A Qualitative Theory of Conflict Resolution and Political Compromise

Joseph M. Abdou, Hans Keiding

Documents de Travail du
Centre d'Economie de la Sorbonne

CES Working Papers

A Qualitative Theory of Conflict Resolution and Political Compromise

Joseph ABDOU, Hans KEIDING

2018.33

Maison des Sciences Économiques, 106-112 boulevard de L'Hôpital, 75647 Paris Cedex 13
https://centredeconomiesorbonne.univ-paris1.fr/
ISSN : 1955-611X

A Qualitative Theory of Conflict Resolution and Political Compromise

Joseph Abdou*, Hans Keiding†

July 25, 2018

Abstract

We view political activity as an interaction between forces seeking to achieve a political agenda. The viability of a situation depends on the compatibility of such agendas. However even in a conflictual situation a compromise may be possible. Mathematically a political structure is modeled as a simplicial complex and a viable configuration as a simplex. A represented compromise is a viable configuration obtained by the withdrawal of some agents in favor of some friendly representatives. A delegated compromise is a sophisticated version of a compromise obtained by the iteration of the withdrawal process. Existence of such solutions depends on the discrete topology of the simplicial complex. In particular we prove that the existence of a delegated compromise is equivalent to the strong contractibility of the simplicial complex.

Keywords: Delegation, compromise, simplicial complex, contiguity, strong homotopy.

JEL Classification: C70, C79, **AMS Classification**: 91A70.

1 Introduction

We study conflict resolution as the search for a compromise in a political context. Political activity is viewed as an interaction between forces seeking to achieve a political agenda. Only a viable government (Commonwealth, State) allows the achievement of political projects. However such a government may not be achievable if the forces do not agree on some common ground. In normal circumstances, the struggle for power is regulated by rules (e.g. Constitution) that guarantee a peaceful and consensual outcome

*Centre d'Economie de la Sorbonne, Université Paris 1, Panthéon-Sorbonne, 106-112 boulevard de l'Hôpital, 75647 Paris Cedex 13-France; email: abdou@univ-paris1.fr.

†Department of Economics, University of Copenhagen, O.Farimagsgade 3, DK-1353 Copenhagen K, Denmark; email: hans.keiding@econ.ku.dk.

of the process. This includes commonly accepted mechanisms for breaking deadlocks once they occur (for instance elections, referendum or justice ruling). However many political entities, under special circumstances, incur a blocked governance process while the current order fails to provide a clue for a solution. This is a political crisis or a stalemate. For instance, one can witness the formation of a crisis in a place where a military conflict left the entity with a pre-state, pre-constitutional configuration, that is, where a universally accepted rule does not prevail; or where the threat of a violent action impedes the regular unfolding of the political process; or else where the risk of disrupting the ongoing process is wielded by a force that accumulates discontent with the current establishment. The general question that we ask is therefore : What happens in a crisis configuration that is unsolvable by the current institutions, or put more explicitly what can be expected if a configuration composed by many parties with incompatible political agendas lacks the institutional mechanism that enforces a settlement?

Modeling politics has a long history starting from early greek philosophers. Foundations of political order in a City or a Commonwealth - the State - have been the main object of political philosophy. The object of political activity being to achieve coexistence, in a common space, of entities with conflicting wills, the notion of conflict is at the center of such a thought. One has to explain locally the emergence of an order between agents, and globally the coexistence of many such orders with different agendas. The difficulty (but also the interest) in modeling the notion of conflict lies in the fact that the latter is, by definition, a situation that erupts as a crisis of the current order and therefore unsolvable by existing institutions: being disruptive the new game has no rules ....

One of the main concepts that marked the modern analysis of conflict is the notion of enemy. According to political thinkers that promote this view the essential moment of politics is the dichotomy friend/enemy (cf. e.g.[11]). This binary choice prevails in situations of disruption, when the political body is in danger, war or civil strife. Although we admit that the conflict is produced in a disrupted situation, we depart from the idea that politics is bi-polar: political action may well be drawing a line of separation from (or the destruction of) an enemy, but it is mainly the search of a viable situation from a disrupted one. Political forces in presence can be either compatible for an inclusive governance or not. In many situations the search for viability may be accomplished through complex processes that could lead to a compromise.

Our model starts by the description of a configuration of forces and the viability relation. This is an abstraction of the typical though extremely stylized situation that prevails after a civil war or an invasion by military forces has destroyed the previous State. A less dramatic but formally similar situation prevails locally in political entities where elections or other rules result in a distribution of forces unable to coexist in a single government. Coexistence means that the coalition would not explode and that a viable order can be established. With the viability structure as only datum we seek to determine whether a compromises exists or not. Mathematically a political structure is represented by a simplicial complex.

A fundamental notion in our approach to crisis resolution in this framework is *friendly delegation:* An agent can delegate power to another agent only when the latter is at least as well situated in the viability distribution as the delegating agent. Friendly delegation by an agent is possible if any viable configuration containing the agent remains viable if the delegate joins the configuration. The idea is that no viable occasion is lost if the agent withdraws in favor of the delegate. The result of withdrawal will be a new configuration which is simplified to some extent, since some of the sources of conflict have been removed. In this model we do not address the question of whether an agent decides to delegate or not since our agents are not endowed with any decision power. Instead we investigate in detail what can emerge as a result of delegation, giving rise to the notion of representation and represented compromise. One can view viability as a sort of potential, distributed in the political configuration with a gradient pointing in the direction of decreasing conflict, and our study consists in the determination of optimal configurations. The process of delegation can be iterated and in this context a new type of solution called a delegated compromise is defined.

As announced in the title, our theory of compromise is qualitative; this is the case because from a formal point of view, the model falls within the field of discrete topology; both the representations and the delegations can be studied using the theory of homotopy in finite simplicial complexes, and the relevant concepts are introduced when needed. The category of simplicial complex is the right category needed in order to define meaningfully notions like viability or compatibility and friendly delegation.

The paper is organized as follows: In the following Section 2, we introduce the basic notion of a political structure which has the the mathematical structure of a simplicial complex, and we define the notion of a delegation, giving rise to representations and represented compromises. As we proceed, examples show how these notions apply to games and to network models of communication. In the following Section 3, the formalism is developed somewhat further so we can use results from homotopy theory of finite complexes to investigate delegation in political structures. The main results on the structure of represented compromises are presented in Section 4, which also points to some of the shortcomings. This leads to the consideration of the somewhat iterated form of delegation and its consequences, the weaker notion of delegated compromises, in Section 5. Finally, Section 6 contains some concluding comments.

2 Basic definitions

We consider a situation described by a nonempty ordered pair $(E, \mathcal{K})$ where E is a finite set and $\mathcal{K}$ is a collection of nonempty subsets of E. The elements of E represent the basic political entities and they are called agents. Subsets of E are called configurations and the elements of $\mathcal{K}$ are called viable configurations. In concrete cases, agents may be individuals, but they may as well be groups of individuals or even political issues or institutions. Agents are the basic forces or entities of the political situation, they are characterized collectively by the viability relations described by $\mathcal{K}$. In our context viability is interpreted as the possibility of coexistence; a configuration is viable if the different political agendas of its components are compatible, for instance a governance including the entities that form the configuration can live without being disrupted; a non viable configuration is conflictual. In this paper we assume that the set $\mathcal{K}$ of viable configuration is given. We shall give concrete examples of situations where the set of viable configurations comes as a result of political bids or strategic interaction. The use of the word "agent" must not be misinterpreted; in fact an agent is not endowed with any choice power or even preferences. The only driving motive will be the search for viability via the principle of decreasing conflict. Our purpose is to study if and how with this principle one can reach a viable configuration or compromise. In what follows we assume that there is given a set $\mathcal{K}$ of *viable* configurations, with the property that any singleton $\{x\}$, where $x \in E$, is viable, and any non empty subset of a viable configuration is viable, so that formally $\mathcal{K}$ is a simplicial complex with set of vertices E. The pair $(E, \mathcal{K})$ is called a *political structure.*

Since the viable configurations are given, our analysis starts at a possibly disrupted situation. In the extreme case where the all-embracing configuration E is viable, our model has nothing to tell. Below we give several examples of political structures by describing first the upstream interactions that lead to them. In the sequel the power set of any set X will be denoted $\mathcal{P}(X)$, and $\mathcal{P}(X) \setminus \{\emptyset\}$ will be denoted $\mathcal{P}_0(X)$.

Example 1 (Effectivity structures)**.** Let N and A be finite nonempty sets (of players and alternatives, respectively). An *effectivity structure* on (N, A) is a subset of $\mathcal{P}_0(N) \times \mathcal{P}_0(A)$, so that elements of E are pairs (S, B) with $S \subseteq N$, $B \subseteq A$. In the interpretation, elements of E are potential cases of exercising power: the coalition S can make sure that outcome of any decision made in the community must belong to B, or equivalently, S can prevent choices outside B. One can consider E as a (partially defined) effectivity function: that is $E : \mathcal{P}_0(N) \to \mathcal{P}(\mathcal{P}_0(A))$ where $E(S)$ stands for $\{B \in \mathcal{P}_0 A | (S, B) \in E\}$ (cf Abdou and Keiding [1]).

