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Population-adjusted egalitarianism*

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Abstract

Egalitarianism focuses on the well-being of the worst-off person. It has attracted a lot of attention in economic theory, for instance when dealing with the sustainable intertemporal allocation of resources. Economic theory has formalized egalitarianism through the Maximin and Leximin criteria, but it is not clear how they should be applied when population size may vary. In this paper, I present possible justifications of egalitarianism when considering populations with variable sizes. I then propose new versions of egalitarianism that encompass many views on how to trade-off population size and well-being. I discuss some implications of egalitarianism for optimal population size. I first describe how population ethical views affects population growth. In a model with natural resources, I then show that utilitarianism always recommend a larger population for low levels of resources, but that this conclusion may not hold true for larger levels.

Keywords: Egalitarianism, Optimal population, Population ethics, Sustainable development, Renewable resources.

JEL Classification numbers: D63, D64, Q56.

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1 Introduction

Egalitarianism is an important principle of social justice that promotes an equal (and efficient) distribution of resources. It has attracted a lot of attention in contemporary moral philosophy since Rawls (1971), even if there have been discussions about what exactly should be equalized (Sen, 1980). In economic theory, egalitarianism has been modeled either through a Maximin criterion or through a lexicographic version of Maximin named Leximin. Many different axiomatic characterizations of such egalitarian criteria can be found in the literature (see for instance Hammond, 1976, Sen, 1986, Barberá and Jackson, 1988, Lauwers, 1997, D'Aspremont and Gevers, 2002, Fleurbaey and Maniquet or 2011).

Egalitarian criteria have been considered by economic theory to deal with the optimal allocations of resources, in particular in an intergenerational context where sustainability issues may arise. Solow (1975) characterized egalitarian intergenerational distributions in a model with an exhaustible resource and showed that they lead sustainable (actually constant) levels of consumption in contrast to utilitarian solutions. The Maximin path, if egalitarian and efficient, indeed satisfies Hartwick's sustainability rule, which requires investing rents from exhaustible resources in reproducible capital to compensate for the depletion of their stocks (Hartwick, 1977).[1]

A key question for sustainable development and the intertemporal allocation of resources is however population size. In particular, concerns about global climate change have renewed the interest in assessing the impacts of policy on population (see for instance IPCC, 2015a, chap. 11) and in the normative aspects of population size (see for instance IPCC, 2015b, chap. 3) The problem then for the egalitarian perspective is to define how the Maximin or Leximin should be applied when population size may vary. There exist few attempts to define such egalitarian rules in a variable population context (Bossert, 1990; Blackorby, Bossert and Donaldson, 1996). However, existing criteria have serious drawbacks (Blackorby, Bossert and Donaldson, 2005; Arrhenius, forth.). According to Critical-level Leximin, as defined by Blackorby, Bossert and Donaldson (1996), any population with excellent lives is worse than a population with one additional person even when the well-being of all the individuals in the latter population is barely above a critical level. According to the Maximin proposed by Bossert (1990) (and the corresponding Leximin suggested by Arrhenius, forth., Sect. 6.8), any population is worse than a population consisting of one individual, provided that the worst-off individual of the former has lower well-being than the single individual of the latter.

In this paper, I propose new versions of egalitarianism that encompass many

[1]See also, for general proofs, Withagen and Asheim (1998) or Mitra (2002).

views about how to trade-off population size and well-being. First, in Section 2, I present arguments to justify egalitarianism when considering populations with variable sizes. One line of argument is similar to that of Fleurbaey and Tungodden (2010): if we satisfy a minimal non-aggregation property that limits the loss by the worst-off for the sake of all best-offs, we are compelled to egalitarian criteria under a consistency requirement. Another (new) line of argument is that, if we accept that the the best-off should make limited sacrifice for the sake of a sufficiently large number of worst-offs, we are also compelled to egalitarian criteria under the consistency requirement.

In Section 3, I discuss how to compare populations with different sizes, provided we use a Maximin criterion. Blackorby, Bossert and Donaldson (1996) have proposed critical-level properties to compare populations with different sizes. I follow this route, and use a weak critical-level property together with a condition on utility measurement to describe a large new class of Maximin social welfare orderings avoiding the repugnant conclusion described by Parfit (1984). The idea is to multiply individuals' well-being by a weight that depends on population size (provided well-being if non-negative) and then to apply a Maximin like in Bossert (1990). Doing so, I am able to cover a variety of attitudes towards the trade-off between population size and (minimal) well-being, avoiding the problems of previous criteria.

In Section 4, I study some implications of egalitarian social welfare orderings for optimal population size. I first provide a general condition on population ethics views (embodied in a function aggregating population size and an equally-distributed equivalent welfare measure) that guarantees that we can order social welfare functions of the same class in terms of optimal population size, whatever the specific underlying economic model of resource allocation. I then compare egalitarian and utilitarian criteria in a simple model with a renewable resource. I show that utilitarian criteria always recommend a larger population than egalitarian criteria for a specific population ethics view or when the level of resources is low. However, a numerical example shows that this finding is not true in general. It is not possible to say that utilitarianism always entails larger population sizes.

Section 5 concludes. The proofs of the main results are in Appendix A. Supplementary materials contain additional results, in particular a proof of the independence of the axioms in Theorem 1 and an analysis of a Leximin counterpart of the Maximin criteria discussed in Section 3.

2 Justifying egalitarianism when population size may vary

Let $\mathbb{N}$ denote the set of positive integers and $\mathbb{R}$ (resp. $\mathbb{R}_+$, $\mathbb{R}_{++}$, $\mathbb{R}_-$, $\mathbb{R}_{--}$) denote the set of real numbers (resp. non-negative, positive, non-positive, negative real numbers). I also let $I_n = \{1, \cdots, n\}$. Let $X = \cup_{n\in\mathbb{N}}\mathbb{R}^n$ be the set of possible finite *allocations* of lifetime well-being. For every $n \in \mathbb{N}$, each allocation $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$ is associated with a finite population size, $n(x) = n$. Following the usual convention in population ethics, a lifetime well-being level equal to 0 represents *neutrality*. Hence, lifetime well-being is normalized so that above neutrality, a life, as a whole, is worth living; below neutrality, it is not. I also define X_+ as the set of allocations such that all individuals have non-negative well-being levels, i.e. $X_+ = \{x \in X | x_i \geq 0, \forall i \in I_{n(x)}\}$. Similarly, let $X_- = \{x \in X | x_i \leq 0, \forall i \in I_{n(x)}\}$.

A social welfare ordering (henceforth SWO) on the set X is a complete, reflexive and transitive binary relation $\succsim$, where for all x, $y \in X$, $x \succsim y$ means that the allocation x is deemed socially at least as good as y. Let $\sim$ and $\succ$ denote the symmetric and asymmetric parts of $\succsim$.

For each $x \in X$, $x_{[\,]} = (x_{[1]}, \ldots, x_{[r]}, \ldots, x_{[n(x)]})$ denotes the non-decreasing allocation, which reorders the components of x; i.e., for each rank $r \in I_{n(x)-1}$, $x_{[r]} \leq x_{[r+1]}$. For every $n \in \mathbb{N}$ and all x, $y \in \mathbb{R}^n$, we write $x_{[\,]} \geq y_{[\,]}$ whenever $x_{[r]} \geq y_{[r]}$ for all $r \in I_{N(x)}$; we write $x_{[\,]} > y_{[\,]}$ whenever $x_{[\,]} \geq y_{[\,]}$ and $x_{[\,]} \neq y_{[\,]}$; and we write $x_{[\,]} \gg y_{[\,]}$ whenever $x_{[r]} > y_{[r]}$ for all $r \in I_{N(x)}$.

For any $a \in \mathbb{R}$ and $n \in \mathbb{N}$, $(a)_n$ denotes the allocation $(a, \cdots, a) \in \mathbb{R}^n$. For any $\lambda \in \mathbb{R}_{++}$ and $x \in X$, let denote λx the allocation y such that $n(y) = n(x)$ and $y_i = \lambda x_i$ for all $i \in I_{n(x)}$. For any $x, y \in X$, (x, y) denotes an allocation $z \in X$ such that $n(z) = n(x) + n(y)$, $z_i = x_i$ for all $i \in I_n(x)$ and $z_i = y_{i-n(x)}$ for all $i \in I_n(z) \setminus I_n(x)$. Hence (x, y) corresponds to a situation where a population with allocation y is added to an existing population with allocation x. In particular $\big(x, (a)_1\big)$ corresponds to a situation where a single person with well-being $a \in \mathbb{R}$ is added the existing population with allocation x.

Let us now introduce the definitions of the two egalitarian social welfare orderings for fixed populations, namely the Maximin and Leximin social welfare orderings.

Definition 1 *For $n \in \mathbb{N}$, the Maximin SWO on $\mathbb{R}^n$, denoted $\succsim^n_M$, is defined as follows. For all x, $y \in \mathbb{R}^n$, $x \succsim^n_M y$ if and only if $x_{[1]} \geq y_{[1]}$.*

We say more generally that an SWO $\succsim$ on X is a Maximin SWO *if for all $n \in \mathbb{N}$ and for all x, $y \in \mathbb{R}^n$, $x \succsim y$ if and only if $x \succsim^n_M y$.*

Definition 2 *For $n \in \mathbb{N}$, the Leximin SWO on $\mathbb{R}^n$, denoted $\succeq^n_L$, is defined as follows. For all x, $y \in \mathbb{R}^n$,*

(a) *$x \sim_L^n y$ if and only if $\big(x_{[1]}, \dots, x_{[n]}\big) = \big(y_{[1]}, \dots, y_{[n]}\big)$.*

(b) *$x \succ_L^n y$ if and only if there exists $R \in I_n$ such that $x_{[r]} = y_{[r]}$ for all $r \in I_{R-1}$ and $x_{[R]} > y_{[R]}$.*

We say more generally that an SWO $\succsim$ on X is a Leximin SWO *if for all $n \in \mathbb{N}$ and for all x, $y \in \mathbb{R}^n$, $x \succsim y$ if and only if $x \succsim_L^n y$.*

These egalitarian social welfare orderings have been justified by Hammond (1976) with the following principle.

Hammond Equity. For all $n \in \mathbb{N}$, for all $x, y \in \mathbb{R}^n$, if $y_i < x_i < x_j < y_j$ for some $i, j \in I_n$ and $x_k = y_k$ for all $I_n \setminus \{i, j\}$ then $x \succ y$.

Hammond Equity is a strong equity requirement, that allows large losses in total utility for the sake of well-being equalization. It is thus often considered too extreme. To obtain justifications of egalitarianism, I will consider equity requirements, stating that limited sacrifice of the best-off(s) are acceptable provided that the gains by the worst-off(s) are sufficient, either because the single worst-off benefit sufficiently or because there are sufficiently many worst-offs benefiting. The first of my equity principle actually also aims at protecting current generations against unlimited sacrifices for the sake of future generations. It states that, whenever the current generation is worse-off, there is a bound on the loss the society can require for a sufficient gain experienced by all future generations.

Limited sacrifice for the rich future. For all $\alpha \in \mathbb{R}{+}{+}$ there exist $\alpha > \beta > 0$ such that, for all $n \in \mathbb{N}$, if $a, b, c, d \in \mathbb{R}$ are such that $b \leq c$, $b - a \geq \alpha$ and $\beta \geq d - c$, then $\big((b)_1, (c)_n\big) \succsim \big((a)_1, (d)_n\big)$.

Limited sacrifice for the rich future is related to the principle of Mild non-aggregation discussed by Fleurbaey and Tungodden (2010). It simplifies their formulation by considering only two classes of people: a single worst-off and the rest of the population, which is equally well-off. Limited sacrifice for the rich future is also a weakening of Hammond Equity because it imposes bounds on how much the best-offs sacrifice for the sake of the worst-off, and how much the worst-off gains.[2]

[2]Limited sacrifice for the rich future can still be considered strong as any level of sacrifice of the current generation is possible provided the future gain is large enough. However, I show in the Supplementary material (Section S.A) that this principle is implied by a weaker principle related to the Weak non-aggregation principle of Fleurbaey and Tungodden (2010) provided we accept a principle of Ratio-scale invariance. Given that I will use Ratio-scale invariance for variable population comparisons, it seems natural to use the stronger Limited sacrifice for the rich future principle at this stage.

Limited sacrifice for the long future. There exists $\gamma \in \mathbb{R}_{++}$ and $k \in \mathbb{N}$ such that, for all $n \in \mathbb{N}$ and $a, b, c, d \in \mathbb{R}$, if $a < b \leq c$, $n \geq k$ and $d - c \leq \gamma$, then $\big((c)_1, (b)_n\big) \succsim \big((d)_1, (a)_n\big)$.

Limited sacrifice for the long future means that if the cost for the best-off is limited (less than γ) and if the number of poor who gain is sufficiently large, we always want to make the transfer (even though the poor may not gain much). It is comparable to the axiom of Hammond Equity for Future introduced by Asheim, Mitra and Tungodden (2007) in the context of the evaluation of infinite utility streams. Hammond Equity for Future states that, if the present is better-off than the future and a sacrifice now improve the well-being of all future generations while leaving the present generation relatively better-off, then such a transfer is socially desirable. The difference with Limited sacrifice for the rich future is that we now have a finite (but large) number of generations; but we limit the sacrifice made by the best-off generation.

To obtain a justification for egalitarianism, I also assume that we endorse the following two principles.

