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Lionel Boisdeffre

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Lionel De BOISDEFFRE

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DROPPING RATIONAL EXPECTATIONS

Lionel de Boisdeffre,[1]

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Abstract

We consider a pure exchange economy, where agents, typically asymmetrically informed, exchange securities, on financial markets, and commodities, on spot markets. Consumers have private characteristics, anticipations and beliefs, and no model to forecast prices. Rational expectation and bounded rationality assumptions are dropped. We show that agents face an incompressible uncertainty, represented by a so-called "minimum uncertainty set". This uncertainty typically adds to the exogenous one, on the state of nature, an 'endogenous uncertainty' over future spot prices. At equilibrium, all agents expect the 'true' price on every spot market as a possible outcome, and elect optimal strategies, ex ante, which clear on all markets, ex post. We show this sequential equilibrium exists whenever agents' prior anticipations embed the minimum uncertainty set. This outcome differs from the standard generic existence results of Hart (1975), Radner (1979), or Duffie-Shaffer (1985), among others, based on the rational expectations of prices.

Key words: sequential equilibrium, temporary equilibrium, perfect foresight, existence, rational expectations, financial markets, asymmetric information, arbitrage.

JEL Classification: D52

[1] University of Paris 1 - Panthéon - Sorbonne, 106-112 Boulevard de l'Hôpital, 75013 Paris, France. Email: lionel.de.boisdeffre@wanadoo.fr

1 Introduction

When agents' information is incomplete or asymmetric, the issue of how markets may reveal information is essential and, yet, debated. Quoting Ross Starr (1989), "*the theory with asymmetric information is not well understood at all. In short, the exact mechanism by which prices incorporate information is still a mystery and an attendant theory of volume is simply missing.*" A traditional response is given by the REE (rational expectations equilibrium) model by assuming, quoting Radner (1979), that "*agents have a 'model' or 'expectations' of how equilibrium prices are determined*". Under this assumption, agents know the relationship between private information signals and equilibrium prices, along a so-called "*forecast function*".

Cornet-De Boisdeffre (2002) suggests an alternative approach, where agents' asymmetric information is represented by private information signals, which correctly inform each agent that tomorrow's state of nature will be in a subset of the state space. The latter paper extends the classical definitions of equilibrium, prices and no-arbitrage condition to asymmetric information. Generalizing Cass (1984), De Boisdeffre (2007) shows the existence of equilibrium on purely financial markets is characterized, in this setting, by that no-arbitrage condition. This existence result differs from Radner's (1979) REE generic one. Finally, Cornet-De Boisdeffre (2009) shows the above no-arbitrage condition may always be reached by agents, with no price model, from observing prices or exchange opportunities on financial markets.

The above papers may picture the information transmission on actual markets and restore a full existence property of equilibrium. But they still retain Arrow's (1953) and Radner's (1972) rational expectation hypothesis (also called the conditional perfect foresight hypothesis), stating that agents know the map between future realized states and equilibrium prices. In such a setting, the states of nature are exogenous and represent all individual ex ante uncertainty.

Yet, actual states typically encompass unobservable variables. Arrow (1953) acknowleges this by noticing that a complete market of exogenous state-contingent claims does not exist and should be replaced by state-contingent financial transfers. In his setting, Kurz and Wu (1996) notice, "*agents need to know the maps from states at future dates to prices in the future and it is entirely unrealistic to assume that agents can find out what this sequence of maps is.*" Quoting Radner (1982) himself, this condition "*seems to require of the traders a capacity for imagination and computation far beyond what is realistic*". So the question of the possibility to discard rational expectations and remain within the sequential equilibrium model.

Radner's (1972-79) rational expectation assumptions would be justified if agents knew the primitives of the economy (endowments, preferences, etc...) and their relations to equilibrium prices, and if they had elected one common price anticipation in each state (under typically many possibilities and opposite interests), with the common knowledge of game theory. Otherwise, any equilibrium result would typically differ from the standard sequential equilibrium. Such conditions are unrealistic.

