $s_1 + s_2 = s'_1 + s'_2$. This is our Proposition 1. Its contribution to the literature is to emphasize a correspondence between risky choices by one DM in two different circumstances involving risk combinations, as long as the sum of stochastic dominance orders linking the risk combinations in each circumstance is the same. Our main motivation is then to check whether this result holds when the two risks combine multiplicatively, instead of additively. We find that the answer is negative in general and we explain why. The answer is positive only in the case where one of the two couples of risks is related by a first-degree stochastic dominance (FSD) relationship, and provided relative risk aversion at order n is larger than n .

However, by turning to a specific case, the case where the DM's preferences are reflected in a Constant Relative Risk Aversion (CRRA) utility function – the function most commonly used in the literature – we obtain that our Proposition 1 is valid for multiplicative risks (Proposition 3). In addition, prior to obtaining this result, we also obtain in Proposition 2 that risk apportionment holds with multiplicative risks and a CRRA utility functions for all such functions where relative risk aversion exceeds one, independently of the stochastic dominance orders relating the two couples of variables X and Y . This much stronger result is driven by a property of CRRA utility functions that we also unveil. If relative risk aversion (at order 1) is larger than one, then relative risk aversion at order n is larger than n , and conversely (The result applies *mutatis mutandis* if relative risk aversion is less than one). This means that the above-mentioned results (Wang and Li, 2010; Chiu et al., 2012), dealing with multiplicative risk apportionment in a specific context, and pointing to the role of relative risk aversion at order n , are down-graded to the role of relative risk aversion at order 1, even in a general stochastic dominance context, if CRRA utility is considered.

Our paper is organized as follows. Section 2 presents the Eeckhoudt et al. (2009b) result on risk apportionment in a univariate additive risks context and derives our result of equivalence between risk apportionment at order $s_1 + s_2$ and risk apportionment at order $s'_1 + s'_2$ ($s_i \neq s'_i$ for $i = 1, 2$) when $s_1 + s_2 = s'_1 + s'_2$. Section 3 addresses risk apportionment in a bivariate and univariate multiplicative risks contexts and explains why our result from section 2 does not hold in the latter context, except in the specific cases $s_2 = 1$ and $s'_1 = 1$, or $s_1 = 1$ and $s'_2 = 1$, and provided relative risk aversion at order n is larger than n . Section 4 turns to the specific case of CRRA utility functions and shows that our result from Section 2 holds generally in a univariate multiplicative risks context, even when $s_1 + s_2 \neq s'_1 + s'_2$, provided relative risk aversion is larger than 1. This result derives from a property of CRRA utility functions so far only known for orders 2 and 3 and that we generalize to order n : if relative risk aversion (order 1) is larger/less than 1, relative risk aversion at order n is larger/less than n . Section 5 concludes briefly.

2 Additive risks and preserved preference ranking

Eeckhoudt et al. (2009b) show that mixed risk aversion (MRA), characterized by a utility function u such that $(-1)^{(1+k)}u^{(k)} \geq 0 \forall k$, means preference for *harms disaggregation* or – equivalently – preference to *combine good with bad and bad with good rather than good with good and bad with bad* where *bad* is defined as *stochastically dominated*. More formally, let us consider four mutually independent risks, X_1, X_2, Y_1, Y_2 such that Y_i dominates X_i by s_i^{th} order stochastic dominance for $i = 1, 2$ ($X_i \preceq_{s_i-SD} Y_i$ for $i = 1, 2$). Eeckhoudt et al. (2009b) show that a MRA DM from 1 to $s_1 + s_2$ (*i.e.* a DM with a utility function u such that $(-1)^{(1+k)}u^{(k)} \geq 0 \forall k = 1, \dots, s_1 + s_2$) will allocate the state-contingent risks in such a way as not to gather the two *bad* risks in the same state. Such an individual prefers the 50 – 50 lottery $[X_1 + Y_2, Y_1 + X_2]$ to the 50 – 50 lottery $[X_1 + X_2, Y_1 + Y_2]$:

$$\frac{1}{2}\mathbb{E}[u(X_1 + Y_2)] + \frac{1}{2}\mathbb{E}[u(Y_1 + X_2)] \geq \frac{1}{2}\mathbb{E}[u(X_1 + X_2)] + \frac{1}{2}\mathbb{E}[u(Y_1 + Y_2)]. \quad (1)$$

Equation (1) means that the DM prefers to disaggregate bad outcomes rather than aggregate them. In what follows, we denote this disaggregation preference relation (Eq. 1) by $\text{PD}(\{(X_1, Y_1); (X_2, Y_2)\}_{(s_1, s_2)})$.

Let us consider a given DM with a MRA utility function from 1 to 4. We consider a first set of four mutually independent risks X_1, X_2, Y_1, Y_2 such that $X_1 \preceq_{2-SD} Y_1$ and $X_2 \preceq_{2-SD} Y_2$, *i.e.* a first set of four mutually independent risks with a total order equal to 4 ($s_1 + s_2 = 2 + 2 = 4$). The EST (2009b) theorem states that $\text{PD}(\{(X_1, Y_1); (X_2, Y_2)\}_{(2,2)})$ holds for the given DM. Let us now consider a second set of four mutually independent risks X'_1, X'_2, Y'_1, Y'_2 such that $X'_1 \preceq_{s'_1-SD} Y'_1$ and $X'_2 \preceq_{s'_2-SD} Y'_2$ with $s'_1 = 1$ and $s'_2 = 3$, *i.e.* such that the total order is also equal to 4. For the same given DM and using Eeckhoudt et al. (2009b), we obtain that $\text{PD}(\{(X'_1, Y'_1); (X'_2, Y'_2)\}_{(1,3)})$ holds since $s'_1 + s'_2 = 4$ and since $u^{(s_1+s_2)} = u^{(s'_1+s'_2)}$ for all s_1, s_2, s'_1, s'_2 such that $s_1 + s_2 = s'_1 + s'_2$ with $s_1 \neq s'_1$ and $s_2 \neq s'_2$.