Suppes-Sen. For all $n \in \mathbb{N}$ and x, $y \in \mathbb{R}^n$, if $x_{[\,]} \geq y_{[\,]}$, then $x \succsim y$; if $x_{[\,]} \gg y_{[\,]}$, then $x \succ y$.

The Suppes-Sen principle represents two ideas. First, that individual and generations should be treated in the same (anonymous) way, so that permuting welfare levels has no impact on the social evaluation. Second, that situations where all individuals have a higher level of welfare can never be worse.

The second principle is a principle of consistency across populations: whenever a population can be split in two subpopulations and that both subpopulations are at least as well-off in one alternative than in another, then the first alternative is weakly better. If both subpopulations are strictly better-off, so is the aggregate population.

Consistency. For all $n, m \in \mathbb{N}$, all x, $y \in \mathbb{R}^n$ and all x', $y' \in \mathbb{R}^n$, if $x \succsim y$ and $x' \succsim y'$ then $(x, x') \succsim (y, y')$. If furthermore $x \succ y$ and $x' \succ y'$ then $(x, x') \succ (y, y')$.

I then obtain a first egalitarian result.

Proposition 1 *Consider an SWO $\succsim$ on X.*

1. *If $\succsim$ satisfies Suppes-Sen, Consistency and Limited sacrifice for the rich future, then, for all $n \in \mathbb{N}$ and x, $y \in \mathbb{R}^n$, $x \succ y$ whenever $x_{[1]} > y_{[1]}$.*

2. *If $\succsim$ satisfies Suppes-Sen, Consistency and Limited sacrifice for the long future, then, for all $n \in \mathbb{N}$ and x, $y \in \mathbb{R}^n$, $x \succ y$ whenever $x_{[1]} > y_{[1]}$.*

To obtain a complete characterization of the maximin social welfare ordering, one only needs to add a continuity requirement.

Continuity. For any $n \in \mathbb{N}$ and any $x \in \mathbb{R}^n$, the sets $\{y \in \mathbb{R}^n | x \succsim y\}$ and $\{y \in \mathbb{R}^n | y \succsim x\}$ are closed.

Theorem 1 *Consider an SWO $\succsim$ on X.*

1. *$\succsim$ satisfies Suppes-Sen, Consistency, Continuity and Limited sacrifice for the rich future, if and only if it is a Maximin SWO.*

2. *$\succsim$ satisfies Suppes-Sen, Consistency, Continuity and Limited sacrifice for the long future, if and only if it is a Maximin SWO.*

Appendix B shows that the two characterizations above are tight in the sense that all principles involved in the characterization are independent from one another.

3 Egalitarianism for variable populations

For the moment, we have only compared populations with the same size. In this section, I introduce and characterize Population-adjusted maximin SWOs that can be used to assess allocations when population size varies. First, let us define a function that transforms well-being numbers according to population size:

> Let $\kappa = (\kappa_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. Π_κ denotes the function $\Pi_\kappa : \mathbb{N} \times \mathbb{R}$ such that, for all $(n, e) \in X \times \mathbb{R}$:
>
> $$\Pi_\kappa(n, e) = \begin{cases} e & \text{if } e \leq 0; \\ \kappa_n \cdot e & \text{if } e > 0. \end{cases}$$

Definition 3 *An SWO $\succsim$ on X is a Population-adjusted maximin SWO if and only if there exists a non-decreasing sequence $(\kappa_k)_{k\in\mathbb{N}}$ such that $\kappa_1 = 1$ and for all $x, y \in X$,*

$$x \succsim y \Longleftrightarrow \Pi_\kappa\big(n(x), x_{[1]}\big) \geq \Pi_\kappa\big(n(y), y_{[1]}\big).$$

Population-adjusted maximin SWOs use the usual maximin procedure by comparing the minimal level well-being in allocations that may have different sizes. However, the contributive value of a life is adjusted by population size before comparing these minimal well-being levels. The adjustment procedure is very simple. If well-being is negative, no adjustment is made. If well-being is positive, it is multiplied by a factor that depends on population size (with larger populations having a larger weight).

To characterize these Population-adjusted maximin SWOs, I introduce two additional principles. A first natural principle is based on the notion of a critical level, that is a level such that adding an individual at that level is a matter of social indifference. The following principle states that such a critical level always exists (but may vary depending on population size and the existing allocation).

Critical level. For any $x \in X$ there exists $a \in \mathbb{R}_+$ such that $\big(x, (a)_1\big) \sim x$.

Another principle is that changes in the scale of the measurement of individual well-being do not affect the social ranking.

Ratio-scale invariance. For any $\lambda \in \mathbb{R}_{++}$ and any x, $y \in X$, $x \succsim y$ if and only if $\lambda x \succsim \lambda y$.

Ratio-scale invariance is a property about the measurement of individual well-being. It implies that only the ratios of well-being levels are directly fully comparable. It also involves picking an interpersonally significant norm, which is the 0. This makes sense in the context of population ethics, where 0 level is a well-being level at which life is no more worthwhile than death, and is supposed to be normatively comparable across people. Note that a similar property has often been consider in the literature with a fixed population (see, e.g. Roberts , 1980; Blackorby and Donaldson, 1982)

Theorem 2 *A Maximin SWO $\succsim$ on X satisfies Critical level and Ratio-scale invariance if and only if it is a Population-adjusted maximin SWO.*

Again, Appendix B shows that the characterization is tight (and more precisely that characterizations using principles in Th. 1 and Th. 2 are tight).

It is interesting to discuss the population ethics properties of the Population-adjusted maximin SWOs characterized in Th. 2. The literature on population ethics has indeed highlighted that variable population social welfare criteria may face several ethical issues. The most discussed issue is the *repugnant conclusion.* According to Parfit (1984), a social welfare ordering leads to the repugnant conclusion if:

> "For any possible population of at least ten billion people, all with a very high quality of life, there must be some much larger imaginable population whose existence, if other things are equal, would be better even though its members have lives that are barely worth living."

The repugnant conclusion has caught much attention in the literature on population ethics as most of the literature have discussed ways to avoid such a conclusion. Formally, one can formulate avoidance of the repugnant conclusion as follows:[3]

Avoidance of the repugnant conclusion. An SWO $\succsim$ avoids the repugnant conclusion if there exist $k \in \mathbb{N}$, $a \in \mathbb{R}_{++}$ and $b \in [0, a]$ such that, for all $m \geq k$, $(a)_k \succsim (b)_m$.

Another problem that an SWO may face is that it may imply the *sadistic conclusion*. According to Arrhenius (2000), an SWO leads to the sadistic conclusion in the following case:

> "When adding people without affecting the original people's welfare, it can be better to add people with negative welfare rather than positive welfare."

Formally, one can formulate avoidance of the sadistic conclusion as follows:[4]

Avoidance of the sadistic conclusion. An SWO $\succsim$ avoids the sadistic conclusion if for all $x \in X$, all $k, m \in \mathbb{N}$, all $a \in \mathbb{R}_{++}$ and $b \in \mathbb{R}_{--}$, $\big(x, (a)_k\big) \succsim \big(x, (b)_m\big)$.

Lastly, we may want to have principles about how the addition of a person changes social welfare depending on whether her well-being level if positive or negative. First, it may seem natural that adding someone with positive well-being to a population without affecting existing people well-being is always acceptable: no one else is affected and the additional person has a good enough life. This is known as the *Mere addition principle*. Second, it seems natural on the contrary that we do not want to add someone with a negative level of well-being to a population: this person has a life not worth living. Arrhenius (forth.) named this principle the *Negative mere addition principle* (it is called the *Negative expansion principle* by Blackorby, Bossert and Donaldson, 2005) The formal statements of these principles are as follows:

Mere addition principle. An SWO $\succsim$ satisfies the Mere addition principle if for all $x \in X$ and all $a \in \mathbb{R}_{++}$, $\big(x, (a)_1\big) \succsim x$.

[3]Other formalizations have been proposed, for instance by Blackorby, Bossert and Donaldson (2005). The formulation is slightly stronger than the one they have, but I think it is in the spirit of Parfit's initial formulation.

[4]Note that we interpret "better" in the statement of the sadistic conclusion as "strictly better".

Negative mere addition principle. An SWO $\succsim$ satisfies the Negative mere addition principle if for all $x \in X$ and all $a \in \mathbb{R}_{--}$, $x \succ \big(x, (a)_1\big)$.

It turns out that Population-adjusted maximin SWOs can avoid both the sadistic conclusion and the repugnant conclusion. On the other hand, they necessarily violate the Mere addition principle and the Negative mere addition principle.

Proposition 2 *Assume that the SWO $\succsim$ is a Population-adjusted maximin SWO. Then it always avoids the sadistic conclusion and it also avoids the repugnant conclusion if and only if the sequence $(\kappa_k)_{k\in\mathbb{N}}$ in Def. 3 is bounded. However, $\succsim$ satisfies neither the Mere addition principle nor the Negative mere addition principle.*

Violating the Negative mere addition principle is probably not a good feature of Population-adjusted maximin SWOs. In the supplementary material (Section S.B), I introduce and characterize Population-adjusted versions of Leximin that satisfy the Negative mere addition principle.

The appeal of the Mere addition principle has been discussed in the literature. In particular, Carlson (1998) proved that the Mere addition principle and a Non-anti egalitarianism principle imply a conclusion akin to the Repugnant conclusion. Indeed adding a person with a low level of well-being to an otherwise well-off population may increase inequality, which may not be an improvement from the social point of view. Note that Population-adjusted maximin SWOs satisfy the following weak version of the Mere addition principle that holds when people in the existing population only have negative levels of well-being.

Weak mere addition principle. An SWO $\succsim$ satisfies the Weak mere addition principle if for all $x \in X_-$ and all $a \in \mathbb{R}_{++}$, $\big(x, (a)_1\big) \succsim x$.

4 Egalitarianism, optimal population size and natural resources

4.1 Optimal population size: a simple model

The question of optimal population has attracted a lot of attention in the economic literature. Already Malthus hypothesized that natural population growth (which is supposedly geometric) is constrained by economic growth (which is only arithmetic) yielding recurrent episodes of extended poverty and migration (Malthus, 1798). Wicksell shared this view and advocated to limit natality through the systematic use of contraceptives within the marriage. The objective was to reach an optimum population size that Wicksell (like many economists) conceived as the one maximizing average well-being (Wicksell, 1893).[5] Later on, Meade (1955) discussed another criteria of optimum population size, namely the maximization of total well-being, and derived what is known as the Sidgwick-Meade rule according to which, at the optimum, the marginal utility of consumption is equal to average well-being level per unit of consumption.

The Sidgwick-Meade rule has been obtained in what Dasgupta (2005) named the *genesis problem*, where there is a total amount of a consumption good available. The problem is to fix the optimal number of individuals such that, sharing the consumption good equally among them, brings the largest social welfare. Asheim and Zuber (2014) studied this similar problem using a more general rank-dependent generalized utilitarian criterion and provided a more general condition than the Sidgwick-Meade rule for optimal population.

Of course, as argued by Dasgupta (2005), the genesis problem is not be the most interesting problem for optimal population theory as it neglects that the resource, for instance a natural resource, has to be used by several successive generations. It may thus regenerate but only if something is left to the next generation. Nerlov, Razin and Sadka (1985) have studied a simple two-periods extension of the genesis model, with a non-renewable resource, assuming that parents have altruistic sentiments towards their children. Spiegel (1993) extended their analysis to the standard maximin case (without the adjustment for population size that we proposed in Section 3).[6] In this paper I study extensions of their framework that allow for more general production processes using a natural resource that may be renewable or non-renewable.

The framework hence involves a current generation, whose size N is exogenously given. This generation has a consumption c and derives utility $u(c)$. There is also a future generation, whose size is rN. The number r is the reproduction rate to be chosen. This future generation's consumption level is d (and utility level $u(d)$). The problem thus consists in choosing the values of c, d and r. To simplify, r is treated as a continuous variable.

An allocation can therefore be seen as a triplet $x = (r, c, d)$. Using the notation of Section 2, let denote $n(x) = (1 + r)N$. Let denote $F \subset \mathbb{R}^3_+$ the set of feasible

[5] A recent questionnaire-experimental study shows that many people actually do not share the view that only average well-being matters for the optimum population, but that the size of the population itself is important (Spears, 2017).

[6] There is also a very large literature focusing on infinite-horizon models of economic growth, for instance Dasgupta (1969), Palivos and Yip (1993), Razin and Yuen (1995), Boucekkine, Fabbri and Gozzi (2014) and Lawson and Spears (2018). This literature studies how per capita well-being and population size is optimized, focusing on total and average utilitarianism. However, they focus on within-generation optimal population, while I focus on the optimal total population size across generations. The results below cannot easily be translated in their context.

allocation. At this stage, I do not impose any restriction of F that may result from the use of natural and non-natural resources, and may involve costs from child rearing.

To determine the optimal allocation, assume that there exists an SWO $\succsim$ on $\mathbb{R}^3_+$. I follow Blackorby, Bossert and Donaldson (2001) and assume that the SWO has the following form:

$$x \succsim y \Longleftrightarrow V\Big(n(x), \Xi(x)\Big) \geq V\Big(n(y), \Xi(y)\Big),$$

with function Ξ a within-generation equally-distributed equivalent function that represent the ordering of allocations for a given total population size, and function V an aggregator function that combines the equally-distributed equivalent welfare (henceforth EDEW) Ξ and population size. Function V describes how population size and equivalent per capita welfare are traded-off and thus embodies the population ethics views of the decision maker.

