Decomposition of games: some strategic considerations

Joseph M. Abdou, Nikolaos Pnevmatikos, Marco Scarsini, Xavier Venel

Documents de Travail du
Centre d'Economie de la Sorbonne

CES Working Papers

Decomposition of games: some strategic considerations

Joseph ABDOU, Nikolaos PNEVMATIKOS
Marco SCARSINI, Xavier VENEL

2019.06

Maison des Sciences Économiques, 106-112 boulevard de L'Hôpital, 75647 Paris Cedex 13
https://centredeconomiesorbonne.univ-paris1.fr/
ISSN : 1955-611X

DECOMPOSITION OF GAMES: SOME STRATEGIC CONSIDERATIONS

JOSEPH ABDOU, NIKOLAOS PNEVMATIKOS, MARCO SCARSINI, AND XAVIER VENEL

Abstract. Candogan et al. (2011) provide an orthogonal direct-sum decomposition of finite games into potential, harmonic and non-strategic components. In this paper we study the issue of decomposing games that are strategically equivalent from a game-theoretical point of view, for instance games obtained via duplications of strategies or suitable linear transformations of payoffs. We consider classes of decompositions and show when two decompositions of equivalent games are coherent.

1. Introduction

Potential games are an interesting class of games that admit pure Nash equilibria and behave well with respect to the most common learning procedures. Some games, although they are not potential games, are close—in a sense to be made precise—to a potential game. It is therefore interesting to examine whether their equilibria are close to the equilibria of the potential game, (see Candogan et al. (2013)). With this in mind, in their seminal paper Candogan et al. (2011) were able to show that the class of strategic-form games having a fixed set of players and a fixed set of strategies for each player is a linear space that can be decomposed into the orthogonal sum of three components, called the *potential*, *harmonic* and *non-strategic* component. Games in the harmonic component have a completely mixed equilibrium where all players mix uniformly over their strategies; games in the non-strategic component are such that the payoff of each player is not affected by her own strategy, but only by other players' strategies. To achieve this decomposition the authors associate to each game a graph where vertices are strategy profiles and edges connect profiles that differ only for the strategy of one player. The analysis is then carried out by studying flows on graphs and using the Helmholtz decomposition theorem.

The decomposition of Candogan et al. (2011) refers to games having all the same set of players and the same set of strategies for each player. In their construction

---

Date: February 1, 2019.

2010 *Mathematics Subject Classification.* Primary 91A70. *OR/MS subject classification.* Games/group decisions, noncooperative.

Key words and phrases. decomposition of games; η-potential games; harmonic games; duplicate strategies; gradient operator; projection operator.

Nikos Pnevmatikos's research was supported by Labex MME-DII. Part of this research was carried out when he was visiting the Engineering Systems and Design pillar at Singapore University of Technology and Design. Marco Scarsini is a member of GNAMPA-INdAM. Xavier Venel acknowledges the support of the Agence Nationale de la Recherche [ANR CIGNE ANR-15-CE38-0007-01]. This work was partially supported by GAMENET COST Action CA 16228. The authors also want to thank Ozan Candogan, Sung-Ha Hwang and Panayotis Mertikopoulos for valuable comments.

nothing connects the decomposition of a specific game $\boldsymbol{g}$ with the decomposition of another game $\breve{\boldsymbol{g}}$ that is obtained from $\boldsymbol{g}$ by adding a strategy to the set S^i of Player i's feasible strategies. One may argue that this is reasonable, since the two games live in linear spaces of different dimension and the new game with an extra strategy may have equilibria that are very different from the ones in the original game, so, in general, the two games may have very little in common. In some situations, though, the two games are indeed strongly related. For instance, consider the case where the payoffs corresponding to the new strategy are just a replica of the payoffs of another strategy. In this case, from a strategic viewpoint, the two games $\breve{\boldsymbol{g}}$ and $\boldsymbol{g}$ are actually the same game and every equilibrium in $\breve{\boldsymbol{g}}$ can be mapped to an equilibrium in $\boldsymbol{g}$. It would be reasonable to expect the decomposition of $\boldsymbol{g}$ and $\breve{\boldsymbol{g}}$ to be strongly related. Unfortunately this is not the case. Consider for instance the matching-pennies game $\boldsymbol{g}$:

	L		R	
T	1	-1	-1	1
B	-1	1	1	-1

$\boldsymbol{g}$

and the game $\breve{\boldsymbol{g}}$, obtained by replicating strategy B of the row-player:

	L		R	
T	1	-1	-1	1
B_1	-1	1	1	-1
B_2	-1	1	1	-1

$\breve{\boldsymbol{g}}$

This game $\breve{\boldsymbol{g}}$ admits a continuum of equilibria where the column player mixes uniformly and the row-player mixes $(1/2, \alpha/2, (1-\alpha)/2)$ with $\alpha \in [0,1]$. Notice that in each of these mixed equilibria the mixed strategy of the row-player assigns probability $1/2$ to $B_1 \cup B_2$. The decomposition result of Candogan et al. (2011) (Theorem 4.1) applied to $\breve{\boldsymbol{g}}$ yields:

	L		R	
T	$4/15$	$3/5$	$-4/15$	$-3/5$
B_1	$-2/15$	$2/10$	$2/15$	$-2/10$
B_2	$-2/15$	$2/10$	$2/15$	$-2/10$

$(\breve{\boldsymbol{g}})_{\mathcal{P}}$

	L		R	
T	$16/15$	$-8/5$	$-16/15$	$8/5$
B_1	$-8/15$	$4/5$	$8/15$	$-4/5$
B_2	$-8/15$	$4/5$	$8/15$	$-4/5$

$(\breve{\boldsymbol{g}})_{\mathcal{H}}$

	L		R	
T	$-1/3$	0	$1/3$	0
B_1	$-1/3$	0	$1/3$	0
B_2	$-1/3$	0	$1/3$	0

$(\breve{\boldsymbol{g}})_{\mathcal{NS}}$

where $(\breve{\boldsymbol{g}})_{\mathcal{P}}$, $(\breve{\boldsymbol{g}})_{\mathcal{H}}$, and $(\breve{\boldsymbol{g}})_{\mathcal{NS}}$ are the potential, harmonic, and non-strategic components of $\breve{\boldsymbol{g}}$, respectively.

The matching-pennies game $\boldsymbol{g}$ admits a unique Nash equilibrium where each player randomizes uniformly between the two available strategies. This game is harmonic, so its decomposition has the potential and non-strategic component identically equal to zero and the harmonic component $\boldsymbol{g}_{\mathcal{H}}$ equal to $\boldsymbol{g}$.

We see that, although the two games $\boldsymbol{g}$ and $\breve{\boldsymbol{g}}$ are strategically equivalent in the sense described before, their decompositions are quite different. In the sequel, when considering a game $\boldsymbol{g}$ with duplicate strategies, we will call the game where duplication of strategies have been eliminated the *reduced version* of $\boldsymbol{g}$ and we will use the notation $\widehat{\boldsymbol{g}}$ for it.

A similar problem appears in games whose payoffs are suitable affine transformation of some other game's payoffs. For instance, given a game $\boldsymbol{g}$, consider the game $\widetilde{\boldsymbol{g}}$ which is obtained by multiplying the payoffs of each player in $\boldsymbol{g}$ by the same positive constant. Multiplying the payoff of a player by a positive constant is innocuous with respect to strategic considerations. To illustrate this, if $\boldsymbol{g}$ is the matching-pennies game, consider the game $\widetilde{\boldsymbol{g}}$ where the payoffs of the row-player in $\boldsymbol{g}$ have been multiplied by 2.

$$
\begin{array}{c|cc|cc|}
 & \multicolumn{2}{c}{L} & \multicolumn{2}{c}{R} \\
\cline{2-5}
T & 2 & -1 & -2 & 1 \\
\cline{2-5}
B & -2 & 1 & 2 & -1 \\
\cline{2-5}
\multicolumn{5}{c}{\widetilde{\boldsymbol{g}}}
\end{array}.
$$

This game admits a unique equilibrium where each player plays uniformly over her set of strategies. The decomposition result of Candogan et al. (2011) applied in $\widetilde{\boldsymbol{g}}$ yields the following decomposition:

$$
\begin{array}{c|cc|cc|}
 & \multicolumn{2}{c}{L} & \multicolumn{2}{c}{R} \\
\cline{2-5}
T & 1/2 & 1/2 & -1/2 & -1/2 \\
\cline{2-5}
B & -1/2 & -1/2 & 1/2 & 1/2 \\
\cline{2-5}
\multicolumn{5}{c}{(\widetilde{\boldsymbol{g}})_{\mathcal{P}}}
\end{array}
\qquad
\begin{array}{c|cc|cc|}
 & \multicolumn{2}{c}{L} & \multicolumn{2}{c}{R} \\
\cline{2-5}
T & 3/2 & -3/2 & -3/2 & 3/2 \\
\cline{2-5}
B & -3/2 & 3/2 & 3/2 & -3/2 \\
\cline{2-5}
\multicolumn{5}{c}{(\widetilde{\boldsymbol{g}})_{\mathcal{H}}}
\end{array}.
$$

Although the games $\boldsymbol{g}$ and $\widetilde{\boldsymbol{g}}$ share the same Nash equilibrium set, they admit different decompositions. Notice that $\boldsymbol{g}$ and $(\widetilde{\boldsymbol{g}})_{\mathcal{H}}$ are both harmonic games and admit the same unique mixed equilibrium, since they are related by an affine transformation that does not affect the harmonic property: one is obtained from the other by multiplying all payoffs by the same positive constant. In the sequel we will be interested in affine transformations where payoffs are multiplied by a positive constant which depends not only on the players but also on their strategies. We will refer to this kind of transformations as *dilations*.

The question that we want to address in this paper is the following: is it possible to conceive a procedure such that the decompositions of $\boldsymbol{g}$ and its duplication $\breve{\boldsymbol{g}}$ and the decompositions of $\boldsymbol{g}$ and its dilation $\widetilde{\boldsymbol{g}}$ are coherent in some sense? To achieve our goal, instead of considering a unique decomposition and the Euclidean inner product in the space of games, as it is the case in Candogan et al. (2011), we deal with a family of decompositions. We use two product measures, μ and η, on the set of strategy profiles to parametrize a class of inner products that are in turn used to define the decompositions. The need for two product measures, rather than one, lies in the fact that duplications and dilations are radically different transformations. One way to see this difference is that the duplication of a strategy of some player affects only her own equilibrium strategy, whereas dilation of some player's payoffs changes the equilibrium strategies of all the other players, but not her own.

First, we generalize the decomposition result of Candogan et al. (2011) using several metrics induced in the space of games by the product measures μ and η on the set of strategy profiles. Then, given a decomposition of the game $\boldsymbol{g}$ having duplicate strategies based on the two product measures μ and η, we show that there exists a new measure $\widehat{\mu}$, which only depends on μ, such that the (μ, η)-decomposition of $\boldsymbol{g}$ is coherent with the $(\widehat{\mu}, \eta)$-decomposition of $\widehat{\boldsymbol{g}}$. Similarly, the introduced family of decompositions further allows to bridge decompositions for dilations. Given a decomposition of the game $\boldsymbol{g}$ based on the two product measures μ and η, we prove that there exists a new measure $\widetilde{\eta}$, which only depends on η, such that the $(\mu, \widetilde{\eta})$-decomposition of $\widetilde{\boldsymbol{g}}$ is coherent with the (μ, η)-decomposition of $\boldsymbol{g}$.

1.1. **Related literature.** In the context of non-cooperative game theory, several approaches have been proposed to decompose a game into simpler games that admit more flexible and attractive equilibrium analysis. Sandholm (2010) proposes a method to decompose n-player normal-form games into 2^n simultaneously-played component games. As a by-product, this decomposition provides a characterization of normal-form potential games. Kalai and Kalai (2013) introduce a novel solution concept founded on a decomposition of two-player games into *zero-sum* and *common-interest* games. This decomposition result is based on the fact that all matrices can be decomposed into the sum of symmetric and antisymmetric matrices. Szabó et al. (2017), in order to study evolutionary dynamic games, refine this dyadic decomposition by further decomposing the antisymmetric component. In a different direction, Jessie and Saari (2013) present a *strategic-behavioral* decomposition of games with two strategies per player and highlight that certain solution concepts are determined by a game's strategic part or influenced by the behavioral portion. More recently, Hwang and Rey-Bellet (2016) study the space of games as a Hilbert space and prove several decomposition theorems for arbitrary games identifying components such as potential games and games that are strategically equivalent—in the sense of sharing the same Nash equilibria—to zero-sum games. They further extend their results to games with uncountable strategy sets. Using this decomposition, they also provide an alternative proof for the well-known characterization of potential games presented by Monderer and Shapley (1996).

The paper of Candogan et al. (2011) on the decomposition of finite games into potential, harmonic, and non-strategic components is a milestone in the field. To achieve their decomposition, they represent an arbitrary game with an undirected graph where nodes stand for strategy profiles and edges connect nodes that differ in the strategy of only one player. In our paper, we follow the same graph representation for a given game. Liu (2018) uses a different graph representation for a finite game where nodes stand for players and edges connect players whose change of strategies influences the other player's payoff. Given a finite game, the author investigates necessary and sufficient conditions for the existence of a pure Nash equilibrium in terms of the structure of its associated directed graph. The decomposition in Candogan et al. (2011) is based on the Helmholtz decomposition theorem.[1] The Helmholtz theorem—a fundamental tool in vector calculus—states that any vector field can be decomposed into a *divergence-free* and a *curl-free* components. Due to the ubiquitous nature of vector fields, this theorem has been applied by various research communities to a wide range of issues. In the context of discrete vector fields, Jiang et al. (2011) provide an implementation of the Helmholtz decomposition in statistical ranking. Stern and Tettenhorst (2017) apply the Helmholtz decomposition to cooperative games and obtain a novel characterization of the Shapley value in terms of the decomposition's components. Various papers related to Candogan et al. (2011) have appeared in the literature. For instance, Liu et al. (2015), Li et al. (2016) focus on the detailed description of the decomposition subspaces by providing some geometric and algebraic expressions and present an explicit formula for the decomposition. More recently, Zhang (2017) provides explicit polynomial expressions for the orthogonal projections onto the subspaces of potential and harmonic games, respectively.

In the terminology of Govindan and Wilson (2005), two strategies of one player are equivalents if they yield every player the same expected payoff for each profile of other players' strategies. A pure strategy of Player i is *redundant* if Player i has another equivalent strategy. In our paper, we study the behavior of the proposed decomposition with respect to redundant strategies and to suitable transformations of payoff vectors that do not alter the strategic structure of the game. The issue of redundant strategies has been dealt with by Govindan and Wilson (1997, 2005, 2008, 2009) in the framework of equilibrium refinement. In particular, the authors show that the degree of a Nash component is invariant under addition or deletion of redundant strategies. As shown, for instance, by Osborne and Rubinstein (1998), some solution concepts are not invariant with respect to addition of redundant strategies. In the framework of decomposition of games, Kalai and Kalai (2013) show that their decomposition is invariant to redundant strategies. Cheng et al. (2016) provide a generalization of the decomposition of Candogan et al. (2011) in terms of *weighted potential* and *weighted harmonic games*. This work is close to ours but still quite different. Precisely, Candogan et al. (2011) assume that the weight of each player is equal to the number of her strategies while Cheng et al. (2016) relax this hypothesis by considering any possible weight. Their approach is coherent with simple dilations like multiplication of the payoffs of some player by a constant that depends only on the other players, but not with more general dilations. Moreover, weighted harmonic games still admit the uniformly strategy profile as equilibrium and therefore their class is not robust to elimination of duplicate strategies.

1.2. **Structure of the paper.** In Section 2, we introduce our decomposition results for games. In Section 3, we deal with the coherence of these decompositions. All proofs can be found in Appendix A.

[1]A generalization of the Helmholtz theorem is known in the literature as the Hodge decomposition theorem, which is defined for differentiable forms on Riemannian manifolds.

2. Decomposition of finite games

In this section, we generalize the decomposition of Candogan et al. (2011) as follows. We first present new classes of games—μ-normalized, η-potential and (μ,η)-harmonic—and study their properties. Then, we provide a first decomposition of games into μ-normalized and non-strategic components. We finally establish a decomposition of the space of finite games into *non-strategic games*, *μ-normalized η-potential* and *μ-normalized (μ, η)-harmonic games*. In order to obtain orthogonality between the components, this decomposition requires the choice of a suitable inner product.

Let $n \geqslant 2$. A *finite game* consists of a finite set of players, denoted by $N = \{1, \dots, n\}$, and, for each Player $i \in N$, a finite set of strategies S^i and a payoff function $g^i : S \to \mathbb{R}$, where $S = \times_{i \in N} S^i$ is the space of strategy profiles $\boldsymbol{s}$. The symbol S^{-i} denotes the set of strategy profiles of all players except Player i. Since, given N and S, every game is uniquely defined by the set of its payoff functions, we call $\boldsymbol{g} = (g^i)_{i \in N}$ a *game*. Hence, if we denote $|A|$ the cardinality of a set A, the space $\mathcal{G}$ of games with set of players N and set of strategy profiles S can be identified with $\mathbb{R}^{n|S|}$. Consequently, we have that $\dim(\mathcal{G}) = n \prod_{i \in N} |S^i|$. Given a positive finite measure ν^i on S^i for every $i \in N$, let ν be the product measure defined for any $\boldsymbol{s} \in S$, by $\nu(\boldsymbol{s}) = \prod_{i \in N} \nu^i(s^i)$. We also use the notation $\nu^{-i}(\boldsymbol{s}^{-i}) = \prod_{j \neq i} \nu^j(s^j)$. Given any positive finite measure ν^i on S^i, we denote its normalization by:

$$\overline{\nu}^i(s^i) = \frac{\nu^i(s^i)}{\sum_{t^i \in S^i} \nu^i(t^i)}. \tag{2.1}$$

For the rest of the paper, we associate to each Player $i \in N$, two positive finite measures on S^i, denoted by μ^i and η^i respectively.

2.1. Special classes of games.

In this section, we introduce the different classes of games that will appear in our decomposition result and we further state their main properties.

Definition 2.1. A game $\boldsymbol{g} \in \mathcal{G}$ is *non-strategic* if, for each $i \in N$, there exists a function $\ell^i : S^{-i} \to \mathbb{R}$, such that, for all $\boldsymbol{s} \in S$,

$$g^i(s^i, \boldsymbol{s}^{-i}) = \ell^i(\boldsymbol{s}^{-i}).$$

The set of non-strategic games is denoted by $\mathcal{NS}$.