In order to make of E a political structure with elements (S, B) as agents we have to describe the viable configurations. For that purpose we need to specify the environment in which the power expressed by (S, B) may be exercised: Let $\mathcal{L}(A)$ denote the set of linear orders on A, and define (preference) profiles as maps $R : N \to \mathcal{L}(A)$, so that the set of profiles is $\mathcal{L}(A)^N$. If for some $i \in N$, the alternative a is (strictly) preferred to the alternative b in the profile R, then we write $a\, R_i^+\, b$, and for $a \in A$, the set of alternatives preferred to a by all members of S in the profile R is

$$P(a, S, R) = \{b \in A \mid b\, R_i^+\, a,\ \text{all } i \in S\}.$$

With this notation, we get that the coalition S is induced to exercise its objection power at the profile R if an alternative $c \in A$ is suggested such that $P(c, S, R)$ contains a subset B with $(S, B) \in E$.

Let $\mathcal{P} \subseteq \mathcal{L}(A)^N$. We shall say that a configuration $\emptyset \neq s \subseteq E$ is stable at $\mathcal{P}$ if it is always able to select an alternative against which none of its members will exercise their objection power, that is if for each profile $R \in \mathcal{P}$ there is some $a \in A$ such that B is not contained in $P(a, S, R)$ for some $(B, S) \in s$. Considering the configuration $s \subseteq E$ as an effectivity function, stability of s at $\mathcal{P}$ amounts to stability of s in the usual sense of the non-emptiness of the core for all profiles in $\mathcal{P}$ (see e.g. Moulin and Peleg [4], Abdou and Keiding [1]). It is easy to see that the family $\mathcal{K}$ of all stable configurations in E is indeed a simplicial complex, so that the ordered pair $(E, \mathcal{K})$ can be considered as a political structure. The *standard political structure associated* with $(E, \mathcal{P})$ is the one where the set of viable configurations is taken as the set of stable configurations.

There is another political structure that can be associated to $(E, \mathcal{P})$. Let $s \subseteq E$ and $(S, B) \in E$. (S, B) is said to be *absorbable* in s if there exists $(S', B') \in s$ such that $S \subseteq S'$ and $B \subseteq B'$. Denote by $\hat{s}$ the set of all (S, B) that are absorbable in s. s is said to be *strongly stable* at $\mathcal{P}$ if the set $\hat{s}$ is stable at $\mathcal{P}$. The *strong political structure associated* to $(E, \mathcal{P})$ is the one where viable sets are precisely strongly stable sets. It will be denoted $(E, \hat{\mathcal{K}})$. Remark that $s \in \hat{\mathcal{K}}$ if and only if $\hat{s} \in \mathcal{K}$. ○

Example 2 (TU games)**.** A TU (Transferable Utility) game is a pair (N, v), where N is a nonempty set of players and v is a map defined on the set $\mathcal{P}_0(N)$ of nonempty subsets of N (coalitions) that assigns to each $S \in \mathcal{P}_0(N)$ a number $v(S)$ interpreted as the money or utility gain that the coalition can obtain for its members. A payoff vector in (N, v) is a vector $x \in \mathbb{R}^N$. An feasible allocation is a payoff vector satisfying $\sum_{i \in N} x_i \leq v(N)$; let $V(N)$ be the set of all feasible allocations. For $S \in \mathcal{P}_0(N)$, let $A(S) = \left\{x \in V(N) \,\middle|\, \sum_{i \in S} x_i \geq v(S)\right\}$ be the set of feasible allocations that cannot be improved by the coalition S, in the sense that its members get at least as much as they could get from the coalition alone in case of secession. A family of coalitions is viable if there is a feasible allocation which cannot be improved by any coalition in the family. A political structure $(\mathcal{P}_0(N), \mathcal{K}_v)$ is defined by positing :

$$\mathcal{K}_v = \{\sigma \subseteq \mathcal{P}_0(N) \mid \cap_{S \in \sigma} A(S) \neq \emptyset\}.$$

Note that $A(S) \neq \emptyset$ and therefore the set of vertices is indeed $\mathcal{P}_0(N)$. Moreover $\mathcal{P}_0(N)$ belongs to $\mathcal{K}_v$ if and only if there is a payoff vector which cannot be improved by any coalition, equivalently if and only if the *core* of (N, v) is nonempty. A closer study of $\mathcal{K}_v$ may however be helpful in the nontrivial case where $\mathcal{P}_0(N) \notin \mathcal{K}_v$. It should be noticed that here, the agents are *coalitions*, and configurations are families of coalitions. ◯

Since our overall purpose is the investigation of conflictual situations and their possible resolution, we are mainly interested in ways in which to eliminate non-viable configurations. Our approach will be to allow agents to delegate their influence to other, more centrally placed agents. In our present setup we formulate this by the notion of *delegation.*

Definition 1. Let $y \in E$ and $d \subseteq E$, $y \notin d$. A *delegation* from d to y is a map $\delta_d : E \to E$ such that $\delta_d(x) = y$ if $x \in d$ and $\delta_{d \to y}(x) = x$ if $x \notin d$. A *simple delegation* is a delegation $\delta_{d \to y}$ where d is a singleton, $d = \{x\}$, and it is written as $\delta_{x \to y}$.

A *delegation* $\delta_{d \to y}$ is said to be *friendly* if $s \cup \{y\} \in \mathcal{K}$ for all $s \in \mathcal{K}$ such that $s \cap d \neq \emptyset$.

A delegation from d to y can be seen as a political action by which the agents in d withdraw from the political interaction in favor of the agent y. A delegation is friendly whenever any agent in d can ensure that every viable configuration to which she participates remains viable after delegation, and moreover, the agent receiving the delegation is present in every configuration which contained the delegating agent. A withdrawal in favor of a delegate is more likely to happen if the delegation is friendly. In our analysis, the possibility of friendly delegations constitutes the only driving principle behind potential moves that reduce conflicts in a political structure.

Example 3 (Networks)**.** The use of formal networks has a long history in the social sciences, with its beginning in the 1930s; for a survey of its history, see e.g. [12]. A social structure is a graph $\mathcal{G} = (V, \mathcal{E})$, where V is a finite set of vertices and $\mathcal{E}$ is a set of two-element subsets $\{v_0, v_1\}$ of $\mathcal{E}$. In the interpretations, V are individuals, and two individuals v_0, v_1 are socially connected if $\{v_0, v_1\} \in \mathcal{E}$. A graph $(V, \mathcal{E})$ can be seen as a simplicial complex $(V, \mathcal{K})$ where $s \in \mathcal{K}$ if and only if s is a nonempty subset of an element of $\mathcal{E}$. This amounts to a simplicial complex where $\mathcal{E}$ is the set of maximal simplexes. Equivalently a graph is a simplicial complex where simplices have dimension 0 or 1. The *degree* of a vertex v is the number of edges with extremity v.

There is a simple friendly delegation from x to y only if $(x, y) \in E$ and if for any z such that $(x, z) \in E$, one has $y = z$ otherwise $\{x, y, z\}$ is a simplex, a contradiction. It follows that an agent can friendly delegate if and only if it has degree 1. On the other hand if the x and y where x delegates friendly to y, have both degree 1, then it follows that there is also a friendly delegation from y to x and this can only happen if $\{x, y\}$ is a connected component of the graph. We conclude that for a connected graph with $|V| \geq 3$ any delegate have degree ≥ 2. ◯

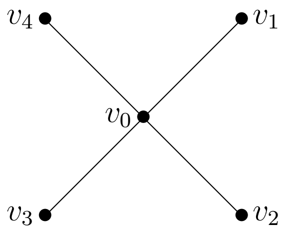

Figure 1: Point centrality in a network: The vertex v_0 with degree 4 is connected to all the other vertices with an edge, whereas the other vertices are connected only to v_0.

The following property of delegations is immediate.

Proposition 1. *For any* $\emptyset \neq d = \{x_1, \ldots, x_n\} \subseteq E$ *and* $y \notin d$*, if* $\delta_{d\to y}$ *is a delegation, then*

$$\delta_{d\to y} = \delta_{x_1\to y} \circ \cdots \circ \delta_{x_k\to y}.$$

Moreover, $\delta_{d\to y}$ *is friendly if and only if* $\delta_{x_j\to y}$ *is friendly for* $j \in \{1, \ldots, n\}$. □

As a result of the delegations, there may be fewer agents with conflicting interests. We introduce the notion of a compromise as a counterpart of real world solutions to political conflicts, where agents (or issues, or power groups) will have to step aside since they will not be accepted as partners in a political deal but on the other hand can be represented by other, more acceptable, agents.

Definition 2. A nonempty subset F of E is a *representation* of E if for any $x \notin F$, there exists some $y \in F$ such that $\delta_{x\to y}$ is a friendly delegation.

Example 4 (Empirical case)**.** A recent stalemate in Lebanese politics and its resolution through compromise provides an illustration of the concepts introduced above. The political configuration inherited by the Lebanese society after the Taef Agreement (October 1989) [15], that put an end to the civil war, prevailed in its principal components until the assassination of Prime Minister Rafic Hariri (February 2005).