For instance, the utilitarian EDEW (expressed in terms of utility) is

$$\Xi^U(r, c, d) = \tfrac{1}{1+r}u(c) + \tfrac{r}{1+r}u(d) \tag{1}$$

while the maximin EDEW is

$$\Xi^E(r, c, d) = \min\big\{u(c), u(d)\big\}. \tag{2}$$

In the utilitarian case, two prominent aggregator functions have been considered. Function $V^T(n, e) = n \times e$ delivers the total utilitarian (or Benthamite) social welfare function:

$$W^{TU}(x) = V^T\Big(n(x), \Xi^U(x)\Big) = Nu(c) + rNu(d).$$

Function $V^A(n, e) = e$ delivers the average utilitarian (or Millian) social welfare function:

$$W^{AU}(x) = V^A\Big(n(x), \Xi^U(x)\Big) = \tfrac{1}{1+r}u(c) + \tfrac{r}{1+r}u(d).$$

Given the form of the SWOs, there are two questions that may be asked. First, given a specific view about how resources should be allocated in a population of a given size (as embodied in function Ξ), how does the population ethics view, that is the shape of function V, influence the optimal population growth rate? This question was raised in the specific case of the total utilitarian and average utilitarian views described above by Edgeworth (1925). Edgeworth conjectured that the socially optimal rate of population growth must be larger for a total than for an average utilitarian social welfare function, and Nerlov, Razin and Sadka (1985) actually proved it in a specific model. In Section 4.2, I show that this is a special case of a more general result, which is not restricted to utilitarianism nor to a specific economic model of resource allocation, and that we can compare more aggregator functions.

A different question, that has attracted less attention, is to understand whether (and how) the ethical view about the allocation for a given population size influences optimal population growth. Does utilitarianism require a larger population than egalitarianism, for a given population ethical view as represented by the aggregator function V? It turns out that this so for the average view and for low levels of resources (Section 4.3), but not in general.

4.2 Population ethical views and optimal population size: a general result

In this section, I study the implications of population ethics views as represented by the aggregator function V on optimal population growth r. I thus take the ethical view about redistribution within a population (embodied in the EDEW Ξ) as given (it may a utilitarian or egalitarian or any other kind of view).

To study the question, let us consider a family $(V_\theta)_{\theta\in\Theta}$ of aggregator functions, with Θ a set of parameters, which is a subset of $\mathbb{R}$ (for instance $\Theta = [0, 1]$). Taking Ξ as given, this delivers a family of SWOs $\succsim_\theta^\Xi$:

$$x \succsim_\theta^\Xi y \Longleftrightarrow V_\theta\Big(n(x), \Xi(x)\Big) \geq V_\theta\Big(n(y), \Xi(y)\Big).$$

with $\theta \in \Theta$ a specific member of this family.

To compare the implications of members of family $(V_\theta)_{\theta\in\Theta}$ for optimal population size, I make the following assumption.

Assumption 1 *There exists a threshold $\gamma \in \mathbb{R}_+$ such that family $(V_\theta)_{\theta\in\Theta}$ satisfies the following conditions for any $\theta \in \Theta$:*

Regularity:

- *for any $n > n'$ and $e > \gamma$, $V_\theta(n, e) > V_\theta(n', e)$;*
- *for any n, n' and $e \leq \gamma < e'$, $V_\theta(n', e') > V_\theta(n, e)$;*
- *for any n and $e > e'$, $V_\theta(n, e) > V_\theta(n, e')$.*

Sorting: *for any $n > n'$ and $\gamma < e < e'$, if $V_\theta(n, e) \geq V_\theta(n', e')$ then $V_{\theta'}(n, e) > V_{\theta'}(n', e')$ for all $\theta' > \theta$.*

The regularity condition means that the social welfare function is increasing in both EDEW and population size (provided welfare is high enough). It also implies that population cannot compensate for welfare if welfare is below a certain threshold: if $\gamma = 0$, this corresponds to what Blackorby, Bossert and Donaldson (2005) name the 'Priority to lives worth living' (we prefer populations with equal positive well-being to populations with equal negative well-being).

The sorting condition means that the family in well-ordered in terms of how the SWOs trade-off population and EDEW (at least for large enough well-being levels): a higher θ means that we want to give up more on well-being to increase population.

Here are two examples of families using the population-weighted approach and satisfying Assumption 1:

- For all $\theta \in [0,1]$, all $n \in \mathbb{N}$ and all $e \in \mathbb{R}$,

$$V^p_\theta(n,e) = \begin{cases} e & \text{if } e \leq 0; \\ n^\theta \cdot e & \text{if } e > 0. \end{cases} \tag{3}$$

- For all $\theta \in [0,1)$, all $n \in \mathbb{N}$ and all $e \in \mathbb{R}$, $V^g_\theta(n,e) = \begin{cases} e & \text{if } e \leq 0; \\ \frac{1-\theta^n}{1-\theta} \cdot e & \text{if } e > 0. \end{cases}$

The following general result is true for all EDEW Ξ.

Proposition 3 *Consider a family $(V_\theta)_{\theta\in\Theta}$ that satisfies Assumption 1 for some threshold $\gamma \in \mathbb{R}_{++}$ and a feasible set F such that, for some $\bar{x} \in F$, $\Xi(\bar{x}) > \gamma$. For each $\theta \in \Theta$ assume that there exists an allocation $x^\star_\theta \in F$ such that $V_\theta\Big(n(x^\star_\theta), \Xi(x^\star_\theta)\Big) \geq V_\theta\Big(n(y), \Xi(y)\Big)$ for all $y \in F$.*
Then, for any $\theta' > \theta$, $n(x^\star_{\theta'}) \geq n(x^\star_\theta)$.

Proof. By definition of $x^\star_\theta$ and $x^\star_{\theta'}$, we have $V_\theta\Big(n(x^\star_\theta), \Xi(x^\star_\theta)\Big) \geq V_\theta\Big(n(\bar{x}), \Xi(\bar{x})\Big)$ and $V_{\theta'}\Big(n(x^\star_{\theta'}), \Xi(x^\star_{\theta'})\Big) \geq V_{\theta'}\Big(n(\bar{x}), \Xi(\bar{x})\Big)$. Hence, by the regularity condition in Assumption 1, $\Xi(x^\star_\theta) > \gamma$ and $\Xi(x^\star_{\theta'}) > \gamma$.

Assume that $n(x^\star_{\theta'}) < n(x^\star_\theta)$.

If $\Xi(x^\star_{\theta'}) \leq \Xi(x^\star_\theta)$, given that $\Xi(x^\star_\theta) > \gamma$, by the the regularity condition in Assumption 1, we would have $V_{\theta'}\Big(n(x^\star_\theta), \Xi(x^\star_\theta)\Big) > V_{\theta'}\Big(n(x^\star_{\theta'}), \Xi(x^\star_{\theta'})\Big)$. This is a contradiction of the definition of $x^\star_{\theta'}$.

If $\Xi(x^\star_{\theta'}) > \Xi(x^\star_\theta)$, we have $n' = n(x^\star_{\theta'}) < n(x^\star_\theta) = n$, $e' = \Xi(x^\star_{\theta'}) > \Xi(x^\star_\theta) = e > \gamma$ and $V_\theta(n,e) \geq V_\theta(n',e')$ by definition of $x^\star_\theta$. By the sorting condition in Assumption 1, this implies $V_{\theta'}\Big(n(x^\star_\theta), \Xi(x^\star_\theta)\Big) > V_{\theta'}\Big(n(x^\star_{\theta'}), \Xi(x^\star_{\theta'})\Big)$. This is again a contradiction of the definition of $x^\star_{\theta'}$. ■

Proposition 3 confirms Edgeworth's conjecture. Indeed, if we take $\Xi = \Xi^U$ and consider the V^p_θ (Eq. 3), with $\theta \in [0,1]$ and $\gamma = 0$, we clearly see that the total utilitarian view corresponds to $\theta = 1$ while the average utilitarian view corresponds to $\theta = 0$. Hence the former induces a larger population than the latter, whatever the economic model inducing the feasible set F.

But Proposition 3 also generalizes Edgeworth's conjecture in several ways. First, and as mentioned just above, the result is true for any feasible set F. Second, the result does not only make it possible to compare a "total" and an "average" view, but many intermediary views corresponding to $0 < \theta < 1$. Last, the result is not restricted to utilitarianism and may apply to other ethical views, including egalitarianism. In Theorem 2, we have shown that population-adjusted egalitarian criteria have appealing properties. In the framework of this section, they are represented by

$$W_\theta^E(x) = \Pi_{\kappa^\theta}\Big((1+r)N, \min\{u(c), u(d)\}\Big),$$

with κ^θ a non-decreasing sequence with $\kappa_1^\theta = 1$. If $u(z) = 0$ for some $z \in \mathbb{R}_{++}$ and the sorting condition is satisfied, we obtain families of social welfare functions that satisfy Assumption 1. We can then apply Proposition 3.

4.3 Optimal population size: egalitarianism vs. utilitarianism

A key question has not been much addressed in the literature: does the egalitarian view imply larger or smaller populations than the utilitarian view? In the genesis problem that has often been studied (Dasgupta, 2005), there is no distinction between utilitarianism and egalitarianism because both of them recommend that consumption should be equalized among individuals, so that they yield the same equally-distributed equivalent for a given population size. To be able to distinguish the two views, we should consider cases where an unequal distribution of consumption (and hence utility) between generations may be optimal. This can be obtained in a simple model involving a renewable natural resource and/or technological progress.

This very simple model specifies the form of the feasible set F and of the utility function u. I assume that there is an initial stock of natural resource Ω. We can use an amount ω_1 of resource in the first period to produce the consumption good, so that $Nc \leq A\,\omega_1$, where $A \in \mathbb{R}_{++}$ is the total factor productivity. The remaining stock of the natural resource will reproduce so that there is an amount $(1+\phi)(\Omega-\omega_1)$ available to the next generation, where $\phi \in \mathbb{R}_+$ is the regeneration rate ($\phi = 0$ corresponds to non-renewable resource). The second generation (of size rN) can use $\omega_2 \leq (1+\phi)(\Omega-\omega_1)$ to produce the consumption good, so that $rNd \leq (1+g)\,A\,\omega_2$, where $g \in \mathbb{R}_+$ is the rate of technological progress (assumed exogenous).

Rewriting the constraints, we have $\omega_1 + \frac{\omega_2}{1+\phi} \leq \Omega$, $c \leq A\,\frac{\omega_1}{N}$ and $r(1+g)^{-1}\,d \leq$

$A\,\frac{\omega_2}{N}$. This gives the following aggregate constraint:

$$c + \tfrac{r}{1+\tilde{g}} d \leq \tilde{\omega},$$

with $\tilde{g} = (1+\phi)(1+g) - 1$ a 'total' productivity growth rate (that combines resource growth and pure technology growth) and $\tilde{\omega} = A\,\frac{\Omega}{N}$ an index of per capita initial resource (that corresponds to the maximum per capita consumption in the first generation if the natural resource is completely exhausted in the first period). Hence the feasible set is $F = \left\{(r,c,d) | c + \frac{r}{1+\tilde{g}} d \leq \tilde{\omega}\right\}$.

The aim of this section is to compare the egalitarian and utilitarian approaches for a given aggregator function. To fit with the population-adjusted egalitarian approach introduced in this paper, let us consider the aggregator function V_θ^g that has already been mentioned above (Eq. 3).

I assume that individual well-being is measured by an iso-elastic function of consumption and I normalize consumption so that a consumption level of 1 corresponds to the neutral level of well-being, i.e. $u(1) = 0$:[7]

$$u(c) = \tfrac{c^{1-\varepsilon}}{1-\varepsilon} - \tfrac{1}{1-\varepsilon}, \qquad \varepsilon > 0.$$

Thus, the utilitarian EDEW is:

$$\Xi^U(r,c,d) = \tfrac{1}{1+r}\tfrac{c^{1-\varepsilon}}{1-\varepsilon} + \tfrac{r}{1+r}\tfrac{d^{1-\varepsilon}}{1-\varepsilon} - \tfrac{1}{1-\varepsilon}.$$

The egalitarian EDEW is $\Xi^E(r,c,d) = \frac{(\min\{c,d\})^{1-\varepsilon}-1}{1-\varepsilon}$.

We look for the optimal allocation according to the utilitarian and egalitarian SWOs, that is a solution to the utilitarian and egalitarian problems. The utilitarian problem is:

$$\max_{(r,c,d)\in F} V_\theta^g \left((1+r)N, \tfrac{1}{1+r}\tfrac{c^{1-\varepsilon}}{1-\varepsilon} + \tfrac{r}{1+r}\tfrac{d^{1-\varepsilon}}{1-\varepsilon} - \tfrac{1}{1-\varepsilon} \right). \tag{4}$$

The egalitarian problem is:

$$\max_{(r,c,d)\in F} V_\theta^g \left((1+r)N, \tfrac{(\min\{c,d\})^{1-\varepsilon}}{1-\varepsilon} - \tfrac{1}{1-\varepsilon} \right). \tag{5}$$

Denote $(r_\theta^{E\star}, c_\theta^{E\star}, d_\theta^{E\star}) \in F$ (resp. $(r_\theta^{U\star}, c_\theta^{U\star}, d_\theta^{U\star}) \in F$) a solution to the egal-

[7]The normalization of the neutral level of consumption to 1 is without loss of generality, given that the general definition with a utility of 0 at consumption level γ is $u(c) = \frac{c^{1-\varepsilon}}{1-\varepsilon} - \frac{\gamma^{1-\varepsilon}}{1-\varepsilon} = \gamma^{1-\varepsilon}\left(\frac{(c/\gamma)^{1-\varepsilon}}{1-\varepsilon} - \frac{1}{1-\varepsilon}\right)$. The multiplication by the positive constant $\gamma^{1-\varepsilon}$ does not change the social ranking.

itarian problem (5) (resp. a solution to the utilitarian problem (4)). To solve these problems, we can proceed in two steps. First, for each possible population growth level r, we find the optimal consumption level for the first and second generations and compute the equally-distributed equivalent. Then we optimize with respect to r.