We can then derive the following result. Preference ranking is preserved when we replace the first set of risks by the second one and vice-versa. We understand that this result can be easily extended to the more general case $s_1 + s_2 = s'_1 + s'_2$ with $s_1 \neq s'_1$ and $s_2 \neq s'_2$. This is our first proposition.

Proposition 1. *Let X_1, X_2, Y_1, Y_2 be a first set of four mutually independent risks such that $X_1 \preceq_{s_1-SD} Y_1$ and $X_2 \preceq_{s_2-SD} Y_2$, and let X'_1, X'_2, Y'_1, Y'_2 be a second set of four mutually independent risks such that $X'_1 \preceq_{s'_1-SD} Y'_1$ and $X'_2 \preceq_{s'_2-SD} Y'_2$ with $s_1 + s_2 = s'_1 + s'_2 = s$, $s_1 \neq s'_1$ and $s_2 \neq s'_2$. For a given mixed risk-averse DM from 1 to s , the disaggregation preference relation is preserved when the first set of risk is replaced by the second set of risks and conversely.*

$$\text{More formally, } \text{PD}(\{(X_1, Y_1); (X_2, Y_2)\}_{(s_1, s_2)}) \iff \text{PD}(\{(X'_1, Y'_1); (X'_2, Y'_2)\}_{(s'_1, s'_2)}).$$

This result deals with additive risks. Our question is then the following: does the result still hold when risks are multiplicative? More generally, let us denote by $\text{PD}_M(\{(X_1, Y_1); (X_2, Y_2)\}_{(s_1, s_2)})$ the disaggregation preference relation when risks interact multiplicatively:

$$\frac{1}{2}\text{E}[u(X_1 \cdot Y_2)] + \frac{1}{2}\text{E}[u(Y_1 \cdot X_2)] \geq \frac{1}{2}\text{E}[u(X_1 \cdot X_2)] + \frac{1}{2}\text{E}[u(Y_1 \cdot Y_2)] , \quad (2)$$

i.e. the preference to disaggregate bad outcomes rather than to aggregate them when outcomes combine in a multiplicative form. Assume that $s'_1 + s'_2 = s_1 + s_2 = s$ with $s_1 \neq s'_1$ and $s_2 \neq s'_2$. Our question is then the following. Is preference ranking preserved when we replace $\{(X_1, Y_1); (X_2, Y_2)\}$ by $\{(X'_1, Y'_1); (X'_2, Y'_2)\}$ and conversely? More formally, is $\text{PD}_M(\{(X_1, Y_1); (X_2, Y_2)\}_{(s_1, s_2)})$ equivalent to $\text{PD}_M(\{(X'_1, Y'_1); (X'_2, Y'_2)\}_{(s'_1, s'_2)})$?

In section 3, we show that the answer is no in the general case of a MRA DM from 1 to s and we explain why. In section 4, we restrict our attention on CRRA utility functions. In this specific case, we show that the result holds and we explain why. We also show that with CRRA utility functions, we obtain a stronger result than the general one obtained in the additive case.

3 Additive and multiplicative contexts: two particular cases of the bivariate context

Consider the preferences for disaggregation displayed in equations (1) and (2) – respectively $\text{PD}(\{(X_1, Y_1); (X_2, Y_2)\}_{(s_1, s_2)})$ in the additive context and $\text{PD}_M(\{(X_1, Y_1); (X_2, Y_2)\}_{(s_1, s_2)})$ in the multiplicative context. Introducing a bivariate utility function V such that $V(x_1, x_2) = u(x_1 + x_2)$ in the additive case and $V(x_1, x_2) = u(x_1 \cdot x_2)$ in the multiplicative case, the two relations (1) and (2) can be summarized by

$$\frac{1}{2}\text{E}[V(X_1, Y_2)] + \frac{1}{2}\text{E}[V(Y_1, X_2)] \geq \frac{1}{2}\text{E}[V(X_1, X_2)] + \frac{1}{2}\text{E}[V(Y_1, Y_2)] . \quad (3)$$

Considering bivariate utility functions quite generally, Denuit and Rey (2013) – see also Jokung (2011) – obtain the following result. The inequality (3) holds for all bivariate utility function V such that $(-1)^{(k_1+k_2+1)}V^{(k_1, k_2)} \geq 0 \forall k_1 = 1, \dots, s_1$ and $\forall k_2 = 1, \dots, s_2$. Eq. (3) states the condition for observing risk apportionment as well for general bivariate functions, as for univariate functions where u reflects $u(x_1 + x_2)$ in the additive case and $u(x_1 \cdot x_2)$ in the multiplicative case. Note that cross-derivatives play a key role in this condition.

Let us make several important remarks.

