Diagnosability and detectability of multi-faults in nonlinear models

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Abstract

This paper presents a novel method for assessing multiple fault diagnosability and detectability of nonlinear parametrized dynamical models. This method is based on computer algebra algorithms which return precomputed values of algebraic expressions characterizing the presence of some constant multiple fault(s). Estimations of these expressions, obtained from input and output measurements, permit then the detection and the isolation of multiple faults acting on the system. The application of this method on a coupled water-tank model attests the relevance of the suggested approach.

Keywords: Diagnosability; Detectability; Algebraic Signature; Nonlinear models; Algorithm

1. Introduction

The problem of fault-diagnosis has received an increasing attention during the recent years in order to increase security of systems, to monitor their performance or to endow them with self diagnostic capabilities. To answer such technological requirements, this problem needs to be taken into account in the system design stage from an a priori diagnosability study on a model. In studying anticipated fault situations from different symptoms of the system, faults or multiple faults can be known as discriminable according to the available sensors in a system. Some procedures for detecting and isolating them may, then, be put in place in the design stage. By this way, diagnosability can permit to anticipate component failures.

In this paper, we assume to be in the model-based framework and, more precisely, that available input and output signals and a nonlinear parametrized dynamical model permit to reach the output trajectories of the system. The problem consists in this framework to evaluate diagnostic performance given a model only. By (multiple) fault, we mean any change(s) of parameter value(s) implying unwanted changes in the behavior of one or more component(s) of the system. The *fault diagnosis* study consists in two subtasks [1]. The first one concerns the *fault detection* (FD) of the malfunction, the second one the *fault isolation* (FI) of the faulty component (that is the determination of its location). The fault diagnosis is done from the comparison between predictions of the model and behaviors of the system. Several methods are proposed in the literature as nonlinear observers [2] or methods based on testable subsets of equations [3]. The issue of subsets generation has been studied by many authors. They can be based on Minimal Structurally Overdetermined sets (MSO) [4, 5], on possible conflicts [6] or on Analytical Redundancy Relations (ARRs) [7]. The latter are relations linking inputs, outputs, their derivatives, the parameters of the model and the faults (See [8, 9, 10, 11, 12] for single faults and [13, 14, 15] for multiple faults). Some of these ARRs are obtained using computer algebra tools such as the Rosenfeld-Groebner algorithm which permits to eliminate the unknown variables of the model (See [16, 17, 18]). With respect to a specific elimination order, this algorithm returns particular differential polynomials classically called *input-output polynomials*. Some recent works have already used these particular polynomials in diagnosis assuming that the model is identifiable with respect to the faults (See [11]). Indeed, identifiability insures that the fault values can be uniquely inferred from input-output measurements. In the case of single faults, authors in [11] prove that if the model is identifiable with respect to the faults then all the faults are discriminable; in other words, the model is diagnosable. Furthermore, they prove that the residuals associated to each ARR permit to detect each identifiable fault in adopting a discriminable behavior. In [19], assuming that the faults act only additively on parameters, detectability is obtained directly from the ARRs. In this last paper, interval analysis is used to estimate the simple faults.

We propose a new approach to exploit such ARRs to discriminate, detect and isolate (multiple) fault(s) in models not necessary identifiable. Starting from the model, the three steps of our method described hereafter can be completely automatized; our contributions consist in the steps (2) and (3).

1. The first one is the computation of ARRs by applying the Rosenfeld-Groebner algorithm to the model.
2. The second step consists in using Groebner basis computations in order to obtain an algebraic application called *algebraic signature*. Each of its components depends only on the parameters and on the coefficients of the ARRs and can be numerically estimated from the input and output measurements of the system. By construction, each component of the algebraic signature vanishes when at least one specific (multiple) fault occurs.
3. For each possible (multiple) fault, the third step consists in using semialgebraic set tools to certify that some components of the signature vanish or never vanish. These expected values can be summarized in a precomputed table.

This table constitutes the input of the numerical treatment which, from the system measurements, returns estimations of the algebraic signatures. Their comparison with their nominal values permits to detect and isolate (multiple) fault(s).

Our method is not based on the direct use of the functional ARRs (see [8, 9, 10, 11, 12, 13, 14, 15]) but only on algebraic relations deduced from ARRs coefficients. This change of point of view is a first advantage since it permits the complete processing of the algebraic relations to study diagnosability. Owing to the use of semialgebraic set tools in this processing, constraints on parameters and multiple faults, such as inequalities satisfied by parameters or constraints deduced from initial conditions, can be taken into account through automatic procedures. These constraints can play a fundamental role in the FDI analysis of a model as it is shown in this paper. The second advantage of the present method is not to require any strong assumption on the model for determining some possible acting multiple fault(s) as i) its identifiability, ii) the value of some model parameters in some particular cases, iii) the additive action of multiple faults on parameters. Finally, from an a priori study on the model, a numerical method based only on estimation of algebraic expressions, and consequently fast, is proposed to do FDI.

The paper is organized as follows. In Section 2, we present the framework of our method. In this section, we precise the assumptions on the dynamical models and on the constraints that must be verified by parameters and faults. Section 3 is devoted to our method consisting in studying the diagnosability of a model, that is the way to compute an algebraic signature and to tabulate its expected values in function of the multiple faults. In Section 4, our method is applied to an example of two coupled water-tanks. Section 5 concludes the paper. In this paper, the symbolic computations had been realized with Maple 18 and the numerical part with Scilab.