Definition 2.2. A game $\boldsymbol{g} \in \mathcal{G}$ is *μ-normalized*, if for all $i \in N$ and for all $\boldsymbol{s}^{-i} \in S^{-i}$,

$$\sum_{s^i \in S^i} \mu^i(s^i) g^i(s^i, \boldsymbol{s}^{-i}) = 0.$$

The set of μ-normalized games is denoted by $\mathcal{NO}$.

We will show that any game can be decomposed into the sum of a non-strategic game and a μ-normalized game. Notice that given $\boldsymbol{g} \in \mathcal{G}$ and $\boldsymbol{\ell} \in \mathcal{NS}$, the games $\boldsymbol{g}$ and $\boldsymbol{g}' = \boldsymbol{g} + \boldsymbol{\ell}$ admit the same best-reply correspondence and hence the same set of Nash equilibria. Hence, they are strategically equivalent and therefore in some sense $\mathcal{NO}$ is just a choice of normalization of games.

We now introduce the classes of η-potential and (μ, η)-harmonic games. The first class of games is related to games where the interests of players are aligned with a weighted potential function and, hence, always admit a pure Nash equilibrium. They are a subclass of ordinal potential games. The second class of games reflects more conflictual situations and we later prove that these games admit a completely mixed equilibrium.

Definition 2.3. A game $\boldsymbol{g} \in \mathcal{G}$ is said to be an *η-potential game* if there exists $\varphi : S \to \mathbb{R}$ such that for any $i \in N$, for all $s^i, t^i \in S^i$, and for all $\boldsymbol{s}^{-i} \in S^{-i}$, we have

$$\eta^{-i}(\boldsymbol{s}^{-i})(g^i(t^i, \boldsymbol{s}^{-i}) - g^i(s^i, \boldsymbol{s}^{-i})) = \varphi(t^i, \boldsymbol{s}^{-i}) - \varphi(s^i, \boldsymbol{s}^{-i}).$$

The function φ is referred to as a *potential function* of the game. We denote $\mathcal{P}$ the set of η-potential games for some η.

In the terminology of Monderer and Shapley (1996), a weighted potential game is an η-potential game, which, in turn, is an ordinal potential game. An immediate consequence of this is the following proposition.

Proposition 2.4. *A game $\boldsymbol{g} \in \mathcal{P}$ admits a pure equilibrium.*

Definition 2.5. A game $\boldsymbol{g}$ is a *(μ, η)-harmonic game* if for all $\boldsymbol{s} \in S$,

$$\sum_{i \in N} \sum_{t^i \in S^i} \mu^i(t^i)\eta^{-i}(\boldsymbol{s}^{-i}) \left(g^i(s^i, \boldsymbol{s}^{-i}) - g^i(t^i, \boldsymbol{s}^{-i})\right) = 0$$

We denote $\mathcal{H}$ the set of (μ, η)-harmonic games for some μ and η.

We can now prove that a (μ, η)-harmonic game admits a completely mixed strategy equilibrium, characterized by μ and η. For that purpose, we define for any $i \in N$,

$$(\mu\eta)^i(s^i) = \mu^i(s^i)\eta^i(s^i) \tag{2.2}$$

and denote $\overline{\mu\eta}^i$ its normalized version, as in Eq. (2.1).

Theorem 2.6. *Let $\boldsymbol{g}$ be a (μ, η)-harmonic game. Then, the completely mixed strategy profile $(\overline{\mu\eta}^i)_{i \in N}$ is an equilibrium, i.e., for all $i \in N$ and for all $r^i, t^i \in S^i$, we have*

$$\sum_{\boldsymbol{s}^{-i} \in S^{-i}} \prod_{j \neq i} \overline{\mu\eta}^j(s^j) g^i(r^i, \boldsymbol{s}^{-i}) = \sum_{\boldsymbol{s}^{-i} \in S^{-i}} \prod_{j \neq i} \overline{\mu\eta}^j(s^j) g^i(t^i, \boldsymbol{s}^{-i}).$$

2.2. **A first decomposition result.** Our first decomposition states that any game can be written as the sum of a μ-normalized game and a non-strategic game. The proof follows Candogan et al. (2011) apart from the fact that we choose μ-normalized games instead of the particular case $\mu^i(s^i) = 1$ for each $s^i \in S^i$ and $i \in N$. In the context of games with continuous strategy sets, a similar decomposition result is proved by Hwang and Rey-Bellet (2016), who view the set of all games as a Hilbert space.

To present our decomposition results, we first define the space $C_0 := \{f : S \to \mathbb{R}\}$ endowed with the following inner product:

$$\forall f, g \in C_0, \quad \langle g, f \rangle_0 = \sum_{\boldsymbol{s} \in S} \mu(\boldsymbol{s}) g(\boldsymbol{s}) f(\boldsymbol{s}). \tag{2.3}$$

Notice that the payoff functions of each player can be viewed as elements of C_0, i.e., $g^i \in C_0$ for any $i \in N$ and so a game $\boldsymbol{g}$ can be seen as an element of C_0^n. Hence, $\mathcal{G} \cong C_0^n$. Given a function $f \in C_0$ and a product measure η on the strategy profiles, we denote $\eta^{-i} f$ the function in C_0 defined for any $\boldsymbol{s} \in S$ by:

$$(\eta^{-i} f)(\boldsymbol{s}) = \eta^{-i}(\boldsymbol{s}^{-i}) f(\boldsymbol{s}).$$

Our next step is to endow the space of games with a suitable inner product. For any $\boldsymbol{g}_1, \boldsymbol{g}_2 \in \mathcal{G}$, using Eq. (2.3), we define:

$$\langle \boldsymbol{g}_1, \boldsymbol{g}_2 \rangle_{\mu,\eta} = \sum_{i \in N} \mu^i(S^i) \left\langle \eta^{-i} g_1^i, \eta^{-i} g_2^i \right\rangle_0 \tag{2.4}$$

and we denote $\oplus_{\mu,\eta}$ the direct orthogonal sum with respect to the above inner product.

Proposition 2.7. *The space of games $\mathcal{G}$ is the direct orthogonal-sum of the μ-normalized and non-strategic subspaces, i.e.,*

$$\mathcal{G} = \mathcal{NO} \oplus_{\mu,\eta} \mathcal{NS}.$$

2.3. (μ, η)**-decomposition result.** In this section we show that any game can be decomposed into the direct sum of three component games: an η-potential μ-normalized game, a (μ, η)-harmonic μ-normalized game, and a non-strategic game. Moreover, this decomposition is orthogonal for the inner product in Eq. (2.4).

Theorem 2.8. *The space of games is the direct orthogonal-sum of the μ-normalized η-potential, μ-normalized (μ, η)-harmonic and non-strategic subspaces, i,e.,*

$$\mathcal{G} = (\mathcal{NO} \cap \mathcal{P}) \oplus_{\mu,\eta} (\mathcal{NO} \cap \mathcal{H}) \oplus_{\mu,\eta} \mathcal{NS}. \tag{2.5}$$

Theorem 2.8 guarantees that, given a game $\boldsymbol{g}$ and a pair of measures (μ, η), we have

$$\boldsymbol{g} = \boldsymbol{g}_{\mathcal{NS}(\mu,\eta)} + \boldsymbol{g}_{\mathcal{P}(\mu,\eta)} + \boldsymbol{g}_{\mathcal{H}(\mu,\eta)}, \tag{2.6}$$

where $\boldsymbol{g}_{\mathcal{NS}(\mu,\eta)}$ is non-strategic, $\boldsymbol{g}_{\mathcal{P}(\mu,\eta)}$ is η-potential μ-normalized, and $\boldsymbol{g}_{\mathcal{H}(\mu,\eta)}$ is (μ, η)-harmonic μ-normalized. For the sake of simplicity, when there is no risk of confusion, we omit the indication of (μ, η).

The key point in the proof of Theorem 2.8 is to associate a given game to a flow on a suitable graph, as it is the case in Candogan et al. (2011). We then characterize η-potential games, non-strategic games, and (μ, η)-harmonic games in terms of their induced flows. Any flow generated by a game can be decomposed into two particular flows using linear algebra tools. From this decomposition result, we can obtain a decomposition in terms of games. The definition of the flow and the decomposition result are built on the gradient operator and the inner product, respectively. Hence, the decomposition is implicitly related to some notion of metric induced in the space of games. Candogan et al. (2011) use the same metric for both the definition of the flow generated by a game and the corresponding decomposition. We generalize their approach in the following ways: first, we deal with families of metrics instead of a unique one and, second, we consider different metrics for the definition of the flow generated by a game and the decomposition.

To prove our decomposition result, we use the Moore-Penrose pseudo-inverse of the gradient operator (see, e.g., Ben-Israel and Greville, 2003). Precisely, since the components of the decomposition are orthogonal, the pseudo-inverse operator allows us to determine the closest η-potential game to an arbitrary game with respect to the induced distance in the space of games. Candogan et al. (2011) provide in fact two approaches to obtain their decomposition result. The first approach relies on the Helmholtz decomposition tool, which becomes a degenerate case when applied to flows induced by games.[2] The second one relies on the Moore-Penrose pseudo-inverse operator. We choose the second approach since it is better suited to study the relation between duplicate strategies and decomposition.

Our result states that, given two product measures μ and η on the set of strategy profiles, the decomposition into μ-normalized η-potential, μ-normalized (μ, η)-harmonic, and non-strategic games is unique. The map that associates to a given game its components will be referred to as the *(μ, η)-decomposition map*. The use of two measures stems from the need to deal simultaneously with duplications and dilations, as highlighted in the introduction. Nevertheless, there is some redundancy by having two product measures as parameters. This gives rise to the following question: what is the set of product finite measures (μ, η) that induce the same (μ, η)-decomposition map?

Proposition 2.9. *Let $\mu, \mathring{\mu}$ and $\eta, \mathring{\eta}$ be four positive product measures. Then, the (μ, η)-decomposition map is identical to the $(\mathring{\mu}, \mathring{\eta})$-decomposition map if and only if there exist $\alpha, \beta > 0$ such that, for all $i \in N$*

$$\mathring{\mu}^i = \alpha\mu^i \quad \textit{and} \quad \mathring{\eta}^i = \beta\eta^i.$$

3. (μ, η)-DECOMPOSITION AND OPERATIONS ON GAMES

We now discuss consistency of (μ, η)-decompositions in games with duplicate strategies and games generated by dilations.

3.1. **Games with duplicate strategies.**

In this section, we investigate how the (μ, η)-decomposition behaves when we deal with duplications. Concerning this type of transformations, we will see that μ is the key parameter that needs to change when eliminating a duplicate strategy. In contrast, in this whole section η is fixed, apart from the change of its domain, inherent to the change of strategy space.

To study how our decomposition behaves in games with duplicate strategies, we first provide an example of a (μ, η)-harmonic game with some duplicate strategies whose reduction is not (μ, η)-harmonic. Then, we prove that duplicate strategies remain duplicate in the components of the decomposition result. That is, given a game with some duplicate strategy, the components of the (μ, η)-decomposition contain the same duplicate strategy. It follows that, when considering a game with some duplicate strategies, one can either first decompose it and then reduce it or vice versa. Each procedure yields a different decomposition. Nevertheless, our notion of (μ, η)-decomposition allows us to obtain a relation between the two approaches, which is described by the commutativity diagram in Fig. 1.

Definition 3.1. A game $\boldsymbol{g} \in \mathcal{G}$ is said to be a *game with duplicate strategies* if there exist $i \in N$ and some strategies $s_0^i, s_1^i \in S^i$, such that for every $j \in N$ and every $\boldsymbol{s}^{-i} \in S^{-i}$, we have: $g^j(s_0^i, \boldsymbol{s}^{-i}) = g^j(s_1^i, \boldsymbol{s}^{-i})$.

[2]According to the graph representation of games, *curl flows* in games are generated only by consecutive deviations of the same player and hence, they are mapped to 0. As a consequence, the divergence-free component of the Helmholtz decomposition tool is reduced to *harmonic flows.*

$$\begin{tikzcd} \boldsymbol{g} \arrow[r, "(\mu,\eta)\text{-decomposition}"] \arrow[d, "\text{reduction}"'] & (\boldsymbol{g}_{\mathcal{NS}}, \boldsymbol{g}_{\mathcal{P}}, \boldsymbol{g}_{\mathcal{H}}) \arrow[d, "\text{reduction}"] \\ \widehat{\boldsymbol{g}} \arrow[r, "(\widehat{\mu},\eta)\text{-decomposition}"'] & \begin{array}{l}((\widehat{\boldsymbol{g}})_{\mathcal{NS}}, (\widehat{\boldsymbol{g}})_{\mathcal{P}}, (\widehat{\boldsymbol{g}})_{\mathcal{H}}) = \\ (\widehat{(\boldsymbol{g}_{\mathcal{NS}})}, \widehat{(\boldsymbol{g}_{\mathcal{P}})}, \widehat{(\boldsymbol{g}_{\mathcal{H}})})\end{array} \end{tikzcd}$$

Figure 1

Definition 3.2. Given a game $\boldsymbol{g}$ with duplicate strategies s_0^i, s_1^i for some Player $i \in N$, the *reduced strategy set* of Player i is denoted by $\widehat{S}^i = S^i \backslash \{s_1^i\}$ and

$$\widehat{S} = \widehat{S}^i \times \left(\bigtimes_{j \neq i} S^j \right)$$

stands for the *reduced strategy profile set*. The *reduced game* is denoted by $\widehat{\boldsymbol{g}} = (\widehat{g}^j)_{j \in N}$, where $\widehat{g}^j : \widehat{S} \to \mathbb{R}$ is such that $\widehat{g}^j(\boldsymbol{s}) = g^j(\boldsymbol{s})$. If μ is a product finite measure on S, we also define the *reduced product measure* $\widehat{\mu}$ as follows:

(i) for every Player $j \neq i$, we have $\widehat{\mu}^j(s^j) = \mu^j(s^j)$,
(ii) for Player i, we have $\widehat{\mu}^i(s_0^i) = \mu^i(s_0^i) + \mu^i(s_1^i)$, and $\widehat{\mu}^i(s^i) = \mu^i(s^i)$ for any $s^i \neq s_0^i$.

Example 3.3. Let $\mu^1 = \mu^2 = \eta^1 = \eta^2 = (1, 1, 1)$. Let $\boldsymbol{g}$ be the following game:

	L		R_1		R_2	
T_1	2	−2	−1	1	−1	1
T_2	2	−2	−1	1	−1	1
B	−4	4	2	−2	2	−2

$\boldsymbol{g}$

The game $\boldsymbol{g}$ is (μ, η)-harmonic and admits a unique equilibrium that is the uniform profile. By eliminating the duplicate strategy of the row-player, we obtain the reduced game $\widehat{\boldsymbol{g}}$:

	L		R_1		R_2	
T	2	−2	−1	1	−1	1
B	−4	4	2	−2	2	−2

$\widehat{\boldsymbol{g}}$

The reduced game $\widehat{\boldsymbol{g}}$ admits the profile $((2/3, 1/3), (1/3, 1/3, 1/3))$ as unique equilibrium. Let $\widehat{\mu}^1 = (2, 1)$, $\widehat{\mu}^2 = \mu^2 = (1, 1, 1)$, $\eta^1 = (1, 1)$ and $\eta^2 = (1, 1, 1)$. Then,

$\boldsymbol{g}$ is a $(\widehat{\mu}, \eta)$-harmonic.[3] It is possible to further eliminate the duplicate strategy of the column-player. We then obtain the reduced game $\widehat{\widehat{\boldsymbol{g}}}$:

$$\begin{array}{c|cc|cc|}
 & \multicolumn{2}{c}{L} & \multicolumn{2}{c}{R} \\
\hline
T & 2 & -2 & -1 & 1 \\
\hline
B & -4 & 4 & 2 & -2 \\
\hline
\end{array}\;.$$
$$\widehat{\widehat{\boldsymbol{g}}}$$

Let $\widehat{\widehat{\mu}}^1 = (2,1)$, $\widehat{\widehat{\mu}}^2 = (1,2)$, $\eta^1 = (1,1)$ and $\eta^2 = (1,1)$. It follows that $\widehat{\widehat{\boldsymbol{g}}}$ is a $(\widehat{\widehat{\mu}}, \eta)$-harmonic game.

We now focus on the elimination of one duplicate strategy. There are games, such as the one in Example 3.3, where several players have duplicate strategies—possibly more than one. In these cases, it is possible to eliminate duplicate strategies one by one. At each iteration of the procedure, one duplicate strategy is eliminated and the measure $\widehat{\mu}$ is updated. The order of elimination does not influence the measure and the game obtained at the end of the iterated procedure.

In the sequel, without loss of generality we always consider a game where Player i has two duplicate strategies s_0^i and s_1^i.

Lemma 3.4. *Let $\boldsymbol{g} \in \mathcal{G}$ be a game with duplicate strategies s_0^i, s_1^i for some $i \in N$. Then, $\boldsymbol{g}_{\mathcal{NS}}$, $\boldsymbol{g}_{\mathcal{P}}$ and $\boldsymbol{g}_{\mathcal{H}}$ are games with duplicate strategies s_0^i, s_1^i.*

We then obtain the following relation between a game with duplicate strategies $\boldsymbol{g}$ and the reduced game $\widehat{\boldsymbol{g}}$.

Lemma 3.5. *Let $\boldsymbol{g}$ be a game with duplicate strategies $s_0^i, s_1^i \in S^i$. Then,*

(i) *if $\boldsymbol{g}$ is η-potential, then $\widehat{\boldsymbol{g}}$ is η-potential,*
(ii) *if $\boldsymbol{g}$ is non-strategic, then $\widehat{\boldsymbol{g}}$ is non-strategic,*
(iii) *if $\boldsymbol{g}$ is μ-normalized, then $\widehat{\boldsymbol{g}}$ is $\widehat{\mu}$-normalized,*
(iv) *if $\boldsymbol{g}$ is (μ, η)-harmonic, then $\widehat{\boldsymbol{g}}$ is $(\widehat{\mu}, \eta)$-harmonic.*

The following theorem is an immediate consequence of the two previous lemmas.