Remark 1. When V takes the additive form $V(x_1, x_2) = u(x_1 + x_2)$, the inequality (3) holds provided that $X_1 \preceq_{s_1-SD} Y_1$ and $X_2 \preceq_{s_2-SD} Y_2$ and the single-attribute utility

function u exhibits MRA from 1 to $s_1 + s_2$. This is the result obtained by Eeckhoudt et al. (2009b). In the additive case, the analysis is simple because all the cross-derivatives of V for a given total order are captured by the expression of a unique derivative of u . Indeed $V^{(s_2, s_1)} = u^{(s_1 + s_2)} = u^{(s'_1 + s'_2)} = V^{(s'_2, s'_1)}$ for all $s_1 + s_2 = s'_1 + s'_2$ and that is why we obtain Proposition 1, *i.e.* the result of preserved preference ranking replacing $\{(X_1, Y_1); (X_2, Y_2)\}_{(s_1, s_2)}$ by $\{(X'_1, Y'_1); (X'_2, Y'_2)\}_{(s'_1, s'_2)}$ and conversely.

Remark 2. When the bivariate utility function V reads $u(x_1 + x_2)$ or $u(x_1 \cdot x_2)$, for all k_1 and k_2 , $\text{sgn } V^{(k_1, k_2)} = \text{sgn } V^{(k_2, k_1)}$ since the utility function is symmetric in its arguments x_1 and x_2 . Indeed, the combination of x_1 and x_2 yields one single variable x , and $u = u(x)$. For instance, in the additive case, we have $V^{(3, 2)}(x_1, x_2) = u^{(5)}(x_1 + x_2) = V^{(2, 3)}(x_1, x_2)$. In the multiplicative case, the equality does not hold, but the signs remain equal, given that x_1 and x_2 are both assumed positive. For instance, $V^{(3, 2)}(x_1, x_2) = x_2 \cdot \Omega(x)$ and $V^{(2, 3)}(x_1, x_2) = x_1 \cdot \Omega(x)$, where $\Omega(x) = x^2 u^{(5)}(x) + 6xu^{(4)}(x) + 6u^{(3)}(x)$.

Remark 3. However, when V takes the multiplicative form, the signs of the cross-derivatives are not necessarily equal if $k_1 + k_2 > 3$. Consider the above example again. With $V^{(3, 2)}$ and $V^{(2, 3)}$, $k_1 + k_2 = 5$. A total derivative order of 5 can also be obtained by 1+4 and 4+1 in the case of cross-derivatives $V^{(1, 4)}$ and $V^{(4, 1)}$. These two derivatives are respectively equal to $(x_1)^3 \Psi(x)$ and to $(x_2)^3 \Psi(x)$ where $\Psi(x) = xu^{(5)}(x) + 4u^{(4)}(x)$. They have same signs. But since $\Omega \neq \Psi$, $\text{sgn } V^{(2, 3)}$ and $\text{sgn } V^{(1, 4)}$ may differ. Proposition 1 does not apply. Of course, the possibilities of divergence increase when the sum $k_1 + k_2$ increases. For instance, when $k_1 + k_2 = 9$, four different pairs of cross-derivatives need to be considered.

Remark 4. Several authors remarked that relative risk aversion plays a role when risks are multiplicative. See Eeckhoudt and Schlesinger (2008), Eeckhoudt et al. (2009a), Wang and Li (2010), Chiu et al. (2012), Denuit and Rey (2013). More specifically, starting from Eeckhoudt and Schlesinger (2006), and using lotteries with multiplicative risks defined by Eeckhoudt et al (2009a), Wang and Li (2010) generalize the latter contribution by showing that risk apportionment of order $n + 1$ occurs if and only if the n^{th} degree relative risk aversion $r_u^{(n)}$ exceeds n :

$$r_u^{(n)} = -\frac{xu^{(n+1)}(x)}{u^{(n)}(x)} \geq n. \quad (4)$$

We observe then the important following point. A MRA DM from 1 to s , who prefers to disaggregate risks X_1, X_2, Y_1, Y_2 with $X_1 \preceq_{s_1-SD} Y_1$ and $X_2 \preceq_{s_2-SD} Y_2$ and $s_1 + s_2 = s$, when risks are additive does not necessarily disaggregate them when they are multiplicative. Indeed, a MRA utility function does not necessarily imply $r_u^{(n)} \geq n$. Denuit and Rey (2013) show that they can adapt their general result above (Eq. 3) to the multiplicative case if they set $s_1 = 1$ (and $s_2 \geq 1$). Indeed, they show that the condition $(-1)^{(1+k+1)}V^{(1, k)} \geq 0$ for all k (which is equivalent to $(-1)^{(k+1+1)}V^{(k, 1)} \geq 0$ following remark 2) reads equivalently $r_u^{(k)} \geq k$ for all k when risks are multiplicative. Specifically, they consider the degenerate lotteries $X_1 = a$ and $Y_1 = b$ where a and b

are two positive constants such that $a < b$ (then $X_1 \preceq_{1-SD} Y_1$ obviously holds). They obtain the following result (see also Eeckhoudt et al. (2009a) and Chiu et al., 2012): the inequality

$$\frac{1}{2}\mathbb{E}[u(aY_2)] + \frac{1}{2}\mathbb{E}[u(bX_2)] \geq \frac{1}{2}\mathbb{E}[u(aX_2)] + \frac{1}{2}\mathbb{E}[u(bY_2)] , \quad (5)$$

holds for all utility function u that satisfies $r_u^{(k)} \geq k$ for all $k = 1, \dots, s_2$.