2. Dynamical models and algebra concepts

We consider nonlinear parametrized models controlled or uncontrolled of the following form:

$$\Gamma_f \begin{cases} \dot{x}(t,p,f) = g(x(t,p,f), u(t), p, f), \\ y(t,p,f) = h(x(t,p,f), u(t), p, f), \\ t_0 \leq t \leq T \end{cases} \quad (1)$$

where:

- the vector of real parameters $p = (p_1, \ldots, p_m)$ belongs to $\mathcal{P} \subseteq \mathbb{R}^m$ where $\mathcal{P}$ is an *a priori* known set of admissible parameters,
- $f = (f_1, \ldots, f_e)$ is a constant fault vector which belongs to a subset $\mathcal{F}$ of $\mathbb{R}^e$. It is equal to 0 when there is no fault. The set $\mathcal{F}$ describes the set of admissible values of the fault vectors f,
- $x(t,p,f) \in \mathbb{R}^n$ denotes the state variables and $y(t,p,f) \in \mathbb{R}^s$ the outputs,
- g and h are real vectors of rational analytical functions in x, p and f [1].
- $u(t) \in \mathbb{R}^r$ is the control vector equal to 0 in the case of uncontrolled models.

Remark 1. *In most practical cases, the faults f_i belong to connected sets of $\mathbb{R}$, and $\mathcal{F}$ is the Cartesian product of these sets. The present work takes place in a more general framework by introducing semialgebraic sets defined hereafter.*

[1] The rational assumption is not restrictive since lots of models can be reduced to a rational model by variable change (See [20]). The analytical assumption is required to obtain ARRs by the first step of our method.

From now on, we suppose that constraints on $p \in \mathcal{P}$ and $f \in \mathcal{F}$, and eventual constraints linking fault and parameter components, can be formulated by the mean of algebraic equations and/or inequalities. This consideration leads naturally to consider semialgebraic sets for which computer algebra tools are developed (See [21, 22, 23] for example):

Definition 1. *(See [24]) Let $\mathbb{R}[X_1, \ldots, X_n]$ the set of polynomials with real coefficients and where $X_1, \ldots, X_n$ are n indeterminates.*
A set of real solutions of a finite set of multi-variable polynomial equations (of the form $P = 0$) and/or polynomial inequalities (of the form $Q \geq 0$) of $\mathbb{R}[X_1, \ldots, X_n]$ is called a semialgebraic set.

Let $C_{p,f}$ be the set of all algebraic equations and inequalities verified by the components of the parameter and fault vectors of the model and $\mathcal{C}_{p,f}$ be the semialgebraic set defined by $C_{p,f}$. In order to take into account initial conditions, the algebraic relations induced by these conditions can be added to the set $C_{p,f}$.

Example 1. *For illustrating the theoretical part, the example of a mass $m = 1$ attached to an elastic spring of force k is considered. Denote u an external force of the system not identically equal to zero and $d \geq 1$ a constant. The move of the mass is described by the following equation*

$$\ddot{x} + k\,x - du = 0 \qquad (2)$$

which can be rewritten as model (1):

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -kx_1 + du. \end{cases} \qquad (3)$$

Assume that two faults $f_1 \in [0, 2)$ and $f_2 \in [0, 2)$ impact respectively the spring k and the parameter d such that the model takes the form:

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -k(f_1 - 1)^2 x_1 + (d + f_2)u. \end{cases} \qquad (4)$$

The parameters, k and d and the faults, f_1 and f_2 verify the following constraints:

$$\begin{cases} 0 < k < 4 \\ 1 \leq d \\ 0 \leq f_1 < 2, \\ 0 \leq f_2 < 2. \end{cases} \qquad (5)$$

These algebraic constraints can be viewed as a set of polynomial equations and inequalities whose indeterminates are k, d, f_1 f_2. Those algebraic inequalities define $C_{p,f}$ and the corresponding set of admissible values of the parameters and faults is $\mathcal{C}_{p,f}$.

Let $\mathcal{N}$ be a subset of $\{1, \ldots, e\}$ and $f_{\mathcal{N}}$ the *multiple fault vector* whose components f_i are not equal to 0 if $i \in \mathcal{N}$ and equal to 0 otherwise. Naturally, $f_{\mathcal{N}}$ belongs to $\mathcal{F}_{\mathcal{N}} = \{f \in \mathcal{F} | f_i \neq 0 \text{ if } i \in \mathcal{N} \text{ and } f_i = 0 \text{ if } i \notin \mathcal{N}\}$ and $\mathcal{F}_{\mathcal{N}}$ is a semialgebraic set by construction. When only one component of f is not null, the fault vector f is called a *simple fault*.

Since diagnosability and detectability of Model (1) may depend on $p = (p_1, \ldots, p_m)$, we consider, afterwards, the set $R = \mathbb{R}[p_1, \ldots, p_m]$ of polynomials in the indeterminates p_i with real coefficients.

The following section, which contains the main contribution of this paper, presents a method based on algebraic tools for doing the diagnosability study of model (1) (diagnosable in the sense that the occurrence of (multiple) faults can be captured by discrepancies between the observed behavior and the output predicted by the model). More precisely, the diagnosability study of Section 3 is based on the definition of an algebraic signature not depending explicitly on the (multiple) faults and assessable from the known quantities of the system.

3. An algebraic diagnosability method

For practical applications, we propose the construction and the study of an algebraic signature in three steps using symbolic computations. The first one requires the implementation of the Rosenfeld-Groebner algorithm in order to obtain algebraic relations linking parameters, faults and real values computable from the outputs and the inputs of the system. From these algebraic relations, the second step consists in using the Groebner basis algorithm to obtain an algebraic signature and the third one uses semialgebraic set tools to discriminate multiple faults from the algebraic signature values.