Theorem 3.6. *Let $\boldsymbol{g}$ be a game with duplicate strategies. Then*

$$(\widehat{\boldsymbol{g}_{\mathcal{NS}(\mu,\eta)}}) = (\widehat{\boldsymbol{g}})_{\mathcal{NS}(\widehat{\mu},\eta)},$$
$$(\widehat{\boldsymbol{g}_{\mathcal{P}(\mu,\eta)}}) = (\widehat{\boldsymbol{g}})_{\mathcal{P}(\widehat{\mu},\eta)},$$
$$(\widehat{\boldsymbol{g}_{\mathcal{H}(\mu,\eta)}}) = (\widehat{\boldsymbol{g}})_{\mathcal{H}(\widehat{\mu},\eta)},$$

i.e., the reduced games of the components are the components of the $(\widehat{\mu}, \eta)$-decomposition of $\widehat{\boldsymbol{g}}$.

The following remarks complete the scope of Theorem 3.6. The first remark presents the reverse operation, i.e., transforming a game into a game with duplicate strategies, whereas the second one replaces the notion of duplication with the more general notion of *redundancy*.

[3]It would be more proper to define $\widehat{\eta}^1$ as the restriction of η^1 to the reduced strategy set of the row-player but for convenience we keep the same notation.

Remark 3.7. An immediate consequence of Theorem 3.6 and of the uniqueness of the decomposition result of Theorem 2.8 is the following. Let $\boldsymbol{g}$ be a (μ, η)-harmonic game and $\breve{\boldsymbol{g}}$ the game where the strategy s^i of Player i has been duplicated into s^i_A and s^i_B. Then the game $\breve{\boldsymbol{g}}$ is $(\breve{\mu}, \breve{\eta})$-harmonic for

$$\begin{aligned} \breve{\eta}^j &= \eta^j \quad \text{for } j \neq i, \quad \breve{\eta}^i(t^i) = \eta^i(t^i) \quad \text{for } t^i \neq s^i, \quad \breve{\eta}^i(s^i_A) = \breve{\eta}^i(s^i_B) = \eta^i(s^i), \\ \breve{\mu}^j &= \mu^j \quad \text{for } j \neq i, \quad \breve{\mu}^i(t^i) = \mu^i(t^i) \quad \text{for } t^i \neq s^i, \quad \breve{\mu}^i(s^i_A) + \breve{\mu}^i(s^i_B) = \mu^i(s^i). \end{aligned} \tag{3.1}$$

Notice that $\breve{\mu}$ is not uniquely defined.

Remark 3.8. The notion of redundant strategy was introduced by Govindan and Wilson (2009). A pure strategy t^i of Player $i \in N$ is *redundant* if, for all players, its payoffs are a mixture of the payoffs of her other strategies, i.e., the payoff vector $(g^j(t^i, .))_{j \in N}$ is a convex combination α of the payoff vectors of the other pure strategies $S^i \backslash \{t^i\}$, i.e.,

$$\forall \boldsymbol{s}^{-i} \in S^{-i},\ \forall j \in N,\ g^j(t^i, \boldsymbol{s}^{-i}) = \sum_{s^i \in S^i \backslash \{t^i\}} \alpha(s^i) g^j(s^i, \boldsymbol{s}^{-i}).$$

Concerning this transformation, one can eliminate the strategy t^i and consider the reduced game, as it was the case in duplications. Considering the (μ, η)-decomposition and then eliminating the redundant strategy is equivalent to first eliminating and then considering the (μ_*, η) decomposition, where

(i) for every player $j \neq i$, we have $\mu^j_*(s^j) = \mu^j(s^j)$,
(ii) for Player i and every strategy $s^i \in S^i \backslash \{t^i\}$, we have

$$\mu^i_*(s^i) = \mu^i(s^i) + \alpha(s^i)\mu^i(t^i).$$

3.2. **Affine transformation and decomposition.** We now investigate then behavior of our decompositions when we deal with affine transformations. We first focus on translations and then on dilations. When dealing with translations, measures μ and η do not change. For dilations we will see that μ is fixed, whereas η changes.

3.2.1. *Translations.* First, we look at translations of the payoffs by a function that depends on the strategies of the other players. We show that, as it is the case in Candogan et al. (2011), there is a natural relation between the (μ, η)-decomposition of a game $\boldsymbol{g}$ and the (μ, η)-decomposition of the translated game.

For any $i \in N$, let $\ell^i : S^{-i} \to \mathbb{R}$. Then, we can define the *translation* of a game $\boldsymbol{g}$ by $\boldsymbol{\ell} = (\ell^i)_{i \in N}$ as follows:

$$\forall i \in N,\ \forall \boldsymbol{s} \in S, \quad g^i_{\boldsymbol{\ell}}(\boldsymbol{s}) = g^i(\boldsymbol{s}) + \ell^i(\boldsymbol{s}^{-i}). \tag{3.2}$$

It is clear that the non-strategic component of $\boldsymbol{g}_{\boldsymbol{\ell}}$ is the translation by $\boldsymbol{\ell}$ of the non-strategic component of $\boldsymbol{g}$. It follows that the (μ, η)-decomposition of $\boldsymbol{g}$ and the (μ, η)-decomposition of $\boldsymbol{g}_{\boldsymbol{\ell}}$ share the same (μ, η)-harmonic μ-normalized, and η-potential μ-normalized components.

Proposition 3.9. *Let $\boldsymbol{g}$ be a game and let $\boldsymbol{g}_{\boldsymbol{\ell}}$ be its translation by $\boldsymbol{\ell} = (\ell^i)_{i \in N}$. Then, the (μ, η)-decompositions of $\boldsymbol{g}$ and $\boldsymbol{g}_{\boldsymbol{\ell}}$ are related as follows:*

- $(\boldsymbol{g}_{\boldsymbol{\ell}})_{\mathcal{P}} = \boldsymbol{g}_{\mathcal{P}}$
- $(\boldsymbol{g}_{\boldsymbol{\ell}})_{\mathcal{H}} = \boldsymbol{g}_{\mathcal{H}}$

- $(\boldsymbol{g_\ell})_{\mathcal{NS}} = \boldsymbol{g}_{\mathcal{NS}} + \boldsymbol{\ell}$

Proof. Given Eq. (3.2) and the fact that $\ell^i \in \mathcal{NS}^i$ for any $i \in N$, the result follows from Theorem 2.8. ■

3.2.2. *Product dilations.* We now consider a particular type of dilations, called product dilations and study their effects on the decomposition of games.

Definition 3.10. For every $i \in N$, let $\beta^i : S^{-i} \to (0, +\infty)$. The *dilation* $\boldsymbol{\beta}$ of a game $\boldsymbol{g}$ is given by

$$\forall i \in N,\ \forall \boldsymbol{s} \in S,\ (\beta \cdot g)^i(\boldsymbol{s}) = \beta^i(\boldsymbol{s}^{-i}) g^i(\boldsymbol{s}).$$

We call $(\boldsymbol{\beta} \cdot \boldsymbol{g})$ the $\boldsymbol{\beta}$*-dilated game.*

Definition 3.11. A dilation $\boldsymbol{\beta}$ is a *product dilation* if for every $j \in N$, there exists $b^j : S^j \to (0, \infty)$ such that

$$\forall i \in N,\ \forall \boldsymbol{s}^{-i} \in S^{-i},\ \beta^i(\boldsymbol{s}^{-i}) = \prod_{j \neq i} b^j(s^j), \tag{3.3}$$

We say that $\boldsymbol{\beta}$ is generated by $\boldsymbol{b}$.

Although these types of dilation may change the Nash equilibrium set, they do it in a very structured way, as the following proposition shows.

Proposition 3.12. *Let $\boldsymbol{\beta}$ be a product dilation generated by $\boldsymbol{b}$. If $\mathcal{E}(\boldsymbol{g})$ is the set of Nash equilibria of the game $\boldsymbol{g}$, then the set of Nash equilibria of $(\boldsymbol{\beta} \cdot \boldsymbol{g})$ is given by:*

$$\mathcal{E}(\boldsymbol{\beta} \cdot \boldsymbol{g}) = \left\{ (\overline{y}^i)_{i \in N} : y^i(s^i) = \frac{x^i(s^i)}{b^i(s^i)} \text{ and } (x^i)_{i \in N} \in \mathcal{E}(\boldsymbol{g}) \right\}.$$

In particular, when $\boldsymbol{\beta}$ is a product dilation, if $\boldsymbol{x}$ is a pure Nash equilibrium of $\boldsymbol{g}$ then $\boldsymbol{x}$ is also a pure Nash equilibrium of $(\boldsymbol{\beta} \cdot \boldsymbol{g})$. If one knows the set of equilibria of some game $\boldsymbol{g}$, one can compute the set of equilibria of $(\boldsymbol{\beta} \cdot \boldsymbol{g})$ without knowing $\boldsymbol{g}$. In the particular case when, for every $i \in N$, $b^i(s^i)$ does not depend on s^i, the Nash equilibrium sets of both games coincide.

Given positive product measures η and b on the strategy profiles, we define the product measure η/b pointwise as follows:

$$\forall i \in N, \forall s^i \in S^i,\ \left(\frac{\eta}{b}\right)^i (s^i) = \frac{\eta^i(s^i)}{b^i(s^i)}.$$

Theorem 3.13. *Let μ, η and b be positive product measures on S. Let $\boldsymbol{g}$ be a finite game. If $\boldsymbol{\beta}$ is the product dilation generated by $\boldsymbol{b}$, then*

$$\begin{aligned}
(\boldsymbol{\beta} \cdot \boldsymbol{g}_{\mathcal{NS}(\mu,\eta)}) &= (\boldsymbol{\beta} \cdot \boldsymbol{g})_{\mathcal{NS}(\mu,\eta/b)} \\
(\boldsymbol{\beta} \cdot \boldsymbol{g}_{\mathcal{P}(\mu,\eta)}) &= (\boldsymbol{\beta} \cdot \boldsymbol{g})_{\mathcal{P}(\mu,\eta/b)} \\
(\boldsymbol{\beta} \cdot \boldsymbol{g}_{\mathcal{H}(\mu,\eta)}) &= (\boldsymbol{\beta} \cdot \boldsymbol{g})_{\mathcal{H}(\mu,\eta/b)}.
\end{aligned}$$

The result of Theorem 3.13 can be represented by the diagram in Fig. 2, where, for the sake of simplicity, we have omitted the reference to the pairs of measures used in the decomposition.

$$
\begin{tikzcd}
\boldsymbol{g} \arrow[r, "(\mu,\eta)\text{-decomposition}"] \arrow[d, "\text{product dilation}"'] & (\boldsymbol{g}_{\mathcal{NS}}, \boldsymbol{g}_{\mathcal{P}}, \boldsymbol{g}_{\mathcal{H}}) \arrow[d, "\text{product dilation}"] \\
(\boldsymbol{\beta}\cdot\boldsymbol{g}) \arrow[r, "(\mu,\eta/b)\text{-decomposition}"'] & ((\boldsymbol{\beta}\cdot\boldsymbol{g})_{\mathcal{NS}}, (\boldsymbol{\beta}\cdot\boldsymbol{g})_{\mathcal{P}}, (\boldsymbol{\beta}\cdot\boldsymbol{g})_{\mathcal{H}})
\end{tikzcd}
$$

Figure 2

The following two facts are easy consequences of Theorem 3.13. First, in the context of two-player games, any $\boldsymbol{\beta}$-dilation is a product dilation since there is only one adversary player. Therefore, in this case Theorem 3.13 covers in fact any type of dilation. Second, consider $\boldsymbol{\beta}$ a dilation such that for every $i \in N$, β^i does not depend on $\boldsymbol{s}^{-i}$ but only on i. Then, there exists a vector $(b^i)_{i\in N}$ such that

$$\forall i \in N,\ \beta^i = \prod_{j\neq i} b^j.$$

Hence, $\boldsymbol{\beta}$ is a product dilation. Therefore, Theorem 3.13 can be applied. Notice that in this particular case, $\boldsymbol{g}$ and $(\boldsymbol{\beta}\cdot\boldsymbol{g})$ share the same set of Nash equilibria.

Cheng et al. (2016) introduced a decomposition of games into two types of games called weighted harmonic and weighted potential games. When restricting to these two classes of games, it is possible to prove a weaker version of Theorem 3.13 with dilations that depend only on the players.

We now provide two examples of product-dilations and a final example combining dilation and duplication. In Example 3.14 the dilation depends only on the players, whereas in Example 3.15 the dilation depends also on the strategy that the players play.

Example 3.14. Let $\mu^1 = \mu^2 = \eta^1 = \eta^2 = (1,1)$, $\beta^1 \equiv 2$, $\beta^2 \equiv 4$. Consider the following μ-normalized game $\boldsymbol{g}$ (on the left) and its $\boldsymbol{\beta}$-dilation $(\boldsymbol{\beta}\cdot\boldsymbol{g})$ (on the right):

$$
\begin{array}{c|cc|cc|}
 & \multicolumn{2}{c}{L} & \multicolumn{2}{c}{R} \\ \cline{2-5}
T & 4 & -1 & -3 & 1 \\ \cline{2-5}
B & -4 & 0 & 3 & 0 \\ \cline{2-5}
\multicolumn{5}{c}{\boldsymbol{g}}
\end{array}
\qquad
\begin{array}{c|cc|cc|}
 & \multicolumn{2}{c}{L} & \multicolumn{2}{c}{R} \\ \cline{2-5}
T & 8 & -4 & -6 & 4 \\ \cline{2-5}
B & -8 & 0 & 6 & 0 \\ \cline{2-5}
\multicolumn{5}{c}{(\boldsymbol{\beta}\cdot\boldsymbol{g})}
\end{array}.
$$

The $(\mu$-$\eta)$-decomposition of $\boldsymbol{g}$ is given by

$$
\begin{array}{c|cc|cc|}
 & \multicolumn{2}{c}{L} & \multicolumn{2}{c}{R} \\ \cline{2-5}
T & 2 & 1 & -1 & -1 \\ \cline{2-5}
B & -2 & -2 & 1 & 2 \\ \cline{2-5}
\multicolumn{5}{c}{\boldsymbol{g}_{\mathcal{P}}}
\end{array}
\qquad
\begin{array}{c|cc|cc|}
 & \multicolumn{2}{c}{L} & \multicolumn{2}{c}{R} \\ \cline{2-5}
T & 2 & -2 & -2 & 2 \\ \cline{2-5}
B & -2 & 2 & 2 & -2 \\ \cline{2-5}
\multicolumn{5}{c}{\boldsymbol{g}_{\mathcal{H}}}
\end{array}
$$

and the $\boldsymbol{\beta}$-dilation of $\boldsymbol{g}_{\mathcal{P}}$ and $\boldsymbol{g}_{\mathcal{H}}$ are given respectively by:

	L		R	
T	4	4	-2	-4
B	-4	-8	2	8

$(\boldsymbol{\beta} \cdot \boldsymbol{g}_{\mathcal{P}})$

	L		R	
T	4	-8	-4	8
B	-4	8	4	-8

$(\boldsymbol{\beta} \cdot \boldsymbol{g}_{\mathcal{H}})$.

Let $\widetilde{\eta}^1 \equiv 1/4$ and $\widetilde{\eta}^2 \equiv 1/2$. It is easy to check that $(\boldsymbol{\beta} \cdot \boldsymbol{g}_{\mathcal{P}})$ is an $\widetilde{\eta}$-potential game and that $(\boldsymbol{\beta} \cdot \boldsymbol{g}_{\mathcal{H}})$ is a $(\mu,\widetilde{\eta})$-harmonic game. On one hand $\boldsymbol{g}_{\mathcal{P}}$ and $(\boldsymbol{\beta} \cdot \boldsymbol{g}_{\mathcal{P}})$ share the same pure Nash equilibria, i.e., (T, L) and (B, R), and the same mixed Nash equilibrium (x^1, x^2), where $x^1 = (2/3, 1/3)$ and $x^2 = (1/3, 2/3)$; as a consequence, $\mathcal{E}(\boldsymbol{g}_{\mathcal{P}}) = \mathcal{E}(\boldsymbol{\beta} \cdot \boldsymbol{g}_{\mathcal{P}})$. On the other hand, there is no pure Nash equilibrium either in $\boldsymbol{g}_{\mathcal{H}}$ or in $(\boldsymbol{\beta} \cdot \boldsymbol{g}_{\mathcal{H}})$ and the mixed strategy profile (x^1, x^2) given by $x^1 = x^2 = (1/2, 1/2)$ is a mixed Nash equilibrium in both $\boldsymbol{g}_{\mathcal{H}}$ and $(\boldsymbol{\beta} \cdot \boldsymbol{g}_{\mathcal{H}})$. It follows that $\mathcal{E}(\boldsymbol{g}_{\mathcal{H}}) = \mathcal{E}(\boldsymbol{\beta} \cdot \boldsymbol{g}_{\mathcal{H}})$.

Next, we consider a dilation that depends on the strategy of the other player and we show how mixed Nash equilibria vary in the transformed game.

Example 3.15. Let $\mu^1 = \mu^2 = \eta^1 = \eta^2 = (1, 1)$, $\beta^1 \equiv b^2 = (2, 1)$ and $\beta^2 \equiv b^1 = (1, 3)$. Consider the μ-normalized game $\boldsymbol{g}$ (appeared also in Example 3.14) (on the left) and its $\boldsymbol{\beta}$-dilation $(\boldsymbol{\beta} \cdot \boldsymbol{g})$ (on the right):

	L		R	
T	4	-1	-3	1
B	-4	0	3	0

$\boldsymbol{g}$

	L		R	
T	8	-1	-3	1
B	-8	0	3	0

$(\boldsymbol{\beta} \cdot \boldsymbol{g})$.

In view of the $(\mu$-$\eta)$-decomposition of $\boldsymbol{g}$ (see Example 3.14), the $\boldsymbol{\beta}$-dilation of $\boldsymbol{g}_{\mathcal{P}}$ and $\boldsymbol{g}_{\mathcal{H}}$ are

	L		R	
T	4	1	-1	-1
B	-4	-6	1	6

$(\boldsymbol{\beta} \cdot \boldsymbol{g}_{\mathcal{P}})$

	L		R	
T	4	-2	-2	2
B	-4	6	2	-6

$(\boldsymbol{\beta} \cdot \boldsymbol{g}_{\mathcal{H}})$.