Probably the first, best known and most radical escape to rational expectations was the temporary equilibrium model, introduced by J. Hicks and later developed by J.-M. Grandmont. It is traditionally presented as dichotomic from the sequential equilibrium model (see Grandmont, 1982). At a temporary equilibrium, agents have exogenous anticipations, which need not be self-fulfilling. Current markets clear at agents' initial plans, which are typically revised, at each period, after observing realized prices and events. Equilibrium allocations need not clear on future spot markets, where agents may face bankruptcy, due to mistaken anticipations. This outcome explains why the temporary equilibrium did not thrive as the perfect foresight's, which lets agents coordinate across periods, on perfectly anticipated prices.

A less radical approach is referred to as bounded rationality. In this line of research, Kurz' (1994) rational belief equilibrium (RBE) allows agents to lack the "*structural knowledge*" of how equilibrium prices are determined. This unawareness may be due to uncertainty about the beliefs, characteristics and actions of other agents. It leads to an additional uncertainty over future variables, which Kurz calls "*endogenous uncertainty*", describes as the major cause of economic fluctuations, and shows to be consistent with heterogenous beliefs.

Bounded rationality models also serve to study learning processes with differential information (alternative to the REE's), and the links between the information structure and equilibrium or core allocations. This is done, in particular, by Koutsougeras and Yannelis (1999), who emphasize "*that the study of cooperative solution concepts (e.g., the core and the (Shapley) value) in differential information economies appears to be a successful alternative to the traditional rational expectations equilibrium, because they provide sensible and reasonable outcomes in situations where any rational expectations equilibrium (REE) notion fails to do so.*"

The current paper departs from both perfect foresight and bounded rationality models. In our view, bounded rationality still demands inference and computational skills, as well as informations, which typically exceed agents' possibilities. In the real world, agents' beliefs, decisions and characteristics are all private and their observations are limited. This restricts their reckonning capacities to a bare minimum and, consequently, their ability to construct any forecast function. The model we propose requires no structural knowledge, nor computation from agents.

As a consequence, agents face an incompressible uncertainty over the set of clearing market prices to expect, represented by a so-called and never empty "*minimum uncertainty set*". This set consists of the possible equilibrium spot prices related to one of many structures of beliefs and decisions today, which are private. It is consistent with Kurz and Wu's (1996) conclusion that price uncertainty and economic fluctuations are "*primarily endogenous and internally propagated phenomena (...) generated by the actions and beliefs of the agents (...) and by their uncertainty about the actions of other agents*".

That minimum uncertainty set (or a bigger set) might be inferred, we argue, by a tradehouse or a financial institution from observing and treating past data on long time series. Consumers themselves would not have such computational capacities, but could be kept informed from public or private institutions. Yet, future equilibrium prices cannot be reckonned precisely by any agent or institution, because this would require to know every agent's beliefs, characteristics and actions. Only a set of possibilities could be assessed ex ante, or the minimum uncertainty set, but not the precise location of future spot prices within that set.

The current model's sequential equilibrium, called "*correct foresight equilibrium*" (CFE), is thus defined as De Boisdeffre's (2007), except for agents' forecasts, which need no longer be unique in any state, but form sets containing the price to prevail. The CFE, we argue, reconciles into one concept the sequential and temporary equilibria. It is sequential, since anticipations are self-fulfilling. It is also temporary since forecasts are exogenously given ex ante. Along our main Theorem, whether the financial stucture be nominal or real, and beliefs be symmetric or not, a CFE exists whenever agents' anticipation sets include the minimum uncertainty set.

The approach to information transmission and equilibrium that we have developed here and in earlier papers seems to model actual behaviours on markets. Endowed with no forecast function, unaware of the primitives of the economy, and with limited observational and reckonning capacities, consumers have exogenous anticipations and face uncertainty over future spot prices. They infer, first, the coarsest arbitrage-free refinement of their prior anticipations from observing trade, along De Boisdeffre (2016). Whence reached, they have no means of further refining their beliefs. Then, market forces, driven by price and demand, lead to equilibrium.