3.1. First step: obtention of ARRs

In [11, 17], the authors give a way to obtain ARRs, that is relations linking inputs, outputs, parameters and faults. The latter are obtained from the Rosenfeld-Groebner algorithm implemented in some computer algebra systems. This elimination algorithm used with an appropriate elimination order permits to eliminate unknown variables from System (1). These input-output representations may act as analytical redundancy relations (ARRs) and have the following forms

$$\begin{aligned} w_i(y, u, p, f) \quad &= m_{0,i}(y, u, p) \\ &+ \sum_{k=1}^{n_i} \gamma_k^i(p, f) m_{k,i}(y, u) = 0\,, \end{aligned} \qquad (6)$$
$$i = 1, \ldots, s$$

where $(\gamma_k^i)_{1\leq k\leq n_i}$ are rational fractions in p and f, $\gamma_k^v \neq \gamma_k^w$ for $v \neq w$, $(m_{k,i}(y,u))_{1\leq k\leq n_i}$ are differential polynomials with respect to y and u and $m_{0,i} \neq 0$. The first part of these polynomials, i.e. the terms $m_{0,i}(y,u,p)$, can be supposed not to be identically equal to zero. It corresponds to the residual *computation form* whereas the second form is known as the residual *internal form*. According to [17], there are as many polynomials of this form as outputs.
The sequence $(\gamma_k^i(p,f))_{k=1,\dots,n_i}$ $(i = 1,\dots,s)$ is called the exhaustive summary of System (1) (See [25]). The function ϕ, used hereafter, is constructed from this sequence and defined by:

$$\begin{array}{cccc} \phi: & \mathbb{R}^e & \longrightarrow & R^N \\ & f & \mapsto & (\gamma_k^i(p,f))_{1\leq i\leq s, 1\leq k\leq n_i} \end{array}$$

where $N = \sum_{i=1}^{s} n_i$.

3.2. Second step: construction of an algebraic signature from the exhaustive summary

Fault signatures are classically functions which associate to a fault the set of indicators called residuals. Residuals and fault signatures can be directly deduced from the ARRs (6) (See [11]). The residuals are defined by $\rho_i = m_{0,i}(y,u,p)$ and the signature by the vector $\left(-\sum_{k=1}^{n_i}\gamma_k^i(p,f)m_{k,i}(y,u)\right)_{i=1,\dots,s}$.
Fault signatures are also functional relations depending on the measured outputs and the inputs of the system.

Example 2. *Let us consider Example 1 and suppose that $y = x$ is a measured output of the system. Equation (2) represents an ARR of model (3). When the faults impact the parameters of this model as described in Example 1, the ARR has the following form:*

$$\ddot{y} + k\,(f_1-1)^2\,y - (d+f_2)u = 0. \qquad (7)$$

The residual is the polynomial $\rho = \ddot{y} + k\,y - du$ and the fault signature $FSig(f) = (-k(f_1^2 - 2f_1)y, f_2 u)$.

In the following definition, we introduce the notion of algebraic signature to characterize multiple faults. This definition differs from the classical ones in the sense that it is composed of algebraic expressions do not depending directly on the measured outputs. It is based on l algebraic expressions, $ASig_i$ $(i = 1,\dots,l)$ which can be deduced from the ARRs defined in (6) as it will be explained in Examples 3 and 5.

Definition 2. *Let $ASig = (ASig_1, \dots, ASig_l)$ be a vector of algebraic expressions admitting $f_1, \cdots, f_e$ as indeterminates with coefficients in R. An algebraic signature is a function $ASig$ defined by:*

$$\begin{array}{cccc} ASig: & \mathcal{F} & \longrightarrow & R^l \\ & f & \mapsto & (ASig_1(f), \dots, ASig_l(f)). \end{array}$$

Example 3. *Considering Example 2, the function ϕ defined by*

$$\phi(f) = \big(k(f_1-1)^2, -d-f_2\big)$$

provides an algebraic signature $ASig(f) = \big(k(f_1-1)^2, -d-f_2\big)$.

The comparison of the images of two multiple faults under the function $ASig$ gives a way to discriminate them. If the model is controlled, we propose to define the strongly and weakly algebraic diagnosability, the first one being true for all inputs and the second one for at least one input.

Definition 3. *Let $\mathcal{N}$ and $\mathcal{N}'$ be two distinct subsets of $\{1, \dots, e\}$. The multiple faults of $\mathcal{F}_{\mathcal{N}}$ and of $\mathcal{F}_{\mathcal{N}'}$ are said input-strongly algebraically discriminable (resp. input-weakly algebraically discriminable) if, there existes an algebraic signature such that, for all input u (resp. one input),*

$$ASig(\mathcal{F}_{\mathcal{N}}) \cap ASig(\mathcal{F}_{\mathcal{N}'}) = \emptyset. \qquad (8)$$

This equality is in particular satisfied when there exists an index i such that $ASig_i(\mathcal{F}_{\mathcal{N}}) \cap ASig_i(\mathcal{F}_{\mathcal{N}'}) = \emptyset$.

If, for any distinct subsets $\mathcal{N}$ and $\mathcal{N}'$ of $\{1, \dots, e\}$, the multiple faults of $\mathcal{F}_{\mathcal{N}}$ and $\mathcal{F}_{\mathcal{N}'}$ are input-strongly algebraically discriminable (resp. input-weakly algebraically discriminable), the model is said input-strongly algebraically diagnosable (resp. input-weakly algebraically diagnosable).