Let $\widetilde{\eta}^1 = \left(1, \frac{1}{3}\right)$ and $\widetilde{\eta}^2 = \left(\frac{1}{2}, 1\right)$. It is easy to check that $(\boldsymbol{\beta} \cdot \boldsymbol{g}_{\mathcal{P}})$ is an $\widetilde{\eta}$-potential game and that $(\boldsymbol{\beta} \cdot \boldsymbol{g}_{\mathcal{H}})$ is a $(\mu, \widetilde{\eta})$-harmonic game. Notice that $\boldsymbol{g}_{\mathcal{P}}$ and $(\boldsymbol{\beta} \cdot \boldsymbol{g}_{\mathcal{P}})$ share the same pure Nash equilibria, however the mixed Nash equilibrium of $\boldsymbol{g}_{\mathcal{P}}$ has been transformed, according to Proposition 3.12, into the mixed strategy profile (x^1, x^2), where $x^1 = (6/7, 1/7)$ and $x^2 = (1/5, 4/5)$, which is a mixed Nash equilibrium in $(\boldsymbol{\beta} \cdot \boldsymbol{g}_{\mathcal{P}})$. Likewise, the mixed Nash equilibrium of $\boldsymbol{g}_{\mathcal{H}}$ has been transformed into the mixed strategy profile (x^1, x^2), where $x^1 = (3/4, 1/4)$ and $x^2 = (2/3, 1/3)$, which is the unique mixed Nash equilibrium in $(\boldsymbol{\beta} \cdot \boldsymbol{g}_{\mathcal{H}})$.

Finally, we describe an example with both dilation and duplication.

Example 3.16. Let $\mu^1 = \mu^2 = \eta^1 = \eta^2 = (1, 1)$, $\beta^1 \equiv b^2 = (2, 1)$ and $\beta^2 \equiv b^1 = (1, 3)$. Consider the following μ-normalized game $\boldsymbol{g}$ (on the left) and let $(\boldsymbol{\beta} \cdot \breve{\boldsymbol{g}})$ (on the right) be its $\boldsymbol{\beta}$-dilation where, moreover, the strategy T is duplicated:

	L	R
T	$4 \quad -1$	$-3 \quad 1$
B	$-4 \quad 0$	$3 \quad 0$

$\boldsymbol{g}$

	L	R
T_1	$8 \quad -1$	$-3 \quad 1$
T_2	$8 \quad -1$	$-3 \quad 1$
B	$-8 \quad 0$	$3 \quad 0$

$(\boldsymbol{\beta} \cdot \breve{\boldsymbol{g}})$.

In view of the (μ-η)-decomposition of $\boldsymbol{g}$ (see Example 3.14), if we apply the same pair of transformations to $\boldsymbol{g}_{\mathcal{P}}$ and $\boldsymbol{g}_{\mathcal{H}}$, we get

	L	R
T_1	$4 \quad 1$	$-1 \quad -1$
T_2	$4 \quad 1$	$-1 \quad -1$
B	$-4 \quad -6$	$1 \quad 6$

$(\boldsymbol{\beta} \cdot (\breve{\boldsymbol{g}})_{\mathcal{P}})$

	L	R
T_1	$4 \quad -2$	$-2 \quad 2$
T_2	$4 \quad -2$	$-2 \quad 2$
B	$-4 \quad 6$	$2 \quad -6$

$(\boldsymbol{\beta} \cdot (\breve{\boldsymbol{g}})_{\mathcal{H}})$.

Let $\breve{\eta}^1_{\boldsymbol{\beta}} = \left(1, 1, \frac{1}{3}\right)$, $\breve{\eta}^2_{\boldsymbol{\beta}} = \left(\frac{1}{2}, 1\right)$, $\breve{\mu}^1_{\boldsymbol{\beta}} = (\alpha, 1-\alpha, 1)$, $\alpha \in (0,1)$ and $\breve{\mu}^2_{\boldsymbol{\beta}} = (1,1)$. It is easy to check that $(\boldsymbol{\beta} \cdot (\breve{\boldsymbol{g}})_{\mathcal{P}})$ is an $\breve{\eta}_{\boldsymbol{\beta}}$-potential game and $(\boldsymbol{\beta} \cdot (\breve{\boldsymbol{g}})_{\mathcal{H}})$ is a $(\breve{\mu}_{\boldsymbol{\beta}}, \breve{\eta}_{\boldsymbol{\beta}})$-harmonic game. Notice that the mixed Nash equilibrium of $\boldsymbol{g}_{\mathcal{P}}$ has been transformed, according to Proposition 3.12, into the mixed strategy profile (x^1, x^2), where $x^1 = (3/7, 3/7, 1/7)$ and $x^2 = (1/5, 4/5)$, which is a mixed Nash equilibrium in $(\boldsymbol{\beta} \cdot (\breve{\boldsymbol{g}})_{\mathcal{P}})$. Likewise, the mixed Nash equilibrium of $\boldsymbol{g}_{\mathcal{H}}$ has been transformed into the mixed strategy profile (x^1, x^2), where $x^1 = (3/8, 3/8, 1/4)$ and $x^2 = (2/3, 1/3)$, which is a mixed Nash equilibrium in $(\boldsymbol{\beta} \cdot (\breve{\boldsymbol{g}})_{\mathcal{H}})$.

3.2.3. *General dilations.* We focused in the previous paragraph on product dilations. When dealing with general dilations, it is impossible to obtain a commutative property such as the one that holds for duplications, translations, and product dilations.

Informally, a dilation that is not a product dilation preserves the set of pure Nash-equilibria but it operates on the set of mixed Nash-equilibria in a non structured way, hence Proposition 3.12 cannot be applied. In Theorem 2.6, we proved that if $\boldsymbol{g}$ is (μ, η)-harmonic, then $(\overline{\mu\eta}^i)_{i \in N}$ is a Nash equilibrium. The two facts yield an incompatibility. We illustrate this incompatibility with a counter-example.

Example 3.17. We present two games $\boldsymbol{g}, \boldsymbol{g}'$, which are (μ, η)-harmonic for $\mu^1 = \mu^2 = \mu^3 = \eta^1 = \eta^2 = \eta^3 = (1,1)$, and a dilation $\boldsymbol{\beta}$ such that there exists no pair $(\widetilde{\mu}, \widetilde{\eta})$ that makes both games $(\boldsymbol{\beta} \cdot \boldsymbol{g})$ and $(\boldsymbol{\beta} \cdot \boldsymbol{g}')$ $(\widetilde{\mu}, \widetilde{\eta})$-harmonic. Consider the following 3-player games. Player 1 chooses the row, Player 2 chooses the column, and Player 3 chooses the matrix. In the first game $\boldsymbol{g}$, Player 3 is dummy and Players 1 and 2 play a matching pennies game, where Player 2 wants to match and Player 1 wants to mismatch:

	H_2	T_2
H_1	$-1 \quad 1 \quad 0$	$1 \quad -1 \quad 0$
T_1	$1 \quad -1 \quad 0$	$-1 \quad 1 \quad 0$

H_3

	H_2	T_2
H_1	$-1 \quad 1 \quad 0$	$1 \quad -1 \quad 0$
T_1	$1 \quad -1 \quad 0$	$-1 \quad 1 \quad 0$

T_3.

In the second game $\boldsymbol{g}'$, Player 1 is a dummy and Players 2 and 3 play a matching pennies game, where Player 2 wants to match and Player 3 wants to mismatch:

H_3:

	H_2			T_2		
H_1	0	1	−1	0	−1	1
T_1	0	1	−1	0	−1	1

T_3:

	H_2			T_2		
H_1	0	−1	1	0	1	−1
T_1	0	−1	1	0	1	−1

.

It is easy to check that these two games are indeed (μ, η)-harmonic. Consider now the following dilation:

- $\beta^2(\cdot, \cdot) = 1$.
- β^1 only depends on player 2: $\beta^1(H_2, H_3) = \beta^1(H_2, T_3) = 1$ and $\beta^1(T_2, H_3) = \beta^1(T_2, T_3) = 2$,
- β^3 only depends on player 2: $\beta^3(H_1, H_2) = \beta^3(T_1, H_2) = 1$ and $\beta^3(H_1, T_2) = \beta^3(T_1, T_2) = 3$

We obtain the new game $(\boldsymbol{\beta} \cdot \boldsymbol{g})$ given by

H_3:

	H_2			T_2		
H_1	−1	1	0	2	−1	0
T_1	1	−1	0	−2	1	0

T_3:

	H_2			T_2		
H_1	−1	1	0	2	−1	0
T_1	1	−1	0	−2	1	0

and $(\boldsymbol{\beta} \cdot \boldsymbol{g}')$ given by

H_3:

	H_2			T_2		
H_1	0	1	−1	0	−1	3
T_1	0	1	−1	0	−1	3

T_3:

	H_2			T_2		
H_1	0	−1	1	0	1	−3
T_1	0	−1	1	0	1	−3

.

On one hand, we see that Player 2 has to play $(2/3, 1/3)$ at the equilibrium in $(\boldsymbol{\beta} \cdot \boldsymbol{g})$ and $(3/4, 1/4)$ in $(\boldsymbol{\beta} \cdot \boldsymbol{g}')$. On the other hand, if there existed a unique pair $(\widetilde{\mu}, \widetilde{\eta})$ such that both $(\boldsymbol{\beta} \cdot \boldsymbol{g})$ and $(\boldsymbol{\beta} \cdot \boldsymbol{g}')$ are $(\widetilde{\mu}, \widetilde{\eta})$-harmonic, then both games would admit $\left(\overline{\widetilde{\mu}\widetilde{\eta}}^i\right)_{i\in N}$ as an equilibrium, i.e., there would exist one strategy of Player 2 that is both an equilibrium in $(\boldsymbol{\beta} \cdot \boldsymbol{g})$ and in $(\boldsymbol{\beta} \cdot \boldsymbol{g}')$. We saw that the equilibrium strategies of Player 2 in $(\boldsymbol{\beta} \cdot \boldsymbol{g})$ and $(\boldsymbol{\beta} \cdot \boldsymbol{g}')$ are different, which produces a contradiction.

Appendix A. Proofs

To prove the characterization of equilibrium in (μ, η)-harmonic game (Theorem 2.6), we need some notation which is naturally introduced in the proof of the decomposition result. Hence, the proofs of Section 2.1 are postponed after the proofs of Sections 2.2 and 2.3.

A.1. **Proofs of Section 2.2.** We prove Proposition 2.7 by describing $\mathcal{NO}$ and $\mathcal{NS}$ through orthogonal projection operators. The notation I_X stands for the identity operator over a set X. First, we define for any $i \in N$, the linear operators Λ^i : $C_0 \to C_0$ and $\Pi^i : C_0 \to C_0$ as follows:

$$\Lambda^i(g^i)(s^i, \boldsymbol{s}^{-i}) = \sum_{t^i \in S^i} \overline{\mu}^i(t^i) g^i(t^i, \boldsymbol{s}^{-i}), \tag{A.1}$$

$$\Pi^i = I_{C_0} - \Lambda^i. \tag{A.2}$$

We call $\mathcal{NS}^i \subset C_0$ the set of functions f for which there exists $\ell \colon S^{-i} \to \mathbb{R}$ such that

$$\forall s^i \in S^i,\ \forall \boldsymbol{s}^{-i} \in S^{-i},\ f(s^i, \boldsymbol{s}^{-i}) = \ell(\boldsymbol{s}^{-i}). \tag{A.3}$$

We call $\mathcal{NO}^i \subset C_0$ the set of functions f such that

$$\forall \boldsymbol{s}^{-i} \in S^{-i},\ \sum_{\boldsymbol{s}^i \in S^i} \mu^i(s^i) f(s^i, \boldsymbol{s}^{-i}) = 0. \tag{A.4}$$

Lemma A.1. (a) *Π^i and Λ^i are projections onto C_0.*

(b) *For every $f_K \in \mathrm{Ker}(\Lambda^i)$ and for every $f_I \in \mathrm{Im}(\Lambda^i)$,*

$$\left\langle \eta^{-i} f_K, \eta^{-i} f_I \right\rangle_0 = 0.$$

(c) *We have $\mathcal{NO}^i = \mathrm{Ker}(\Lambda^i)$ and $\mathcal{NS}^i = \mathrm{Im}(\Lambda^i)$.*

Proof. (a) In view of Eqs. (A.1) and (A.2), we have

$$\mathrm{Ker}(\Pi^i) = \mathrm{Im}(\Lambda^i) \quad \text{and} \quad \mathrm{Ker}(\Lambda^i) = \mathrm{Im}(\Pi^i). \tag{A.5}$$

Thus, we only need to prove that $\Lambda^i \circ \Lambda^i = \Lambda^i$. Indeed,

$$\begin{aligned}
\Lambda^i \circ \Lambda^i(g^i)(s^i, \boldsymbol{s}^{-i}) &= \sum_{t^i \in S^i} \overline{\mu}^i(t^i) \left(\Lambda^i(g^i)(t^i, \boldsymbol{s}^{-i})\right) \\
&= \sum_{t^i \in S^i} \overline{\mu}^i(t^i) \sum_{r^i \in S^i} \overline{\mu}^i(r^i) g^i(r^i, \boldsymbol{s}^{-i}) \\
&= \sum_{r^i \in S^i} \overline{\mu}^i(r^i) \left(\sum_{t^i \in S^i} \overline{\mu}^i(t^i) \right) g^i(r^i, \boldsymbol{s}^{-i}) \\
&= \sum_{r^i \in S^i} \overline{\mu}^i(r^i) g^i(r^i, \boldsymbol{s}^{-i}),
\end{aligned}$$

where the last equality stems from the fact that $\overline{\mu}^i$ is probability distribution. Hence, Λ^i is a projection operator.

(b) Let $f_K \in \mathrm{Ker}(\Lambda^i)$ and $f_I \in \mathrm{Im}(\Lambda^i)$. By definition of Λ^i, any function in $\mathrm{Im}(\Lambda^i)$ does not depend on s^i, i.e., $\mathrm{Im}(\Lambda^i) \subset \mathcal{NS}^i$. Hence, there exists $\ell : S^{-i} \to \mathbb{R}$ such that $\ell(\boldsymbol{s}^{-i}) = f_I(s^i, \boldsymbol{s}^{-i})$. Therefore,

$$\begin{aligned}
\left\langle \eta^{-i} f_K, \eta^{-i} f_I \right\rangle_0 &= \sum_{\boldsymbol{s} \in S} \mu(\boldsymbol{s}) (\eta^{-i}(\boldsymbol{s}^{-i}))^2 f_K(\boldsymbol{s}) f_I(\boldsymbol{s}) \\
&= \mu^i(S^i) \sum_{\boldsymbol{s}^{-i} \in S^{-i}} \mu^{-i}(\boldsymbol{s}^{-i}) (\eta^{-i}(\boldsymbol{s}^{-i}))^2 \ell(\boldsymbol{s}^{-i}) \left(\sum_{s^i \in S^i} \overline{\mu}^i(s^i) f_K(s^i, \boldsymbol{s}^{-i}) \right) \\
&= 0,
\end{aligned}$$

since $f_K \in \mathrm{Ker}(\Lambda^i)$.

(c) In view of Eqs. (A.1) and (A.4),

$$\sum_{s^i \in S^i} \mu^i(s^i) f(s^i, \boldsymbol{s}^{-i}) = \mu^i(S^i) \left(\sum_{s^i \in S^i} \overline{\mu}^i(s^i) f(s^i, \boldsymbol{s}^{-i}) \right) = \mu^i(S^i) \Lambda^i(\boldsymbol{s}).$$

Hence, we have $\mathrm{Ker}(\Lambda^i) = \mathcal{NO}^i$. We saw in the previous paragraph that $\mathrm{Im}(\Lambda^i) \subset \mathcal{NS}^i$. Reciprocally, if f a function in C_0 does not depend on s^i, an immediate computation yields $\Lambda^i(f) = f$ and the equality between the two sets. ■

We now define $\Lambda : C_0^n \to C_0^n$ and $\Pi : C_0^n \to C_0^n$ as

$$\Lambda(\boldsymbol{g}) = \left(\Lambda^1(g^1), \ldots, \Lambda^n(g^n) \right), \tag{A.6}$$

$$\Pi(\boldsymbol{g}) = \left(\Pi^1(g^1), \ldots, \Pi^n(g^n) \right), \tag{A.7}$$

with $\boldsymbol{g} = (g^i)_{i \in N}$.

Lemma A.2. (a) *We have*

$$\mathcal{NO} = \left\{ \boldsymbol{g} \in \mathcal{G} : \forall i \in N, g^i \in \mathcal{NO}^i \right\} = \mathrm{Ker}(\Lambda), \tag{A.8}$$

$$\mathcal{NS} = \left\{ \boldsymbol{g} \in \mathcal{G} : \forall i \in N, g^i \in \mathcal{NS}^i \right\} = \mathrm{Im}(\Lambda). \tag{A.9}$$

(b) Π *and* Λ *are orthogonal projections onto* $\mathcal{G}$ *for* $\langle \cdot, \cdot \rangle_{\mu,\eta}$.

Proof. (a) This follows immediately from Lemma A.1(c).

(b) We have $\mathrm{Ker}(\Lambda) = \mathcal{NO}$ and $\mathrm{Im}(\Lambda) = \mathcal{NS}$, where Λ is defined as in Eq. (A.6). Let $\boldsymbol{g}_{\mathcal{NO}} \in \mathcal{NO}$ and $\boldsymbol{g}_{\mathcal{NS}} \in \mathcal{NS}$. Then, using the definition of the scalar product of Eq. (2.4) and Lemma A.1(b), we obtain

$$\langle \boldsymbol{g}_{\mathcal{NO}}, \boldsymbol{g}_{\mathcal{NS}} \rangle_{\mu,\eta} = \sum_{i \in N} \mu^i(S^i) \left\langle \eta^{-i} g^i_{\mathcal{NO}}, \eta^{-i} g^i_{\mathcal{NS}} \right\rangle_0 = 0.$$

since $g^i_{\mathcal{NO}} \in \mathrm{Ker}(\Lambda^i)$ and $g^i_{\mathcal{NS}} \in \mathrm{Im}(\Lambda^i)$ for every $i \in N$. ■

Proof of Proposition 2.7. This is an immediate corollary of Lemma A.2. We showed that $\mathcal{NO}$ and $\mathcal{NS}$ are orthogonal subspaces and further that any game $\boldsymbol{g}$ can be decomposed into $\Lambda(\boldsymbol{g}) \in \mathcal{NS}$ and $(I_{\mathcal{G}} - \Lambda)(\boldsymbol{g}) \in \mathcal{NO}$. ■

We first give the definition of the flow associated to a game and, in Proposition A.5, characterize the different classes of games. Then, from these characterizations and from the properties of the Moore-Penrose pseudo-inverse, we deduce the proof of Theorem 2.8.