4.3.1 Optimum EDEW and the average view

Routine reasoning implies that, for a given r, the optimal egalitarian allocation requires $c = d$. The resource constraint $c + \frac{r}{1+\tilde{g}}d \leq \tilde{\omega}$ (given that utility is increasing in consumption) implies that

$$\tfrac{1+\tilde{g}+r}{1+\tilde{g}}c = c + \tfrac{r}{1+\tilde{g}}d = \tilde{\omega},$$

so that $c = \frac{1+\tilde{g}}{1+\tilde{g}+r}\tilde{\omega}$. The optimum egalitarian EDEW for a population growth r is therefore:

$$EDEW^E(r) = \left(\tfrac{1+\tilde{g}}{1+\tilde{g}+r}\right)^{1-\varepsilon} \tfrac{\tilde{\omega}^{1-\varepsilon}}{1-\varepsilon} - \tfrac{1}{1-\varepsilon}. \tag{6}$$

This is clearly decreasing in r: increasing population size necessarily comes to a cost, which is a decrease in the egalitarian EDEW.

Similarly, is is easy to show that, for a given r, the optimal utilitarian allocation is such that $d = (1+\tilde{g})^{\frac{1}{\varepsilon}}c$. The resource constraint implies that

$$\left(1 + r(1+\tilde{g})^{\frac{1}{\varepsilon}-1}\right) c = c + \tfrac{r}{1+\tilde{g}}d = \tilde{\omega},$$

so that $c = \left(1 + r(1+\tilde{g})^{\frac{1}{\varepsilon}-1}\right)^{-1} \tilde{\omega}$ and $d = (1+\tilde{g})^{\frac{1}{\varepsilon}} \left(1 + r(1+\tilde{g})^{\frac{1}{\varepsilon}-1}\right)^{-1} \tilde{\omega}$. The optimum utilitarian EDEW for a population growth r is therefore:

$$\begin{aligned} EDEW^U(r) &= \left[\tfrac{1+r(1+\tilde{g})^{\frac{1}{\varepsilon}-1}}{1+r}\right] \left(1 + r(1+\tilde{g})^{\frac{1}{\varepsilon}-1}\right)^{-(1-\varepsilon)} \tfrac{\tilde{\omega}^{1-\varepsilon}}{1-\varepsilon} - \tfrac{1}{1-\varepsilon} \\ &= \left(1 + r(1+\tilde{g})^{\frac{1}{\varepsilon}-1}\right)^{\varepsilon} (1+r)^{-1} \tfrac{\tilde{\omega}^{1-\varepsilon}}{1-\varepsilon} - \tfrac{1}{1-\varepsilon} \end{aligned} \tag{7}$$

Contrary to the egalitarian case, this utilitarian EDEW may not be always decreasing. Indeed, if the total productivity growth rate is large enough, the utilitarian EDEW may be first increasing and then decreasing so that there exists a unique non zero population growth rate r that maximizes $EDEW^U$.[8]

The behavior of the EDEW at the optimal allocation for a given population growth gives a first result, in the case where $\theta = 0$ in Eq.(3), that is when no additional weight is given to larger population. We call this case the average view, given that it corresponds to average utilitarianism when a utilitarian criterion is used. Recall that the optimal levels of population growth when $\theta = 0$ are denoted $r_0^{E\star} = 0$ and $r_0^{U\star}$.

Proposition 4 *For the average view, $r_0^{E\star} = 0$ while $r_0^{U\star} > 0$ if $\ln(1+\tilde{g}) > \frac{\varepsilon}{\varepsilon-1}\ln(\varepsilon)$ (and $r_0^{U\star} = 0$ otherwise).*

Prop. 4 states that egalitarianism always recommends no population growth in the average view, in order to have the largest possible level of minimal welfare. This is not true of utilitarianism that may recommend population growth if it contributes to increasing average welfare. When the total productivity growth rate is high, it is possible to have a rich enough second generation without hurting the first one too much, so as to increase average welfare.

4.3.2 The general case

Let us now consider the case where $\theta > 0$ in Eq. (3): we multiply EDEW by an increasing function of population size (provided that EDEW is positive). Using the results from Section 4.3.1, we can rewrite the egalitarian objective function in problem (5) as follows:

$$U_\theta^E(r) = V_\theta^p\left((1+r)N, \left(\frac{1+\tilde{g}}{1+\tilde{g}+r}\right)^{1-\varepsilon}\frac{\tilde{\omega}^{1-\varepsilon}}{1-\varepsilon} - \frac{1}{1-\varepsilon}\right).$$

If $\tilde{\omega} \leq 1$, then $\left(\frac{1+\tilde{g}}{1+\tilde{g}+r}\right)^{1-\varepsilon}\frac{\tilde{\omega}^{1-\varepsilon}}{1-\varepsilon} < \frac{1}{1-\varepsilon}$ (EDEW is always negative) and $U_\theta^E(r) = EDEW^E(r)$ for all $r > 0$. We know from Section Section 4.3.1 that maximizing

[8] The derivative of $EDEW^U$ with respect to r is indeed:

$$(1+\tilde{g})^{\frac{1}{\varepsilon}-1}\varepsilon\left(1+r(1+\tilde{g})^{\frac{1}{\varepsilon}-1}\right)^{\varepsilon-1}(1+r)^{-1}\frac{\tilde{\omega}^{1-\varepsilon}}{1-\varepsilon} - \left(1+r(1+\tilde{g})^{\frac{1}{\varepsilon}-1}\right)^{\varepsilon}(1+r)^{-2}\frac{\tilde{\omega}^{1-\varepsilon}}{1-\varepsilon}$$
$$= \frac{(1+\tilde{g})^{\frac{1}{\varepsilon}-1}\varepsilon(1+r)-1+r(1+\tilde{g})^{\frac{1}{\varepsilon}-1}}{1-\varepsilon}\left(1+r(1+\tilde{g})^{\frac{1}{\varepsilon}-1}\right)^{\varepsilon-1}(1+r)^{-2}\,\tilde{\omega}^{1-\varepsilon},$$

which depends on the sign of $\frac{(1+\tilde{g})^{\frac{1}{\varepsilon}-1}\varepsilon-1}{1-\varepsilon} - (1+\tilde{g})^{\frac{1}{\varepsilon}-1}r$. This sign is eventually negative but may first be positive whenever $\frac{(1+\tilde{g})^{\frac{1}{\varepsilon}-1}\varepsilon-1}{1-\varepsilon} > 0$, or equivalently $\ln(1+\tilde{g}) > \frac{\varepsilon}{\varepsilon-1}\ln(\varepsilon)$.

$EDEW^E$ always yield $r_\theta^{E\star} = 0$. In that case, the utilitarian approach will always induces an (at least weakly) larger population. I will thus focus on cases where $\tilde{\omega} > 1$.

Proposition 5 *If $\varepsilon > 1$, $\tilde{\omega} > 1$ and $\ln(1+\tilde{g}) \leq 1$ then there exists unique solutions $r_\theta^{E\star}$ and $r_\theta^{U\star}$.*

Furthermore, there exist $1 < \underline{\omega} < \bar{\omega}$ such that:

1. *If $\tilde{\omega} \leq \underline{\omega}$, then $r_\theta^{U\star} = r_\theta^{E\star} = 0$,*
2. *if $\underline{\omega} < \tilde{\omega} \leq \bar{\omega}$, then $r_\theta^{U\star} > r_\theta^{E\star} = 0$.*

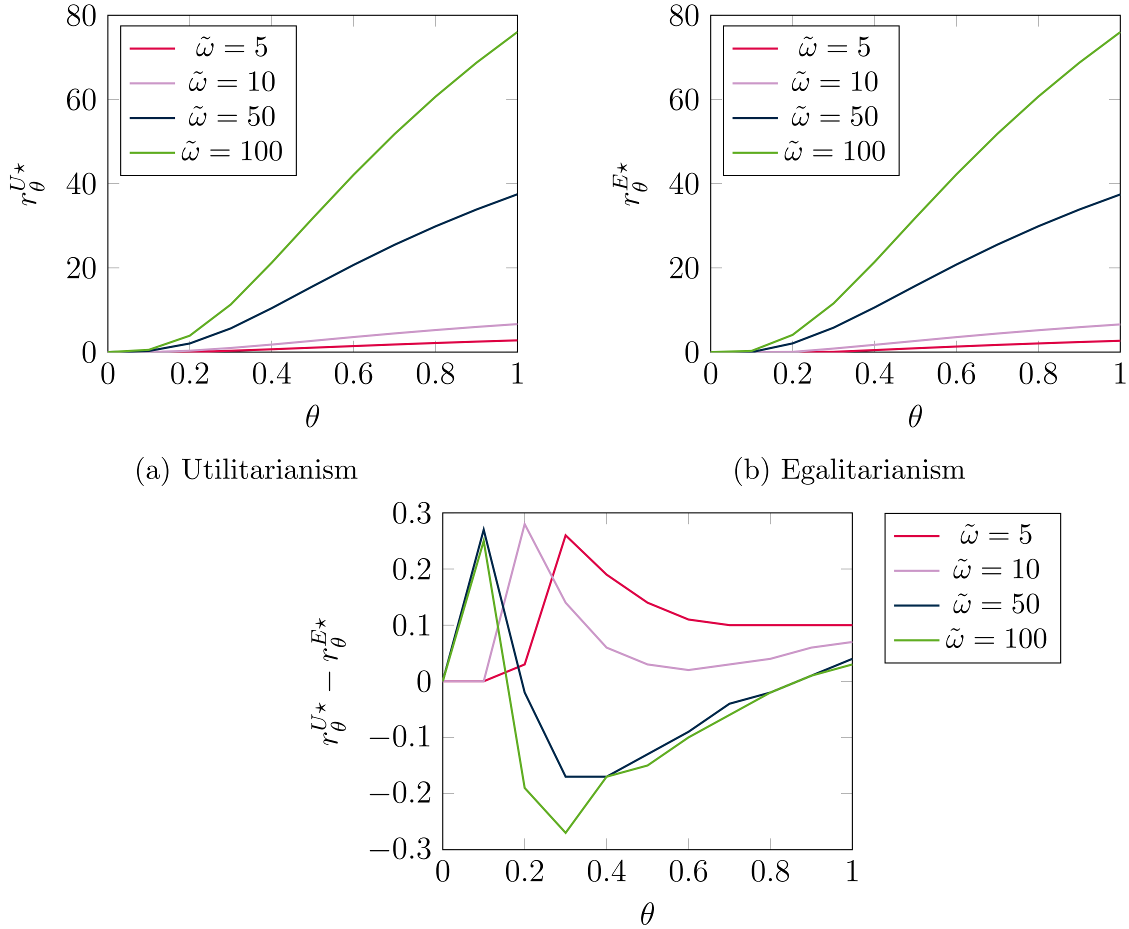
Prop. 5 shows that utilitarianism may recommend a larger population growth than egalitarianism when resources are low. Actually, it recommends a strictly larger population growth whenever the initial level of resources is at an intermediate level (not lower than $\underline{\omega} > 1$ but not too large either).

This seemingly hints at a potential general result that utilitarianism always recommend larger population growth. However, there is no such general result as illustrated in Figure 1.

Figure 1 provides numerical results of optimal population growth for various levels of initial resources $\tilde{\omega}$ in the case where $\varepsilon = 1.1$ and $\tilde{g} = 1$ (so that $\ln(1+\tilde{g}) < 1$). Panel (a) of Figure 1 gives optimal population growth $r_\theta^{U\star}$ for the utilitarian case and Panel (b) gives optimal population growth $r_\theta^{E\star}$ for the egalitarian case. The main first insight is that $r_\theta^{U\star}$ and $r_\theta^{E\star}$ exhibit very similar patterns when θ and $\tilde{\omega}$ vary. In particular, Figure 1 illustrates the result in Prop. 3: an increase in θ always yield a larger optimal population both for the utilitarian and egalitarian approaches. Similarly, an increase in initial resources $\tilde{\omega}$ always increases optimal population sizes.[9] It is perhaps more surprising that the exact levels of optimal population growth seems very close in the two approaches.

To look closer into the exact ranking of utilitarianism and egalitarianism in terms of optimal population growth, Panel (c) of Figure 1 draws the difference $(r_\theta^{U\star} - r_\theta^{E\star})$. It shows that the difference is usually very small, even when population growth is large. More strikingly compared to Prop. 5, Panel (c) shows that there are cases where the difference is negative, that is the egalitarian optimal population growth is larger than the utilitarian one. This happens when $\tilde{\omega}$ is large enough ($\tilde{\omega} = 50$ and $\tilde{\omega} = 100$) and θ has intermediates values (in the range $[0.2, 0.8]$). In any case, this proves that there is no hope to obtain general results regarding the relative utilitarian and egalitarian optimal population sizes.