These authors explain that they cannot extend their result to higher values of s_1 (or s_2), because for a given total order s ($s_1 + s_2 = s$), the signs of the higher cross-derivatives of V are not necessarily the same, *i.e.* $(-1)^{(1+s_2+1)}V^{(1,s_2)} \geq 0$ is not equivalent to $(-1)^{(2+(s_2-1)+1)}V^{(2,s_2-1)} \geq 0$ for all $s_2 \geq 3$ (see remark 3).

Let us now return to our question of Section 2. If the total order $s_1 + s_2 = s'_1 + s'_2 \leq 3$ then we can extend Proposition 1 to multiplicative risks. Indeed $\text{sgn } V^{(1,2)} = \text{sgn } V^{(2,1)}$ (see remarks 2 and 3). The result still holds when the total order is equal to $1 + n$ with $s_1 = 1$ and $s_2 = n$ on the one hand, and $s'_1 = n$ and $s'_2 = 1$ on the other hand. But when the total order is greater than or equal to four ($s \geq 4$) with $s_1 > 1$ and $s_2 > 1$, then we cannot extend the result.

4 Multiplicative risks and CRRA utility functions

CRRA utility functions u verify

$$\frac{d}{dx} \left(\frac{-xu''(x)}{u'(x)} \right) = 0 \quad \forall x , \quad (6)$$

and they read

$$u(x) = \begin{cases} \frac{x^{1-\gamma}}{1-\gamma}, & \gamma > 0, \gamma \neq 1, \\ \ln x, & \gamma = 1. \end{cases} \quad (7)$$

In this case, the relative risk aversion index, $r_u^{(1)}$, is equal to the constant γ :

$$\frac{-xu''(x)}{u'(x)} = \gamma . \quad (8)$$

We can state the following (see proof in appendix).

Lemma 1. *Consider multiplicative risks and a CRRA utility function with $\gamma > 0$.*

(a) *For all $n \geq 2$, the relation*

$$\text{sgn } V^{(1,n)} = \text{sgn } V^{(2,n-1)} = \text{sgn } V^{(3,n-2)} = \dots = \text{sgn } V^{(n,1)} ,$$

holds

(b) Additionally, $V^{(m-p,p)}$ and $V^{(m,0)}$, with $m > p$ for all $p \geq 1$, have the same (opposite) sign if and only if $\gamma > 1$ ($\gamma < 1$).

As $V^{(m,0)}$ is negative when m is even and positive when m is odd, we obtain from item (b) of Lemma 1:

$$(-1)^{(1+(m-p)+p)}V^{(m-p,p)} > 0 \Leftrightarrow r_u^{(1)} = \gamma > 1. \quad (9)$$

Using Denuit and Rey (2013) – see equation (3) above – this means that risk apportionment holds, by item (b) of Lemma 1, for all $p \geq 1$ and $m > p$, if and only if $\gamma > 1$. The restrictions spelt out in equation (4) about the value of s_1 (or s_2) and about the value of relative risk aversion at orders greater than 1 do not apply any more. For instance, if $s_2 = p = 10$, and $s_1 = m - p = 5$, multiplicative risk apportionment holds assuming a CCRA utility function with $X_1 \preceq_{5-SD} Y_1$ and $X_2 \preceq_{10-SD} Y_2$ provided $\gamma > 1$. This result is expressed in the following proposition.

Proposition 2. *Let us assume that (i) Y_i dominates X_i in the sense of the s_i^{th} order stochastic dominance for $i = 1, 2$ ($X_1 \preceq_{s_1-SD} Y_1$ and $X_2 \preceq_{s_2-SD} Y_2$) and (ii) X_1, X_2, Y_1, Y_2 are mutually independent risks. The inequality*

$$\frac{1}{2}\mathbb{E}[u(X_1 \cdot Y_2)] + \frac{1}{2}\mathbb{E}[u(Y_1 \cdot X_2)] \geq (\leq) \frac{1}{2}\mathbb{E}[u(X_1 \cdot X_2)] + \frac{1}{2}\mathbb{E}[u(Y_1 \cdot Y_2)],$$

holds for all CRRA utility function u verifying $r_u^{(1)} > (<)1$.

Turning now to part (a) of Lemma 1, we see that, if risk apportionment holds for a total order $s = 1 + n$, with $s_1 = 1$ and $s_2 = n$, it also holds for the same total order with $s_1 = 2$ and $s_2 = n - 1$, or $s_1 = 3$ and $s_2 = n - 2$, and so on. We can then reproduce, for multiplicative risks and CCRA utility, the same result as in Proposition 1 dealing with additive risks. This is our Proposition 3.

Proposition 3. *Given a decision maker with a CRRA utility function u such that $r_u^{(1)} > 1$. Given a first set of four mutually independent risks X_1, X_2, Y_1, Y_2 such that $X_1 \preceq_{s_1-SD} Y_1$ and $X_2 \preceq_{s_2-SD} Y_2$ and given a second set of four mutually independent risks X'_1, X'_2, Y'_1, Y'_2 such that $X'_1 \preceq_{s'_1-SD} Y'_1$ and $X'_2 \preceq_{s'_2-SD} Y'_2$ for all s_1, s_2, s'_1, s'_2 with $s_1 \neq s'_1$ and $s_2 \neq s'_2$. When risks are multiplicative, the disaggregation preference relation is preserved when the first set of risk is replaced by the second set of risks and conversely.*

More formally, $\text{PD}_M(\{(X_1, Y_1); (X_2, Y_2)\}_{(s_1, s_2)}) \iff \text{PD}_M(\{(X'_1, Y'_1); (X'_2, Y'_2)\}_{(s'_1, s'_2)})$.