Let $C_1 := \{X : S \times S \to \mathbb{R} | X(\boldsymbol{s}, \boldsymbol{t}) = -X(\boldsymbol{t}, \boldsymbol{s}), \forall \boldsymbol{s}, \boldsymbol{t} \in S\}$ be the set of flows. We endow C_1 with the following inner product:

$$\forall X, Y \in C_1, \quad \langle X, Y \rangle_1 = \frac{1}{2} \sum_{\boldsymbol{s}, \boldsymbol{t} \in S} \mu(\boldsymbol{s}) \mu(\boldsymbol{t}) X(\boldsymbol{s}, \boldsymbol{t}) Y(\boldsymbol{s}, \boldsymbol{t}). \tag{A.10}$$

To any game $\boldsymbol{g}$, we associate an undirected graph as follows. Given a pair of strategy profiles $(\boldsymbol{s}, \boldsymbol{t}) \in S \times S$, if there exists a unique $i \in N$, such that $s^i \neq t^i$ then $(\boldsymbol{s}, \boldsymbol{t})$ will be referred to as an *i-comparable profile pair*. We denote $E^i \subset S \times S$ the set of i-comparable profile pairs. For any two different players i and j we have: $E^i \cap E^j = \emptyset$. We call E the set of *comparable profile pairs*, i.e., $E = \cup_i E_i$. Following Candogan et al. (2011), for any given game $\boldsymbol{g}$, we associate to Player i

an undirected graph defined as $\Gamma^i := (S, E^i)$ and we further associate to the game $\boldsymbol{g}$ the disjoint union of the graphs Γ^i defined as $\Gamma := (S, E)$.

For any $i \in N$, let $W^i \colon S \times S \to \mathbb{R}$ be the non-negative symmetric function defined as

$$W^i(\boldsymbol{s}, \boldsymbol{t}) = \begin{cases} \dfrac{1}{\sqrt{\mu^{-i}(\boldsymbol{s}^{-i})}} & \text{if } (\boldsymbol{s}, \boldsymbol{t}) \in E^i, \\ 0 & \text{otherwise.} \end{cases} \tag{A.11}$$

Recalling that any pair of strategy profiles cannot be comparable for more than one player, we have:

$$W^i(\boldsymbol{s}, \boldsymbol{t}) W^j(\boldsymbol{s}, \boldsymbol{t}) = 0, \quad \text{for all } j \neq i \text{ and } \boldsymbol{s}, \boldsymbol{t} \in S. \tag{A.12}$$

To any Player i, we associate the *partial gradient operator* $\delta_0^i : C_0 \to C_1$, defined for any $f \in C_0$ as follows:

$$\delta_0^i(f)(\boldsymbol{s}, \boldsymbol{t}) = W^i(\boldsymbol{s}, \boldsymbol{t}) \left(f(\boldsymbol{t}) - f(\boldsymbol{s}) \right). \tag{A.13}$$

The gradient operator δ_0 on Γ is defined as $\delta_0 = \sum_{i \in N} \delta_0^i$.

We now introduce the adjoint operators. Recall that we have considered the following inner product on C_0:

$$\forall\ g, f \in C_0,\ \langle g, f \rangle_0 = \sum_{\boldsymbol{s} \in S} \mu(\boldsymbol{s}) g(\boldsymbol{s}) f(\boldsymbol{s}).$$

The adjoint of δ_0^i, denoted by $\delta_0^{i*} : C_1 \to C_0$ is the unique linear operator satisfying:

$$\left\langle \delta_0^i f, X \right\rangle_1 = \left\langle f, \delta_0^{i*} X \right\rangle_0, \tag{A.14}$$

for any $f \in C_0$, $X \in C_1$. By linearity of the dual operations, we obtain that the dual of δ_0 satisfies $\delta_0^* = \sum_{i \in N} \delta_0^{i*}$. Moreover, we have the following explicit expression for δ_0^{i*}.

Proposition A.3. *The adjoint of the gradient operator, $\delta_0^{i*} : C_1 \to C_0$ is given for any $X \in C_1$ by*

$$\forall \boldsymbol{s} \in S, \quad \delta_0^{i*} X(\boldsymbol{s}) = - \sum_{\boldsymbol{t} \in S} \mu(\boldsymbol{t}) W^i(\boldsymbol{s}, \boldsymbol{t}) X(\boldsymbol{s}, \boldsymbol{t}) \tag{A.15}$$

$$= - \sum_{\boldsymbol{t} : (\boldsymbol{s}, \boldsymbol{t}) \in E^i} \mu^i(t^i) \sqrt{\mu^{-i}(\boldsymbol{s}^{-i})} X(\boldsymbol{s}, \boldsymbol{t}). \tag{A.16}$$

Proof. We introduce the basis $(\varepsilon_{\boldsymbol{r}})_{\boldsymbol{r} \in S}$ of C_0, defined as

$$\varepsilon_{\boldsymbol{r}}(\boldsymbol{s}) = \begin{cases} \dfrac{1}{\sqrt{\mu(\boldsymbol{r})}} & \text{if } \boldsymbol{s} = \boldsymbol{r}, \\ 0 & \text{otherwise.} \end{cases} \tag{A.17}$$

This basis is orthonormal with respect to the inner product in Eq. (2.3). For any $X \in C_1$, we have

$$\delta_0^{i*} X = \sum_{\boldsymbol{r} \in S} \left\langle \varepsilon_{\boldsymbol{r}}, \delta_0^{i*} X \right\rangle_0 \varepsilon_{\boldsymbol{r}}$$

and thus,

$$\delta_0^{i*} X(\boldsymbol{s}_0) = \frac{1}{\sqrt{\mu(\boldsymbol{s}_0)}} \left\langle \varepsilon_{\boldsymbol{s}_0}, \delta_0^{i*} X \right\rangle_0.$$

By using the relation between δ_0^{i*} and δ_0 in Eq. (A.14) and then the definition of δ_0 in Eq. (A.13), we get

$$\begin{aligned}
\delta_0^{i*} X(\boldsymbol{s}_0) &= \frac{1}{\sqrt{\mu(\boldsymbol{s}_0)}} \langle \delta_0 \varepsilon_{\boldsymbol{s}_0}, X \rangle_1 \\
&= \frac{1}{2\sqrt{\mu(\boldsymbol{s}_0)}} \sum_{\boldsymbol{s},\boldsymbol{t} \in S} \mu(\boldsymbol{s})\mu(\boldsymbol{t}) \left(\delta_0 \varepsilon_{\boldsymbol{s}_0}\right)(\boldsymbol{s},\boldsymbol{t}) X(\boldsymbol{s},\boldsymbol{t}) \\
&= \frac{1}{2\sqrt{\mu(\boldsymbol{s}_0)}} \left(\sum_{\boldsymbol{s},\boldsymbol{t} \in S} \mu(\boldsymbol{s})\mu(\boldsymbol{t}) W^i(\boldsymbol{s},\boldsymbol{t}) \varepsilon_{\boldsymbol{s}_0}(\boldsymbol{t}) X(\boldsymbol{s},\boldsymbol{t}) \right. \\
&\qquad \left. - \sum_{\boldsymbol{s},\boldsymbol{t} \in S} \mu(\boldsymbol{s})\mu(\boldsymbol{t}) W^i(\boldsymbol{s},\boldsymbol{t}) \varepsilon_{\boldsymbol{s}_0}(\boldsymbol{s}) X(\boldsymbol{s},\boldsymbol{t}) \right).
\end{aligned}$$

Using the definition of $\varepsilon_{\boldsymbol{s}_0}$ (Eq. (A.17)), we obtain

$$\begin{aligned}
\delta_0^{i*} X(\boldsymbol{s}_0) &= \frac{1}{2\sqrt{\mu(\boldsymbol{s}_0)}} \left(\sum_{\boldsymbol{s} \in S} \mu(\boldsymbol{s})\mu(\boldsymbol{s}_0) \frac{1}{\sqrt{\mu(\boldsymbol{s}_0)}} W^i(\boldsymbol{s},\boldsymbol{s}_0) X(\boldsymbol{s},\boldsymbol{s}_0) \right. \\
&\qquad \left. - \sum_{\boldsymbol{t} \in S} \mu(\boldsymbol{t})\mu(\boldsymbol{s}_0) \frac{1}{\sqrt{\mu(\boldsymbol{s}_0)}} W^i(\boldsymbol{s}_0,\boldsymbol{t}) X(\boldsymbol{s}_0,\boldsymbol{t}) \right) \\
&= \frac{1}{2} \left(\sum_{\boldsymbol{s} \in S} \mu(\boldsymbol{s}) W^i(\boldsymbol{s},\boldsymbol{s}_0) X(\boldsymbol{s},\boldsymbol{s}_0) - \sum_{\boldsymbol{t} \in S} \mu(\boldsymbol{t}) W^i(\boldsymbol{s}_0,\boldsymbol{t}) X(\boldsymbol{s}_0,\boldsymbol{t}) \right) \\
&= -\frac{1}{2} \sum_{\boldsymbol{s} \in S} \mu(\boldsymbol{s}) W^i(\boldsymbol{s}_0,\boldsymbol{s}) X(\boldsymbol{s}_0,\boldsymbol{s}) - \frac{1}{2} \sum_{\boldsymbol{t} \in S} \mu(\boldsymbol{t}) W^i(\boldsymbol{s}_0,\boldsymbol{t}) X(\boldsymbol{s}_0,\boldsymbol{t}) \\
&= - \sum_{\boldsymbol{t} \in S} \mu(\boldsymbol{t}) W^i(\boldsymbol{s}_0,\boldsymbol{t}) X(\boldsymbol{s}_0,\boldsymbol{t}),
\end{aligned}$$

where the third equality is due to the skew-symmetric structure of X and the last equality is simply obtained by merging the two summations. ■

To Γ we associate the *joint embedding operator* $D : C_0^n \to C_1$. The operator D maps a game $\boldsymbol{g}$ into a flow $D(\boldsymbol{g})$. It is defined for any $\boldsymbol{g} \in C_0^n$ as

$$D(\boldsymbol{g}) = \sum_{i \in N} \delta_0^i(\eta^{-i} g^i). \tag{A.18}$$

Lemma A.4. *The following hold:*

$$\delta_0^{i*} \delta_0^j = 0, \quad \textit{for all } i \neq j, \tag{A.19}$$

$$\delta_0^* D(\boldsymbol{g}) = \sum_{i \in N} \delta_0^{i*} \delta_0^i(\eta^{-i} g^i), \quad \textit{for all } \boldsymbol{g} \in C_0^n, \tag{A.20}$$

$$\delta_0^{i*} \delta_0^i(\eta^{-i} f) = \eta^{-i} \delta_0^{i*} \delta_0^i(f), \quad \textit{for all } f \in C_0. \tag{A.21}$$

Proof. To prove Eq. (A.19), let $f \in C_0$. By the explicit formula of Eq. (A.15), for all $\boldsymbol{s} \in S$, we have

$$\begin{aligned}\delta_0^{i*}(\delta_0^j f)(\boldsymbol{s}) &= -\sum_{\boldsymbol{t}\in S}\mu(\boldsymbol{t})W^i(\boldsymbol{s},\boldsymbol{t})(\delta_0^j f)(\boldsymbol{s},\boldsymbol{t})\\ &= -\sum_{\boldsymbol{t}\in S}\mu(\boldsymbol{t})W^i(\boldsymbol{s},\boldsymbol{t})W^j(\boldsymbol{s},\boldsymbol{t})\left(f(\boldsymbol{t})-f(\boldsymbol{s})\right)\\ &= 0,\end{aligned}$$

since for $i \neq j$, we have $W^i(\boldsymbol{s},\boldsymbol{t})W^j(\boldsymbol{s},\boldsymbol{t}) = 0$.

To prove Eq. (A.20), let $\boldsymbol{g} \in \mathcal{G}$. We then get

$$\delta_0^* D(\boldsymbol{g}) = \sum_{i\in N}\delta_0^{i*}\left(\sum_{j\in N}\delta_0^j(\eta^{-j}g^j)\right) = \sum_{i\in N}\delta_0^{i*}\delta_0^i(\eta^{-i}g^i),$$

since, by Eq. (A.19), all cross-products are equal to 0.

To prove Eq. (A.21), let $f \in C_0$. We have, for all $\boldsymbol{s} \in S$,

$$\begin{aligned}\delta_0^{i*}(\delta_0^i\eta^{-i}f)(\boldsymbol{s}) &= -\sum_{\boldsymbol{t}\in S}\mu(\boldsymbol{t})W^i(\boldsymbol{s},\boldsymbol{t})(\delta_0^i\eta^{-i}f)(\boldsymbol{s},\boldsymbol{t})\\ &= -\sum_{t^i\in S^i}\mu^i(t^i,\boldsymbol{s}^{-i})W^i(\boldsymbol{s},\boldsymbol{t})W^i(\boldsymbol{s},\boldsymbol{t})\eta^{-i}(\boldsymbol{s}^{-i})\left(f(s^i,\boldsymbol{s}^{-i})-f(t^i,\boldsymbol{s}^{-i})\right)\\ &= \eta^{-i}(\boldsymbol{s}^{-i})\left(-\sum_{t^i\in S^i}\mu^i(t^i,\boldsymbol{s}^{-i})W^i(\boldsymbol{s},\boldsymbol{t})W^i(\boldsymbol{s},\boldsymbol{t})\left(f(s^i,\boldsymbol{s}^{-i})-f(t^i,\boldsymbol{s}^{-i})\right)\right)\\ &= \eta^{-i}(\boldsymbol{s}^{-i})\delta_0^{i*}(\delta_0^i f)(\boldsymbol{s}).\end{aligned}$$

■

At this point, we can relate the classes of games presented in Section 2.1 to the previously defined operators.

Proposition A.5. *We have:*

(a) $\mathcal{NS} = \{\boldsymbol{g} \in \mathcal{G}, D(\boldsymbol{g}) = 0\}$.
(b) $\mathcal{P} = \{\boldsymbol{g} \in \mathcal{G}, D(\boldsymbol{g}) \in \operatorname{Im}\delta_0\} = (D)^{-1}(\operatorname{Im}(\delta_0))$.
(c) $\mathcal{H} = \{\boldsymbol{g} \in \mathcal{G},\ D(\boldsymbol{g}) \in \operatorname{Ker}\delta_0^*\}$.

Proof. (a) Let $\boldsymbol{g} \in \mathcal{G}$. Eq. (A.18) implies that $D(\boldsymbol{g}) \in 0$ if and only if, for all $i \in N$ and for all $\boldsymbol{s},\boldsymbol{t} \in S$, we have

$$W^i(\boldsymbol{s},\boldsymbol{t})\left(\eta^{-i}(\boldsymbol{s}^{-i})g^i(\boldsymbol{s}) - \eta^{-i}(\boldsymbol{s}^{-i})g^i(\boldsymbol{t})\right) = 0. \tag{A.22}$$

Since $W^i(\boldsymbol{s},\boldsymbol{t})$ is strictly positive on E^i and η^{-i} is strictly positive, Eq. (A.22) holds if and only if, for all $i \in N$, for all $\boldsymbol{s}^{-i} \in S^{-i}$, and for all $s^i, t^i \in S^i$, we have

$$g^i(s^i,\boldsymbol{s}^{-i}) - g^i(t^i,\boldsymbol{s}^{-i}) = 0.$$

Hence g^i does not depend on Player i's strategy, which, by Definition 2.1, means that the game is non-strategic.

(b) Let $\boldsymbol{g} \in \mathcal{G}$. Then, by Eq. (A.18), $D(\boldsymbol{g}) \in \operatorname{Im}(\delta_0)$ if and only if there exists φ such that for every $i \in N$, and $\boldsymbol{s},\boldsymbol{t} \in S$,

$$W^i(\boldsymbol{s},\boldsymbol{t})\left(\eta^{-i}(\boldsymbol{s}^{-i})g^i(\boldsymbol{s}) - \eta^{-i}(\boldsymbol{s}^{-i})g^i(\boldsymbol{t})\right) = W^i(\boldsymbol{s},\boldsymbol{t})\left(\varphi(\boldsymbol{s}) - \varphi(\boldsymbol{t})\right).$$

We equivalently have, for any $s^i, t^i \in S^i$ and any $\boldsymbol{s}^{-i} \in S^{-i}$,

$$\varphi(s^i, \boldsymbol{s}^{-i}) - \varphi(t^i, \boldsymbol{s}^{-i}) = \eta^{-i}(\boldsymbol{s}^{-i}) \left(g^i(s^i, \boldsymbol{s}^{-i}) - g^i(t^i, \boldsymbol{s}^{-i})\right).$$

Hence, the result follows from Definition 2.3 of an η-potential game.

(c) Let $\boldsymbol{g} \in \mathcal{H}$ be a (μ, η)-harmonic game. In view of Eq. (A.20) we can write the condition in terms of gradient and adjoint operator. By replacing them with their explicit expression, we obtain that, for any $\boldsymbol{s} \in S$,

$$\begin{aligned}
\delta_0^*(D(\boldsymbol{g}))(\boldsymbol{s}) &= \sum_{i \in N} \delta_0^{i*} \delta_0^i (\eta^{-i} g^i)(\boldsymbol{s}) \\
&= \sum_{i \in N} \left(- \sum_{\boldsymbol{t} \in S} \mu(\boldsymbol{t}) W^i(\boldsymbol{s}, \boldsymbol{t}) W^i(\boldsymbol{s}, \boldsymbol{t}) \left((\eta^{-i} g^i)(\boldsymbol{s}) - (\eta^{-i} g^i)(\boldsymbol{t})\right) \right) \\
&= - \sum_{i \in N} \sum_{t^i \in S^i} \mu^i(t^i, \boldsymbol{s}^{-i}) \frac{1}{\mu^{-i}(\boldsymbol{s}^{-i})} \eta^{-i}(\boldsymbol{s}^{-i}) \left(g^i(s^i, \boldsymbol{s}^{-i}) - g^i(t^i, \boldsymbol{s}^{-i})\right) \\
&= - \sum_{i \in N} \sum_{t^i \in S^i} \mu^i(t^i) \eta^{-i}(\boldsymbol{s}^{-i}) \left(g^i(s^i, \boldsymbol{s}^{-i}) - g^i(t^i, \boldsymbol{s}^{-i})\right)
\end{aligned}$$

and the result follows from Definition 2.5 of a (μ, η)-harmonic game. ■

The aim of the next results is to prove that the space of games is the orthogonal sum of potential μ-normalized games, (μ, η)-harmonic μ-normalized games, and non-strategic games. We first introduce the Moore-Penrose pseudo-inverse. Then, we prove several results leading to the above-mentioned orthogonality. Finally, we prove that any game can be decomposed in such a triple by providing an explicit formula that uses the pseudo-inverse. These results yield Theorem 2.8.