[9] A proof that this is true in general in the present setting when $\varepsilon > 1$ and $\ln(1+\tilde{g}) < 1$ is available upon request.



(a) Utilitarianism

(b) Egalitarianism

(c) Difference

Figure 1: Optimal population for different levels of initial resources

5 Concluding remarks

In this paper, I have provided arguments why egalitarianism may be an attractive view in intergenerational models where resources have to be allocated between successive generations, whose number and size may be affected by the policy. If we want to limit the sacrifices made by the current generation for the sake of the future, or if we want to promote small sacrifices that benefit many future people, egalitarianism is appealing. I have also provided a new class of egalitarian criteria that may be used to compare populations with different sizes, while embodying many views about population ethics.

Applying these criteria to the question of optimal future population size, when we use renewable or non-renewable resources, I have exhibited a condition (the sorting condition) that orders population ethics views in terms of the optimal population size they recommend. We can thus define a 'total egalitarian' and an 'average egalitarian' view (much like the 'total utilitarian' and 'average utilitarian' views) and show that the former induces more population than the latter. But we can also define many intermediate views that are worth studying in more details. In a simple model I have also showed that egalitarianism may recommend less population growth than utilitarianism in specific cases (average view and low level of resources) but that this is not true in general. It would be interesting to study the implications of Population-adjusted egalitarianism in more complex models. This will be the topic of future work.

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A Appendix A: Proofs

A.1 Preliminary result

For any $x \in X$ and any $k \in \mathbb{N}$, let $k \star x$ denote $y \in X$ such that $n(y) = kn(x)$ and, for any $\ell \in \{1, \cdots, k\}$ and any $i \in \{1, \cdots, n(x)\}$, $y_{(\ell-1)+i} = x_i$. This is a k-replica of x.

Consider the following principle, which is very common in the literature. It states that replicating several times the same welfare distributions should not alter social judgements.

Replication Invariance. For all $n \in \mathbb{N}$, all $x, y \in \mathbb{R}^n$ and all $k \in \mathbb{N}$, $k\star x \succsim k\star y$ if and only if $x \succsim y$.

This principle is implied by Consistency.

Lemma 1 *If an SWO $\succsim$ on X satisfies Consistency then it satisfies Replication Invariance.*

Proof. Assume that the SWO $\succsim$ on X satisfies Consistency. Consider any $n \in \mathbb{N}$, $x, y \in \mathbb{R}^n$ and $k \in \mathbb{N}$. Assume that $x \succsim y$. Then by k applications of Consistency $k \star x \succsim k \star y$.

Conversely assume that $k \star x \succsim k \star y$ but $x \prec y$. By k applications of Consistency, we should have $k \star x \prec k \star y$, which is a contradiction. ■

A.2 Proof of Proposition 1

Assume that the SWO $\succsim$ satisfies Suppes-Sen, Pigou-Dalton and Consistency (and thus Replication invariance by Lemma 1). If $n = 1$, for any $x, y \in \mathbb{R}$ if $x_{[1]} > y_{[1]}$ then $x \succ y$ by Suppes-Sen. Below we focus on cases where $n > 1$.

A.2.1 Proof of statement 1.

Assume that $\succsim$ also satisfies Limited sacrifice for the rich future.

Consider any $n \in \mathbb{N} \setminus \{1\}$ and $x, y \in \mathbb{R}^n$, such that $x_{[1]} > y_{[1]}$. If $x_{[1]} > y_{[n]}$, then $x \succ y$ by Suppes-Sen. If $x_{[1]} \leq y_{[n]}$, consider the real numbers $a, b, c \in \mathbb{R}$ such that $y_{[1]} < a < b < x_{[1]} \leq y_{[n]} < c$.

Define $\alpha = b - a$ and let β be the corresponding term in the statement of Limited sacrifice for the rich future. Let $0 < \gamma < \beta$ and $m \in \mathbb{N}$ be such that $b = c - m\gamma$. Consider a collection of allocations $(w^0, w^1, \cdots, w^m)$ such that $w^0 = \big((a)_m, (c)_{m(n-1)}\big)$, $w^m = (b)_{mn}$ and for each $k \in I_{m-1}$ $w^k = \big((b)_k, (a)_{m-k}, (c-k\gamma)_{m(n-1)}\big)$.

By Limited sacrifice for the rich future, $\big((b)_1, (c-k\gamma)_{m(n-1)}\big) \succsim \big((a)_1, (c-(k-1)\cdot\gamma)_{m(n-1)}\big)$ for all $k \in I_m$. Denoting $z^1 = (a)_{m-1}$ and $z^k = \big((b)_{k-1}, (a)_{m-k}\big)$, by Consistency this implies

$$w^k = \big(z^k, (b)_1, (c-k\gamma)_{m(n-1)}\big) \succsim \big(z^k, (a)_1, (c-(k-1)\cdot\gamma)_{m(n-1)}\big) = w^{k-1}$$

for all $k \in I_m$. By transitivity, we obtain $w^m \succsim w^0$ and by Replication invariance $b_n \succsim \big((a)_1, (c)_{n-1}\big)$.

But, given that $b < x_{[1]}$, Suppes-Sen implies that $x \succ (b)_n$. Similarly, given that $y_{[1]} < a$ and $y_{[n]} < c$, Suppes-Sen implies that $\big((a)_1, (c)_{(n-1)}\big) \succ y$. By transitivity, $x \succ y$.

A.2.2 Proof of statement 2.

Assume that $\succsim$ also satisfies Limited sacrifice for the long future.
Step 1: there exists $k \in \mathbb{N}$ such that for any $a, b, c, d \in \mathbb{R}$ with $a < b \leq c < d$ and any $n \geq k$, $\big((c)_1, (b)_n\big) \succsim \big((d)_1, (a)_n\big)$.

Let k be the number in the statement of Limited sacrifice for the long future. Consider any $a, b, c, d \in \mathbb{R}$ such that $a < b \leq c < d$ and any $n \geq k$. Let $m \in \mathbb{N}$ and $\theta \in \mathbb{R}_{++}$ be such that $d - m\theta = a$ and $\theta \leq \gamma$, where γ is the number in the statement of Limited sacrifice for the long future. Denote $\varepsilon = \frac{b-a}{m}$.

By Limited sacrifice for the long future, $\big((a+(\ell+1)\cdot\varepsilon)_n, (d-(\ell+1)\cdot\theta)_1\big) \succsim \big((a+\ell\cdot\varepsilon)_n, (d-\ell\cdot\theta)_1\big)$ for all $\ell = 0, \cdots, m-1$, so that, by transitivity, $\big((b)_n, (c)_1\big) \succsim \big((a)_n, (d)_1\big)$.[10]
Step 2: there exists $k \in \mathbb{N}$ such that for any $a, b, c \in \mathbb{R}$ with $a < b \leq c$ and any $n \geq k$ and $m \in \mathbb{N}$, $\big((c)_1, (b)_{m+n-1}\big) \succsim \big((c)_m, (a)_n\big)$.

For any $a, b, c \in \mathbb{R}$ with $a < b \leq c$ and $m \in \mathbb{N}$, consider a collection of real numbers $(d^1, \cdots, d^m)$ such that $d^1 = b$, $d^m = a$ and $d^1 > d^2 > \cdots > d^m$. Let $n \geq k$, with $k \in \mathbb{N}$ the number in Step 1. Consider a collection of allocations $(w^1, \cdots, w^m)$ such that, for each $k \in I_m$ $w^k = \big((c)_k, (d^k)_{m+n-k}\big)$.

By Step 1, for each $k \in I_{m-1}$, we have $(d^k)_{m+n-k} \succsim \big((c)_1, (d^{k+1})_{m+n-k-1}\big)$. By Consistency, this implies $\big((c)_k, (d^k)_{m+n-k}\big) \succsim \big((c)_k, (c)_1, (d^{k+1})_{m+n-k-1}\big)$, which can be written $w^k \succsim w^{k+1}$. By transitivity, this implies that $\big((c)_1, (b)_{m+n-1}\big) = w^1 \succsim w^m = \big((c)_m, (a)_n\big)$.
Step 3: Conclusion.

Consider any $n \in \mathbb{N} \setminus \{1\}$ and $x, y \in \mathbb{R}^n$, such that $x_{[1]} > y_{[1]}$. If $x_{[1]} > y_{[n]}$, then $x \succ y$ by Suppes-Sen. If $x_{[1]} \leq y_{[n]}$, consider the real numbers $a, b, c, d, e \in \mathbb{R}$ such that $y_{[1]} < a < b < c < x_{[1]} \leq y_{[n]} < d$.

[10] Note that $a + m \cdot \varepsilon = b$ and $d - m \cdot \varepsilon = c$.

Let $k \in \mathbb{N}$ be the number in the statement of Step 1, which is the same as the k in Step 2. By Step 1, $(c)_{kn} \succsim \big((d)_1, (b)_{kn-1}\big)$. By Step 2, $\big((d)_1, (b)_{kn-1}\big) \succsim \big((d)_{k(n-1)}, (a)_k\big)$. Hence, by transitivity, $(c)_{kn} \succsim \big((d)_{k(n-1)}, (a)_k\big)$. By Replication invariance, this means that $(c)_n \succsim \big((d)_{(n-1)}, (a)_1\big)$. Given that $c < x_{[1]}$, Suppes-Sen implies that $x \succ (c)_n$. Similarly, given that $y_{[1]} < a$ and $y_{[n]} < d$, Suppes-Sen implies that $\big((d)_{(n-1)}, (a)_1\big) \succ y$. By transitivity, $x \succ y$.

A.3 Proof of Theorem 1

It is straightforward to check that Maximin SWOs satisfy Suppes-Sen, Consistency, Continuity, Limited sacrifice for the rich future and Limited sacrifice for the long future.

Assume that an SWO $\succsim$ on X satisfies Suppes-Sen, Pigou-Dalton, Consistency, Continuity and Limited sacrifice for the rich future (resp. Limited sacrifice for the long future). By Proposition 1, for all $n \in \mathbb{N}$ and x, $y \in \mathbb{R}^n$, $x \succ y$ whenever $x_{[1]} > y_{[1]}$.

Thus, consider any $n \in \mathbb{N}$ and $x, y \in \mathbb{R}^n$, and assume without loss of generality that $x_{[1]} \geq y_{[1]}$. If $x_{[1]} > y_{[1]}$ we know that $x \succ y$ as required by the Maximin ordering. The only remaining case is $x_{[1]} = y_{[1]}$. We need to show that in that case $x \sim y$.

To do so, let us prove that, for any $z \in \mathbb{R}^n$, $z \sim (z_{[1]})_n$. For any sequence of real numbers $(a_1, a_2, \cdots, a_k, \cdots)$ that converges to $z_{[1]}$ and such that $a_k > z_{[1]}$, we have $(a_k)_n \succ z$ because $a_k > z_{[1]}$. Hence, by Continuity $(z_{[1]})_n \succsim z$. Similarly, for any sequence of real numbers $(b_1, b_2, \cdots, b_k, \cdots)$ that converges to $z_{[1]}$ and such that $b_k < z_{[1]}$, we have $z \succ (b_k)_n$ because $b_k < z_{[1]}$. Hence, by continuity $z \succsim (z_{[1]})_n$. Therefore $z \sim (z_{[1]})_n$.

So, when $x_{[1]} = y_{[1]} = a$, $x \sim (a)_n$ and $y \sim (a)_n$. By transitivity, $x \sim y$.

A.4 Proof of Theorem 2

A.4.1 Extended continuity: A Lemma

Let us first introduce the following property proposed by Blackorby, Bossert and Donaldson (2001).

Extended continuity. For all $k, \ell \in \mathbb{N}$ and all $x \in \mathbb{R}^k$, the sets $\{y \in \mathbb{R}^\ell \mid y \succsim x\}$ and $\{y \in \mathbb{R}^\ell \mid x \succsim y\}$ are closed in $\mathbb{R}^\ell$.

Note that Extended continuity implies Continuity. The next lemma proves that Extended continuity is implied by Continuity when Critical level holds.

Lemma 2 *If an SWO $\succsim$ on X satisfies Continuity and Critical level then it satisfies Extended continuity.*

Proof. Consider any k, $\ell \in \mathbb{N}$ and all $x \in \mathbb{R}^k$. By repeated applications of Critical level, there exists $z_x \in \mathbb{R}^\ell$ such that $x \sim z_x$. By transitivity, $\{y \in \mathbb{R}^\ell \mid y \succsim x\} = \{y \in \mathbb{R}^\ell \mid y \succsim z_x\}$ and $\{y \in \mathbb{R}^\ell \mid x \succsim y\} = \{y \in \mathbb{R}^\ell \mid z_x \succsim y\}$. By continuity, the sets $\{y \in \mathbb{R}^\ell \mid y \succsim z_x\}$ and $\{y \in \mathbb{R}^\ell \mid z_x \succsim y\}$ are closed in $\mathbb{R}^\ell$, and therefore so are the sets $\{y \in \mathbb{R}^\ell \mid y \succsim x\}$ and $\{y \in \mathbb{R}^\ell \mid x \succsim y\}$. ■

A.4.2 A general class of Maximin criteria

Assuming that a Maximin SWO also satisfies Critical level, we obtain the following Proposition.