The polarizing issue that divided political forces can be described schematically as East-West, although viability is not exclusively determined by this issue. East is a shorthand for a political view supported by neighboring countries Iran and Syria and locally by the Islamic Resistance, while West is a shorthand for US-Saudi-Israeli-led policy supported locally by the Future Movement. We assume that the viability of the agendas of the local forces was mainly dictated by that issue. Independently of viability, the National Pact, partially renewed by the Taef Agreement, requires that all the three major confessions participate in the governance. The complexity of modern Lebanese politics is widely viewed as a result of the entanglement of these two levels of

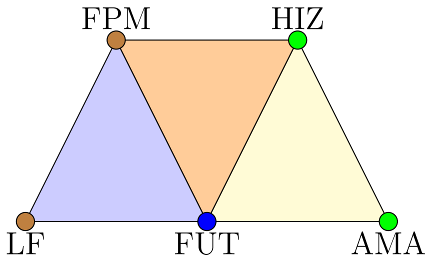

Figure 2: Friendly delegation and political compromise in Lebanese politics

political interaction. This configuration can be described by the simplicial complex of Fig.2. The major forces present in the conflict are the Future Movement (FUT), AMAL (AMA), Hezbollah (HIZ), Free Patriotic Movement (FPM), and Lebanese Forces (LF). In Fig.2, brown color has been assigned to Christians (Maronites), blue to Sunnis, and green to Shias. Shaded triangles represent the maximal viable configurations (simplices). Viability of a configuration means that its components can coexist in some form of governance. A non viable configuration cannot form a stable government because clashes would soon occur and block the main institutions.

The clearest expression of the deadlock that we are analyzing was the vacancy of the Lebanese Presidency after the end of the term of President Michel Suleiman (24 May 2014): Lebanon had no President after 25 May 2014, and no consensus existed about who would be elected as President of the Lebanese Republic. It is very instructive to understand why a consensus is needed to elect a president in the Lebanese political configuration. In fact the parliament convened for the purpose of electing a President, but there were 45 failed attempts to achieve a parliamentary quorum. No law has been broken regarding the constitution, and it is clear that the failure could be repeated indefinitely since a Parliament Member has the right not to show up for a meeting. In many countries the constitution provides for this type of failure by requiring weaker conditions for convening after a few failed attempts with the required quorum. But the Lebanese constitution does not allow such a provision. The country entered in a deadlock with no solution dictated by the law in sight. If the quorum was to be reached there must be a consensus about the name of the President before the meeting of the Parliament.

After a long search, a compromise was reached that respects the National Pact, as represented by the middle shaded triangle. Presumably this compromise is the result of a friendly delegation from LF to either FPM or FUT, and from AMA to either FUT or HIZ. This compromise was formally implemented by the election of Michel Aoun as President (October 2016) [16] and the formation of a government by Saad Hariri (December 2016). ◯

In the following we shall study delegation and compromise using tools of finite homotopy theory, and for this purpose it is convenient to consider delegation from another angle, emphasizing that agents are removed from direct participation.

Definition 3. Let $\emptyset \neq F \subseteq E$. A *retraction* to F is a map $r : E \to E$ such that $r(E) = F$ and $r(x) = x$ for all $x \in F$. F is said to be a *retract* if there is a retraction to F.

A retraction r is *friendly* if $x \in s \in \mathcal{K}$ implies $\{r(x)\} \cup s \in \mathcal{K}$, and a *friendly retract* is a retract F such that there exists a friendly retraction to F.

A retraction is a political action by which the conflictual political structure $(E, \mathcal{K})$ is reduced to the (sub) configuration $(F, \mathcal{K}_F)$, withdrawing agents not in F in favor of suitable agents in F, their representatives. A retraction is friendly if any viable configuration containing a retracting agent as a member remains viable when he is replaced by his representative. Clearly, a delegation $\delta_{d\to y}$ is a retraction to $E\backslash d$, and a retraction r to F such that $r(E\backslash F) = \{y\}$ is a delegation from $E\backslash F$ to y. We have the following characterization:

Proposition 2. *$F \subseteq E$ is a representation if and only if F is a friendly retract.*

Proof: Let F be a representation. Define a map $r : E \to E$ as follows: r is the identity on F, and if $x \notin F$ then put $r(x) = y$ where y is some element of F such that there is a friendly delegation from x to y. It is clear that r is a retraction on F. Moreover if $x \notin F$ for any $s \in \mathcal{K}$ such that $x \in s$, we have that$\{r(x)\} \cup s \in \mathcal{K}$; and for any $x \in F$ and $s \in \mathcal{K}$ such that $x \in s$, we have that $\{r(x)\} \cup s = s \in \mathcal{K}$. The converse is straightforward. □

The notion of friendly retraction characterizes a representation as the result of withdrawal of some agents, in what follows we consider whether a representation can result from of a sequential process of delegations.

Definition 4. A *progressive delegation* is a sequence $\delta_{x_1\to y_1}, \dots, \delta_{x_p\to y_p}$ of simple delegations where the elements $x_1, \cdots, x_p$ are distinct and $y_k \notin \{x_1, \dots, x_k\}$ for all $k = 1, \dots, p$. A progressive delegation is *friendly* if each of the simple delegations that compose it is friendly. The configuration $c = \delta_{x_p\to y_p} \circ \cdots \circ \delta_{x_1\to y_1}(E)$ is called the *outcome* of the progressive delegation.

Thus, when performing a progressive delegation, agents are successively delegating their influence to other agents, and in this case it seems natural that once an agent has renounced on influence through delegation, she cannot herself be an object of delegation.

Proposition 3. *Let $\delta_{x_1\to y_1}, \dots, \delta_{x_p\to y_p}$ be a progressive delegation with $\phi = \delta_{x_p\to y_p} \circ \cdots \circ \delta_{x_1\to y_1}$. Then ϕ is a retraction to $E \setminus \{x_1, \dots, x_p\}$, and*

$$\mathrm{fix}_\phi = \{x \in E \mid \phi(x) = x\} = \phi(E) = E \setminus \{x_1, \dots, x_p\} \neq \emptyset. \tag{1}$$

Conversely, any retraction $\phi : E \to E$ to some $F \subseteq E$ can be obtained through a progressive delegation. Moreover all the involved delegations can be chosen to be friendly if and only if the retraction is friendly.

Proof: (i) Put $F = E \setminus \{x_1, \ldots, x_p\}$. It is clear that $F \subseteq \mathrm{fix}_\phi$. Conversely, for any $k \in \{1, \ldots, p\}$ $\phi(x_k) = y_\ell$ with $\ell \geq k$ and by progressivity, $y_\ell \neq x_k$, so that $\mathrm{fix}_\phi \subseteq F$ and consequently $\mathrm{fix}_\phi = F$.

Now, $F \subseteq \phi(E)$. In order to prove the opposite inclusion, let $x \in E$. Then either $x \in F$ and therefore $\phi(x) = x \in F$, or $x = x_k$ and $\phi(x) = y_\ell$ for $\ell \geq k$. It follows that $y_\ell \neq x_{k'}$ for $k' \geq \ell$ and by progressivity of the sequence $y_\ell \neq x_{k'}$ for $k' < \ell$ and finally $y_\ell \in F$. We conclude that $F = \phi(E)$.

For any $k \in \{1, \ldots, p\}$ let $\phi_k = \delta_{x_k \to y_k} \circ \cdots \circ \delta_{x_1 \to y_1}$ and $F_k = E \setminus \{x_1, \ldots, x_k\}$. By convention $F_0 = E$ and $\phi_0 = \mathrm{Id}_E$. Then, by the first part of the proof ϕ_k is a retraction onto F_k and one has $\phi_k = \delta_{x_k \to y_k} \circ \phi_{k-1}$. We prove by induction that ϕ_k is friendly. Clearly $\phi_0 = \mathrm{Id}_E$ is friendly. Assume that ϕ_{k-1} is friendly. If $x \in s \in \mathcal{K}$ then $\{\phi_{k-1}(x)\} \cup s \in \mathcal{K}$ by the induction hypothesis and $C \equiv \{\delta_{x_k \to y_k}(\phi_{k-1}(x))\} \cup (\phi_{k-1}(x) \cup s) \in \mathcal{K}$ since $\delta_{x_k \to y_k}$ is a friendly delegation. But $\{\phi_k(x)\} \cup s$ is a subset of C, therefore $\{\phi_k(x)\} \cup s$ is a simplex. Thus our claim is proved.