The paper is so organized: Section 2 presents the model. Section 3 states the existence Theorem. Section 4 proves the Theorem. An Appendix proves Lemmas.

2 The basic model

We consider, throughout, a two-period economy, with private information signals, a consumption market and a financial market. The sets, I, S, L and J, respectively, of consumers, states of nature, goods and assets are all finite. The first period is also referred to as $t = 0$ and the second, as $t = 1$. At $t = 0$, there is an uncertainty upon which state of nature, $s \in S$, will prevail tomorrow. The non random state at $t = 0$ is denoted by $s = 0$ and, whenever $\Sigma \subset S$, we let $\Sigma' := \{0\} \cup \Sigma$. Similarly, we denote by $l = 0$ the unit of account and let $L' = \{0\} \cup L$.

2.1 Markets, information and beliefs

Agents consume and may exchange the same consumption goods, $l \in L$, on the spot markets of each period. The generic i^{th} agent's welfare is measured, ex post, by a utility index, $u_i : \mathbb{R}_+^{L \times L} \to \mathbb{R}_+$, over her consumptions at both dates.

At the first period ($t = 0$), each agent, $i \in I$, receives a private information signal, $S_i \subset S$, about which states of the world may occur at $t = 1$. That is, she knows that no state, $s \in S \backslash S_i$, will prevail tomorrow. Each set S_i is assumed to contain the true state. Hence, the pooled information set, denoted by $\underline{\mathbf{S}} := \cap_{i \in I} S_i$, is non-empty and we let, w.l.o.g., $S = \cup_{i \in I} S_i$. The information structure, (S_i), is henceforth given.

Agents are unaware of the primitives of the economy and of other agents' beliefs and actions. They fail to know how market prices are determined and, therefore, face uncertainty over future spot prices. Thus, at $t = 0$, the generic i^{th} agent elects a private set of anticipations, out of the price set, $P := \{p \in \mathbb{R}^L_{++} : \|p\| = 1\}$, in each state $s \in S_i$. We refer to $\Omega := S \times P$ as the set of forecasts and denote by ω its generic element, and by $\mathcal{B}(\Omega)$ its Borel σ-algebra. A forecast, $\omega := (s, p) \in \Omega$, is thus a pair of a random state, $s \in S$, and a conditional spot price, $p \in P$, expected in that state.

Remark 1 Strictly positive prices in P are related to strictly increasing preferences, as assumed below. For simplicity, but w.l.o.g., the set, P, normalizes all agents' price expectations to one. In each state, this common value of one could be replaced by any other positive value without changing the model's properties.

Agents may operate financial transfers across states in S' (actually in $\underline{\mathbf{S}}'$) by exchanging, at $t = 0$, finitely many assets, $j \in J$, which pay off, at $t = 1$, conditionally on the realization of forecasts. Payoffs may be nominal (i.e., in cash) or real (i.e., in goods) or a mix of both. All assets' payoffs define a $(S \times L') \times J$ return matrix, V, whose generic row across forecasts, $\omega \in \Omega$, is denoted $V(\omega) \in \mathbb{R}^J$. Thus, at asset price, $q \in Q$, agents may buy or sell unrestrictively portfolios of assets, $z = (z_j) \in \mathbb{R}^J$, for $q \cdot z$ units of account at $t = 0$, against the promise of delivery of a flow, $V(\omega) \cdot z$, of conditional cash payoffs across forecasts, $\omega \in \Omega$. We now define anticipations.

Definition 1 *An anticipation set is a closed subset of* $\Omega := S \times P$. *A collection of anticipation sets,* $\Omega_i := \cup_{s \in S_i}\{s\} \times P_s^i$, *for each* $i \in I$, *is an anticipation structure if:*

(a) $P_s^i \neq \varnothing$, *for every* $(i,s) \in I \times S_i$, *and* $\cap_{i \in I}\ P_s^i \neq \varnothing$, *for every* $s \in \underline{\mathbf{S}}$.