Proposition 6 *If a Maximin SWO $\succsim$ on X satisfies Critical level then there exists a function $V : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$, which is non-decreasing in its first argument (and constant when the second argument is negative), continuous and increasing in its second argument, and such that, for all $x, y \in X$,*

$$x \succsim y \Longleftrightarrow V\Big(n(x), x_{[1]}\Big) \geq V\Big(n(y), y_{[1]}\Big).$$

Proof. Given that $\succsim$ is a Maximin SWO, for any $k \in \mathbb{N}$, there exists a representative welfare function $e_k : \mathbb{R}^k \to \mathbb{R}$ satisfying $x \succsim y \Longleftrightarrow e_k(x) \geq e_k(y)$ for all $x, y \in \mathbb{R}^k$ and $x \sim (a)_k$ whenever $a = e_k(x)$. Specifically, function e_k is such that $e_k(x) = x_{[1]}$ for all $x \in \mathbb{R}^k$.

By Lemma 2, the SWO $\succsim$ satisfies Extended continuity (because Maximin SWOs satisfy Continuity, and $\succsim$ satisfies Critical-level). As shown by Blackorby, Bossert and Donaldson (2001), given that there exists a representative welfare function for each $k \in \mathbb{N}$ and that $\succsim$ satisfies Extended continuity, there exists a function $V : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$, which is continuous and increasing in its second argument, such that, for all $x, y \in X$,

$$x \succsim y \Longleftrightarrow V\big(n(x), e_{n(x)}(x)\big) \geq V\big(n(y), e_{n(y)}(y)\big).$$

Given that $e_{n(x)}(x) = x_{[1]}$ for all $x \in X$ and that V is increasing in its second argument, this implies that for all $x, y \in X$,

$$x \succsim y \Longleftrightarrow V\Big(n(x), x_{[1]}\Big) \geq V\Big(n(y), y_{[1]}\Big) \tag{8}$$

Let us prove that V is not decreasing in its first argument. Consider any $a \in \mathbb{R}$ and $n \in \mathbb{N}$. By Critical level, there exists $b \in \mathbb{R}_{++}$ such that $(a)_n \sim \big((a)_n, b\big)$. If $b > a$, because $\succsim$ is Maximin, $\big((a)_n, b\big) \sim (a)_{n+1}$. If $b \leq a$, because $\succsim$ is Maximin, $\big((a)_n, b\big) \precsim (a)_{n+1}$. So in any case, by transitivity, $(a)_{n+1} \succsim (a)_n$. By Eq. (8), this implies that $V(n+1,a) \geq V(n,a)$ for any any $a \in \mathbb{R}$ and $n \in \mathbb{N}$: V is non-decreasing in its first argument.

Furthermore, if $a \in \mathbb{R}_{--}$, for any $n \in \mathbb{N}$, Critical level implies that there exists $b \in \mathbb{R}_{++}$ such that $(a)_n \sim \big((a)_n, b\big)$. But given that $b \in \mathbb{R}_{++}$ and $a \in \mathbb{R}_{--}$, because $\succsim$ is Maximin, $\big((a)_n, b\big) \sim (a)_{n+1}$. Hence $(a)_n \sim (a)_{n+1}$. By Eq. (8), this implies that $V(n+1,a) = V(n,a)$ for any any $a \in \mathbb{R}_{--}$ and $n \in \mathbb{N}$: V is constant in its first argument when the second argument is negative. ■

A.4.3 Proof of Theorem 2

It is straightforward to check that Population-adjusted maximin SWOs are Maximin SWOs that satisfy Critical-level and Scale invariance.

Assume that a Maximin SWO $\succsim$ on X satisfies Critical level and Scale invariance. By Prop. 6, we know that there exists a function function $V : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$, which is non-decreasing in its first argument (and constant when the second argument is negative), continuous and increasing in its second argument, and such that, for all $x, y \in X$,

$$x \succsim y \iff \min_{i \in I_{n(x)}} V\big(n(x), x_i\big) \geq \min_{i \in I_{n(y)}} V\big(n(y), y_i\big). \tag{9}$$

Without loss of generality, let normalize function V so that $V(1,a) = a$ for all $a \in \mathbb{R}$ (this is possible given that $V(1,.)$ is increasing and continuous). Given that V is constant when its second argument is negative, we thus have $V(n,a) = a$ for all $n \in \mathbb{N}$ and all $a \in \mathbb{R}_{--}$.

On the other hand, by repeated application of Critical level and because $\succsim$ is Maximin, for any $k \in \mathbb{N}$ there exists $a_k \in \mathbb{R}_+$ such that $(1)_1 \sim (a_k)_k$. By Scale invariance, for any $b \in \mathbb{R}_+$, we have $(b)_k \sim \left(\frac{b}{a_k}\right)_1$ (indeed, $b = \lambda a_k$, with $\lambda = b/a_k > 0$). For any $k \in \mathbb{N}$, denote $\kappa_k = 1/a_k > 0$ (with $a_1 = 1$ so that $\kappa_1 = 1$). By Eq. (9) and the normalization of the V function, we obtain $V(k,a) = \kappa_k \cdot a$ for any $a \in \mathbb{R}_+$. Given that V is non-decreasing in its first argument, the sequence $(\kappa_k)_{k \in \mathbb{N}}$ must be a non-decreasing sequence.

A.5 Proof of Proposition 2

Assume that an SWO $\succsim$ on X is a Population-adjusted maximin SWO with population weights $(\kappa_k)_{k \in \mathbb{N}}$.

Let us first show that $\succsim$ avoids the sadistic conclusion. Take any $x \in X$, $k, \ell \in \mathbb{N}$, $a \in \mathbb{R}_{++}$ and $b \in \mathbb{R}_{--}$. We need to show that $\big(x, (a)_k\big) \succsim \big(x, (b)_m\big)$. There are two cases. If $x_{[1]} \leq b < 0$, then, by definition of Population-adjusted maximin SWOs, $\big(x, (a)_k\big) \sim (x_{[1]})_{n(x)+k}$, $(x_{[1]})_{n(x)+m} \sim \big(x, (b)_m\big)$ and $(x_{[1]})_{n(x)+k} \sim$

$(x_{[1]})_{n(x)+m}$, so that, by transitivity, $\big(x,(a)_k\big) \sim \big(x,(b)_m\big)$. On the other hand, if $x_{[1]} > b$ and letting $c = \min\{x_{[1]}, a\} > b$, by definition of Population-adjusted maximin SWOs, $\big(x,(a)_k\big) \sim (c)_{n(x)+k}$, $(b)_{n(x)+m} \sim \big(x,(b)_m\big)$ and $(c)_{n(x)+k} \succ (b)_{n(x)+m}$, so that, by transitivity, $\big(x,(a)_k\big) \succ \big(x,(b)_m\big)$.

Let us then assume that $\succsim$ also satisfies avoidance of the repugnant conclusion. Hence, there exist $k \in \mathbb{N}$, $a \in \mathbb{R}_{++}$ and $b \in [0,a]$ such that, for all $m \geq k$, $(a)_k \succsim (b)_m$. By definition of Population-adjusted maximin SWOs, this implies that $\kappa_m \cdot b \leq \kappa_k \cdot a$, that is $\kappa_m \leq \kappa_k \cdot \frac{a}{b}$, for all $m \geq k$. This is the definition of the sequence $(\kappa_k)_{k\in\mathbb{N}}$ being bounded. It is straightforward to see that, reciprocally, if sequence $(\kappa_k)_{k\in\mathbb{N}}$ is bounded, then there exist $k \in \mathbb{N}$, $a \in \mathbb{R}_{++}$ and $b \in [0,a]$ such that, for all $m \geq k$, $(a)_k \succsim (b)_m$.

Let us now show that $\succsim$ cannot satisfy the Mere addition principle. Indeed, there necessarily exist $a > b > 0$ such that $a > (1+\kappa_2)b$. Thus, by definition of Population-adjusted maximin SWOs, $\big((a)_1,(b)_1\big) \sim (b)_2$ and $(a)_1 \succ (b)_2$ so that by transitivity $(a)_1 \succ \big((a)_1,(b)_1\big)$. This is a violation of the Mere addition principle.

Lastly, let us now show that $\succsim$ cannot satisfy the Negative mere addition principle. Indeed, by definition of Population-adjusted maximin SWOs, $(b)_1 \sim (b)_2$ for any $b \in \mathbb{R}_{--}$. This is a violation of the Negative mere addition principle.

A.6 Proof of Proposition 5

A.6.1 Egalitarian case

Let us consider the following function of $r \in \mathbb{R}_+$:

$$V^E_\theta(r) = \tfrac{N^\theta(1+r)^\theta}{1-\varepsilon} \times \Big[\Big(\tfrac{1+\tilde{g}}{1+\tilde{g}+r}\Big)^{1-\varepsilon}\ \tilde{\omega}^{1-\varepsilon} - 1 \Big].$$

Function V^E_θ has the same value as function U^E_θ provided that the egalitarian EDEW is larger than 0. Given that $EDEW^E(0) = u(\tilde{\omega}) > 0$ (because $tildew > 1$), there exists values of r where V^E_θ and U^E_θ are the same and only such values can be optima (when $EDEW^E(r) \leq 0$, social welfare is lower than when $r = 0$). Hence solving problem (5) is equivalent to maximizing V^E_θ.

Now, we have:

$$\begin{aligned}\frac{\partial V^E_\theta}{\partial r} &= \tfrac{N^\theta(1+r)^{\theta-1}}{1-\varepsilon} \times \Big[\theta\Big(\tfrac{1+\tilde{g}}{1+\tilde{g}+r}\Big)^{1-\varepsilon}\ \tilde{\omega}^{1-\varepsilon} - \theta\Big] + \tfrac{N^\theta(1+r)^\theta}{1-\varepsilon} \times \Big(\tfrac{\varepsilon-1}{1+\tilde{g}+r}\Big)\ \Big(\tfrac{1+\tilde{g}}{1+\tilde{g}+r}\Big)^{1-\varepsilon}\ \tilde{\omega}^{1-\varepsilon} \\ &= \tfrac{N^\theta(1+r)^{\theta-1}}{1-\varepsilon} \times \Big[\Big(\theta + (\varepsilon-1)\tfrac{1+r}{1+\tilde{g}+r}\Big)\ \Big(\tfrac{1+\tilde{g}}{1+\tilde{g}+r}\Big)^{1-\varepsilon}\ \tilde{\omega}^{1-\varepsilon} - \theta\Big]\end{aligned}$$

When $\varepsilon > 1$, the function $H^E_\theta(r) = \Big(\theta + (\varepsilon-1)\tfrac{1+r}{1+\tilde{g}+r}\Big)\ \Big(\tfrac{1+\tilde{g}}{1+\tilde{g}+r}\Big)^{1-\varepsilon}\ \tilde{\omega}^{1-\varepsilon} - \theta$

is increasing in r and tends to $+\infty$ when r tends to $+\infty$. Thus:

1. if $H_\theta^E(0) < 0$, then H_θ^E is first negative and then positive, so that $\frac{\partial V_\theta^E}{\partial r}$ is first positive and then negative. Function V_θ^E is first increasing and then decreasing and thus admits a unique maximum $r_\theta^{E\star} > 0$, which such that $H_\theta^E(r_\theta^{E\star}) = 0$.

2. if $H_\theta^E(0) \geq 0$, then H_θ^E is always positive (except perhaps at $r = 0$), so that $\frac{\partial V_\theta^E}{\partial r}$ is negative. Function V_θ^E is always decreasing so that its maximum is reached at $r_\theta^{E\star} = 0$.

To sum up, the egalitarian optimal population growth level $r_\theta^{E\star}$ is strictly positive provided $\tilde{\omega} > \left[1 + \frac{\varepsilon-1}{\theta\,(1+\tilde{g})}\right]^{\frac{1}{\varepsilon-1}}$.[11]

A.6.2 Utilitarian case

Using the results from Section 4.3.1, we can rewrite utilitarian objective function in Eq. (4) when $\theta > 1$ as follows:

$$U_\theta^U(r) = V_\theta^p\left((1+r)N, \left(1 + r(1+\tilde{g})^{\frac{1}{\varepsilon}-1}\right)^{\varepsilon}(1+r)^{-1}\frac{\tilde{\omega}^{1-\varepsilon}}{1-\varepsilon} - \frac{1}{1-\varepsilon}\right).$$

Let us consider the following function of $r \in \mathbb{R}_+$:

$$V_\theta^U(r) = \frac{N^\theta(1+r)^\theta}{1-\varepsilon} \times \left[\left(1 + r(1+\tilde{g})^{\frac{1}{\varepsilon}-1}\right)^{\varepsilon}(1+r)^{-1}\,\tilde{\omega}^{1-\varepsilon} - 1\right].$$

We can use a line of arguments similar to the one developed for the egalitarian case to show that solving problem (4) is equivalent to maximizing V_θ^U when $\tilde{\omega} > 1$.