For the converse implication, if ϕ is a retraction to F and $E \backslash F = \{x_1, \ldots, x_p\}$, we consider $\delta_{x_1 \to x_p}, \ldots, \delta_{x_p \to y_p}$ where $y_k = \phi(x_k)$ for $k = 1, \ldots, p$. It is easily seen that $\phi = f_{x_p \to y_p} \circ \cdots \circ \delta_{x_1 \to y_1}$. For any $x \in E$ such that $x \neq \phi(x)$ one has $x = x_k$ for some k, so that $\delta_{x_\ell \to y_\ell}(x) = \phi(x)$ if $\ell = k$ and $\delta_{x_\ell \to y_\ell}(x) = x$ if $\ell \neq k$. It is easily seen that if $x \in s \in \mathcal{K}$, one has $s \cup \delta_{x_\ell \to y_k}(x) \subseteq s \cup \phi(x)$. It follows that $\delta_{x_k \to y_k}$ is friendly if ϕ is friendly. □

Corollary 4. *$F \subseteq E$ is a representation if and only if there exists a friendly progressive delegation with outcome F.*

3 Topology of representations

Let $\mathcal{K}$ be a simplicial complex on a set E and let $s = \{x_0, \ldots, x_r\}$ be a simplex in $\mathcal{K}$. The *closed simplex* $\overline{s}$ is the set of formal convex combinations $\sum_{i=0}^{r} \lambda_i x_i$ with $\lambda_i \geq 0$ for $i = 0, \ldots, r$ and $\sum_{i=0}^{r} \lambda_i = 1$. Each closed simplex is a metric space under the metric based on Euclidean distance. The *geometric realization* of $|\mathcal{K}|$ of the simplicial complex $\mathcal{K}$ is the union of all closed simplices $\overline{s}$ for $s \in \mathcal{K}$ endowed with the topology for which $U \subseteq |\mathcal{K}|$ is open if $U \cap \overline{s}$ is open in $\overline{s}$ for each $s \in \mathcal{K}$. Given simplicial complexes $\mathcal{K}$ on a set E and $\mathcal{L}$ on F, a map $\phi : E \to F$ is *simplicial* if it takes simplices in $\mathcal{K}$ to simplices in $\mathcal{L}$, that is if $\phi(s) \in \mathcal{L}$ for each $s \in \mathcal{K}$, or, otherwise put, if it extends to a map $\phi : \mathcal{K} \to \mathcal{L}$. We associate to φ the map $|\varphi| : |\mathcal{K}| \to |\mathcal{L}|$ obtained from φ by linear extension.

It is easily seen that friendly retractions are simplicial.

Two simplicial maps $\varphi, \psi : (E, \mathcal{K}) \to (F, \mathcal{L})$ are said to be *contiguous* if for any $s \in \mathcal{K}$, $\phi(s) \cup \psi(s) \in \mathcal{L}$. Denote by $\approx$ the contiguity relation in E. $\approx$ is symmetric but generally not transitive. Let $\sim$ be the transitive closure of $\approx$. By definition, $f \sim g$ if there exists a sequence $f = f_0, \ldots, f_p = g$ such that $f_k \approx f_{k+1}$ for $k = 0, \ldots, p-1$. Then $\sim$ is transitive and symmetric, hence an equivalence relation, the classes of which are called *contiguity classes*. Moreover $\approx$ and therefore also $\sim$ are compatible with composition. A simplicial map $\varphi : (E, \mathcal{K}) \to (F, \mathcal{L})$ is a *strong equivalence* if there exists $\psi : (F, \mathcal{L}) \to (E, \mathcal{K})$ such that $\psi \circ \varphi \sim \mathrm{Id}_E$ and $\varphi \circ \psi \sim \mathrm{Id}_F$. It is worthwhile noting that two simplicial maps that have the same contiguity class are homotopic, but that the converse is not true (cf. e.g. Spanier [14] Chap. 3).

Let $(E, \mathcal{K})$ be a political structure. For any configuration F we define $\mathcal{K}_F$ as the set of all elements of $\mathcal{K}$ that are included in F. It is clear that $(F, \mathcal{K}_F)$ is a political structure, the agents of which are the elements of F. Moreover $\mathcal{K}_F$ is a full subcomplex of $\mathcal{K}$. Clearly one has $\mathcal{K} = \mathcal{K}_E$.

Lemma 5. *A map $\varphi : E \to E$ is contiguous to Id_E if and only if for all $x \in s \in \mathcal{K}$ one has $\{\varphi(x)\} \cup s \in \mathcal{K}$.*

Proof. Assume that φ has the announced property, and $\{x_1, \ldots x_p\} = s \in \mathcal{K}$. Then by induction on k, $s \cup \{\varphi(x_1), \ldots, \varphi(x_k)\} = s \cup \{\varphi(x_1), \cdots, \varphi(x_{k-1})\} \cup \{\varphi(x_k)\}$ is in $\mathcal{K}$, therefore $s \cup \varphi(s) \in \mathcal{K}$ and φ is contiguous to Id_E. Conversely If φ is contiguous to Id_E and if $x \in s \in \mathcal{K}$ then $\{\varphi(x)\} \cup s \subseteq \varphi(s) \cup s \in \mathcal{K}$ □

In particular we have the following:

Proposition 6. *A retraction is friendly if and only if it is contiguous to* Id_E.

Remark 1. To a retraction r to F we associate the map $r_F : E \to F$ which is just r with restricted range F. If $i : F \to E$ is the inclusion map, then by definition $r_F \circ i = \mathrm{Id}_F$. Conversely, any $r_F : E \to F$ such that $r_F \circ i = \mathrm{Id}_F$ gives rise to the retraction r by putting $r = i \circ r_F$. We use the same terminology of a retraction for r_F when no confusion is possible.

In view of Proposition 2 and Lemma 5, we can have the following characterization of representations:

Proposition 7. *F is a representation of $(E, \mathcal{K})$ if and only if $r_F : (E, \mathcal{K}) \to (F, \mathcal{K}_F)$ is a strong equivalence.*

Proof: In view of Lemma 5, r is friendly if and only if r is contiguous to Id_E, and in view of the Remark 1 this happens if and only if

$$i \circ r_F = r \sim \mathrm{Id}_E.$$

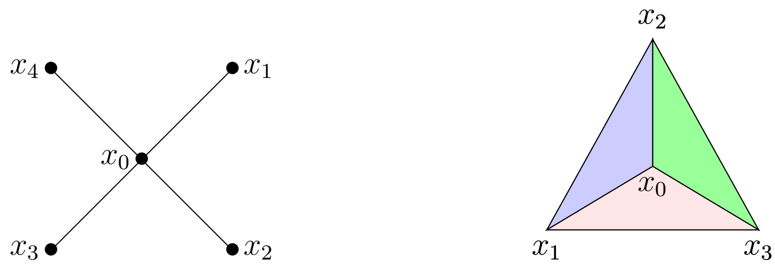

Figure 3: Political structures represented as points connected by lines (1-simplices) or triangles (2-simplices). In both of the above political structures, the retraction onto x_0 is contiguous to the identity. Consequently $\{x_0\}$ is a representation.

Since $r_F \circ i = \mathrm{Id}_F$, this implies that r_F is a strong equivalence. □

The set of simple friendly delegations in a political structure can be used to define a binary relation $\rhd$ on E as follows,

$$x, y \in E, x \rhd y \text{ if either } x = y \text{ or the delegation } \delta_{x \to y} \text{ is friendly.}$$

Lemma 8. *For any $x \in E$, let σ_x be the set of all maximal simplices containing x. Then $x \rhd y$ if and only if $\sigma_x \subseteq \sigma_y$.*

Proof: If $x \rhd y$ and $x \neq y$, then there is a friendly delegation $\delta_{x \to y}$, so that every maximal simplex containing x contains y as well. Conversely, if $\sigma_x \subseteq \sigma_y$, then the map δ taking x to y and leaving all other elements of E unchanged is a friendly delegation. □

It follows from Lemma 8 that that $\rhd$ is transitive and reflexive so that $(E, \rhd)$ is a preordered set.

Minimal representations. As usual it is of special interest to investigate the simplest possible cases of a particular object, in this case a representation.

Definition 5. A representation F of $(E, \mathcal{K})$ is said to be *minimal* if there is no representation that is strictly included in F.

Proposition 9. *A representation F of $(E, \mathcal{K})$ is minimal if and only if for any $x, y \in F$, $x \rhd y$ implies $x = y$.*

Proof: If F is not minimal for inclusion then there exists some $F' \subseteq F$, $x \in F \setminus F'$, and $y \in F'$ such that $x \rhd y$. Conversely If there exists $x, y \in F$ $x \neq y$ such that $x \rhd y$ then $F' = F \setminus \{x\}$ is easily seen to be a representation contradicting minimality of F. □

Lemma 10. *Let $\emptyset \neq F \subseteq E$ be arbitrary.*

(i) If $x \rhd y$ for all $x, y \in F$, then $F \in \mathcal{K}$.

(ii) $E \in \mathcal{K}$ if and only if for any $x, y \in E$, $x \rhd y$.

Proof. Assume that $F \notin \mathcal{K}$ and let s be a maximal simplex included in F. Since $s \neq F$ there exists some $y \in F, y \notin s$. Let $x \in s$. Since, by the maximality of s, $s \cup \{y\} \notin \mathcal{K}$ one has that $\delta_{x \to y}$ is not a friendly delegation. This proves (i). (ii) follows immediately from (i) □

One may exploit constructions developed for simplicial complexes to obtain a measure of the possibility of delegation. For this purpose, we introduce (cf. Barmak [2]) the *nerve* $\mathcal{N}(\mathcal{K})$ of $\mathcal{K}$ as the simplicial complex in which the set of vertices is the set of maximal simplices in $\mathcal{K}$, say $\mathcal{K}^M$, and the set of simplices is the set of subsets $\{s_0, \dots, s_r\} \subseteq \mathcal{K}^M$ such that $\cap_{i=0}^r s_i \neq \emptyset$. Repeating the construction, one gets $\mathcal{N}^2(\mathcal{K}) = \mathcal{N}(\mathcal{N}(\mathcal{K}))$ and in generel, $\mathcal{N}^k(K_E) = \mathcal{N}(\mathcal{N}^{k-1}(\mathcal{K}))$.