Remark 5. Note that the inequality with \geq (\leq) of proposition 2 holds for all CRRA utility function u verifying $r_u^{(1)} > (<)1$ whatever the total derivative order $s_1 + s_2 = s$. Consequently, the inequality holds for $s_1 + s_2 = s$, it holds for $s'_1 + s'_2 = s$ with $s_1 \neq s'_1$, $s_2 \neq s'_2$, but it also holds for $s'_1 + s'_2 \neq s$, i.e. for $s'_1 + s'_2 < s_1 + s_2$ and for $s'_1 + s'_2 > s_1 + s_2$.

Thus, the disaggregation preference relation is preserved when the first set of risk is replaced by the second set of risks and vice-versa: $\text{PD}_M(\{(X_1, Y_1); (X_2, Y_2)\}_{(s_1, s_2)}) \iff \text{PD}_M(\{(X'_1, Y'_1); (X'_2, Y'_2)\}_{(s'_1, s'_2)})$ for all total order level *i.e.* such that $s'_1 + s'_2 = s_1 + s_2$ or $s'_1 + s'_2 \neq s_1 + s_2$. For this last reason, with CRRA utility functions, the result obtained when risks are multiplicative (Proposition 3) is stronger than the one obtained with additive risks (Proposition 1). Indeed, in the additive case, the condition $(-1)^{(1+s)}u^{(s)} \geq 0$, that holds for a s -order MRA DM, does not necessarily hold at order $s + 1$, if the DM is only MRA up to order s . Note, however, that CRRA utility yields MRA at any order. In this sense, even in the additive case, a preference for harm disaggregations at total order s implies a preference for disaggregation at order $s + n$, whatever n , provided the DM is endowed with a CRRA utility function. This remark underlines the *power* of CRRA utility functions.

Let us denote by $\text{PA}_M(\{(X_1, Y_1); (X_2, Y_2)\}_{(s_1, s_2)})$ the aggregation preference relation when risks are multiplicative:

$$\frac{1}{2}\text{E}[u(X_1 \cdot Y_2)] + \frac{1}{2}\text{E}(u(Y_1 \cdot X_2)) \leq \frac{1}{2}\text{E}[u(X_1 \cdot X_2)] + \frac{1}{2}\text{E}[u(Y_1 \cdot Y_2)] . \quad (10)$$

We can derive the following result.

Proposition (3'). *Given a decision maker with a CRRA utility function u such that $r_u^{(1)} < 1$. Given a first set of four mutually independent risks X_1, X_2, Y_1, Y_2 such that $X_1 \preceq_{s_1-SD} Y_1$ and $X_2 \preceq_{s_2-SD} Y_2$ and given a second set of four mutually independent risks X'_1, X'_2, Y'_1, Y'_2 such that $X'_1 \preceq_{s'_1-SD} Y'_1$ and $X'_2 \preceq_{s'_2-SD} Y'_2$ for all s_1, s_2, s'_1, s'_2 with $s_1 \neq s'_1$ and $s_2 \neq s'_2$. When risks are multiplicative, the aggregation preference relation is preserved when the first set of risk is replaced by the second set of risks and conversely.*

More formally, $\text{PA}_M(\{(X_1, Y_1); (X_2, Y_2)\}_{(s_1, s_2)}) \iff \text{PA}_M(\{(X'_1, Y'_1); (X'_2, Y'_2)\}_{(s'_1, s'_2)})$.

Finally, having in mind that the condition $(-1)^{(1+k+1)}V^{(1,k)} \geq 0$ for all k (which is equivalent to $(-1)^{(k+1+1)}V^{(k,1)} \geq 0$ following remark 2) reads equivalently $r_u^{(k)} \geq k$ for all k when risks are multiplicative (see remark 4), we derive two noteworthy properties of CRRA utility functions.

Property 1 of CRRA utility function u , with $\gamma > 0$.

$$r_u^{(n+1)} = r_u^{(n)} + 1 \text{ for all } n \geq 1.$$

(See proof in the appendix). From Property 1, one easily derives Property 2.

Property 2 of CRRA utility function u , with $\gamma > 0$.

$$(a) r_u^{(1)} > 1 \iff r_u^{(n)} > n \text{ for all } n \geq 2,$$

$$(b) r_u^{(1)} < 1 \iff r_u^{(n)} < n \text{ for all } n \geq 2.$$

Note that the equivalence $r_u^{(1)} > 1 \iff r_u^{(2)} > 2$ already appears in the literature (see, for instance, Eeckhoudt et al., 2005), but the generalization to order n is new

to our knowledge and is a direct implication of Property 1. Note also that these two properties are in the background of our Proposition 2, that holds with the simple condition $r_u^{(1)} > (<)1$, given CRRA utility.