In the case of uncontrolled model, the definition of algebraic diagnosability can be proposed too in omitting the notion of input in the previous definitions.

Detectability consists in discriminating the faulty situation whatever the faults or multiple faults acting on the system. A natural approach in the framework of ARRs is to use residuals or fault signatures to define detectability (See [10, 11]). Indeed, in the case of a faulty situation, they are no more identically equal to zero. In the same way, the notion of detectability of a set of multiple faults can be defined from the algebraic signature. In that case, the detectability consists, from the algebraic signature, to compare its value to the one obtained when the set $\mathcal{N}$ is empty, that is when no fault occurs in the system. This definition is given below.

Definition 4. *A set of multiple faults vectors $\mathcal{F}_{\mathcal{N}}$ is algebraically detectable if*

$$ASig(\mathcal{F}_{\mathcal{N}}) \cap ASig(\mathcal{F}_{\emptyset}) = \emptyset,$$

$ASig(\mathcal{F}_{\emptyset})$ being the algebraic signature evaluated when no fault occurs in the system.

Example 4. *In Example 3, the algebraic signatures of the possible multiple faults are $ASig(f_{\emptyset}) = (k, -d)$, $ASig(f_{\{1\}}) = (k(f_1 - 1)^2, -d)$, $ASig(f_{\{2\}}) = (k, -d-f_2)$ and $ASig(f_{\{1,2\}}) = (k(f_1-1)^2, -d-f_2)$. Constraints on parameters and faults imply that the images of these algebraic signatures do not intersect: for example, $ASig(f_{\{1\}}) \cap ASig(f_{\{1,2\}}) = \emptyset$ since for any $f_2 \in (0, 2)$, $-d \neq -d - f_2$. Consequently, the model is algebraic diagnosable since the multiple faults can be discriminated.*
In this example, the function ϕ is not injective: the values $1/2$ and $3/2$ of f_1 will give the same value of $\phi(f)$. The injectivity of ϕ is strongly connected to the notion of identifiability of the model. Recall that a model is identifiable if the model parameters are uniquely determined by the model inputs and outputs. In [25], under some technical assumptions, the authors prove that if the function ϕ is injective the model is identifiable. Consequently, full identifiability of the fault parameters implies algebraic diagnosability since any fault vector instance will give a distinct value of $\phi(f)$. However, algebraic diagnosability does not imply identifiability. Indeed, even if ϕ is not injective, (multiple) faults discrimination may be possible as shown in this example.

The algebraic signature defined by the exhaustive summary is not sufficient since two distinct faults acting on its same components may not be discriminated. A natural approach to exploit the exhaustive summary consists in obtaining an explicit expression of the fault components in function of the model parameters and the components, $\phi_1, \ldots, \phi_N$, of $\phi(f)$. This approach focusing on the inversion of an algebraic system fails in general. That is why we propose a method to obtain algebraic expressions not depending on the faults and characterizing their presence. Such expressions can be computed by automatic procedures based on Groebner basis algorithms (See [26, 27]) and are used below to define an algebraic signature.

To lighten our approach, we suppose that $(\gamma_k^i)_{1\leq k\leq n_i}$ are polynomials of $R[f_1, \ldots, f_e]$. Actually, when these expressions are rational fractions, they can be expressed as polynomials by introducing new variables equal to the inverses of denominators. Nonvanishing conditions of these denominators can be added to $C_{p,f}$.

Given a multiple fault $f \in \mathcal{F}_{\mathcal{N}}$ ($\mathcal{N} \subset \{1, \cdots, e\}$), let $E_{\mathcal{N}}$ be the set of polynomials

$$E_{\mathcal{N}} = \{\gamma_1^1(p, f) - \phi_1, \ldots, \gamma_s^{n_s}(p, f) - \phi_N\} \cup \{v_i f_i - 1 | i \in \mathcal{N}\} \cup \{f_i | i \notin \mathcal{N}\}$$

where v_i are new indeterminates. In the definition of $E_{\mathcal{N}}$, the sets $\{v_i f_i - 1 | i \in \mathcal{N}\}$ and $\{f_i | i \notin \mathcal{N}\}$ characterize multiple faults of $\mathcal{F}_{\mathcal{N}}$. Let us consider the polynomial ideal $I_{\mathcal{N}}$ generated by $E_{\mathcal{N}}$, that is the set of all linear combinations of elements of $E_{\mathcal{N}}$ in $R[v_1, \ldots, v_e, f_1, \ldots, f_e, \phi_1, \ldots, \phi_N]$.

A Groebner basis of this ideal $I_{\mathcal{N}}$ is computed with respect to an elimination order chosen to eliminate first the indeterminates v_i and f_i. The intersection $G_{\mathcal{N}}$ of this Groebner basis and of $R[\phi_1, \ldots, \phi_N]$ generates the elimination ideal $J_{\mathcal{N}} = I_{\mathcal{N}} \cap R[\phi_1, \ldots, \phi_N]$ (See [26]). Clearly, any polynomial of $G_{\mathcal{N}}$ vanishes when a multiple fault $f \in \mathcal{F}_{\mathcal{N}}$ occurs.
For all the possible multiple faults $f_{\mathcal{N}}$, the sets $G_{\mathcal{N}}$ are computed. Polynomials of $\cup_{\mathcal{N}\subset\{1,\ldots,m\}} G_{\mathcal{N}}$ vanishing for all multiple faults, i.e. polynomials of $\cap_{\mathcal{N}\subset\{1,\ldots,m\}} I_{\mathcal{N}}$, are removed of this set. The remaining polynomials are kept to define the components of an algebraic signature.