We consider the second nerve $\mathcal{N}^2(\mathcal{K})$ in more detail since it plays a role in the characterization of minimal representations. A vertex in $\mathcal{N}^2(\mathcal{K})$ is a maximal set $\sigma = \{s_0, \dots, s_r\}$ of maximal simplices in $\mathcal{K}$ such that $\cap_{i=0}^r s_i \neq \emptyset$. Let Φ be the set of all maps φ from $\mathcal{N}^2(\mathcal{K})$ to E assigning to $\sigma = \{s_0, \dots, s_r\}$, a vertex of $\mathcal{N}^2(\mathcal{K})$, some element of $\cap_{i=0}^r s_i$. Notice that the set Φ is nonempty since $\cap_{s \in \sigma} s \neq \emptyset$ for every $\sigma \in \mathcal{N}^2(\mathcal{K})$.

Proposition 11. *The set Φ has the following properties:*

(i) Any $\varphi \in \Phi$ is a simplicial map from $\mathcal{N}^2(\mathcal{K})$ to $\mathcal{K}$ and an isomorphism onto its image.

(ii) For any $\varphi \in \Phi$ the image $\varphi(\mathcal{N}^2(\mathcal{K}))$ is a minimal representation.

(iii) For each minimal representation $F \subseteq E$, there is $\varphi \in \Phi$ such that $F = \varphi(\mathcal{N}^2(\mathcal{K}))$.

Proof: (i) Let $\varphi \in \Phi$. If $\{\sigma_0, \dots, \sigma_k\}$ is a simplex in $\mathcal{N}^2(\mathcal{K})$, then $\cap_{i=0}^k \sigma_i \neq \emptyset$, so that there exists a maximal simplex s in $\mathcal{K}$ which belong to all the sets $\sigma_0, \dots, \sigma_k$. By its definition, $\varphi(\sigma_i)$ belongs to s for $i = 0, \dots, k$, therefore $\{\varphi(\sigma_1), \dots, \varphi(\sigma_k)\}$ is a simplex in $\mathcal{K}$. This shows that φ is a simplicial map.

Next, we show that φ is an injective map. Suppose that $\varphi(\sigma_1) = \varphi(\sigma_2) = x$. If $\sigma_1 = (s_0^1, \dots, s_r^1)$, $\sigma_2 = (s_0^2, \dots, s_p^2)$, then by our construction $x \in s_i^1$ for $i = 0, 1, \dots, r$ and $x \in s_i^2$ for $i = 0, 1, \dots, p$, so that $\sigma_1 \cup \sigma_2$ is a family of maximal simplices from $\mathcal{K}$ with nonempty intersection. Since σ^1 and σ^2 are already maximal collections of simplices from $\mathcal{K}$ with nonempty intersection, we have that $\sigma^1 = \sigma^2$, so that f is indeed injective.

Finally we prove that the inverse map φ^{-1} defined on the image F of φ is simplicial. Let $s = \{x_1, \dots, x_k\} \subseteq F$ be a simplex (i.e. an element of $\mathcal{K}_F$) and let $\sigma_i = \varphi^{-1}(x_i)$ $i = 1, \dots k$. Denote by s' a maximal element of $\mathcal{K}$ containing s. For any $i \in \{1, \dots, n]$, any element of σ_i has x_i as a member, therefore any element of $\sigma_i \cup \{s'\}$ has x_i as a member. In particular $\sigma_i \cup \{s'\}$ is a simplex in $\mathcal{N}(\mathcal{K})$. Since σ_i is maximal, we have that $s' \in \sigma_i$. Since every σ_i $i = 1, \dots, k$, contains s' it follows that $\{\sigma_1, \dots, \sigma_k\}$ is a simplex.

(ii) Let $\varphi \in \Phi$ and let F be the image by φ of the vertices of $\mathcal{N}^2(\mathcal{K})$. For any $x \in E$, let σ_x be the set of all maximal simplices in $\mathcal{K}$ containing x. σ_x is thus a simplex of $\mathcal{N}(\mathcal{K})$. If $x \notin F$, let σ be a maximal element of $\mathcal{N}(\mathcal{K})$ containing σ_x, and let $y = \varphi(\sigma)$. Then $\sigma_x \subseteq \sigma = \sigma_y$, so that by Lemma 8, $x \rhd y$. This proves that F is a representation. If $x \in F$, there exists σ maximal such that $x = \varphi(\sigma)$ so that $\sigma \subseteq \sigma_x$ and therefore $\sigma = \sigma_x$. Similarly, if $y \in F$ then σ_y is maximal and $\varphi(\sigma_y) = y$, but if $x \rhd y$, then $\sigma_x \subseteq \sigma_y$, and by maximality of σ_x one has $\sigma_x = \sigma_y$ and therefore $y = x$. It follows from Proposition 9 that F is a minimal representation.

(iii) Conversely, let F be a minimal representation. We claim that for any $y \in F$ the set σ_y is a maximal element of $\mathcal{N}^2(\mathcal{K})$. Suppose not; then there must be $x \neq y$ with $\sigma_y \subseteq \sigma_x$. By Lemma 8, $y \rhd x$, and by Proposition 9, $x \notin F$. But then there must be $z \in F$, such that $x \rhd z$. By transitivity of $\rhd$ we get that $y \rhd z$, and by Proposition 9 $y = z$. It follows that $\sigma_x \subseteq \sigma_y$, a contradiction. We conclude that σ_y is indeed maximal for each $y \in F$.

Remark that for any maximal σ one has $\sigma_x = \sigma$ for all $x \in \cap_{s \in \sigma} s$. We claim that there exists some $x \in F$ such that $\sigma = \sigma_x$. If not then $\sigma = \sigma_y$ only for some $y \notin F$. But $y \rhd z$ for some $z \in F$ and therefore $\sigma \subseteq \sigma_z$ and since σ is maximal $\sigma_z = \sigma$, a contradiction. In fact such an x is unique since $\sigma_x = \sigma_y$ where $x, y \in F$ implies $x = y$ (Proposition 9). One therefore may define φ by putting $\varphi(\sigma) = x$ where x is the unique F such that $\sigma = \sigma_x$. By our construction, φ belongs to Φ and by the preceding paragraph every element of F is an image of some maximal σ, that is $\varphi(\mathcal{N}^2(\mathcal{K})) = F$. □

It follows from Proposition 11 that there is a bijection between Φ and the set of minimal representations of $(E, \mathcal{K})$.

Irreducible political structures. We conclude this section by a study of the particular case where no friendly delegations exist.

Definition 6. A political structure $(E, \mathcal{K})$ is *irreducible* if it has no friendly delegation.

Example 5. Let $E = \{1, \dots, n\}$ and $\mathcal{K} = \mathcal{P}_0(E) \setminus \{E\}$. Then $(E, \mathcal{K})$ is irreducible. Indeed if $x \neq y$ a delegation from x to y cannot be friendly, otherwise $x \in E \setminus \{y\} \in \mathcal{K}$ would imply $E = E \setminus \{y\} \cup \{y\} \in \mathcal{K}$, a contradiction. In Example 1, a configuration E is irreducible if and only if E is not stable but any proper subset of E is stable. ◯

It is clear that $(E, \mathcal{K})$ is irreducible if and only if its only friendly retract is the identity or equivalently if E is a minimal representation in $(E, \mathcal{K})$. The following result can be found in Barmak [2], Chap.5.

Lemma 12. *Let $(E, \mathcal{K})$ be irreducible and let $f : E \to E$ be contiguous to* Id_E. *Then* $f = \mathrm{Id}$.

Proof: Let $x \in E$ and let s be a maximal simplex containing x. Then $f(s) \cup s$ is a simplex, and $f(x) \in f(s) \cup s = s$, where the last equality follows from the maximality of s. Thus, every maximal simplex containing x contains $f(x)$ as well. If $x \neq f(x)$, there would be a friendly delegation $\delta_{x \to f(x)}$ and $E \backslash \{x\}$ contradicting the irreducibility. We conclude that $f = \mathrm{Id}_E$. □

The lemma has a useful consequence, which is stated below as a corollary.

Corollary 13. *A strong equivalence between two irreducible political structures* $(F_1, \mathcal{K}_{F_1})$ *and* $(F_2, \mathcal{K}_{F_2})$ *is an isomorphism.*

As a corollary of Proposition 11 we have :

Corollary 14. *A political structure* $(E, \mathcal{K})$ *is irreducible if and only if it is isomorphic to* $\mathcal{N}^2(\mathcal{K})$.

In particular if $(E, \mathcal{K})$ is irreducible then Φ is a singleton.

4 Represented compromises

Definition 7. Let $(E, \mathcal{K})$ be a political structure. A subset F of E is a *represented compromise* (shorthand: an *R-compromise*) for $(E, \mathcal{K})$ if F is a representation and F is viable.

A *minimal R-compromise* is a compromise that does not contain a strictly smaller R-compromise for inclusion.