Let $\delta_0^{i\dagger} : C_1 \to C_0$ be defined as follows:

- On $\operatorname{Im}(\delta_0^i)$, $\delta_0^{i\dagger}$ is the inverse of the restriction of δ^i on $\mathcal{NO}^i$, i.e.,
$$\delta_0^{i\dagger}\restriction_{\operatorname{Im}(\delta_0^i)} = \left[\delta^i \restriction_{\mathcal{NO}^i}\right]^{-1}.$$
- on $\operatorname{Im}(\delta_0^i)^\perp$, $\delta_0^{i\dagger} = 0$.
- on C_1, $\delta_0^{i\dagger}$ is defined linearly, i.e.,
$$\delta_0^{i\dagger}(X) = \delta_0^{i\dagger}(X_I + X_I^\perp) = \delta_0^{i\dagger}(X_I),$$
where $X_I \in \operatorname{Im}(\delta_0^i)$.

The operator $\delta_0^{i\dagger}$ is in fact the Moore-Penrose pseudo-inverse of δ_0^i. In particular, $\delta_0^{i\dagger}\delta_0^i$ is the orthogonal projection onto $\mathcal{NO}^i$ and thus,

$$\delta_0^{i\dagger}\delta_0^i = \Pi^i. \tag{A.23}$$

Moreover, $\delta_0^i\delta_0^{i\dagger}\delta_0^i = \delta_0^i$. Furthermore, $(I_{C_0} - \delta_0^{i\dagger}\delta_0^i)$ is the orthogonal projection onto $\mathcal{NS}^i$ and thus, $(I_{C_0} - \delta_0^{i\dagger}\delta_0^i) = \Lambda^i$. Moreover, $\delta_0^i\delta_0^{i\dagger}$ is the orthogonal projection onto $\operatorname{Im}(\delta_0^i)$ and $(I_{C_1} - \delta_0^i\delta_0^{i\dagger})$ is the orthogonal projection onto $\operatorname{Ker}(\delta_0^{i*})$.

We further define $\delta_0^\dagger : C_1 \to C_0$ as $\delta_0^\dagger = \sum_{i \in N} \delta_0^{i\dagger}$.

Lemma A.6. *Let $\varphi \in \mathcal{NO}^i$ and $\widetilde{\varphi} \in C_0$ and assume that $\delta_0^i\varphi = \delta_0^i\widetilde{\varphi}$. Then, for all $\psi \in \mathcal{NO}^i$ we have: $\langle \varphi, \psi \rangle_0 = \langle \widetilde{\varphi}, \psi \rangle_0$.*

Proof. Since $\varphi \in \mathcal{NO}^i$ and $\delta_0^{i\dagger}\delta_0^i$ is the identity operator on $\mathcal{NO}^i$, we have:

$$\langle \varphi, \psi \rangle_0 = \left\langle \delta_0^{i\dagger}\delta_0^i \varphi, \psi \right\rangle_0 = \left\langle \delta_0^{i\dagger}\delta_0^i \widetilde{\varphi}, \psi \right\rangle_0 ,$$

where the last equality follows from the initial assumption. Using the fact that $\delta_0^{i\dagger}\delta_0^i$ is self-adjoint, that the operator $\delta_0^{i\dagger}\delta_0^i$ is the identity on $\mathcal{NO}^i$, and that $\psi \in \mathcal{NO}^i$, we get

$$\langle \varphi, \psi \rangle_0 = \left\langle \widetilde{\varphi}, \delta_0^{i\dagger}\delta_0^i \psi \right\rangle_0 = \langle \widetilde{\varphi}, \psi \rangle_0 . \qquad \blacksquare$$

Lemma A.7. *For any $i \in N$, we have*

$$\delta_0^{i*} \circ \delta_0^i = \mu^i(S^i)\Pi^i. \tag{A.24}$$

Proof. Using the explicit formula for δ_0^{i*} of Eq. (A.15) and replacing W^i with its value, for all $f \in C_0$ and all $\boldsymbol{s} \in S$, we have

$$\begin{aligned}\left(\delta_0^{i*}\left(\delta_0^i f\right)\right)(\boldsymbol{s}) &= \sum_{\boldsymbol{t}\in S} \mu(\boldsymbol{t}) W^i(\boldsymbol{s},\boldsymbol{t}) W^i(\boldsymbol{s},\boldsymbol{t}) \left(f(\boldsymbol{s}) - f(\boldsymbol{t})\right) \\ &= \sum_{t^i \in S^i} \mu^i(t^i) \left(f(s^i, \boldsymbol{s}^{-i}) - f(t^i, \boldsymbol{s}^{-i})\right).\end{aligned}$$

By introducing the probability distribution $\overline{\mu}$ associated to μ, we obtain

$$\begin{aligned}\left(\delta_0^{i*}\left(\delta_0^i f\right)\right)(\boldsymbol{s}) &= \mu^i(S^i) \sum_{t^i \in S^i} \overline{\mu}^i(t^i) \left(f(s^i, \boldsymbol{s}^{-i}) - f(t^i, \boldsymbol{s}^{-i})\right) \\ &= \mu^i(S^i) \left(\Pi^i(f)\right)(\boldsymbol{s}),\end{aligned}$$

which concludes the proof. $\blacksquare$

Proposition A.8. *A game $\boldsymbol{g}$ is a μ-normalized (μ, η)-harmonic game if and only if*

$$\sum_{i\in N} \mu^i(S^i)\eta^{-i} g^i = 0 \quad \textit{and} \quad \Pi(\boldsymbol{g}) = \boldsymbol{g}$$

Proof. Let $\boldsymbol{g} \in \mathcal{NO}$. By Lemma A.1(c), this holds if and only if $\Lambda(\boldsymbol{g}) = 0$, which is equivalent to $\Pi(\boldsymbol{g}) = \boldsymbol{g}$. Therefore, for all $\boldsymbol{s} \in S$, we get

$$\begin{aligned}\sum_{i\in N} \mu^i(S^i)\eta^{-i}(\boldsymbol{s}^{-i}) g^i(\boldsymbol{s}) &= \sum_{i\in N} \mu^i(S^i)\eta^{-i}(\boldsymbol{s}^{-i})\Pi^i(g^i)(\boldsymbol{s}) \\ &= \sum_{i\in N} \eta^{-i}(\boldsymbol{s}^{-i})\delta_0^{i*}\delta_0^i g^i(\boldsymbol{s}) \\ &= \sum_{i\in N} \delta_0^{i*}\delta_0^i(\eta^{-i} g^i)(\boldsymbol{s}) \\ &= \delta_0^*\left(D(\boldsymbol{g})\right)(\boldsymbol{s}),\end{aligned}$$

where the second equality follows from Lemma A.7, the third from Eq. (A.21), and the last from Eq. (A.20). This, together with Proposition A.5(c), completes the proof. $\blacksquare$

Proposition A.9. *The sets of games $\mathcal{NO} \cap \mathcal{H}$, $\mathcal{NO} \cap \mathcal{P}$, and $\mathcal{NS}$ are orthogonal with respect to the inner product in Eq. (2.4).*

Proof. We know by Proposition 2.7 that $\mathcal{NO}$ and $\mathcal{NS}$ are orthogonal spaces. It is sufficient to prove that $\mathcal{NO} \cap \mathcal{H}$ and $\mathcal{NO} \cap \mathcal{P}$ are also orthogonal spaces.

To prove orthogonality between μ-normalized potential games and μ-normalized (μ, η)-harmonic games, let $\boldsymbol{g}_{\mathcal{P}} \in \mathcal{NO} \cap \mathcal{P}$ and $\boldsymbol{g}_{\mathcal{H}} \in \mathcal{NO} \cap \mathcal{H}$. In view of Eq. (2.4), we get

$$\begin{aligned}
\langle \boldsymbol{g}_{\mathcal{P}}, \boldsymbol{g}_{\mathcal{H}} \rangle_{\mu,\eta} &= \sum_{i\in N} \mu^i(S^i) \left\langle \eta^{-i} g^i_{\mathcal{P}}, \eta^{-i} g^i_{\mathcal{H}} \right\rangle_0 \\
&= \sum_{i\in N} \left\langle \eta^{-i} g^i_{\mathcal{P}}, \mu^i(S^i) \eta^{-i} g^i_{\mathcal{H}} \right\rangle_0 .
\end{aligned}$$

Since $\boldsymbol{g}_{\mathcal{P}} \in \mathcal{NO} \cap \mathcal{P}$, there exists φ such that for any $i \in N$, $\delta^i_0(\eta^{-i} g^i_{\mathcal{P}}) = \delta^i_0 \varphi$. Hence using first Lemma A.6 and then Proposition A.8, we obtain

$$\begin{aligned}
\langle \boldsymbol{g}_{\mathcal{P}}, \boldsymbol{g}_{\mathcal{H}} \rangle_{\mu,\eta} &= \sum_{i\in N} \left\langle \varphi, \mu^i(S^i) \eta^{-i} g^i_{\mathcal{H}} \right\rangle_0 \\
&= \left\langle \varphi, \sum_{i\in N} \mu^i(S^i) \eta^{-i} g^i_{\mathcal{H}} \right\rangle_0 \\
&= \langle \varphi, 0 \rangle_0 \\
&= 0.
\end{aligned}$$

∎

The following lemma states that the flow induced by a game $\boldsymbol{g}$ and the flow induced by its projection $\Pi(\boldsymbol{g})$ on normalized games are equal.

Lemma A.10. *Let $\boldsymbol{g} \in \mathcal{G}$, then $D(\Pi(\boldsymbol{g})) = D(\boldsymbol{g})$.*

Proof. The proof relies on two properties of the pseudo-inverse. Let $\boldsymbol{g} \in \mathcal{G}$. By Eq. (A.23), we have $\delta^{i\dagger}_0 \delta^i_0 = \Pi^i$. Hence,

$$\begin{aligned}
D(\Pi(\boldsymbol{g})) &= \sum_{i\in N} \delta^i_0(\eta^{-i} \Pi^i(g^i)) \\
&= \sum_{i\in N} \delta^i_0 \eta^{-i} \left(\delta^{i\dagger}_0 \delta^i_0 g^i \right) \\
&= \sum_{i\in N} \eta^{-i} \delta^i_0 \left(\delta^{i\dagger}_0 \delta^i_0 g^i \right) .
\end{aligned}$$

By definition of the pseudo-inverse, $\delta^i_0 \delta^{i\dagger}_0 \delta^i_0 = \delta^i_0$, hence we can simplify the right-hand side to obtain

$$D(\Pi(\boldsymbol{g})) = \sum_{i\in N} \eta^{-i} \delta^i_0 g^i = \sum_{i\in N} \delta^i_0(\eta^{-i} g^i) = D(\boldsymbol{g}).$$

∎

Lemma A.11. *Given a game $\boldsymbol{g}$, let $\boldsymbol{g}_{\mathcal{P}}, \boldsymbol{g}_{\mathcal{H}}, \boldsymbol{g}_{\mathcal{NS}}$ be the games defined as follows:*

$$\boldsymbol{g}_{\mathcal{P}} = \Pi(\boldsymbol{f}), \quad \boldsymbol{g}_{\mathcal{H}} = \Pi\left(\boldsymbol{g} - \boldsymbol{f}\right), \quad \boldsymbol{g}_{\mathcal{NS}} = \Lambda(\boldsymbol{g}), \tag{A.25}$$

where

$$\boldsymbol{f} = \left((1/\eta^{-1}) \delta^\dagger_0 D(\boldsymbol{g}), \dots, (1/\eta^{-n}) \delta^\dagger_0 D(\boldsymbol{g}) \right). \tag{A.26}$$

Then, $\boldsymbol{g}_{\mathcal{P}}$ is η-potential μ-normalized, $\boldsymbol{g}_{\mathcal{H}}$ is (μ, η)-harmonic μ-normalized, and $\boldsymbol{g}_{\mathcal{NS}}$ is non-strategic. Hence, $\boldsymbol{g}_{\mathcal{P}}$,$\boldsymbol{g}_{\mathcal{H}}$, and $\boldsymbol{g}_{\mathcal{NS}}$ are the components of the (μ, η)-decomposition of $\boldsymbol{g}$.

Proof. It is clear that $\boldsymbol{g}_{\mathcal{P}} + \boldsymbol{g}_{\mathcal{H}} + \boldsymbol{g}_{\mathcal{NS}} = \boldsymbol{g}$. By Lemma A.2, we know that $\boldsymbol{g}_{\mathcal{NS}}$ is non-strategic whereas $\boldsymbol{g}_{\mathcal{P}}$ and $\boldsymbol{g}_{\mathcal{H}}$ are μ-normalized. Then, we need to verify that $\boldsymbol{g}_{\mathcal{P}}$ and $\boldsymbol{g}_{\mathcal{H}}$ are potential and harmonic, respectively.

Let $\varphi : S \to \mathbb{R}$ be such that $\varphi = \delta_0^{\dagger} D(\boldsymbol{g})$. We start with the η-potential component. By using Lemma A.10, we obtain

$$D(\boldsymbol{g}_{\mathcal{P}}) = D(\Pi(\boldsymbol{f})) = D(\boldsymbol{f}).$$

It follows from Eq. (A.18) that

$$\begin{aligned} D(\boldsymbol{g}_{\mathcal{P}}) &= \sum_{i \in N} \delta_0^i \left(\eta^{-i} \frac{1}{\eta^{-i}} \varphi \right) \\ &= \sum_{i \in N} \delta_0^i(\varphi) \\ &= \delta_0(\varphi). \end{aligned}$$

We get for the (μ, η)-harmonic component the same simplification of Π:

$$\begin{aligned} \delta_0^* \left(D(\boldsymbol{g}_{\mathcal{H}}) \right) &= \delta_0^* \left(D(\Pi \left(\boldsymbol{g} - \boldsymbol{f} \right)) \right) \\ &= \delta_0^* \left(D(\boldsymbol{g}) - D(\boldsymbol{f}) \right) \\ &= \delta_0^* \left(D(\boldsymbol{g}) - \sum_{i \in N} \delta_0^i \frac{\eta^{-i}}{\eta^{-i}} \delta_0^{i\dagger} D(\boldsymbol{g}) \right) \\ &= \delta_0^* (I_{C_1} - \delta_0 \delta_0^{\dagger}) D(\boldsymbol{g}) \\ &= 0, \end{aligned}$$

where the third equality is obtained by replacing $\boldsymbol{f}$ with its definition, as in Eq. (A.26), and the last one is due to the fact that $(I_{C_1} - \delta_0 \delta_0^{\dagger})$ is the orthogonal projection onto $\mathrm{Ker}(\delta_0^*)$. ∎

We now characterize the set of measures which yield the same (μ, η)-decomposition.

Proof of Proposition 2.9. Let $\alpha, \beta > 0$. For every $i \in N$, let $\mu_\alpha^i = \alpha \mu^i$ and $\eta_\beta^i = \beta \eta^i$. Clearly, (μ_α, η_β) yields the same decomposition as (μ, η). We now check that there is no other pair of measures which induces the same decomposition.

Let $\mu, \mathring{\mu}$ and $\eta, \mathring{\eta}$ be four product measures such that the (μ, η)-decomposition and the $(\mathring{\mu}, \mathring{\eta})$-decomposition coincide. Equivalently, the particular classes of games that appear as components of each decomposition are the same: any η-potential game is $\mathring{\eta}$-potential, any μ-normalized game is also $\mathring{\mu}$-normalized, and any (μ, η)-harmonic game is also $(\mathring{\mu}, \mathring{\eta})$-harmonic.

We now consider suitable games to obtain the relation between $\mu, \mathring{\mu}, \eta$, and $\mathring{\eta}$. We first focus on μ-normalized games. Fix $i \in N$ and $t^i, r^i \in S^i$ and consider the game $\boldsymbol{g} \in \mathcal{G}$ such that, for all $\boldsymbol{s}^{-i} \in S^{-i}$,

$$g^i(t^i, \boldsymbol{s}^{-i}) = \frac{1}{\mu^i(t^i)} \quad \text{and} \quad g^i(r^i, \boldsymbol{s}^{-i}) = -\frac{1}{\mu^i(r^i)}.$$

Let the payoffs for every other player and the payoffs of Player i in any other profile be equal to 0. The game $\boldsymbol{g}$ is μ-normalized and therefore, it is also $\mathring{\mu}$-normalized:

$$\frac{\mathring{\mu}^i(t^i)}{\mu^i(t^i)} - \frac{\mathring{\mu}^i(r^i)}{\mu^i(r^i)} = 0$$

and thus,

$$\frac{\mathring{\mu}^i(t^i)}{\mu^i(t^i)} = \frac{\mathring{\mu}^i(r^i)}{\mu^i(r^i)}.$$

By changing the game, it follows that all the quotients are equal to some positive real number and thus, for every $i \in N$, there exists $\alpha_i > 0$ such that $\mathring{\mu}^i = \alpha_i \mu^i$.