We denote by $\mathcal{AS}$ *the set of anticipation structures. Given* $(\Omega_i) \in \mathcal{AS}$, *a structure,* $(\Omega_i') \in \mathcal{AS}$, *which is smaller (for the inclusion relation) than* (Ω_i), *is called a refinement of* (Ω_i), *and denoted by* $(\Omega_i') \leq (\Omega_i)$. *A belief is a probability distibution over* $(\Omega, \mathcal{B}(\Omega))$, *whose support is an anticipation set. A collection of beliefs,* (π_i), *whose supports define an anticipation structure,* (Ω_i), *is called a structure of beliefs, said to support* (Ω_i) *and denoted by* $(\pi_i) \in \Pi_{(\Omega_i)}$. *Their overall set is denoted by* $\mathcal{SB}$.

2.2 The agent's behaviour and the concept of equilibrium

The generic i^{th} agent receives an endowment, $e_i := (e_{is}) \in \mathbb{R}_{++}^{L \times S_i'}$, promising the commodity bundles, $e_{i0} \in \mathbb{R}_{++}^L$ at $t = 0$, and $e_{is} \in \mathbb{R}_{++}^L$, in each state $s \in S_i$ if it obtains.

Agents' forecasts are represented by an anticipation structure, $(\Omega_i) \in \mathcal{AS}$, henceforth given, which is reached when they elect their strategies at $t = 0$, jointly with beliefs, $(\pi_i) \in \Pi_{(\Omega_i)}$, along Definition 1. The i^{th} the agent's consumption set, denoted by X_i, is that of continuous mappings, $x : \Omega_i' \to \mathbb{R}_+^L$ (where $\Omega_i' := \{0\} \cup \Omega_i$).

We restrict first period prices to $P_0 = \{p \in \mathbb{R}_+^L : \|p\| \leqslant 1\} \times \{q \in \mathbb{R}^J : \|q\| \leqslant 1\}$. Given a price $\omega_0 := (p_0, q) \in P_0$, at $t = 0$, the generic i^{th} agent's consumptions, $x \in X_i$, are mappings, relating $s = 0$ to a consumption decision, $x_{\omega_0} := x_0 \in \mathbb{R}_+^L$, at $t = 0$, and, continuously on Ω_i, every forecast, $\omega \in \Omega_i$, to a consumption decision, $x_\omega \in \mathbb{R}_+^L$, at $t = 1$, which is conditional on the realization of the forecast ω. Her budget set is:

$$B_i(\omega_0) := \{(x, z) \in X_i \times \mathbb{R}^J : \ p_0 \cdot (x_0 - e_{i0}) \leqslant -q \cdot z, \ \ p_s \cdot (x_\omega - e_{is}) \leqslant V(\omega) \cdot z, \ \forall \omega := (s, p_s) \in \Omega_i\}.$$

Given agents' structure of beliefs at the time of trading, $(\pi_i) \in \Pi_{(\Omega_i)}$, each consumer, $i \in I$, has preferences represented by the V.N.M. utility function:

$x \in X_i \mapsto U_i^{\pi_i}(x) := \int_{\omega \in \Omega_i} u_i(x_0, x_\omega) d\pi_i(\omega)$.

The above economy, denoted $\mathcal{E}_{(\pi_i)} = \{(I, S, L, J), V, (S_i), (\Omega_i), (\pi_i), (e_i), (u_i)\}$, retains the small consumer price-taker hypothesis, by which no single agent may, alone, have a significant impact on prices. It is called standard under the following Conditions:

- ***Assumption A1*** *(strong survival): for each* $i \in I,\ e_i \in \mathbb{R}_{++}^{L \times S'_i}$;
- ***Assumption A2****: for each* $i \in I$, u_i *is continuous, strictly concave and increasing:* $[(x, y, x', y') \in \mathbb{R}_+^{4L},\ (x, y) \leqslant (x', y'),\ (x, y) \neq (x', y')] \Rightarrow [u_i(x', y') > u_i(x, y)]$;
- ***Assumption A3****: the system* $\{V(\omega)\}_{\omega \in \cap_{i \in I} \Omega_i}$ *contains* $\#J$ *independent vectors.*

Remark 2 Strict concavity is retained in Assumption *A2* to alleviate the proof of a correspondence selection amongst optimal strategies (see proof of Lemma 4).