5 Conclusion

In this paper, we extend the Eeckhoudt et al. (2009b) results on risk apportionment with two additive risks by showing that, for one DM, risk apportionment at total order $s = s_1 + s_2$ implies risk apportionment at total order $s' = s'_1 + s'_2$, with $s_1 \neq s'_1$ and $s_2 \neq s'_2$, provided $s = s'$. Our main motivation is then to check whether this simple property holds in a multiplicative risk context. We show that this is not the case, in general. We also show that the property is however recovered, when we restrict the analysis to CRRA utility functions. In this case, the property is even stronger, as it holds also, and more generally, for all $s \neq s'$, and as well for multiplicative risks as for additive risks, provided relative risk aversion is larger than 1. We are able to link these last results to a particular feature of constant relative risk aversion functions ignored so far: relative risk aversion at order $n + 1$ is equal to relative risk aversion at order n , plus one.

References

- Attema, A. E., L'Haridon, O., van de Kuilen, G. 2019. Measuring multivariate risk preferences in the health domain. *Journal of Health Economics* 64, 15–24.
- Caballé, J., Pomansky, A., 1996. Mixed risk aversion, *Journal of Economic Theory* 71, 485–513.
- Chiu, W.H., Eeckhoudt, L., Rey, B., 2012. On relative and partial risk attitudes: Theory and implications. *Economic Theory* 50, 151–167.
- Deck C., Schlesinger H., 2010. Exploring higher-order risk effects. *Review of Economic Studies* 77, 1403–1420.
- Deck C., Schlesinger H., 2014. Consistency of higher order risk preferences. *Econometrica* 82, 1913–1943.
- Deck C., Schlesinger H., 2018. On the robustness of higher-order risk preferences. *Journal of Risk and Insurance* 85, 313–333.
- Denuit, M., Lefevre, C., Mesfioui, M., 1999. A class of bivariate stochastic orderings with applications in actuarial sciences. *Insurance: Mathematics and Economics* 24, 31–50.
- Denuit, M., Rey, B., 2013. Another look at risk apportionment. *Journal of Mathematical Economics* 49, 335–343.
- Eeckhoudt, L., Etner, J., Schroyen, F., 2009a. The values of relative risk aversion and prudence: A context-free interpretation. *Mathematical Social Sciences* 58, 1–7.
- Eeckhoudt, L., Gollier, C., Schlesinger, H., 2005. *Economic and Financial Decisions under Risk*. Princeton University Press.
- Eeckhoudt, L., Schlesinger, H., 2006. Putting risk in its proper place. *American Economic Review* 96, 280–289.
- Eeckhoudt, L., Schlesinger, H., 2008. Changes in risk and the demand for saving. *Journal of Monetary Economics* 55, 1329–1336.
- Eeckhoudt, L., Schlesinger H., Tsetlin, I. 2009b. Apportioning risks via stochastic dominance. *Journal of Economic Theory* 144, 994–1003.
- Ekern, S. 1980. Increasing n^{th} degree risk. *Economics Letters* 6, 329–333.
- Friedman, M., Savage, L.J., 1948. The utility analysis of choices involving risk. *Journal of Political Economy* 56, 279–304.
- Hadar, J. and Russel, W.R., 1969. Rules for ordering uncertain prospects. *American Economic Review* 59, 25–34.

- Hanoch, G., Levy, H., 1969. The efficiency analysis of choices involving risk. *Review of Economic Studies* 36, 335–346.
- Ingersoll, J., 1987. *Theory of Financial Decision Making*. Rowman & Littlefield, New Jersey.
- Jokung, O., 2011. Risk apportionment via bivariate stochastic dominance. *Journal of Mathematical Economics* 47, 448–452.
- Kimball, M.S., 1990. Precautionary saving in the small and in the large. *Econometrica* 58, 53–73.
- Kimball, M.S., 1992. Precautionary motives for holding assets, in Newman, P., Milgate, M. and Falwell, J. (eds), *The New Palgrave Dictionary of Money and Finance*, London: MacMillan.
- Lajeri-Chaherli, F., 2004. Proper prudence, standard prudence and precautionary vulnerability. *Economics Letters* 82, 29–34.
- Pratt, J.W., 1964. Risk aversion in the small and in the large. *Econometrica* 32, 122–136.
- Rothschild, M., Stiglitz J.E., 1970. Increasing risk: 1. A definition. *Journal of Economic Theory* 2, 225–243.
- Wang, J., Li, J. 2010. Multiplicative risk apportionment. *Mathematical Social Sciences* 60 79–81.

Appendix

Proof of Property 1 of CRRA utility function.

Let us show that, for any CRRA utility function $u(\cdot)$

$$r_u^{(n+1)} = r_u^{(n)} + 1. \quad (11)$$

Let us recall that

$$r_u^{(n)}(x) := -x \cdot \frac{u^{(n+1)}(x)}{u^{(n)}(x)}. \quad (12)$$

For all $n \geq 0$, with u given by eq. (7)

$$u^{(n)}(x) = (-1)^{n+1} \gamma(\gamma + 1)(\gamma + 2) \cdots (\gamma + n - 2) \cdot x^{-\gamma-n+1}, \quad (13)$$

$$= (-1)^{n+1} \left[\prod_{j=0}^{n-2} (\gamma + j) \right] \cdot x^{-\gamma-n+1}. \quad (14)$$

Hence

$$-x \cdot \frac{u^{(n+1)}(x)}{u^{(n)}(x)} = -x \cdot \frac{(-1)^{n+2} \left[\prod_{j=0}^{n-1} (\gamma + j) \right] \cdot x^{-\gamma-n}}{(-1)^{n+1} \left[\prod_{j=0}^{n-2} (\gamma + j) \right] \cdot x^{-\gamma-n+1}}, \quad (15)$$

$$= \gamma + n - 1. \quad (16)$$

So

$$-x \cdot \frac{u^{(n+2)}(x)}{u^{(n+1)}(x)} = \gamma + n - 1 + 1, \quad (17)$$

$$= -x \cdot \frac{u^{(n+1)}(x)}{u^{(n)}(x)} + 1, \quad (18)$$

that is

$$r_u^{(n+1)} = r_u^{(n)} + 1, \quad (19)$$

which concludes the proof.