We now consider η-potential games. We construct a game where we focus only on two players and two strategies for each player. Let $i, j \in N$, $t^i \in S^i$ and $t^j \in S^j$. Define the potential function φ on S as follows:

$$\forall \boldsymbol{s} \in S,\ \varphi(\boldsymbol{s}) = \begin{cases} \eta^{-(i,j)}(\boldsymbol{s}^{-(i,j)}) & \text{if } s^i = t^i \text{ and } s^j = t^j, \\ 0 & \text{otherwise,} \end{cases}$$

where $\eta^{-(i,j)}(\boldsymbol{s}^{-(i,j)}) = \prod_{l \neq i,j} \eta^l(s^l)$. Fix $\boldsymbol{s}^{-(i,j)} \in S^{-(i,j)}$. Let $r^i \neq t^i$ and $r^j \neq t^j$. We focus on the profiles $(r^i, r^j, \cdot)$, $(r^i, t^j, \cdot)$, $(t^i, t^j, \cdot)$ and $(t^i, r^j, \cdot)$ where $\cdot$ is a short notation for $\boldsymbol{s}^{-(i,j)}$. The following matrices represent the potential function φ and an η-potential game $\boldsymbol{g}$ associated to φ:

$$\begin{array}{c|c|c|} & t^j & r^j \\ \hline t^i & \eta^{-(i,j)}(\boldsymbol{s}^{-(i,j)}) & 0 \\ \hline r^i & 0 & 0 \\ \hline \end{array} \qquad \begin{array}{c|cc|cc|} & t^j & & r^j & \\ \hline t^i & 0 & 0 & 0 & -1/\eta^i(t^i) \\ \hline r^i & -1/\eta^j(t^j) & 0 & 0 & 0 \\ \hline \end{array}\,.$$

φ $\qquad$ $\boldsymbol{g}$

Player i chooses the row and her payoff is the first coordinate whereas Player j chooses the column and her payoff is the second coordinate. By assumption, $\boldsymbol{g}$ is an $\mathring{\eta}$-potential game and so we have

$$\begin{aligned} &\mathring{\eta}^{-i}(t^j, \cdot)(g^i(r^i, t^j, \cdot) - g^i(t^i, t^j, \cdot)) + \mathring{\eta}^{-j}(r^i, \cdot)(g^j(r^i, r^j, \cdot) - g^j(r^i, t^j, \cdot)) \\ &\quad + \mathring{\eta}^{-i}(r^j, \cdot)(g^i(t^i, r^j, \cdot) - g^i(r^i, r^j, \cdot)) + \mathring{\eta}^{-j}(t^i, \cdot)(g^j(t^i, t^j, \cdot) - g^j(t^i, r^j, \cdot)) = 0. \end{aligned}$$

Dividing by $\mathring{\eta}^{-(i,j)}(\boldsymbol{s})$ and replacing the corresponding payoff in $\boldsymbol{g}$, we get:

$$\mathring{\eta}^j(t^j)\left(-\frac{1}{\eta^j(t^j)} - 0\right) + \mathring{\eta}^i(r^i)(0-0) + \mathring{\eta}^j(r^j)(0-0) + \mathring{\eta}^i(t^i)\left(0 + \frac{1}{\eta^i(t^i)}\right) = 0$$

Hence, for every strategy $t^i \in S^i$ and every strategy $t^j \in S^j$, we obtain

$$\frac{\mathring{\eta}^j(t^j)}{\eta^j(t^j)} = \frac{\mathring{\eta}^i(t^i)}{\eta^i(t^i)}.$$

By changing the game, we obtain that the equality is true for every pair of players and for every pair of strategies. In particular, it is also true for two strategies of a player since they have to be equal to the same quotient for another player. It follows that there exists $\beta > 0$, such that for every $i \in N$, $\mathring{\eta}^i = \beta \eta^i$.

Finally, we consider (μ, η)-harmonic games. Let $i, j \in N$, $t^i \in S^i$ and $t^j \in S^j$. We construct a particular (μ, η)-harmonic game $\boldsymbol{g}$. The payoff of player $i \in N$ is defined as follows:

$$g^i(\boldsymbol{s}) = \begin{cases} 0 & \text{if } s^i = t^i, \\ \dfrac{1-\mu^j(t^j)}{\eta^{-i}(\boldsymbol{s}^{-i})} & \text{if } s^i \neq t^i \text{ and } s^j = t^j, \\ \dfrac{-\mu^j(t^j)}{\eta^{-i}(\boldsymbol{s}^{-i})} & \text{if } s^i \neq t^i \text{ and } s^j \neq t^j. \end{cases}$$

Likewise, for player $j \in N$, we have

$$g^j(\boldsymbol{s}) = \begin{cases} 0 & \text{if } s^j = t^j, \\ \dfrac{1-\mu^i(t^i)}{\eta^{-j}(\boldsymbol{s}^{-j})} & \text{if } s^j \neq t^j \text{ and } s^i = t^i, \\ \dfrac{\mu^i(t^i)}{\eta^{-j}(\boldsymbol{s}^{-j})} & \text{if } s^j \neq t^j \text{ and } s^i \neq t^i. \end{cases}$$

The payoff of all the other players is assumed to be 0.

This game could be reduced to a game with two strategies for each player, where a player chooses either her strategy labeled by t or any other strategy. This yields the following representation:

	t^j		$s^j \neq t^j$	
t^i	0	0	0	$-(1-\mu^i(t^i))$
$s^i \neq t^i$	$(1-\mu^j(t^j))$	0	$-\mu^j(t^j)$	$\mu^i(t^i)$

$\eta^{-i}g^i$ (left) and $\eta^{-j}g^j$ (right)

One can check that $\boldsymbol{g}$ is indeed (μ,η)-harmonic. Therefore, $\boldsymbol{g}$ is also $(\mathring{\mu},\mathring{\eta})$-harmonic. We have seen previously that $\mathring{\eta}$ is a multiple of η, hence $\boldsymbol{g}$ is also $(\mathring{\mu},\eta)$-harmonic. Let $r^i \neq t^i$, $r^j \neq t^j$ and $\boldsymbol{s}^{-(i,j)} \in S^{-(i,j)}$. Since all actions different from t^i (resp. t^j) are duplicate, the $(\mathring{\mu},\eta)$-harmonicity of $\boldsymbol{g}$ at $(r^i, r^j, \cdot)$ yields, by Definition 2.5,

$$\begin{aligned} \mathring{\mu}^j(t^j)\eta^{-j}(r^i,\cdot)\left(g^j(r^i,t^j,\cdot) - g^j(r^i,r^j,\cdot)\right) + \\ \mathring{\mu}^i(t^i)\eta^{-i}(r^j,\cdot)\left(g^i(t^i,r^j,\cdot) - g^i(r^i,r^j,\cdot)\right) = 0, \end{aligned}$$

where $\cdot$ is a short notation for $\boldsymbol{s}^{-(i,j)}$. Replacing $\eta^{-i}g^i$ and $\eta^{-j}g^j$ with their definitions, we obtain

$$\mathring{\mu}^j(t^j)(0-\mu^i(t^i)) + \mathring{\mu}^i(t^i)(0-(-\mu^j(t^j))) = 0$$

and, hence,

$$\frac{\mathring{\mu}^j(t^j)}{\mu^j(t^j)} = \frac{\mathring{\mu}^i(t^i)}{\mu^i(t^i)}.$$

By changing the game, we obtain that the equality is true for every pair of players and for every pair of strategies. It follows that there exists a unique $\alpha > 0$, such that for every $i \in N$, $\mathring{\mu}^i = \alpha\mu^i$. This concludes the proof. ■

A.2. **Proofs of Section 2.1.** The following lemma states that the global flow around a set T only depends on the flow on $T^c := S\backslash T$ since the flows inside T compensate each others.

Lemma A.12. *Let $X \in C_1$ and T be a subset of S. Then,*

$$\sum_{\boldsymbol{s}\in T}\sum_{\boldsymbol{t}\in S}\mu(\boldsymbol{s})\mu(\boldsymbol{t})X(\boldsymbol{s},\boldsymbol{t}) = \sum_{\boldsymbol{s}\in T}\sum_{\boldsymbol{t}\in T^c}\mu(\boldsymbol{s})\mu(\boldsymbol{t})X(\boldsymbol{s},\boldsymbol{t}).$$

Proof. We have:

$$\sum_{\boldsymbol{s}\in T}\sum_{\boldsymbol{t}\in S}\mu(\boldsymbol{s})\mu(\boldsymbol{t})X(\boldsymbol{s},\boldsymbol{t}) = \sum_{\boldsymbol{s}\in T}\sum_{\boldsymbol{t}\in T}\mu(\boldsymbol{s})\mu(\boldsymbol{t})X(\boldsymbol{s},\boldsymbol{t}) + \sum_{\boldsymbol{s}\in T}\sum_{\boldsymbol{t}\in T^c}\mu(\boldsymbol{s})\mu(\boldsymbol{t})X(\boldsymbol{s},\boldsymbol{t})$$

and due to the skew-symmetric structure of X for any $\boldsymbol{s},\boldsymbol{t}\in S$, $X(\boldsymbol{s},\boldsymbol{t})+X(\boldsymbol{t},\boldsymbol{s}) = 0$ and thus the first term of the right hand side is equal to 0. ■

Proof of Theorem 2.6. Let $\boldsymbol{g}$ be a (μ,η)-harmonic game. Then, $\delta_0^* D(\boldsymbol{g})(\boldsymbol{s}) = 0$ for all $\boldsymbol{s}\in S$. Let $i\in N$ and $r^i \in S^i$. Call T the subset of strategy profiles $\left\{(r^i,\boldsymbol{s}^{-i})|\boldsymbol{s}^{-i}\in S^{-i}\right\}$. Then, multiplying by $\mu(r^i,\boldsymbol{s}^{-i})$ and summing over $\boldsymbol{s}^{-i}\in S^{-i}$ we get

$$\begin{aligned}
0 &= \sum_{\boldsymbol{s}^{-i}\in S^{-i}}\mu(r^i,\boldsymbol{s}^{-i})\delta_0^* D(\boldsymbol{g})(r^i,\boldsymbol{s}^{-i})\\
&= -\sum_{\boldsymbol{r}\in T}\mu(\boldsymbol{r})\sum_{j\in N}\delta_0^{j*}\delta_0^j(\eta^{-j}g^j)(\boldsymbol{r})\\
&= \sum_{\boldsymbol{r}\in T}\sum_{\boldsymbol{s}\in S}\mu(\boldsymbol{r})\mu(\boldsymbol{s})\sum_{j\in N}W^j(\boldsymbol{r},\boldsymbol{s})\delta_0^j(\eta^{-j}g^j)(\boldsymbol{r},\boldsymbol{s}).
\end{aligned}$$

Thus, in view of Lemma A.12, we can eliminate some terms of the summation and then notice that $\boldsymbol{s}\in T$ and $\boldsymbol{t}\in T^c$ are not j-comparable if $j\neq i$, hence

$$\begin{aligned}
0 &= \sum_{\boldsymbol{r}\in T}\sum_{\boldsymbol{s}\in T^c}\mu(\boldsymbol{r})\mu(\boldsymbol{s})\sum_{j\in N}W^j(\boldsymbol{r},\boldsymbol{s})\delta_0^j(\eta^{-j}g^j)(\boldsymbol{r},\boldsymbol{s})\\
&= \sum_{r\in T}\sum_{\boldsymbol{s}\in T^c}\mu(\boldsymbol{r})\mu(\boldsymbol{s})\left(W^i(\boldsymbol{r},\boldsymbol{s})\delta_0^i(\eta^{-i}g^i)(\boldsymbol{s},\boldsymbol{t}) + \sum_{j\neq i}W^j(\boldsymbol{r},\boldsymbol{s})\delta_0^j(\eta^{-j}g^j)(\boldsymbol{s},\boldsymbol{t})\right)\\
&= \sum_{r\in T}\sum_{\boldsymbol{s}\in T^c}\mu(\boldsymbol{r})\mu(\boldsymbol{s})W^i(\boldsymbol{r},\boldsymbol{s})\delta_0^i(\eta^{-i}g^i)(\boldsymbol{r},\boldsymbol{s}) + 0.
\end{aligned}$$

We can now replace δ_0^i and W^i with their definitions to obtain

$$\begin{aligned}
0 &= \sum_{r\in T}\sum_{\boldsymbol{s}\in T^c}\mu(\boldsymbol{r})\mu(\boldsymbol{s})W^i(\boldsymbol{r},\boldsymbol{s})^2((\eta^{-i}g^i)(\boldsymbol{s}) - (\eta^{-i}g^i)(\boldsymbol{r}))\\
&= \sum_{\boldsymbol{s}^{-i}\in S^{-i}}\sum_{s^i\in S^i}\mu(r^i,\boldsymbol{s}^{-i})\mu(s^i,\boldsymbol{s}^{-i})\frac{\eta^{-i}(\boldsymbol{s}^{-i})}{\mu^{-i}(\boldsymbol{s}^{-i})}\left(g^i(r^i,\boldsymbol{s}^{-i}) - g^i(s^i,\boldsymbol{s}^{-i})\right)\\
&= \mu^i(r^i)\sum_{\boldsymbol{s}^{-i}\in S^{-i}}\eta^{-i}(\boldsymbol{s}^{-i})\mu^{-i}(\boldsymbol{s}^{-i})\left(\sum_{s^i\in S^i}\mu^i(s^i)\left(g^i(r^i,\boldsymbol{s}^{-i}) - g^i(s^i,\boldsymbol{s}^{-i})\right)\right).
\end{aligned}$$

Dividing by $\mu^i(r^i)$, which is strictly positive, we obtain

$$0 = \sum_{\boldsymbol{s}^{-i}\in S^{-i}}\eta^{-i}(\boldsymbol{s}^{-i})\mu^{-i}(\boldsymbol{s}^{-i})\left(\sum_{s^i\in S^i}\mu^i(s^i)\left(g^i(r^i,\boldsymbol{s}^{-i}) - g^i(s^i,\boldsymbol{s}^{-i})\right)\right).$$

It follows that, for all $r^i \in S^i$,

$$\sum_{\boldsymbol{s}^{-i}\in S^{-i}} \eta^{-i}(\boldsymbol{s}^{-i})\mu^{-i}(\boldsymbol{s}^{-i}) \sum_{s^i\in S^i} \mu^i(s^i)g^i(r^i,\boldsymbol{s}^{-i}) = \sum_{\boldsymbol{s}^{-i}\in S^{-i}} \eta^{-i}(\boldsymbol{s}^{-i})\mu^{-i}(\boldsymbol{s}^{-i}) \sum_{s^i\in S^i} \mu^i(s^i)g^i(s^i,\boldsymbol{s}^{-i}),$$

and thus, equivalently,

$$\left(\sum_{s^i\in S^i} \mu^i(s^i)\right) \sum_{\boldsymbol{s}^{-i}\in S^{-i}} \eta^{-i}(\boldsymbol{s}^{-i})\mu^{-i}(\boldsymbol{s}^{-i})g^i(r^i,\boldsymbol{s}^{-i}) = \sum_{\boldsymbol{s}^{-i}\in S^{-i}} \eta^{-i}(\boldsymbol{s}^{-i})\mu^{-i}(\boldsymbol{s}^{-i}) \sum_{s^i\in S^i} \mu^i(s^i)g^i(s^i,\boldsymbol{s}^{-i}).$$

Define

$$(\mu\eta)^{-i}(S^{-i}) = \prod_{j\neq i}\sum_{s^j\in S^j} \mu^j(s^j)\eta^j(s^j).$$

Then, dividing by $\mu^i(S^i)(\mu\eta)^{-i}(S^{-i})$, we have

$$\sum_{\boldsymbol{s}^{-i}\in S^{-i}} \overline{\mu\eta}^{-i}(\boldsymbol{s}^{-i})g^i(r^i,\boldsymbol{s}^{-i}) = \sum_{\boldsymbol{s}^{-i}\in S^{-i}} \overline{\mu\eta}^{-i}(\boldsymbol{s}^{-i}) \sum_{s^i\in S^i} \overline{\mu}^i(s^i)g^i(s^i,\boldsymbol{s}^{-i}).$$

Notice that the right-hand side is independent of r^i. This conclude the proof since r^i is arbitrary. ∎

A.3. **Proofs of Section 3.1.** To prove Theorem 3.6, it is sufficient to establish both Lemmas 3.4 and 3.5. Uniqueness of the decomposition then implies the theorem.

Proof of Lemma 3.4. Let $\boldsymbol{g} \in \mathcal{G}$ be a game with duplicate strategies $s_0^i, s_1^i \in S^i$. It is easy to see that s_0^i and s_1^i remain duplicate in $\Pi(\boldsymbol{g})$ and in $\Lambda(\boldsymbol{g})$. For the rest of the proof, we will use the explicit formulas for the decomposition introduced in Eq. (A.25). Recall that

$$\boldsymbol{g}_{\mathcal{P}} = \Pi(\boldsymbol{f}), \quad \boldsymbol{g}_{\mathcal{H}} = \Pi\left(\boldsymbol{g} - \boldsymbol{f}\right), \quad \boldsymbol{g}_{\mathcal{NS}} = \Lambda(\boldsymbol{g}),$$

where $\boldsymbol{f} = \Big((1/\eta^{-1})\delta_0^{\dagger}D(\boldsymbol{g}), \dots, (1/\eta^{-n})\delta_0^{\dagger}D(\boldsymbol{g})\Big)$.

Using Eq. (A.18), for any $i \in N$, we have

$$\Pi^i(f) = \Pi^i\left(\sum_{i\in N}\delta_0^{i\dagger}\delta_0^i(\eta^{-i}g^i)\right) = \Pi^i\left(\sum_{i\in N}\eta^{-i}\delta_0^{i\dagger}\delta_0^i g^i\right) = \Pi^i\left(\sum_{i\in N}\eta^{-i}\Pi^i(g^i)\right),$$

where the last equality holds true since $\delta_{0,W}^{\dagger}\delta_0$ is the orthogonal projection onto $\mathcal{NO}$. Moreover, for any $\boldsymbol{s}^{-i} \in S^{-i}$,

$$\eta^{-i}(\boldsymbol{s}^{-i})\Pi^i(g^i)(s_0^i,\boldsymbol{s}^{-i}) = \eta^{-i}(\boldsymbol{s}^{-i})\Pi^i(g^i)(s_1^i,\boldsymbol{s}^{-i}),$$

and, thus, s_0 and s_1 are duplicate in the η-potential component. Since $\boldsymbol{g}_{\mathcal{H}} = \Pi(\boldsymbol{g}) - \Pi(\boldsymbol{f})$, it follows that s_0 and s_1 are also duplicate in the (μ,η)-harmonic component. ∎

Proof of Lemma 3.5. (i) Assume that $\boldsymbol{g}$ is η-potential. By Proposition A.5, there exists $\varphi : S \to \mathbb{R}$ such that $D(\boldsymbol{g}) = \delta_0(\varphi)$.

The restriction of φ to $\widehat{S}$ is still a potential function for $\widehat{\boldsymbol{g}}$ and hence $\widehat{\boldsymbol{g}}$ is η-potential.

(ii) Assume now that $\boldsymbol{g}$ is non-strategic. We know that $D(\boldsymbol{g}) = 0$. The flow of $\widehat{\boldsymbol{g}}$ is equal to the restriction of the flow of $\boldsymbol{g}$ to $\widehat{S}$ hence $D(\widehat{\boldsymbol{g}}) = 0$ and $\widehat{\boldsymbol{g}}$ is non-strategic.