Let us summarize our algorithm returning an algebraic signature.

`Algebraic_signature`

1. For each subset $\mathcal{N}$ of $\{1, \cdots, e\}$, we consider a generic multiple fault $f_{\mathcal{N}}$ and we apply the following steps to this multiple fault.
   (a) Computation of the Groebner basis of the ideal $I_{\mathcal{N}}$ generated by $E_{\mathcal{N}}$ with respect to the lexicographical order $v_{i_1} \succ \ldots \succ v_{i_l} \succ f_1 \succ \ldots \succ f_m \succ \phi_1 \succ \ldots \succ \phi_N \succ p_1 \succ \cdots \succ p_m$.
   (b) Determination of the intersection, $G_{\mathcal{N}}$, of this last Groebner basis and of $R[\phi_1, \ldots, \phi_N]$.
2. Remove to $\cup_{\mathcal{N}\subset\{1,\ldots,m\}} G_{\mathcal{N}}$ polynomials vanishing for any multiple fault, in other words, polynomials of the ideal $\cap_{\mathcal{N}\subset\{1,\ldots,m\}} I_{\mathcal{N}}$.
3. Order arbitrarily all the polynomials of the last obtained set in a sequence $ASig = (ASig_1, \ldots, ASig_l)$.
4. Return $ASig$.

By this way, we obtain an *algebraic signature of the multiple faults* used afterwards:

$$\begin{array}{rccc} ASig: & \mathbb{R}^e & \longrightarrow & (R[\phi_1, \ldots, \phi_N])^l \\ & f & \mapsto & (ASig_1(\phi), \ldots, ASig_l(\phi)). \end{array}$$

Example 5. *Let us consider the exhaustive summary* $\phi(f_1, f_2) = (k(f_1 - 1)^2, -d - f_2)$ *given in Example 3. The algorithm* `Algebraic_signature` *returns the signature* $ASig$ *defined by* $ASig(f_1, f_2) = (\phi_1 - k, \phi_2 + d)$ *whose components vanish for at least one mutiple fault.*

By construction, the signature $ASig(f)$ does not depend explicitly on f. However, the presence of multiple fault(s) is reflected in the numerical values $(\phi_1, \ldots, \phi_N)$ of ϕ and, consequently, of $ASig$. From the comparison between an estimation of $ASig(f)$ and the expected null components of the lists $ASig(f_{\mathcal{N}})$, some possible multiple faults can be discarded. Nevertheless, such a comparison may not be sufficient to discriminate some multiple fault signatures. Indeed, polynomials of $G_{\mathcal{N}}$ appearing in $ASig(f_{\mathcal{N}})$ are insured to vanish when the fault $f_{\mathcal{N}}$ occurs but the other components of the signature $ASig(f_{\mathcal{N}})$ may also vanish for some particular values of the parameters and faults. That is why supplementary criterions are needed to improve the multiple faults discrimination.

3.3. Third step : Criterions to differentiate multiple fault signatures

In order to elaborate additional criterions, the *semialgebraic approach* (See [24]), focusing on real solutions of polynomial equations and inequalities, is adapted. Indeed, this approach permits to take into account the set of constraints on parameters and on faults, $C_{p,f}$, of System (1) which can play an important role for the discrimination of multiple fault signatures (See Example 6).

The three following results lies on the emptyness of semialgebraic sets which can be tested by using computer algebra tools (See [22, 23]).

The first criterion (resp. the second) consists in determining whether the k-th component of $ASig(f_{\mathcal{N}})$ vanishes for at least one real value of a multiple fault $f \in \mathcal{F}_{\mathcal{N}}$ (resp. never vanishes).

For any $\mathcal{N} \subset \{1, \cdots, m\}$, let us consider the set $S_{\mathcal{N}}$ of polynomial equations and inequalities defined by

$$S_{\mathcal{N}} = \{\gamma_1^1(p, f) = \phi_1, \ldots, \gamma_s^{n_s}(p, f) = \phi_N\} \cup C_{p,f} \cup \{v_i f_i = 1 | i \in \mathcal{N}\} \cup \{f_i = 0 | i \notin \mathcal{N}\}$$

where v_i are new indeterminates.

Criterion 1. *If the semialgebraic set defined by* $S_{\mathcal{N}} \cup \{ASig_k(f_{\mathcal{N}}) = 0\}$ *is empty then the kth component of* $ASig(f_{\mathcal{N}})$ *never vanishes.*

Criterion 2. *If the semialgebraic set defined by* $S_{\mathcal{N}} \cup \{v_k ASig_k(f_{\mathcal{N}}) - 1 = 0\}$ *is empty then the kth component of* $ASig(f_{\mathcal{N}})$ *is equal to 0.*

For some particular systems, a vanishing component of the signature charaterizes multiple faults f whose ith component is null.

Criterion 3. *Let* S *be the semialgebraic set defined by* $S = \{\gamma_1^1(p, f) = \phi_1, \ldots, \gamma_s^{n_s}(p, f) = \phi_N\} \cup C_{p,f}$. *If the sets of real solutions* $S \cup \{ASig_j(f) = 0, v_i f_i - 1 = 0\}$ *and* $S \cup \{v_j ASig_j(f) - 1 = 0, f_i = 0\}$ *are empty then* $ASig_j(f) = 0$ *is equivalent to* $f_i = 0$.