Proof of Lemma 1

We consider a CRRA utility function u with $\gamma \neq 1$ (if $\gamma = 1$, all the cross-derivatives of $V(x_1, x_2) = \ln(x_1 \cdot x_2)$ vanish). Let us express the cross-derivatives of V .

We start with the first order derivatives:

$$V^{(1,0)}(x_1, x_2) = \frac{\partial}{\partial x_1} u(x_1 x_2), \quad (20)$$

$$= x_2 \cdot u'(x_1 x_2), \quad (21)$$

$$= x_1^{-\gamma} \cdot x_2^{1-\gamma}, \quad (22)$$

and

$$V^{(0,1)}(x_1, x_2) = x_1 \cdot u'(x_1 x_2). \quad (23)$$

It shows that $V^{(1,0)}(x_1, x_2)$ and $V^{(0,1)}(x_1, x_2)$ have both the same sign and have the sign of u' , *i.e.* are positive.

For the second order derivatives, we get

$$V^{(2,0)}(x_1, x_2) = x_2^2 \cdot u^{(2)}(x_1 x_2), \quad (24)$$

$$V^{(1,1)}(x_1, x_2) = -\frac{1-\gamma}{\gamma} x_1 x_2 \cdot u^{(2)}(x_1 x_2), \quad (25)$$

$$V^{(0,2)}(x_1, x_2) = x_1^2 \cdot u^{(2)}(x_1 x_2), \quad (26)$$

meaning that $V^{(2,0)}$ and $V^{(0,2)}$ always have the same signs (are negative as $u^{(2)}$) but have the same sign as $V^{(1,1)}$ if and only if $\gamma > 1$.

Let us now consider the n^{th} order derivatives. A straightforward recursion shows that

$$V^{(n,0)}(x_1, x_2) = x_2^n \cdot u^{(n)}(x_1 x_2), \quad (27)$$

from which we obtain, with $n \geq 2$,

$$V^{(n-1,1)}(x_1, x_2) = \frac{\partial}{\partial x_2} V^{(n-1,0)}(x_1, x_2), \quad (28)$$

$$\stackrel{(27)}{=} (n-1) \cdot x_2^{n-2} \cdot u^{(n-1)}(x_1 x_2) + x_1 x_2^{n-1} \cdot u^{(n)}(x_1 x_2), \quad (29)$$

$$\stackrel{(16)}{=} \frac{\gamma-1}{\gamma+n-2} \cdot x_1 x_2^{n-1} \cdot u^{(n)}(x_1 x_2), \quad (30)$$

$$\stackrel{(27)}{=} \frac{\gamma-1}{\gamma+n-2} \cdot \frac{x_1}{x_2} \cdot V^{(n,0)}(x_1, x_2), \quad (31)$$

so that $V^{(n,0)}$ and $V^{(n-1,1)}$ have same signs if and only if $\frac{\gamma-1}{\gamma+n-2} > 0$ that is $\gamma > 1$ whenever $n \geq 2$.

We then get, with $n \geq 3$,

$$V^{(n-2,2)}(x_1, x_2) = \frac{\partial}{\partial x_2} V^{(n-2,1)}(x_1, x_2), \quad (32)$$

$$\stackrel{(27,31)}{=} \frac{(\gamma-1) \cdot (n-2)}{\gamma+n-3} \cdot x_1 x_2^{n-3} \cdot u^{(n-1)}(x_1 x_2) + \frac{\gamma-1}{\gamma+n-3} \cdot x_1^2 x_2^{n-2} \cdot u^{(n)}(x_1 x_2), \quad (33)$$

$$\stackrel{(16)}{=} \frac{(\gamma-1) \cdot \gamma}{(\gamma+n-3) \cdot (\gamma+n-2)} \cdot x_1^2 x_2^{n-2} \cdot u^{(n)}(x_1 x_2), \quad (34)$$

$$\stackrel{(27)}{=} \frac{(\gamma-1) \cdot \gamma}{(\gamma+n-3) \cdot (\gamma+n-2)} \cdot \left(\frac{x_1}{x_2}\right)^2 \cdot V^{(n,0)}(x_1, x_2), \quad (35)$$

$$\stackrel{(31)}{=} \frac{\gamma}{\gamma+n-3} \cdot \frac{x_1}{x_2} \cdot V^{(n-1,1)}(x_1, x_2), \quad (36)$$

so that

- $V^{(n,0)}$ and $V^{(n-2,2)}$ have the same sign *if and only if* $\frac{(\gamma-1) \cdot \gamma}{(\gamma+n-3) \cdot (\gamma+n-2)} > 0$ that is $\gamma > 1$,
- $V^{(n-2,2)}$ and $V^{(n-1,1)}$ always have the same sign (irrespective of $\gamma \leq 1$),

whenever $n \geq 3$.