Now, we have:

$$\begin{aligned}
\frac{\partial V_\theta^U}{\partial r} &= \frac{N^\theta(1+r)^{\theta-1}}{1-\varepsilon} \times \left[\theta\left(1 + r(1+\tilde{g})^{\frac{1}{\varepsilon}-1}\right)^{\varepsilon}(1+r)^{-1}\,\tilde{\omega}^{1-\varepsilon} - \theta\right] \\
&\quad + \frac{N^\theta(1+r)^\theta}{1-\varepsilon} \times \varepsilon(1+\tilde{g})^{\frac{1}{\varepsilon}-1}\left(1 + r(1+\tilde{g})^{\frac{1}{\varepsilon}-1}\right)^{\varepsilon-1}(1+r)^{-1}\,\tilde{\omega}^{1-\varepsilon} \\
&\quad - \frac{N^\theta(1+r)^{\theta-1}}{1-\varepsilon} \times \left(1 + r(1+\tilde{g})^{\frac{1}{\varepsilon}-1}\right)^{\varepsilon}(1+r)^{-1}\,\tilde{\omega}^{1-\varepsilon} \\
&= \frac{N^\theta(1+r)^{\theta-1}}{1-\varepsilon} \times H_\theta^U(r),
\end{aligned}$$

[11] This condition corresponds to $H_\theta^E(0) < 0$.

where

$$H_\theta^U(r) = \left(\theta - 1 + \varepsilon(1+\tilde{g})^{\frac{1}{\varepsilon}-1}\frac{1+r}{1+r(1+\tilde{g})^{\frac{1}{\varepsilon}-1}}\right)\left(1+r(1+\tilde{g})^{\frac{1}{\varepsilon}-1}\right)^{\varepsilon}(1+r)^{-1}\,\tilde{\omega}^{1-\varepsilon}-\theta.$$

When $\varepsilon > 1$ and $\ln(1+\tilde{g}) \leq 1$, then $\varepsilon(1+\tilde{g})^{\frac{1}{\varepsilon}-1} \geq 1$ and the functions $x \to \varepsilon(1+\tilde{g})^{\frac{1}{\varepsilon}-1}\frac{1+r}{1+r(1+\tilde{g})^{\frac{1}{\varepsilon}-1}}$ and $x \to \left(1+r(1+\tilde{g})^{\frac{1}{\varepsilon}-1}\right)^{\varepsilon}(1+r)^{-1}$ are increasing. Function H_θ^U is increasing in r and tends to $+\infty$ when r tends to $+\infty$. Thus:

1. if $H_\theta^U(0) < 0$, then H_θ^U is first negative and then positive, so that $\frac{\partial V_\theta^U}{\partial r}$ is first positive and then negative. Function V_θ^U is first increasing and then decreasing and thus admits a unique maximum $r_\theta^{U\star} > 0$, which such that $H_\theta^U(r_\theta^{U\star}) = 0$.

2. if $H_\theta^U(0) \geq 0$, then H_θ^U is always positive (except perhaps at $r = 0$), so that $\frac{\partial V_\theta^U}{\partial r}$ is negative. Function V_θ^U is always decreasing so that its maximum is reached at $r_\theta^{U\star} = 0$.

To sum up, the egalitarian optimal population growth level $r_\theta^{E\star}$ is strictly positive provided $\tilde{\omega} > \left[1 + \frac{\varepsilon(1+\tilde{g})^{\frac{1}{\varepsilon}-1}-1}{\theta}\right]^{\frac{1}{\varepsilon-1}}$.[12]

A.6.3 A sufficient condition for the egalitarian approach to yield a lower optimal population growth

In the egalitarian case, population growth is positive if $\tilde{\omega} > \bar{\omega} = \left[1 + \frac{\varepsilon-1}{\theta\,(1+\tilde{g})}\right]^{\frac{1}{\varepsilon-1}}$, otherwise it is nil.

In the utilitarian case, population growth is positive if $\tilde{\omega} > \underline{\omega} = \left[1 + \frac{\varepsilon(1+\tilde{g})^{\frac{1}{\varepsilon}-1}-1}{\theta}\right]^{\frac{1}{\varepsilon-1}}$, otherwise it is nil.

It can be showed that $\frac{\varepsilon-1}{\theta\,(1+\tilde{g})} \geq \frac{\varepsilon(1+\tilde{g})^{\frac{1}{\varepsilon}-1}-1}{\theta}$ when $\varepsilon > 1$ and $\tilde{g} \geq 0$ so that $\bar{\omega} \geq \underline{\omega}$. Therefore:

- If $\tilde{\omega} \leq \underline{\omega}$, then $r_\theta^{U\star} = r_\theta^{E\star} = 0$.
- If $\underline{\omega} < \tilde{\omega} \leq \bar{\omega}$, then $r_\theta^{U\star} > 0$ and $r_\theta^{E\star} = 0$.

[12] This condition corresponds to $H_\theta^U(0) < 0$.

B Independence of the axioms in the characterization of the maximin SWO (Theorem 1) and Population-adjusted maximin SWO (Theorem 2)

Consider the following SWOs:

1. Negative utilitarianism $\succsim_{NU}$: for all $x, y \in X$, $x \succsim_{NU} y \iff -\sum_{i=1}^{n(x)} x_i \geq -\sum_{i=1}^{n(y)} y_i$;
2. Partial utilitarianism $\succsim_{PU}$: for all $x \in X$, let $S : X \to \mathbb{R}$ be such that $S(x) = x_{[1]}$ if $n(x) = 1$ and $S(x) = x_{[1]} + x_{[2]}$ if $n(x) > 1$. For all $x, y \in X$, $x \succsim_{PU} y \iff S(x) \geq S(y)$;
3. Leximin $\succsim_L$: see Definition S.B.1 in the Supplementary material with $\kappa_k = 1$ for all $k \in \mathbb{N}$;
4. Utilitarianism $\succsim_U$: for all $x, y \in X$, $x \succsim_U y \iff \sum_{i=1}^{n(x)} x_i \geq \sum_{i=1}^{n(y)} y_i$;
5. Maximin with population priority $\succsim_{MPP}$: for all $x, y \in X$, if $n(x) > n(y)$ then $x \succ y$, if $n(y) > n(x)$ then $y \succ x$, and if $n(x) = n(y)$ then $x \succsim_{MPP} y \iff x_{[1]} \geq y_{[1]}$;
6. Power population maximin $\succsim_{PPM}$: For all $x \in X$, let $\Gamma : X \to \mathbb{R}$ be such that $\Gamma(x) = x_{[1]}$ if $x_{[1]} \leq 0$ and $\Gamma(x) = (1 + x_{[1]})^{n(x)} - 1$ if $x_{[1]} > 0$. For all $x, y \in X$, $x \succsim_{PPM} y \iff \Gamma(x) \geq \Gamma(y)$.

The Negative utilitarian SWO $\succsim_{NU}$ satisfies Consistency, Continuity, Limited sacrifice for the rich future and Limited sacrifice for the long future, Critical level and Ratio-scale invariance but not Suppes-Sen. The Partial utilitarian SWO $\succsim_{PU}$ satisfies Suppes-Sen, Continuity, Limited sacrifice for the rich future, Limited sacrifice for the long future (for $k = 3$), Critical level and Ratio-scale invariance but not Consistency. The leximin SWO $\succsim_L$ satisfies Suppes-Sen, Consistency, Limited sacrifice for the rich future, Limited sacrifice for the long future, Critical level and Ratio-scale invariance but not Continuity. The utilitarian SWO $\succsim_U$ satisfies Suppes-Sen, Consistency, Continuity, Critical level and Ratio-scale invariance but neither Limited sacrifice for the rich future nor Limited sacrifice for the long future. The Maximin with population priority SWO $\succsim_{MPP}$ satisfies Suppes-Sen, Consistency, Continuity, Limited sacrifice for the rich future, Limited sacrifice for the long future and Ratio-scale invariance but not Critical level. The Power population maximin SWO $\succsim_{PPM}$ satisfies Suppes-Sen, Consistency, Continuity, Limited sacrifice for the rich future, Limited sacrifice for the long future and Critical level but not Ratio-scale invariance.

Population-adjusted egalitarianism
Supplementary material

Stéphane Zuber[a]

S.A Weak limited sacrifice for the rich future.

The next principle states that if the (single-person of the) current generation is poorer and all future generations equally well-off, then there is a maximal amount of sacrifice we can require of this generation provided all future generations gain enough. Conversely, there is a small loss that is tolerable for all the well-off future generations, no matter how numerous they are, provided the current poorer generation gains enough.

Weak limited sacrifice for the rich future. There exist $\tilde{\alpha}, \tilde{\beta} \in \mathbb{R}_{++}$ such that $\tilde{\alpha} > \tilde{\beta}$ and, for all $n \in \mathbb{N}$, if $a, b, c, d \in \mathbb{R}$ are such that $b \leq c$, $b - a \geq \tilde{\alpha}$ and $\tilde{\beta} \geq d - c$, then $\big((b)_1, (c)_n\big) \succsim \big((a)_1, (d)_n\big)$.

Weak limited sacrifice for the rich future is related to a principle named Weak non-aggregation by Fleurbaey and Tungodden (2010). The next Proposition states if this principle is satisfied together with Ratio-Scale invariance then we obtain the principle of Limited sacrifice for the rich future used in the main text.

Proposition S.A.1 *Consider an SWO $\succsim$ on X. If $\succsim$ satisfies Weak limited sacrifice for the rich future and Ratio-scale invariance then it satisfies Limited sacrifice for the rich future.*

Proof. Consider any $\alpha \in \mathbb{R}_{++}$. Let $\tilde{\alpha}$ be the number in the statement of Weak limited sacrifice for the rich future, $\lambda = \frac{\tilde{\alpha}}{\alpha} > 0$ and $\beta = \tilde{\beta}/\lambda$ where $\tilde{\beta} \in \mathbb{R}_{++}$ be the number in the statement of Weak limited sacrifice for the rich future (hence $\alpha > \beta > 0$).

Consider any $n \in \mathbb{N}$ and any $a, b, c, d \in \mathbb{R}$ are such that $b \leq c$, $b - a \geq \alpha$ and $\beta \geq d - c$. Thus $\lambda\, a, \lambda\, b, \lambda\, c, \lambda\, d$ are real numbers such that $\lambda\, b \leq \lambda\, c$, $\lambda\, b - \lambda\, a \geq \lambda\, \alpha = \tilde{\alpha}$ and $\tilde{\beta} = \lambda\, \beta \geq \lambda\, d - \lambda\, c$. By Weak limited sacrifice for the rich future, this implies that $\big((\lambda b)_1, (\lambda c)_n\big) \succsim \big((\lambda a)_1, (\lambda d)_n\big)$. And by Ratio-scale invariance, this yields $\big((b)_1, (c)_n\big) \succsim \big((a)_1, (d)_n\big)$. ■

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S.B Population-adjusted leximin social welfare orderings

In this section, I show how the axiomatic analysis of Section 3 can be extended to the case of Leximin SWOs. A difficulty is that leximin SWOs are nor continuous, so that we cannot define an EDEW and combine it with population size like in Blackorby, Bossert and Donaldson (2001). To compare populations with different sizes, I thus use a different approach.

A key property of leximin SWOs is that they satisfy a strong notion of Consistency that requires that if the situation is strictly socially better for one subpopulation it is also socially better for the whole population. This can be expressed as an independence axiom.

Weak independence For all $n \in \mathbb{N}$, all $x, y \in \mathbb{R}^n$ and all $a, b \in \mathbb{R}$, $(a, x) \succsim (a, y)$ if and only if $(b, x) \succsim (b, y)$.

Furthermore, for all x, $y \in X$ and all $a, b \in \mathbb{R}_-$, $(a, x) \succsim (a, y)$ if and only if $(b, x) \succsim (b, y)$.

This a weak existence independence principle because independence holds only for allocations with the same population size or when independence is with respect to individuals with non-positive level of well-being.

Proposition S.B.1 *Consider an SWO $\succsim$ on X.*

1. *If $\succsim$ satisfies Suppes-Sen, Weak independence and Limited sacrifice for the rich future, then it is a Leximin SWO.*

2. *If $\succsim$ satisfies Suppes-Sen, Weak independence and Limited sacrifice for the long future, then it is a Leximin SWO.*

Proof. Let us first show that Weak independence implies Consistency. Consider any $n, m \in \mathbb{N}$, any x, $y \in \mathbb{R}^n$ and any x', $y' \in \mathbb{R}^m$. Assume that $x \succsim y$ and $x' \succsim y'$. By repeated applications of Weak independence, $(x, x') \succsim (x, y')$ and $(x, y') \succsim (y, y')$. By transitivity, $(x, x') \succsim (y, y')$. The same line of reasoning implies that if $x \succ y$ and $x' \succ y'$ then $(x, x') \succ (y, y')$.

Now, by Prop. 1, given the axioms in the two statements, we know that for all $n \in \mathbb{N}$ and x, $y \in \mathbb{R}^n$, if $x_{[1]} > y_{[1]}$ then $x \succ y$.

Consider any $n \in \mathbb{N}$ and any x, $y \in \mathbb{R}^n$. By Suppes-Sen, if $x_{[\,]} = y_{[\,]}$, then $x \sim y$. If $x_{[\,]} \neq y_{[\,]}$, there must exist $R \in I_n$ such that $x_{[r]} = y_{[r]}$ for all $r \in I_{R-1}$ and either $x_{[R]} > y_{[R]}$ or $y_{[R]} > x_{[R]}$. Consider without loss of generality the case $x_{[R]} > y_{[R]}$ and let $a, b \in \mathbb{R}$ be such that $x_{[R]} > a > y_{[R]}$ and $b > \max\{x_{[R]}, y_{[n]}\}$. Define $z \in \mathbb{R}^{R-1}$, $\tilde{x}, \tilde{y} \in \mathbb{R}^{n-R+1}$ in the following way:

- For all $i \in I_{R-1}$, $z_i = x_{[i]} = y_{[i]}$;
- For all $j \in I_{n-R+1}$, $\tilde{x}_j = x_{[R]}$;
- $\tilde{y}_1 = a$ and for all $j \in \{2, \cdots, n-R+1\}$, $\tilde{y}_j = b$.