In the case where Criterion 3 is satisfied for all the components f_i of $f = (f_1, \ldots, f_m)$, it is clearly useless to apply criterions 1 and 2 on all the $m!$ possible multiple faults since it permits to determine (non) null components of f.

With the help of these three criterions, the expected values of $ASig(f)$ when a multiple fault f occurs can be tabulated. In the next example and in Section 4, the following convention is used in these tables: for any multiple fault f,

- A cell containing ∅ means that Criterion 1 or 3 insures that the component of $ASig(f)$ never vanishes when the multiple fault occurs;
- A 0 in a cell means that Criterion 2 or 3 insures that the $ASig_i(f)$ is necessarily equal to 0 when the multiple fault occurs;
- An empty cell indicates that the component of the signature vanishes for some values of (p, f) and does not vanish for some other values of (p, f).

Example 6. *Let us continue Example 1 and consider the algebraic signature* $ASig(f_1, f_2) = (\phi_1 - k, \phi_2 + d)$ *returned by the second step of our method (See Example 5).*

If the set of constraints $C_{p,f} = \{0 < k < 4, 1 \leq d, 0 \leq f_1 < 2, 0 \leq f_2 < 2\}$ *is taken into account, the two first criterions provide some characteristics of the algebraic signature for the possible mutiple faults. They are summarized in the following table:*

f	$ASig_1(f)$	$ASig_2(f)$
$f_{\{\}}$	0	0
$f_{\{1\}}$	∅	0
$f_{\{2\}}$	0	∅
$f_{\{1,2\}}$	∅	∅

Clearly, the values of $ASig(f)$*, and, more precisely, the values of* $ASig_1(f)$ *and* $ASig_2(f)$ *are sufficient to discriminate all the possible multiple faults. This result can also be obtained by applying Criterion 3 to these two signature components: this criterion permits to show* *the equivalence between $f_1 = 0$ (resp. $f_2 = 0$) and $ASig_1 = 0$ (resp. $ASig_2 = 0$).*

Without considering constraints on parameters and faults, the following table of signatures is obtained.

f	$ASig_1(f)$	$ASig_2(f)$
$f_{\{\}}$	0	0
$f_{\{1\}}$		0
$f_{\{2\}}$	0	$\emptyset$
$f_{\{1,2\}}$		$\emptyset$

This last table indicates that these constraints play an important role for studying, a priori, the values of $ASig(f)$ in function of the multiple faults. More precisely, the semialgebraic set tools insure that, for some particular values of the parameters, the fault $f_{\{1\}}$ can not be detected. The same remark holds for the discrimination of the multiple faults $f_{\{1\}}$ and $f_{\{1,2\}}$.

Remark 2. *Algebraic criterions, using Groebner basis computations, can also be developed to obtain information about the possible values of $ASig(f_{\mathcal{N}})$ when a multiple fault $f_{\mathcal{N}}$ occurs. For example,*

1. *if the Groebner basis of $E_{\mathcal{N}} \cup \{ASig_k(f_{\mathcal{N}})\}$ is equal to $\{1\}$ then the kth component of $ASig(f_{\mathcal{N}})$ never vanishes.*
   Indeed, in this case, polynomials of $E_{\mathcal{N}} \cup \{ASig_k(f_{\mathcal{N}})\}$ has no common complex zeros (See [26]) and, consequently, no real zeros.
2. *by construction of $I_{\mathcal{N}}$, we can state that if $ASig_i(f_{\mathcal{N}})$ belongs to $I_{\mathcal{N}}$ then $ASig_i(f_{\mathcal{N}}) = 0$.*

Even if constraints on parameters involving inequalities can not be taken into account, these criterions can be tested more rapidly in practice than Criterions 1 or 2.

4. Application

The construction of an algebraic signature and the criterions of Section 3.3 had been implemented in the computer algebra system Maple 18. The table giving the expected values of the signature in function of the possible multiple faults constitutes the input of a Scilab program. The latter software is used to estimate numerically the algebraic signature from the simulated noisy output. The comparison between the numerical values and the expected values of the signature permits to discriminate multiple faults. In order to determine thresholds for detecting and isolating multiple faults from an algebraic signature, several methods can be applied. For its simplicity, we have chosen to use constant thresholds. Nevertheless, some other methods can be applied to enhance the robustness of FDI in our case (See [28]).

Our method is applied on a model of two coupled water tanks (See [11, 29]) given by

$$\begin{cases} \dot{x}_1(t,p) = p_1\, u(t) - p_2\,\sqrt{x_1(t,p)},\ x_1(0) = 0.3, \\ \dot{x}_2(t,p) = p_3\,\sqrt{x_1(t,p)} - p_4\,\sqrt{x_2(t,p)},\ x_2(0) = 0.6, \\ y(t,p) = p_5\,\sqrt{x_1(t,p)}, \end{cases} \tag{9}$$

where $p_1, \ldots, p_5$ are nonnegative model parameters, $x = (x_1, x_2)^T$ represents the state vector and corresponds to the level in each tank, and u is the input vector assumed not identically equal to zero. The water level in the tanks can vary between 0 and 10. Contrary to [11, 29], we suppose that there is only one output, y, on the first water-tank.

Let f_1 denote an unknown additive fault on the actuator signal, f_2 an additive fault on the sensor at the output of the first water tank, and $f_3 \in [0;1]$ a clogging fault. The fully clogged pipe situation corresponds to $f_3 = 1$ and $0 < f_3 < 1$ represents a partial clogging. Afterwards, the clogged pipe situation is supposed partial.