A *pure R-compromise* is an element $y \in E$ such that $\{y\}$ is an R-compromise.

In this section we are interested in the structure of R-compromises and the existence of such solutions.

Lemma 15. *If* Z *is a representation and if* $Z \subseteq Z'$ *then* Z' *is representation.*

Proof: (i) Let r be a friendly retraction to Z, and let r' be the map defined by :

$$r'(x) = \begin{cases} r(x) & x \in E \backslash Z', \\ x & x \in Z'. \end{cases}$$

Clearly r' is a retraction to Z'. Moreover for any $x \in s \in \mathcal{K}$, either $x \in E \backslash Z'$ and $s \cup \{r('x)\} = s \cup \{r(x)\} \in \mathcal{K}$, or $x \in F$ and $s \cup \{r('x)\} = s \cup \{x\} = s \in \mathcal{K}$. We conclude that r is friendly. □

Corollary 16. *Any viable configuration that includes an R-compromise is an R-compromise.*

Remark 2. It is not true that a subset of an R-compromise is an R-compromise. Figure 4 presents an example where there is a represented compromise but where there is no pure represented compromise.

Let $\mathfrak{R}_E$ be the set of all R-compromises and let R_E be the set of all pure R-compromises.

Proposition 17. *If $\mathfrak{R}_E \neq \emptyset$ then minimal R-compromises are exactly minimal R-representations.*

Proof: Let F be a minimal R-compromise and let $F' \subseteq$ be a representation, then $F' \in \mathcal{K}$ and therefore F' is an R-compromise. Sine F is minimal $F' = F$ so that F is a minimal representation. Conversely let F be a minimal representation and let C be an R-compromise (by assumption there is one). Then C contains a minimal R-compromise, say C', which by the first part of the proof is also a minimal representation. In view of Proposition 11, F and C' are isomorphic. In particular $F \in \mathcal{K}$. We conclude that F is also an R-compromise, and even a minimal one. □

Proposition 18. *If $R_E \neq \emptyset$, then $R_E \in \mathcal{K}$, that is the set of all pure R-compromises is an R-compromise.*

Proof: Assume that R_E is nonempty, and let s be a maximal (for inclusion) subset of R_E that also belongs to $\mathcal{K}$. Let $x \in R_E$ be arbitrary, and let r be the retraction to $\{x\}$. Since r is contiguous to Id_E, one has that $r(s) \cup s$ is a simplex in $\mathcal{K}$. Since $r(s) = \{x\}$, it follows that $\{x\} \cup s$ is a simplex contained in R_E. By the maximality of s, we have that $x \in s$. Since x is arbitrary we conclude that $s = R_E$. □

Corollary 19. *$R_E = E$ if and only if $E \in \mathcal{K}$.*

The following result gives some more information about the structure of the R-compromises even in the absence of pure R-compromises. It is interesting politically in the search for an inclusive R-compromise. It shows that when an R-compromise exists then there is one that intersect all R-compromises.

Proposition 20. *Let $\mathcal{I}$ be an arbitrary subset of $\mathfrak{R}_E$. Then there exists a subset of $\cup_{F\in\mathcal{I}}F$ which belongs to $\mathfrak{R}_E$ and intersects all members of $\mathcal{I}$.*

In particular, if $\mathfrak{R}_E \neq \emptyset$, then there exists an element of $\mathfrak{R}_E$ that intersects all elements of $\mathfrak{R}_E$.

Proof: Let $F \in \mathcal{I}$ and denote by r_F a friendly retraction on F. The collection of simplices included in $\cup_{F'\in\mathcal{I}}F'$ and containing F is nonempty and has a maximal (for inclusion) element which we denote c. Let $F' \in \mathcal{I}$ and let r be a retraction to F' that is contiguous to Id_E, so that $r(c) \cup c$ is a simplex. Since $r(c) \cup c \subseteq \cup_{F'\in\mathcal{I}}F'$, it follows from the maximality of c that $r(c) \subseteq c$. Since $\emptyset \neq r(c) \subseteq F'$, we conclude that $c \cap F' \neq \emptyset$. Now we prove that c is a friendly retract. Let r_c be defined by:

$$r_c(x) = \begin{cases} x & x \in c \\ r_F(x) & x \notin c. \end{cases}$$

If $s \in \mathcal{K}$, then $r_c(s) \cup s = r_c(s \backslash c) \cup s \subseteq r_F(s) \cup s \in \mathcal{K}$. We conclude that r_c is contiguous to Id_E, so that $c \in \mathfrak{R}_E$. □

Example 6 (Networks continued)**.** Let $(V, \mathcal{K})$ be the political structure of a graph $(V, \mathcal{E})$. We want to find those political structures, of this category, that have an R-compromise. Assume that the graph is connected and $|V| \geq 3$ then, by the analysis conducted in Example 3, it has a unique minimal representation C composed by all vertices of degree ≥ 2. If $(V, \mathcal{K})$ has an R-compromise, then C is a minimal R-compromise (as follows from Proposition 17). Therefore either C is a singleton $\{x\}$, or $C = \{x_0, y_0\}$ and $(x_0, y_0) \in \mathcal{E}$. It follows that such a graph is of the type that we now describe (see figure 4) :

Type Γ : $\Gamma(x_0, y_0, x_1, \ldots x_m, y_1, \ldots y_n)$ $m \geq 0, n \geq 0$ where $\mathcal{E} = \{(x_0, x_1), \ldots (x_0, x_m)\} \cup \{(x_0, y_0)\} \cup \{(y_0, y_1), \ldots (y_0, y_n)\}$

By convention the graph with a single node (and no edges) will be called *trivial.*

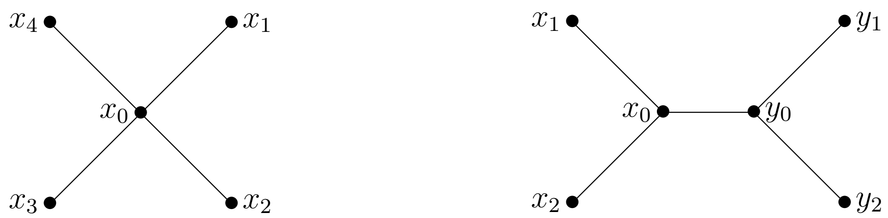

Figure 4: The graph in the left panel represents the class of graphs with a pure R-compromise, and the one in the right panel represents those with an R-compromise containing two agents. Any graphs with a nonempty R-compromise belongs to one of the two classes.

Proposition 21. *The political structure of a non trivial graph* $(V, \mathcal{E})$ *has an R-compromise if and only if it is of type* Γ.

Proof. It is clear that any graph of type Γ has a minimal representation: if $m \geq 1$ and $n \geq 1$ then $\{x_0, y_0\}$ is the unique minimal representation; if $m = 0$ and $n \geq 1$ then $\{y_0\}$ is the unique minimal representation; if $m \geq 1$ and $n = 0$ then $\{x_0\}$ is the unique minimal representation, and finally if $m = 0$ and $n = 0$ then both $\{x_0\}$ and $\{y_0\}$ are two minimal representations. We see that in all cases minimal representations are viable, therefore they are R-compromises. Conversely, by the above comments, if $(V, \mathcal{K})$ has an R-compromise, then it is connected and it is of type Γ. ○

We now introduce an important concept of the theory of simplicial complexes: The political structure $(E, \mathcal{K})$ is *strongly contractible* if $(E, \mathcal{K}) \sim \{x_0\}$, for some $x_0 \in E$; this is equivalent to saying that the identity map on E belongs to the same contiguity class as some constant map. It is interesting to compare this notion with the standard concept of contractibility. $(E, \mathcal{K})$ is *contractible* if there exists a homotopy from the identity map of $|\mathcal{K}|$ to some constant map. It is known that strong contractibility implies contractibility (cf. [14] Chap. 3, Section 5) but not the converse (cf. [3], Example 2.13).

Proposition 22. *If E has a R-compromise then E is strongly contractible.*

Proof. Let $x \in F$, where F is an R-compromise. Let r be a contraction to F that is contiguous to Id_E, and let r_x be the contraction to $\{x\}$, that is r_x is the constant map taking all elements in E to x. For any $s \in \mathcal{K}$, $r(s) \cup r_x(s) \subseteq F$, so $r \approx r_x$. Since $r \approx \mathrm{Id}_E$, one has $r_x \sim \mathrm{Id}_E$. We conclude that E is strongly contractible. □

The converse of Proposition 22 is not true. A configuration may be strongly contractible but fail to have an R-compromise. In the example to the right in Fig.5, no represented compromise exists in E, however it is clear that E is strongly contractible.

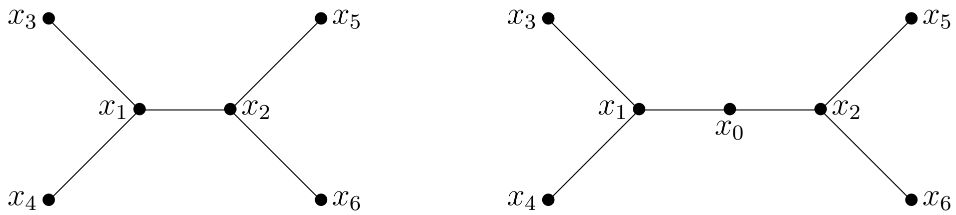

Figure 5: In the political structure to the left there is no pure represented compromise, but the configuration $\{x_1, x_2\}$ is an R-compromise. In the political structure to the right no R-compromise exists. Both structures are strongly contractible.