By Suppes-Sen, $x \succsim (z, \tilde{x})$ and $(z, \tilde{y}) \succsim y$. Let $\tilde{x}' = \big((b)_{R-1}, \tilde{x}\big)$ and $\tilde{y}' = \big((b)_{R-1}, \tilde{y}\big)$, so that $\tilde{x}'_{[1]} = x_{[R]} > a = \tilde{y}'_{[1]}$. Hence, $\tilde{x}' \succ \tilde{y}'$. By repeated applications of Weak independence this implies $(z, \tilde{x}) \succ (z, \tilde{y})$ and by transitivity that $x \succ y$. ■

To apply Leximin SWOs to variable populations comparisons, I propose to accept the following two principles. The first principles ensures that there exists a trade-off between average welfare and population size.

Sensible trade-off. For any $n \in \mathbb{N}$, there exist a, $b \in \mathbb{R}_{++}$ such that $a \geq b$ and $(a)_n \sim (b)_{n+1}$.

A stronger property than the Critical level used in the main text requires all critical levels to be equal to the utility level representing neutrality. This is what Blackorby, Bossert and Donaldson (2005) name the Zero critical level principle. As they argue, this may not be a compelling axiom in general, but here we restrict it to 'bad lives' (below neutrality). In that case, the addition of people with non-negative well-being seems to be an improvement. Combined with Suppes-Sen, the axiom also ensures that the Weak mere addition principle (defined in the main text) is satisfied.

Zero critical level for bad lives. For any $x \in X_-$, $\big(x, (0)_1\big) \sim x$.

I also propose a equivalence condition that resembles the Suppes-Sen principle but can be applied to populations with different sizes. To do so, we need to take into account population size when assessing the contributive value of lives.

Population-adjusted Suppes-Sen equivalence. For all x, $y \in X$, if $n(x) > n(y)$, $(x_{[\ell]})_{n(x)} \sim (y_{[\ell]})_{n(y)}$ for all $\ell \leq n(y)$, $x_{[n(y)]} \geq 0$ and $x_{[n(x)]} = x_{[n(y)]}$, then $x \sim y$.

To define Population-adjusted leximin SWOs, let me introduce a new piece of notation. For an allocation $x \in \mathbb{R}^n$, a sequence $(\kappa_k)_{k \in \mathbb{N}} \in \mathbb{R}_+^{\mathbb{N}}$ and a positive integer $m \in \mathbb{N}$ such that $m \geq n$, we denote $x^{\kappa,m}$ the allocation such that:

- If $m = n$, then $x_i^{\kappa,m} = \Pi_\kappa(n, x)$ for all $i \in I_n$;[2]
- If $m > n$ then $x_i^{\kappa,m} = \Pi_\kappa(n, x_{[i]})$ for all $i \in I_n$ and $x_j^{\kappa,m} = \Pi_\kappa\Big(n, \max\{x_{[n]}, 0\}\Big)$ for all $j \in \{n+1, \cdots, m\}$.

Definition S.B.1 *An SWO $\succsim$ on X is a Population-adjusted leximin SWO if and only if there exists a non-decreasing sequence $(\kappa_k)_{k\in\mathbb{N}}$ such that $\kappa_1 = 1$ and for all $x, y \in X$ with $n(x) \geq n(y)$,*

$$x \succsim y \Longleftrightarrow x^{\kappa,n(x)} \succsim_L^{n(x)} y^{\kappa,n(x)}$$

and

$$y \succsim x \Longleftrightarrow y^{\kappa,n(x)} \succsim_L^{n(x)} x^{\kappa,n(x)}.$$

The next Theorem is a characterization of Population-adjusted leximin SWOs.

Theorem S.B.1 *Consider an SWO $\succsim$ on X.*

1. *$\succsim$ satisfies Suppes-Sen, Limited sacrifice for the rich future, Ratio-scale invariance, Weak independence, Sensible trade-off, Zero critical level for bad lives and Population-adjusted Suppes-Sen equivalence if and only if it is a Population-adjusted leximin SWO.*

2. *$\succsim$ satisfies Suppes-Sen, Limited sacrifice for the long future, Ratio-scale invariance, Weak independence, Sensible trade-off, Zero critical level for bad lives and Population-adjusted Suppes-Sen equivalence if and only if it is a Population-adjusted leximin SWO.*

Proof. It is straightforward to check that Population-adjusted leximin SWOs satisfy all the principles in the two statements.

Assume that an SWO $\succsim$ on X satisfies the axioms in principles in the two statements. Let us show that it is a Population-adjusted leximin SWO.

Step 1: $\succsim$ is a Leximin SWO.

By Prop. S.B.1, given the principles in the two statements, we know that $\succsim$ is a Leximin SWO.

Step 2: For any $x \in X_-$, for any $m > n(x)$, $x \sim x^{\hat{\kappa},m}$ for any sequence $(\hat{\kappa}_k)_{k\in\mathbb{N}} \in \mathbb{R}_+^{\mathbb{N}}$.

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[2]Recall that function $\Pi_\kappa : \mathbb{N} \times \mathbb{R}$ is defined for all $(n, e) \in X \times \mathbb{R}$ by:

$$\Pi_\kappa(n, e) = \begin{cases} e & \text{if } e \leq 0; \\ \kappa_n \cdot e & \text{if } e > 0. \end{cases}$$

By repeated applications of Zero critical level for bad lives $x \sim \big(x, (0)_{m-n(x)}\big)$. By definition, $\big(x, (0)_{m-n(x)}\big) = x^{\hat{\kappa},m}$ for any sequence $(\hat{\kappa}_k)_{k\in\mathbb{N}} \in \mathbb{R}_+^{\mathbb{N}}$. Remark that this implies that $(0)_m \sim (0)_n$ for all $m, n \in \mathbb{N}$.

Step 3: There exists a non-decreasing sequence $(\kappa_k)_{k\in\mathbb{N}}$ such that $\kappa_1 = 1$ and for any $a, b \in R_{++}$ and $n, m \in \mathbb{N}$ if $\kappa_n \cdot a = \kappa_n \cdot b$ then $(a)_n \sim (b)_m$.

By Sensible trade-off, for any $n \in \mathbb{N}$ there exists $a, b \in \mathbb{R}_{++}$ such that $a \geq b$ and $(a)_n \sim (b)_{n+1}$. Denote $\mu_n = \frac{a}{b} \geq 1$ so that $(\mu_n b)_n \sim (b)_{n+1}$. By Scale invariance, for any $c \in \mathbb{R}_{++}$, there exists $d \in \mathbb{R}_{++}$ such that $d = \frac{c}{b}a = \mu_n c$ such that $(d)_n \sim (c)_{n+1}$ (indeed, $c = \lambda b$ and $d = \lambda a$ with $\lambda = \frac{c}{b}$). This implies that there exists $\mu_n \geq 1$ such that, for $c \in \mathbb{R}_{++}$, $(\mu_n c)_n \sim (c)_{n+1}$.

By transitivity and repeated application of the above procedure, for any $a \in \mathbb{R}_{++}$ and $n \in \mathbb{N}$ we have $(a)_n \sim (\kappa_n a)_1$, where $\kappa_1 = 1$ and for any $n > 1$, $\kappa_n = \left(\prod_{k=1}^{n-1} \mu_k\right)$. Remark that $\kappa_{n+1} \geq \kappa_n$ for all $n \in \mathbb{N}$ given that $\mu_k \geq 1$ for all $k \in \mathbb{N}$. By transitivity, we obtain that for any $a, b \in \mathbb{R}_{++}$ and any $n, k \in \mathbb{N}$, if $\kappa_n \cdot a = \kappa_n \cdot b$ then $(a)_n \sim (b)_m$.

Step 4: There exists a non-decreasing sequence $(\kappa_k)_{k\in\mathbb{N}}$ such that $\kappa_1 = 1$ and, for all $n, m \in \mathbb{N}$ with $m \geq n$, for all $x \in \mathbb{R}^n$, $x \sim x^{\kappa^m,m}$, where $\kappa^m = (\kappa_1/\kappa_m, \cdots, \kappa_{m-1}/\kappa_m, 1, 1, \cdots)$.

By Step 2, we know that this is true for all $x \in X_-$. Thus consider any $n, m \in \mathbb{N}$ with $m \geq n$ and any $x \in \mathbb{R}^n \setminus \mathbb{R}^n_{--}$. Let $R \in I_n$ be the integer such that $x_{[r]} < 0$ for all $r < R$ and $x_{[r]} \geq 0$ for all $r \geq R$. Let us assume that $R > 1$ (the proof for $R = 1$ is similar but does not need the complication of dealing with negative welfare levels).

For $n = m$, the statement is true as $x \sim x^{\kappa^m,m}$. Assume that $m > n$ and let $(\kappa_k)_{k\in\mathbb{N}}$ be the sequence in Step 3. Define $z \in \mathbb{R}^{R-1}$, $\tilde{x} \in \mathbb{R}^{n-R+1}$ and $\tilde{y} \in \mathbb{R}^{m-R+1}$ in the following way:

- For all $i \in I_{R-1}$, $z_i = x_{[i]}$;
- For all $j \in I_{n-R+1}$, $\tilde{x}_j = x_{[R-1+j]}$ and $\tilde{y}_j = \frac{\kappa_n}{\kappa_m} x_{[R-1+j]}$;
- For all $j \in \{n-R+2, \cdots, m-R+1\}$, $\tilde{y}_j = \frac{\kappa_n}{\kappa_m} x_{[n]}$.

Consider allocations $\hat{x} = \big((0)_{R-1}, \tilde{x}\big)$ and $\hat{y} = \big((0)_{R-1}, \tilde{y}\big)$. Clearly, for all $r \in I_{R-1}$ $(\hat{x}_r)_n \sim (\hat{y}_r)_m$ because $x_{[r]} = y_{[r]} = 0$ (see Step 2). For all $r \in \{R, \cdots, n\}$, we have $\hat{x}_r = x_{[r]}$ and $\hat{y}_r = \frac{\kappa_n}{\kappa_m} x_{[r]}$. Hence $\kappa_n \hat{x}_r = \kappa_m \hat{y}_r$. Thus, by Step 3, $(\hat{x}_r)_n \sim (\hat{y}_r)_m$ for all $r \in \{R, \cdots, n\}$. Lastly, $\hat{y}_{[m]} = \hat{y}_{[n]}$. Hence, by Population-adjusted Suppes-Sen equivalence, $\hat{x} \sim \hat{y}$.

By repeated applications of Weak independence, $\big(z, \tilde{x}\big) \sim \big(z, \tilde{y}\big)$. Let $\kappa^m = (\kappa_1/\kappa_m, \cdots, \kappa_{m-1}/\kappa_m, 1, 1, \cdots)$. By definition, $\big(z, \tilde{y}\big)_{[\,]} = x_{[\,]}^{\kappa^m,m}$ and $\big(z, \tilde{x}\big)_{[\,]} = x_{[\,]}$. By Suppes-Sen and transitivity, this implies $x \sim x^{\kappa^m,m}$.

Step 5: Conclusion. Consider any x, $y \in X$ with $n(x) \geq n(y)$.

By Step 4, $x \sim x^{\kappa^{n(x)},n(x)}$ and $y \sim y^{\kappa^{n(x)},n(x)}$ where $(\kappa_k)_{k\in\mathbb{N}}$ is the sequence in the statement of Step 3 and (κ^n, n) is defined in the statement of Step 4. Both $x^{\kappa^{n(x)},n(x)}$ and $y^{\kappa^{n(x)},n(x)}$ are allocations in $\mathbb{R}^{n(x)}$ so, by Step 1, $x^{\kappa^{n(x)},n(x)} \succsim y^{\kappa^{n(x)},n(x)} \iff x^{\kappa^{n(x)},n(x)} \succsim_L^{n(x)} y^{\kappa^{n(x)},n(x)}$ and $y^{\kappa^{n(x)},n(x)} \succsim x^{\kappa^{n(x)},n(x)} \iff y^{\kappa^{n(x)},n(x)} \succsim_L^{n(x)} x^{\kappa^{n(x)},n(x)}$. Remark that for all $i \in I_{n(x)}$ $x_i^{\kappa,n(x)} = \Pi\Big(n(x), x_i^{\kappa^{n(x)},n(x)}\Big)$ and $y_i^{\kappa,n(x)} = \Pi\Big(n(x), y_i^{\kappa^{n(x)},n(x)}\Big)$. Given $\Pi(n(x), .)$ is an increasing function and that Leximin orderings are invariant with respect to transformations of utility by a common increasing function we obtain that $x^{\kappa^{n(x)},n(x)} \succsim y^{\kappa^{n(x)},n(x)} \iff x^{\kappa,n(x)} \succsim_L^{n(x)} y^{\kappa,n(x)}$ and $y^{\kappa^{n(x)},n(x)} \succsim x^{\kappa^{n(x)},n(x)} \iff y^{\kappa,n(x)} \succsim_L^{n(x)} x^{\kappa,n(x)}$. Then transitivity yields the result. ■