In order to use the Rosenfled-Groebner algorithm, a change of variables is necessary. By setting $z_1(t,p) = \sqrt{x_1(t,p)}$ and $z_2(t,p) = \sqrt{x_2(t,p)}$, the model hereafter is obtained:

$$\Gamma_f \begin{cases} \dot{x}_1 = p_1(u + f_1) - p_2(1 - f_3)z_1, \\ \dot{x}_2 = p_3(1 - f_3)z_1 - p_4 z_2, \\ z_1^2 = x_1, z_2^2 = x_2, \\ y = p_5(1 - f_3)z_1 + f_2 \end{cases} \tag{10}$$

The first step of our approach can then be applied to obtained the following ARR:
$2\,y\,\dot{y} - p_5\,(f_3 - 1)^2\,(p_1\,p_5\,f_1 + p_2\,f_2) - p_1\,p_5^2\,(f_3 - 1)^2\,u + p_2\,p_5\,(f_3 - 1)^2\,y - 2\,f_2\,\dot{y} = 0$ and the corresponding exhaustive summary:

$$\begin{aligned} \phi(f_1, f_2, f_3) = (&-p_5\,(f_3 - 1)^2\,(p_1\,p_5\,f_1 + p_2\,f_2), \\ &- p_1\,p_5^2\,(f_3 - 1)^2,\ p_2\,p_5\,(f_3 - 1)^2,\ -2\,f_2)\,. \end{aligned}$$

The second step of our method, which is the application of the Algorithm `Algebraic_signature`, provides the following signature:

$$\begin{aligned} ASig(f) = (&\phi_1,\ \phi_4,\ p_1\,p_5^2 + \phi_2,\ -p_2\,p_5 + \phi_3, \\ &- \phi_3\,\phi_4 + 2\,\phi_1,\ -p_2\,p_5\,\phi_4 + 2\,\phi_1). \end{aligned}$$

The third step consists in computing the expected values of $ASig(f)$. For this application, we start by defining the set of constraints:

$$C_{p,f} = \{0 < p_1, \ldots, 0 < p_5,\ 0 \le f_3 < 1\}\,.$$

which corresponds to the physical signification of the model parameters and to the assumption of a non fully clogging pipe. Next, Criterions 1 and 2 are used to discriminate (multiple) faults. The expected values of $ASig(f)$ are summarized in Table 1.

	$ASig_1(f)$	$ASig_2(f)$	$ASig_3(f)$	$ASig_4(f)$	$ASig_5(f)$	$ASig_6(f)$
$f_{\{\}}$	0	0	0	0	0	0
$f_{\{1\}}$	$\emptyset$	0	0	0	$\emptyset$	$\emptyset$
$f_{\{2\}}$	$\emptyset$	$\emptyset$	0	0	0	0
$f_{\{3\}}$	0	0	$\emptyset$	$\emptyset$	0	0
$f_{\{1,2\}}$		$\emptyset$	0	0	$\emptyset$	$\emptyset$
$f_{\{1,3\}}$	$\emptyset$	0	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$f_{\{2,3\}}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	0	$\emptyset$
$f_{\{1,2,3\}}$		$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	

Table 1: Numerical Expected Values of the Algebraic Signatures

Table 1 shows that the components $ASig_2(f)$, $ASig_4(f)$ and $ASig_5(f)$ permit the discrimination of all the multiple faults for any input u. Indeed, $ASig_2(f)$, $ASig_4(f)$ and $ASig_5(f)$ do not depend on the component ϕ_2 which is the coefficient of the only term depending on u in the ARR. Consequently, the model is input-strongly algebraically diagnosable.

Remark 3. 1. *This result holds even if the values of p_1, p_3 and p_4 are not known. In other terms, the knowledge of the values of all the internal parameters is not needed for detecting and discriminating the possible multiple faults.*

2. *The fact that the model is input-strongly algebraically diagnosable can be obtained by applying Criterion 3 to $ASig_2(f)$, $ASig_4(f)$ and $ASig_5(f)$.*

In the simulations, a simple controller is used to control the water level in the upper tank to follow a square reference signal. The parameters of the model are equal to $p_1 = p_2 = p_3 = p_4 = 0.3$, $p_5 = 1$. The simulated output is disturbed by a truncated Gaussian noise η such that $\eta(t) \in [-0.01; 0.01]$. Thus, $y(t) = \bar{y}(t) + \eta(t)$ where $\bar{y}$ is the exact output corresponding to the exact value of parameters. The observations are supposed to be done at the discrete time $(t_i)_{i=1,\ldots,M}$ on the interval $[0, 50]$ with a sampling period equal to 0.5. In the faulty scenarios, we assume that the faults are introduced at time $t = 22s$. Figures 1 and 2 represent different cases of simple and multiple faults respectively acting on the system.

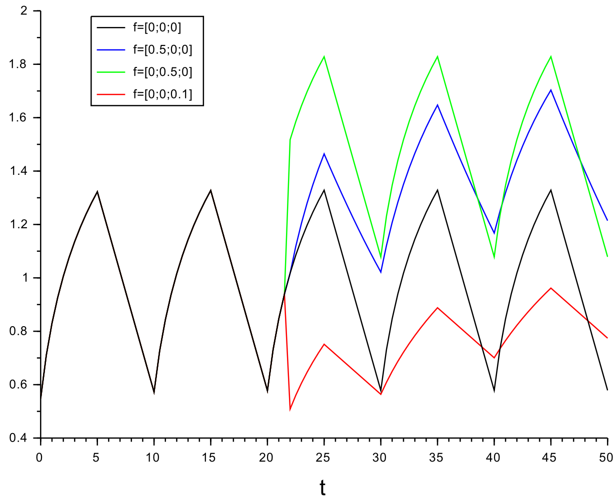

Figure 1: Water level in the upper tank, y, during fault-free simulation and during simple fault simulations introduced at time $t = 22s$.