A configuration may have an R-compromise but fail to have a pure R-compromise: In the example to left in Fig.5 neither $\delta_{x_1 \to x_2}$ nor $\delta_{x_2 \to x_1}$ are contiguous to identity, so one can verify that no retraction to a singleton exists. Therefore no pure R-compromise exists in E. Let $\phi = \delta_{x_4 \to x_1} \circ \delta_{x_3 \to x_1}$ and $\eta = \delta_{x_6 \to x_2} \circ \delta_{x_5 \to x_2}$. Then $\eta \circ \phi$ is a retraction to $\{x_1, x_2\}$. It can be checked that ϕ and Id_E are contiguous, and that also η and Id_E are contiguous, so that $\eta \circ \phi$ is contiguous to Id_E. Therefore $\eta \circ \phi$ is a friendly retraction to $\{x_1, x_2\}$, and $\{x_1, x_2\}$ is an R-compromise.

Some other simple cases of political structures are shown in Fig.6. Again it may be noticed that all are strongly contractible but that R-compromises exist only in two of them, and only one has a pure R-compromise. As a consequence of Lemma 11 we have necessary and sufficient conditions for the existence of an R-compromise. Precisely:

Proposition 23. *Let $(E, \mathcal{K})$ be a political structure. The following assertions are equivalent:*

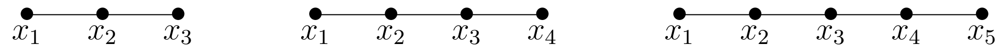

Figure 6: To the left the central point x_2 is a represented pure compromise; in the middle the central 1-simplex $\{x_2, x_3\}$ is an R-compromise and no pure R-compromise exists; to the right there is no R-compromise.

(i) There exists an R-compromise,

(ii) One (and therefore any) minimal representation is viable,

(iii) The second nerve $\mathcal{N}^2(\mathcal{K})$ is a simplex.

Corollary 24. *In a political structure $(E, \mathcal{K})$, a pure R-compromise exists if and only if $\mathcal{N}^2(\mathcal{K})$ is a singleton.*

More generally, when an R-compromise exists, the dimension of any minimal R-compromise is equal to the dimension of $\mathcal{N}^2(\mathcal{K})$.

5 Delegated compromise

In the previous section we have seen that R-compromises, where the agents that delegate from the political contentions remain represented by suitable representatives in the sense that the representing agent enters into all the viable configurations where the delegating agent might have been present, leaving the latter out of the new configuration, be it a representation or an R-compromise. Unfortunately, R-compromises may not exist; indeed we saw that the existence of R-compromises implies that the political structure, viewed as a simplicial complex, is strongly contractible, in itself a rather restrictive property, but this was not enough, since even a strongly contractible political structure may fail to exhibit R-compromises. This shortcoming has to do with the restrictions that we have put on a delegation. In an R-compromise, the delegation $\delta_{x \to y}$ amounts to removing x from the outcome, but it does not imply the withdrawal of x from the search for a compromise. This happens since any other other delegation $\delta_{x' \to y'}$, in order to be friendly, must secure that $\{y'\} \cup s$ be viable for all s that contain x', and the simplex s may contain x. In this sense x is not yet absent from the search for an R-compromise even after the delegation. In the following we present a slightly weaker notion called *delegated* compromise, in which a delegation from x to y implies that x withdraws from the process, so that further delegations need not take x into account. This clearly makes the search for a solution easier.

Definition 8. A configuration F of $(E, \mathcal{K})$ is a *narrowing* of E if there exists a decreasing sequence $E = F_0 \supset F_1 \supset \cdots \supset F_n = F$ and friendly retractions r_i of $(F_{i-1}, \mathcal{K}_{F_{i-1}})$ onto F_i for $i = 1, \ldots, n$.

Notice that the retractions r_i are relative to the simplicial complex $(F_{i-1}, \mathcal{K}_{F_{i-1}})$. It follows from Proposition 7 that this is equivalent to requiring that the associated $\hat{r}_i : (F_{i-1}, \mathcal{K}_{F_{i-1}}) \to (F_i, \mathcal{K}_{F_i})$ is a strong equivalence for $i = 1, \ldots, n$. Therefore we have the following:

Proposition 25. *If F is a narrowing of $(E, \mathcal{K})$ then $(F, \mathcal{K}_F)$ is strongly equivalent to $(E, \mathcal{K})$.*

In view of Proposition 3 any narrowing F can be obtained by a progressive delegation $\delta_{x_1 \to y_1}, \ldots, \delta_{x_p \to y_p}$ such that $\delta_{x_p \to y_p} \circ \cdots \circ \delta_{x_1 \to y_1}(E) = F$ and such that the restrictions of $\delta_{x_k \to y_k}$ to $E \setminus \{x_1, \ldots, x_{k-1}\}$, $k = 1, \ldots, p$, are friendly. We are ready to introduce the main concept of solution of this section:

Definition 9. A configuration F of $(E, \mathcal{K})$ is a *delegated compromise* (D-compromise for short) if F is a narrowing of $(E, \mathcal{K})$ and $F \in \mathcal{K}$.

The obvious way of looking at delegated compromises is to see them as repeated compromises; in the first step, F_1 is an R-compromise for E, in the next step an R-compromise F_2 in F_1 is obtained, etc. It should be noted however, that the sets F_i themselves need not be R-compromises in the original structure since we do not demand that they belong to $\mathcal{K}$. It is clear that an R-compromise is a delegated compromise, but the converse need not be true. Note that a D-compromise must be a viable configuration (a simplex).

The D-core. Given a political structure, one may search for a D-compromise following a process of successive friendly delegations, but is there a narrowing that leads to a positive outcome, namely a viable configuration? In order to capture the best narrowing that can be obtained from such a search, we need to know what are the minimal configurations that can be achieved as outcome.

The idea of best narrowing is captured by the following:

Definition 10. A configuration F of $(E, \mathcal{K})$ is a *delegated core* (shorthand: a D-core[1]) of $(E, \mathcal{K})$ if F is a narrowing of $(E, \mathcal{K})$ and $(F, \mathcal{K}_F)$ is irreducible.

Now we can state and prove the main result about D-cores of political structures:

Proposition 26. *Let $(E, \mathcal{K})$ be a political structure. Then the following hold:*

(i) $(E, \mathcal{K})$ has a D-core,

(ii) $(E, \mathcal{K})$ is strongly equivalent to its D-core,

(iii) the D-core is unique up to isomorphism.

[1]The term ”core” is used by Barmak [2], Chap.5, for the same notion in the context of abstract finite spaces.

Proof: (i) Either $(E, \mathcal{K})$ is irreducible, in which case it is itself a D-core, or one can find a non trivial friendly retract F_1. If $F_1, \mathcal{K}_{F_1}$ is irreducible, otherwise we repeat the operation on $(F_1, \mathcal{K}_{F_1})$ and so on we thus construct a narrowing $E = F_0 \supset F_1 \supset \cdots \supset F_k \supset \cdots$. Since the sequence is strictly decreasing and E is finite we reach eventually an index n with irreducible simplicial complex $(F_n, \mathcal{K}_{F_n})$. Clearly this is a D-core. (ii) follows from Proposition 25. It remains only to prove (iii): Suppose that $(F_1, \mathcal{K}_{F_1})$ and $(F_2, \mathcal{K}_{F_2})$ are two D-cores of $(E, \mathcal{K})$. Then by (ii) we have that

$$(F_1, \mathcal{K}_{F_1}) \sim (E, \mathcal{K}) \sim (F_2, \mathcal{K}_{F_2}),$$

and by the Corollary to Lemma 12, $(F_1, \mathcal{K}_{F_1})$ and $(F_2, \mathcal{K}_{F_2})$ are isomorphic. □

Using the D-core as a formal description of conflict resolution may seem attractive in view of these results. On the other hand, it should be remembered that the D-core does not point to a single configuration, rather it is an indication of the limits beyond which conflicts cannot be solved by delegation alone.

Existence of D-compromises. We now investigate the existence of a D-compromise in a political structure. By definition, if the D-core is viable, then it is a D-compromise. The D-core always exists (Proposition 26); on the other hand, being irreducible, if it is viable it must be a singleton. These considerations are made precise by the following propostiion.

Proposition 27. *Let $(E, \mathcal{K})$ be a political structure. Then the following hold:*

(i) $(E, \mathcal{K})$ has a D-compromise,

(ii) $(E, \mathcal{K})$ has a pure D-compromise,

(iii) $(E, \mathcal{K})$ is strongly contractible,

(iv) The D-core is a singleton.

Proof: (i)⇔(ii). If F is a D-compromise, then, since F is a simplex, there exists a friendly retraction to any of its points $\{x\}$. It follows that $\{x\}$ is a narrowing of $(E, \mathcal{K})$ an therefore that $\{x\}$ is a pure D-compromise.