We can proceed one step further, with $n \geq 4$,

$$V^{(n-3,3)}(x_1, x_2) = \frac{\partial}{\partial x_2} V^{(n-3,2)}(x_1, x_2), \quad (37)$$

$$\stackrel{(34)}{=} \frac{(\gamma-1) \cdot \gamma \cdot (n-3)}{(\gamma+n-4) \cdot (\gamma+n-3)} \cdot x_1^2 x_2^{n-4} \cdot u^{(n-1)}(x_1 x_2) \\ + \frac{(\gamma-1) \cdot \gamma}{(\gamma+n-4) \cdot (\gamma+n-3)} \cdot x_1^3 x_2^{n-3} \cdot u^{(n)}(x_1 x_2), \quad (38)$$

$$\stackrel{(16)}{=} \frac{(\gamma-1) \cdot \gamma \cdot (\gamma+1)}{(\gamma+n-4) \cdot (\gamma+n-3) \cdot (\gamma+n-2)} \cdot x_1^3 x_2^{n-3} \cdot u^{(n)}(x_1 x_2), \quad (39)$$

$$\stackrel{(27)}{=} \frac{(\gamma-1) \cdot \gamma \cdot (\gamma+1)}{(\gamma+n-4) \cdot (\gamma+n-3) \cdot (\gamma+n-2)} \cdot \left(\frac{x_1}{x_2}\right)^3 \cdot V^{(n,0)}(x_1, x_2), \quad (40)$$

$$\stackrel{(36)}{=} \frac{\gamma+1}{\gamma+n-4} \cdot \frac{x_1}{x_2} \cdot V^{(n-2,2)}(x_1, x_2), \quad (41)$$

so that

- $V^{(n,0)}$ and $V^{(n-3,3)}$ have the same sign *if and only if* $\frac{(\gamma-1) \cdot \gamma \cdot (\gamma+1)}{(\gamma+n-4) \cdot (\gamma+n-3) \cdot (\gamma+n-2)} > 0$ that is $\gamma > 1$,
- $V^{(n-3,3)}$ and $V^{(n-2,2)}$ always have the same sign (irrespective of $\gamma \leq 1$),

whenever $n \geq 4$.

The recursion is now straightforward and we derive the following results.

- By generalization of equation (40), we obtain the relation between $V^{(n-p,p)}$ and $V^{(n,0)}$ for all $n > p \geq 1$:

$$V^{(n-p,p)}(x_1, x_2) = \prod_{k=0}^{p-1} \frac{\gamma+k-1}{\gamma+n-k-2} \cdot \left(\frac{x_1}{x_2}\right)^p \cdot V^{(n,0)}(x_1, x_2). \quad (42)$$

It shows that $V^{(n-p,p)}$ and $V^{(n,0)}$ have the same sign *if and only if* $\prod_{k=0}^{p-1} \frac{\gamma+k-1}{\gamma+n-k-2} > 0$. Since $\gamma > 0$, all the terms of the product are necessarily positive excepted the term corresponding to $k=0$, that is the term $\frac{\gamma-1}{\gamma+n-2}$, which is positive *if and only if* $\gamma > 1$. This result establishes statement (b) of lemma 1.

- By generalization of equation (41), we have

$$V^{(n-(p+1),p+1)}(x_1, x_2) = \frac{\gamma+p-1}{\gamma+n-p-2} \cdot \frac{x_1}{x_2} \cdot V^{(n-p,p)}(x_1, x_2), \quad \forall n > p. \quad (43)$$

Replacing n by $n+1$, we obtain

$$V^{(n-p,p+1)}(x_1, x_2) = \frac{\gamma+p-1}{\gamma+n-p-1} \cdot \frac{x_1}{x_2} \cdot V^{(n+1-p,p)}(x_1, x_2), \quad \forall n \geq p, \quad (44)$$

and, with $p = n - k$, $k \leq n$, we finally get

$$V^{(k,n-k+1)}(x_1, x_2) = \frac{\gamma + n - k - 1}{\gamma + k - 1} \cdot \frac{x_1}{x_2} \cdot V^{(k+1,n-k)}(x_1, x_2), \quad \forall n \geq k. \quad (45)$$

Hence $V^{(k,n-k+1)}$ and $V^{(k+1,n-k)}$ have the same sign *if and only if* $\gamma + n - k - 1 > 0$, that is, *if and only if* $n - k > 1 - \gamma$. When $\gamma > 1$, this condition holds for all $n \geq k$ since n and k are integers. When $\gamma \leq 1$, the condition still holds for all $n > k$ but is not met when $n = k$. It is however sufficient for our purpose since statement (a) in lemma 1 does not involve derivatives of the form $V^{(0,n)}$ nor $V^{(n,0)}$.

Eventually irrespective of $\gamma \geq 1$, we conclude that $\text{sgn } V^{(k,n-k+1)} = \text{sgn } V^{(k+1,n-k)}$ while, with $k+1$ instead of k , $\text{sgn } V^{(k+1,n-k)} = \text{sgn } V^{(k+2,n-k-1)}$ so that $\text{sgn } V^{(k,n-k+1)} = \text{sgn } V^{(k+1,n-k)} = \text{sgn } V^{(k+2,n-k-1)}$. Repeating this substitution for all k ranging from 1 to $n - 1$, we obtain the set of equalities that establishes statement (a) in lemma 1.