(iii) For the rest, assume that $\boldsymbol{g}$ is μ-normalized. By definition, we know that for all $j \in N$ and for all $\boldsymbol{s}^{-j} \in S^{-j}$,

$$\sum_{s^j \in S^j} \mu^j(s^j) g^j(s^j, \boldsymbol{s}^{-j}) = 0.$$

We need to distinguish two cases. If $j \neq i$, then, for all $\boldsymbol{s}^{-j} \in \widehat{S}^{-j} \subset S^{-j}$, we have

$$\sum_{s^j \in S^j} \widehat{\mu}^j(s^j) g^j(s^j, \boldsymbol{s}^{-j}) = \sum_{s^j \in S^j} \mu^j(s^j) g^j(s^j, \boldsymbol{s}^{-j}) = 0.$$

If $j = i$, then, for all $\boldsymbol{s}^{-i} \in S^{-i}$, we have

$$\begin{aligned}
\sum_{s^i \in \widehat{S}^i} \widehat{\mu}^i(s^i) g^i(s^i, \boldsymbol{s}^{-i}) &= \widehat{\mu}^i(s^i_0) g^i(s^i_0, \boldsymbol{s}^{-i}) + \sum_{s^i \in \widehat{S}^i \setminus \{s^i_0\}} \widehat{\mu}^i(s^i) g^i(s^i, \boldsymbol{s}^{-i}) \\
&= \mu^i(s^i_0) g^i(s^i_0, \boldsymbol{s}^{-i}) + \mu^i(s^i_1) g^i(s^i_1, \boldsymbol{s}^{-i}) \\
&\quad + \sum_{s^i \in S^i \setminus \{s^i_0, s^1_1\}} \mu^i(s^i) g^i(s^i, \boldsymbol{s}^{-i}) \\
&= \sum_{s^i \in S^i} \mu^i(s^i) g^j(s^i, \boldsymbol{s}^{-i}) \\
&= 0.
\end{aligned}$$

Therefore, $\widehat{\boldsymbol{g}}$ is $(\widehat{\mu}, \eta)$-normalized.

(iv) Finally, assume that $\boldsymbol{g}$ is (μ, η)-harmonic. Let $\boldsymbol{s} = (s^j)_{j \in N}$ and for all $j \in N$ and $t^j \in S^j$, put $g^j_{t^j}(\boldsymbol{s}) = g^j(s^j, \boldsymbol{s}^{-j}) - g^j(t^j, \boldsymbol{s}^{-j})$. Then, since $\boldsymbol{g}$ is assumed to be (μ, η)-harmonic, for any $\boldsymbol{s} \in S$, we have

$$\begin{aligned}
\delta^*_0 D(\boldsymbol{g})(\boldsymbol{s}) &= \sum_{j \in N} \sum_{t^j \in S^j} \mu^j(t^j) \eta^{-j}(\boldsymbol{s}^{-j}) g^j_{t^j}(\boldsymbol{s}) \\
&= \sum_{t^i \in S^i} \mu^i(t^i) \eta^{-i}(\boldsymbol{s}^{-i}) g^i_{t^i}(\boldsymbol{s}) + \sum_{j \neq i} \sum_{t^j \in S^j} \mu^j(t^j) \eta^{-j}(\boldsymbol{s}^{-j}) g^j_{t^j}(\boldsymbol{s}) \\
&= 0.
\end{aligned}$$

Since $g^i_{s^i_0}(\boldsymbol{s}) = g^i_{s^i_1}(\boldsymbol{s})$, it follows that

$$\begin{aligned}
\left(\mu^i(s^i_0) + \mu^i(s^i_1)\right) \eta^{-i}(\boldsymbol{s}^{-i}) g^i_{s^i_0}(\boldsymbol{s}) + \sum_{t^i \neq s^i_0, s^i_1} \eta^{-i}(\boldsymbol{s}^{-i}) \mu^i(t^i) g^i_{t^i}(\boldsymbol{s}) \\
+ \sum_{j \neq i} \sum_{t^j \in S^j} \eta^{-j}(\boldsymbol{s}^{-j}) \mu^j(t^j) g^j_{t^j}(\boldsymbol{s}) = 0.
\end{aligned}$$

Likewise, in the reduced game $\widehat{\boldsymbol{g}} = (g^j)_{j\in N}$ we get:

$$\delta_0^* D(\widehat{\boldsymbol{g}})(\boldsymbol{s}) = \widehat{\mu}^i(s_0^i)\eta(\boldsymbol{s}^{-i})g^i_{s_0^i}(\boldsymbol{s}) + \sum_{t^i\neq s_0^i} \widehat{\mu}^i(t^i)\eta^{-i}(\boldsymbol{s}^{-i})g^i_{t^i}(\boldsymbol{s}) + \sum_{j\neq i}\sum_{t^j\in S^j} \widehat{\mu}^j(t^j)\eta(s^{-j})g^j_{t^j}(\boldsymbol{s}). \quad \text{(A.27)}$$

We obtain that $\delta_0^* D(\widehat{\boldsymbol{g}})(\boldsymbol{s}) = 0$ and it thus follows that $\widehat{\boldsymbol{g}}$ is $(\widehat{\mu},\eta)$-harmonic. ■

A.4. **Proofs of Section 3.2.2.**

Proof of Proposition 3.12. Let $\boldsymbol{g} \in \mathcal{G}$ and let $\boldsymbol{x} = (x^i)_{i\in N}$ be a Nash equilibrium of $\boldsymbol{g}$. Let $\operatorname{supp}(x^i) = \{s^i \in S^i : x^i(s^i) > 0\}$. By definition of y^i, we have that $\operatorname{supp}(x^i) = \operatorname{supp}(y^i)$. Then, for any $i \in N$ and any $s^i, t^i \in \operatorname{supp}(x^i)$, we have $x^{-i}(\boldsymbol{s}^{-i})g^i(s^i,\boldsymbol{s}^{-i}) = x^{-i}(\boldsymbol{s}^{-i})g^i(t^i,\boldsymbol{s}^{-i})$. By Definition 3.10 and Eq. (3.3), we get

$$\begin{aligned}
y^{-i}(\boldsymbol{s}^{-i})(\boldsymbol{\beta}\cdot\boldsymbol{g})^i(s^i,\boldsymbol{s}^{-i}) &= \frac{x^{-i}(\boldsymbol{s}^{-i})}{\Pi_{j\neq i} b^j(s^j)}(\boldsymbol{\beta}\cdot\boldsymbol{g})^i(s^i,\boldsymbol{s}^{-i}) \\
&= x^{-i}(\boldsymbol{s}^{-i})g^i(s^i,\boldsymbol{s}^{-i}) \\
&= x^{-i}(\boldsymbol{s}^{-i})g^i(t^i,\boldsymbol{s}^{-i}) \\
&= \frac{x^{-i}(\boldsymbol{s}^{-i})}{\Pi_{j\neq i} b^j(s^j)}(\boldsymbol{\beta}\cdot\boldsymbol{g})^i(t^i,\boldsymbol{s}^{-i}) \\
&= y^{-i}(\boldsymbol{s}^{-i})(\boldsymbol{\beta}\cdot\boldsymbol{g})^i(t^i,\boldsymbol{s}^{-i}).
\end{aligned}$$

Let $s^i \in \operatorname{supp}(x^i)$ and $t^i \notin \operatorname{supp}(x^i)$. Then,

$$\begin{aligned}
y^{-i}(\boldsymbol{s}^{-i})(\boldsymbol{\beta}\cdot\boldsymbol{g})^i(s^i,\boldsymbol{s}^{-i}) &= \frac{x^{-i}(\boldsymbol{s}^{-i})}{\Pi_{j\neq i} b^j(s^j)}(\boldsymbol{\beta}\cdot\boldsymbol{g})^i(s^i,\boldsymbol{s}^{-i}) \\
&= x^{-i}(\boldsymbol{s}^{-i})\boldsymbol{g}^i(s^i,\boldsymbol{s}^{-i}) \\
&\geqslant x^{-i}(\boldsymbol{s}^{-i})\boldsymbol{g}^i(t^i,\boldsymbol{s}^{-i}) \\
&= \frac{x^{-i}(\boldsymbol{s}^{-i})}{\Pi_{j\neq i} b^j(s^j)}(\boldsymbol{\beta}\cdot\boldsymbol{g})^i(t^i,\boldsymbol{s}^{-i}) \\
&= y^{-i}(\boldsymbol{s}^{-i})(\boldsymbol{\beta}\cdot\boldsymbol{g})^i(t^i,\boldsymbol{s}^{-i}).
\end{aligned}$$

The same equality holds for the normalized strategy $\bar{y}^i$ and it thus follows that $\bar{y} = \left(\bar{y}^i\right)_{i\in N}$ is a Nash equilibrium. ■

Proof of Theorem 3.13. Let μ, η and b be three product measures on S and $\boldsymbol{g}$ a finite game. By Theorem 2.8, there exist $\boldsymbol{g}_{\mathcal{P}}$, $\boldsymbol{g}_{\mathcal{H}}$ and $\boldsymbol{g}_{\mathcal{NS}}$ such that $\boldsymbol{g} = \boldsymbol{g}_{\mathcal{P}} + \boldsymbol{g}_{\mathcal{H}} + \boldsymbol{g}_{\mathcal{NS}}$, where $\boldsymbol{g}_{\mathcal{P}}$ is η-potential μ-normalized, $\boldsymbol{g}_{\mathcal{H}}$ is (μ,η)-harmonic μ-normalized and $\boldsymbol{g}_{\mathcal{NS}}$ is non-strategic. For any $i \in N$, $s^i, t^i \in S^i$ and $\boldsymbol{s}^{-i} \in S^{-i}$, we have:

$$\varphi(t^i,\boldsymbol{s}^{-i}) - \varphi(s^i,\boldsymbol{s}^{-i}) = \eta^{-i}(\boldsymbol{s}^{-i})g^i_{\mathcal{P}}(t^i,\boldsymbol{s}^{-i}) - \eta^{-i}(\boldsymbol{s}^{-i})g^i_{\mathcal{P}}(s^i,\boldsymbol{s}^{-i}),$$

therefore,

$$\begin{aligned}\varphi(t^i,\boldsymbol{s}^{-i})-\varphi(s^i,\boldsymbol{s}^{-i}) &= \frac{\eta^{-i}(\boldsymbol{s}^{-i})}{\beta^i(\boldsymbol{s}^{-i})}\left(\beta^i(\boldsymbol{s}^{-i})g^i_{\mathcal{P}}(t^i,\boldsymbol{s}^{-i})-\beta^i(\boldsymbol{s}^{-i})g^i_{\mathcal{P}}(s^i,\boldsymbol{s}^{-i})\right)\\ &= \left(\prod_{j\neq i}\frac{\eta^j(s^j)}{b^j(s^j)}\right)\left((\boldsymbol{\beta}\cdot\boldsymbol{g}_{\mathcal{P}})^i(t^i,\boldsymbol{s}^{-i})-(\boldsymbol{\beta}\cdot\boldsymbol{g}_{\mathcal{P}})^i(s^i,\boldsymbol{s}^{-i})\right)\\ &= \left(\frac{\eta}{b}\right)^{-i}(\boldsymbol{s}^{-i})\left((\boldsymbol{\beta}\cdot\boldsymbol{g}_{\mathcal{P}})^i(t^i,\boldsymbol{s}^{-i})-(\boldsymbol{\beta}\cdot\boldsymbol{g}_{\mathcal{P}})^i(s^i,\boldsymbol{s}^{-i})\right).\end{aligned}$$

Hence $(\boldsymbol{\beta}\cdot\boldsymbol{g}_{\mathcal{P}})$ is η/b-potential. Moreover one can check easily that $(\boldsymbol{\beta}\cdot\boldsymbol{g}_{\mathcal{P}})$ is still μ-normalized.

Moreover, we also have

$$\begin{aligned}0 &= \sum_{i\in N}\sum_{t^i\in S^i}\mu^i(t^i)\eta^{-i}(\boldsymbol{s}^{-i})\left(g^i_{\mathcal{H}}(s^i,\boldsymbol{s}^{-i})-g^i_{\mathcal{H}}(t^i,\boldsymbol{s}^{-i})\right)\\ &= \sum_{i\in N}\sum_{t^i\in S^i}\frac{\mu^i(t^i)\eta^{-i}(\boldsymbol{s}^{-i})}{\beta^i(\boldsymbol{s}^{-i})}\left(\beta^i(\boldsymbol{s}^{-i})g^i_{\mathcal{H}}(s^i,\boldsymbol{s}^{-i})-\beta^i(\boldsymbol{s}^{-i})g^i_{\mathcal{H}}(t^i,\boldsymbol{s}^{-i})\right)\\ &= \sum_{i\in N}\sum_{t^i\in S^i}\mu^i(t^i)\left(\frac{\eta}{b}\right)^{-i}(\boldsymbol{s}^{-i})\left((\boldsymbol{\beta}\cdot\boldsymbol{g}_{\mathcal{H}})^i(s^i,\boldsymbol{s}^{-i})-(\boldsymbol{\beta}\cdot\boldsymbol{g}_{\mathcal{H}})^i(t^i,\boldsymbol{s}^{-i})\right).\end{aligned}$$

Hence, $(\boldsymbol{\beta}\cdot\boldsymbol{g}_{\mathcal{H}})$ is $(\mu,\eta/b)$-harmonic. Moreover one can check easily that $(\boldsymbol{\beta}\cdot\boldsymbol{g}_{\mathcal{H}})$ is still μ-normalized. Since $(\boldsymbol{\beta}\cdot\boldsymbol{g})=(\boldsymbol{\beta}\cdot\boldsymbol{g}_{\mathcal{P}})+(\boldsymbol{\beta}\cdot\boldsymbol{g}_{\mathcal{H}})+(\boldsymbol{\beta}\cdot\boldsymbol{g}_{\mathcal{NS}})$, the result follows from uniqueness of the $(\mu,\eta/b)$-decomposition. ■

References

Ben-Israel, A. and Greville, T. N. E. (2003) *Generalized Inverses*. Springer-Verlag, New York, second edition.

Candogan, O., Menache, I., Ozdaglar, A., and Parrilo, P. A. (2011) Flows and decompositions of games: harmonic and potential games. *Math. Oper. Res.* **36**, 474–503. URL https://doi.org/10.1287/moor.1110.0500.

Candogan, O., Ozdaglar, A., and Parrilo, P. A. (2013) Dynamics in near-potential games. *Games Econom. Behav.* **82**, 66–90. URL https://doi.org/10.1016/j.geb.2013.07.001.

Cheng, D., Liu, T., Zhang, K., and Qi, H. (2016) On decomposed subspaces of finite games. *IEEE Trans. Automat. Control* **61**, 3651–3656. URL https://doi.org/10.1109/TAC.2016.2525936.

Govindan, S. and Wilson, R. (1997) Equivalence and invariance of the index and degree of Nash equilibria. *Games Econom. Behav.* **21**, 56–61. URL https://doi.org/10.1006/game.1997.0516.

Govindan, S. and Wilson, R. (2005) Essential equilibria. *Proc. Natl. Acad. Sci. USA* **102**, 15706–15711. URL https://doi.org/10.1073/pnas.0506796102.

Govindan, S. and Wilson, R. (2008) Refinements of Nash equilibrium. In Durlauf, S. and Blume, L. (eds.), *The New Palgrave Dictionary of Economics, 2nd edition*. Palgrave Macmillan, London.

Govindan, S. and Wilson, R. (2009) On forward induction. *Econometrica* **77**, 1–28.
URL https://doi.org/10.3982/ECTA6956.

Hwang, S. and Rey-Bellet, L. (2016) Strategic decompositions of normal form games: zero-sum games and potential games. arXiv 1602.06648.
URL http://arxiv.org/abs/1602.06648.

Jessie, D. and Saari, D. G. (2013) Strategic and behavioral decomposition of $2 \times 2 \times \cdots \times 2$ games. Technical report, Institute for Mathematical Behavioral Sciences, University of California, Irvine.
URL https://www.imbs.uci.edu/files/docs/technical/2013/mbs_03-13.pdf.

Jiang, X., Lim, L.-H., Yao, Y., and Ye, Y. (2011) Statistical ranking and combinatorial Hodge theory. *Math. Program.* **127**, 203–244.
URL https://doi.org/10.1007/s10107-010-0419-x.

Kalai, A. and Kalai, E. (2013) Cooperation in strategic games revisited. *Q. J. Econ.* **128**, 917–966.
URL http://dx.doi.org/10.1093/qje/qjs074.

Li, C., Liu, T., He, F., Cheng, D., Qi, H., and Hong, Y. (2016) On finite harmonic games. In *2016 IEEE 55th Conference on Decision and Control (CDC)*, 7024–7029.
URL https://doi.org/10.1109/CDC.2016.7799351.

Liu, S. (2018) Directed graphical structure, Nash equilibrium, and potential games. *Oper. Res. Lett.* **46**, 273–277.
URL https://doi.org/10.1016/j.orl.2018.02.002.

Liu, T., Qi, H., and Cheng, D. (2015) Dual expressions of decomposed subspaces of finite games. In *2015 34th Chinese Control Conference (CCC)*, 9146–9151.
URL https://doi.org/10.1109/ChiCC.2015.7261086.

Monderer, D. and Shapley, L. S. (1996) Potential games. *Games Econom. Behav.* **14**, 124–143.
URL https://doi.org/10.1006/game.1996.0044.

Osborne, M. J. and Rubinstein, A. (1998) Games with procedurally rational players. *Amer. Econ. Rev.* **88**, 834–847.
URL http://www.jstor.org/stable/117008.

Sandholm, W. H. (2010) Decompositions and potentials for normal form games. *Games Econom. Behav.* **70**, 446–456.
URL https://doi.org/10.1016/j.geb.2010.01.008.

Stern, A. and Tettenhorst, A. (2017) Hodge decomposition and the Shapley value of a cooperative game. arXiv:1709.08318.
URL https://arxiv.org/abs/1709.08318.

Szabó, G., Bodó, K. S., and Samani, K. A. (2017) Separation of cyclic and starlike hierarchical dominance in evolutionary matrix games. *Phys. Rev. E* **95**, 012320.
URL https://doi.org/10.1103/PhysRevE.95.012320.

Zhang, K. (2017) Polynomial representation for orthogonal projections onto subspaces of finite games. In *2017 36th Chinese Control Conference (CCC)*, 11267–11272.
URL https://doi.org/10.23919/ChiCC.2017.8029154.

Centre d'économie de la Sorbonne, Université Paris 1, Panthéon-Sorbonne, 106-112 boulevard de l'Hôpital, 75647 Paris Cedex 13-France; email: abdou@univ-paris1.fr.

Lemma, Université Paris 2, Panthéon-Assas, 4 Rue Desgoffe, 75006, Paris, France; email: nikolaos.pnevmatikos@u-paris2.fr.

Dipartimento di Economia e Finanza, LUISS, Viale Romania 32, 00197 Rome, Italy, email: marco.scarsini@luiss.it.

Ecole d'Economie de Paris, Université Paris 1, Panthéon-Sorbonne, 106-112 boulevard de l'Hôpital, 75647 Paris Cedex 13-France; email: xavier.venel@univ-paris1.fr