(ii)⇒(iii). Since $\{x\}$ is a narrowing, we have that $(E, \mathcal{K}) \sim \{x\}$, so that $(E, \mathcal{K})$ is strongly contractible.

(iii)⇒(iv). Let F be a D-core. From $(E, \mathcal{K}) \sim (F, \mathcal{K}_F)$ and $(E, \mathcal{K}) \sim \{x\}$, we get that $(F, \mathcal{K}_F) \sim \{x\}$. Since both $(F, \mathcal{K}_F)$ and $\{x\}$ are irreducible, they are isomorphic by the Corollary to Lemma 12.

(iv)⇒(ii) is straightforward, since the D-core being a singleton is also a pure D-compromise. □

We may exploit the results on nerves of simplicial complexes obtained earlier to obtain additional information about narrowings of the political structure.

If $(E,\mathcal{K})$ is not irreducible, $\mathcal{N}^2(\mathcal{K})$ is not isomorphic to $(E,\mathcal{K})$, but the number of vertices in $\mathcal{N}^2(\mathcal{K})$ is no greater than that of $(E,\mathcal{K})$, and one may proceed to the derived nerves $\mathcal{N}^k(\mathcal{K}))$ for $k>3$. Clearly, there will be some number $d\geq 1$ such that $\mathcal{N}^{2d+2}(\mathcal{K})$ and $\mathcal{N}^{2d}(\mathcal{K})$ have the same number of vertices. By convention $\mathcal{N}^0(\mathcal{K})=\mathcal{K}$.

Definition 11. Let $(E,\mathcal{K})$ be a political structure. The *depth of delegation* of $(E,\mathcal{K})$, written $d_{(E,\mathcal{K})}$, is the smallest number d such that the $\mathcal{N}^{2d+2}(\mathcal{K})$ and $\mathcal{N}^{2d}(\mathcal{K})$ have the same number of vertices.

Using Lemma 11, we get the following proposition.

Proposition 28. $\mathcal{N}^{2k}(\mathcal{K})$ *is isomorphic to the D-core of* $(E,\mathcal{K})$ *if and only if* $k\geq d_{(E,\mathcal{K})}$.

Clearly the depth provides the smallest number of iterated representations in a narrowing to achieve the D-core.

In the remaining part of this section, we consider political structures associated to graphs, as in Example 3. We established that such political structures have R-compromises if and only they are trees with a simple structure (Proposition 21). Here the existence of a delegated compromise is equivalent to being a tree.

Proposition 29. *Let* $(E,\mathcal{K})$ *be a political structure of a graph. The following are equivalent:*

(i) $(E,\mathcal{K})$ *is a tree,*

(ii) each $x\in E$ *is a pure delegated compromise in* $(E,\mathcal{K})$*,*

(iii) the simplicial complex $\mathcal{K}$ *is strongly contractible,*

(iv) the topological space $|\mathcal{K}|$ *is contractible.*

Proof: (i)⇒(ii). Let $x\in E$ be arbitrary, and for each $y\in E$, define the distance from y to x, $d(y,x)$, as the number of edges in a path from x to y; since $(E,\mathcal{K})$ is a tree, $d(x,y)$ is well-defined. Let $D=\{x_1,\dots,x_k\}\subseteq E$ be the set of all vertices in E with maximal distance from x; then each $x\in D$ has degree 1. Let $E_1=E\backslash D$. Each $x\in D$ is connected to some $y(x)\in E$ which does not belong to D since $(E,\mathcal{K})$ is a tree. There is a friendly delegation $\delta_{x\to y(x)}$ from x to the vertex $y(x)$, and the composition $\delta_{x_k\to y(x_k)}\circ\cdots\circ\delta_{x_1\to y(x_1)}$ is progressive and defines a narrowing from $(E,\mathcal{K})$ to $(E_1,\mathcal{K}_{E_1})$. The political structure $(E_1,\mathcal{K}_{E_1})$ is a graph which is connected and has no cycles, hence it is a tree. Moreover E_1 has strictly less elements than E. Repeating the above procedure, we successively obtain substructures $(E_i,\mathcal{K}_{E_i})$ and narrowings $(E_i,\mathcal{K}_{E_i})\to(E_{i+1},\mathcal{K}_{E_{i+1}})$, for $i=1,\dots,p$, untill we obtain a structure $(E_{p+1},\mathcal{K}_{E_{p+1}})$ in which no agent has degree 1. But a tree without vertices of degree 1 consists of the point x, which consequently is a delegated compromise of $(E,\mathcal{K})$.

(ii)⇔(iii) is a consequence of Proposition 27, (iii)⇒(iv) is a consequence of Lemma 2 in Chapter 3, Section 5 in Spanier [14], and finally (iv)⇒(i) is Lemma 1 in Chapter 3, Section 7 in Spanier [14]. □

It is seen that delegated compromises exist if and only if the political structure is strongly contractible. The topology of the political structure, and precisely its strong homotopy, is the basic determinant for the possibility of a compromise. When delegated compromises exist, they may take the form of single agents (pure D-compromise) or a viable configuration. The question whether some D-compromise may seem unacceptable (dictatorial) is irrelevant in our setting. This is due to the fact that in our model we are concerned only with the viability of the solution. In this sense the model may look underdetermined: It may be the case that the political environment requires some further conditions for viable configurations to be acceptable, as seen in the empirical case in Example 4, where the confessional composition of any compromise is an indispensable pre condition which must be satisfied by a viable configuration. Future research must therefore consider viability with some formal kind of acceptability.

6 Concluding remarks

In the present paper, we have presented a formal model of compromise in contexts of political conflict. This model is that of a political structure, mathematically a simplicial complex in which vertices are interpreted as political agents and simplices as viable configurations. In this context, the fundamental motive of action is founded on the notion of delegation. Delegation can be formulated as a mapping taking the delegating agent to another agent, to whom power is to be transferred, and the idea of handling over influence to another agent is formalized through the notion of friendliness. The analysis of delegation and its consequences can now exploit the theory of maps between simplicial complexes, and due the discrete character of the model the relevant mathematics appear to be that of strong homotopy.

Two fundamental types of delegation are of interest. In the first one, influence may be transferred but the receiver must be able to respond back and cannot delegate further. This gives rise to the notions of representations and R-compromises. Allowing for repeated delegation opens up for further concentration of power, expressed by notions of D-cores and D-compromise.

Since a general theory must be assessed in terms of the insights, which it offers in particular applications, we have briefly considered some such applications of the theory, namely to TU and in particular simple games, to effectivity functions, and to network models of political influence. The notion of delegation indicates how to analyse situations where decisions are not immediately obtainable so that compromises are called for. We have only touched upon this area of research where there are several directions of future developments of the theory. One of such directions is to consider situations where acceptability conditions in addition to viability are required for a compromise. Those conditions appear as restrictive rules such as plurality, inclusiveness, parity etc. Another direction must address the limits of our notion of delegation. In our model the delegate is not modified by the delegation; the delegating agent is removed but the delegate does not keep any mark of the delegation. This drawback is due to the extreme simplicity of the model (only a simplicial complex). We think that any future research on compromise must address this question by enriching the model, for instance by introducing an operation of creation of new forces as well as an operation of annihilation, thus acting on both sides of "delegation".

References

[1] Abdou, J. and H.Keiding (1991). Effectivity functions in social choice. Kluwer, Dordrecht.

[2] Barmak, J.A. (2011), Algebraic topology of finite topological spaces and applications, Springer, Heidelberg.

[3] Barmak, J. A., and E.G.Minian (2012). Strong homotopy types, nerves and collapses. Discrete and Computational Geometry, 47(2), 301-328.

[4] Moulin, H. and B.Peleg (1982). Cores of effectivity functions and implementation theory. Journal of Mathematical Economics 10, 115-145.

[5] Freeman, L.C. (1978). Centrality in social networks: conceptual clarification, Social Networks 1, 215-239.

[6] Nakamura, K. (1979). The vetoers in a simple game with ordinal preferences, International Journal of Game Theory 8, 55-61.

[7] von Neumann, J. and O.Morgenstern (1944). Theory of games and economic behavior, Princeton.

[8] Peleg, B. (1984). Game theoretic analysis of voting in committees, Cambridge University Press, Cambridge.

[9] (2010). Strategic social choice: Stable representations of constitutions. Springer, Heidelberg.

[10] Reny, P.J. and M.H.Wooders (1996), The partnered core of a game without side payments, Journal of Economic Theory 70, 298-311.

[11] Schmitt, C. (2008). The concept of the political: Expanded edition. University of Chicago Press.

[12] Scott, J. (2011). Social network analysis: developments, advances, and prospects. Social Networks Analysis and Mining 1, 21-26.

[13] (1962). Simple games: An outline of the descriptive theory. Systems Research and Behavioral Science 7, 59-66.

[14] Spanier, E.H. (1966). Algebraic topology, McGraw-Hill, New York.

[15] Ibrahim, Y.M. (1989, October 29). Peace at last, on paper at least, for Lebanon. The New York Times, E3.

[16] Cabmanis T. (2016, October 21). Michel Aoun Rises to Lebanese Presidency, Ending Power Vacuum, The New York Times, https://www.nytimes.com/2016/11/01/world/middleeast/michel-aoun-lebanon-president.html