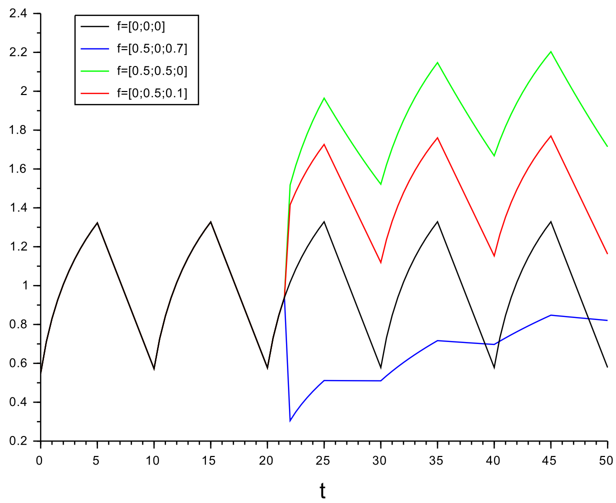

Figure 2: Water level in the upper tank, y, during fault-free simulation and during simple multiple fault simulations introduced at time $t = 22s$.

The derivatives are estimated in using a method based on the B-splines [30]. In order to estimate ϕ, the method developped in [18] is taken again. Rewriting the ARR at each discrete time t_i, M linear relations with respect to the components of ϕ are obtained leading to a linear system. If we denote $y_p(t_i)$ the estimate of $\dot{y}(t_i)$, the system has the following form:

- in the faulty situation,

$$A^f X_f = b \tag{11}$$

  with

$$X_f = (-p_5\,(f_3-1)^2\,(p_1\,p_5\,f_1 + p_2\,f_2),\\ -\,p_1\,p_5^2\,(f_3-1)^2,\ p_2\,p_5\,(f_3-1)^2,\ -2\,f_2),$$
  $A_i^f = (1, u(t_i), y(t_i), y_p(t_i))$ and
  $b_i = -2\,y(t_i)\,y_p(t_i)$,
  Note that $\phi(f) = (X_f(1), X_f(2), X_f(3), X_f(4))$.

- in the fault-free situation,

$$AX_0 = b \tag{12}$$

  with $X_0 = (p_1\,p_5^2\,; p_2\,p_5),\ A_i = (u(t_i), y(t_i))$ and $b_i = -2\,y(t_i)\,y_p(t_i)$.

These systems are solved with the QR factorization which does not require any initial guess.
System (12) is used to detect the time point t_d at which the multiple fault acts. From the 10 first time points, matrix A and vector b are constructed. Then, at each iteration, they are completed in considering one more time and system (12) is solved with this new matrix A and this new vector b. The estimate of X_0 is compared to the nominal value obtained with the real parameter values. If their difference in norm 2 is upper than $\varepsilon = 10^{-3}$, we consider that a multiple fault acts; in other terms, the fault is algebraically detectable.

Once the fault detected, System (11) serves to discriminate the (multiple) fault. At least four time points after the detecting time point t_d are needed since X_f is of dimension 4. Remark that the multiple faults can be detected and discriminated only every 0.5 second corresponding to output measurements. $ASig$ is then estimated and Table 1 is used to discriminate the multiple fault acting. The results are summarized in Table 2.

Even if we have chosen the simplest way to define the threshold ε, that is a constant parameter, all the multiple faults have been detected and discriminated. The results of Table 2 could be improved in using another method for designing thresholds permitting to better consider noise during FDI.

(Multi-)faults f	Detection times (s)	Discrimination times (s)
$f_{\{1\}} = (0.5\,,0\,,0)$	0	3
$f_{\{2\}} = (0\,,0.5\,,0)$	0.5	1.5
$f_{\{3\}} = (0\,,0\,,0.5)$	0.5	2
$f_{\{1,3\}} = (0.5\,,0\,,0.1)$	0	1.5
$f_{\{1,3\}} = (0.5\,,0\,,0.7)$	0.5	11*
$f_{\{1,2\}} = (0.5\,,0.5\,,0)$	0	1.5
$f_{\{2,3\}} = (0\,,0.5\,,0.1)$	0	1.5
$f_{\{2,3\}} = (0\,,0.5\,,0.7)$	0	1.5

Table 2: Detection and discrimination times.

* $f_3 \neq 0$ is first detected at $t = 22.5s$ and the multiple fault $f_{\{1,3\}}$ is discriminated at $t = 33s$.

5. Conclusion

In this paper, an algebraic method based on ARRs for assessing (multiple) faults diagnosability and detectability of non linear parametrized dynamical models is proposed. This method combines computer algebra tools leading to efficient discriminatory relations for all the possible multiple faults. At our knowledge, it is the first time that the ARRs are used in this way and permit, from the processing of the ARRs coefficients through our procedure, to discriminate multiple faults of a system. The application on the coupled water-tanks example highlights the potential of the proposed method even if some limitations need to be considered.
Our approach requires a model which can be written on the form of a rational system composed of polynomials. The precomputed part of our method depends directly on the performance of the needed algebraic tools which are used to compute the algebraic signatures and its expected values. A better consideration of the noise in the numerical part could be the goal of a future work. For the last point, we propose to extend our numerical work to set-membership models permitting to consider faults whose values are unknown but bounded.

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