Remark 3 Assumption *A3* is unnecessary if the financial structure is nominal. With real assets, the condition of Assumption *A3* is met, e.g., if assets yield non-zero payoffs in $\#J$ realizable states and if agents expect the true price within a small neighbourhood of possibilities. These conditions are typical of the model's equilibrium, where no agent has a function to forecast prices precisely.

The consumer elects an optimal strategy in her budget set. So the equilibrium:

Definition 2 *A collection of prices,* $\omega_0 := (p_0, q) \in P_0$ *and* $p := (p_s) \in \mathcal{P} := P^{\underline{\mathbf{S}}}$, *and decisions,* $(x_i, z_i) \in B_i(\omega_0)$, *for each* $i \in I$, *is a sequential equilibrium of the economy,* $\mathcal{E}_{(\pi_i)}$, *or correct foresight equilibrium (C.F.E.), if the following Conditions hold:*

(a) $\forall i \in I,\ \forall s \in \underline{\mathbf{S}},\ \omega_s := (s, p_s) \in \Omega_i$;

(b) $\forall i \in I,\ (x_i, z_i) \in \arg\max_{(x,z) \in B_i(\omega_0)}\ U_i^{\pi_i}(x)$;

(c) $\sum_{i \in I} (x_{i\omega_s} - e_{is}) = 0,\ \forall s \in \underline{\mathbf{S}'}$;

(d) $\sum_{i \in I} z_i = 0$.

3 The existence Theorem

With the model's endogenous uncertainty, only the set of possible equilibrium prices could be assessed. No agent or institution would know the true price's location within that set, because this would require to know, ex ante, all agents' characteristics, beliefs and decisions, which are private in a non-cooperative setting.

3.1 Endogenous uncertainty and the existence of equilibrium

The above incompressible uncertainty is embedded into the so-called "*minimum uncertainty set*", defined below and shown to be never empty. The following Theorem 1 shows that equilibrium exists, in a standard economy, whenever agents' anticipation sets include the latter set. This existence result holds, whatever agents' beliefs, and for any financial structure. This full existence result seems worth noticing, so it differs from the generic ones of the classical sequential equilibrium models.

Definition 3 *Let Λ be the set of prices, $p := (p_s) \in \mathcal{P} := P^{\underline{\mathbf{S}}}$, which support the equilibrium (i.e., are equilibrium prices) of a standard economy, $\mathcal{E}_{(\widetilde{\pi}_i)}$, for some arbitrary structure of beliefs, $(\widetilde{\pi}_i) \in \mathcal{SB}$. The set, $\Delta := \{\omega \in \Omega : \exists p := (p_s) \in \Lambda,\ \exists s \in \underline{\mathbf{S}},\ \omega = (s, p_s)\}$, of forecasts which support an equilibrium, is called the minimum uncertainty set.*

Lemma 1 *Under Assumptions A1-A2, the following Assertion holds:*

(i) $\exists \delta > 0 : \Delta \subset \underline{\mathbf{S}} \times [\delta, 1]^L$.

Proof See the Appendix. □

Assumption A4 *(correct foresight): for each $i \in I$, the relation $\Delta \subset \Omega_i$ holds, in which $(\Omega_i) \in \mathcal{AS}$ is the given anticipation structure of the economy, $\mathcal{E}_{(\pi_i)}$.*

Theorem 1 *Under Assumptions A1-A2-A3-A4, the economy, $\mathcal{E}_{(\pi_i)}$, admits a C.F.E.*

3.2 Endogenous uncertainty and how to reach correct anticipations

Along Theorem 1, above, as long as agents have correct foresight (i.e., meet Assumption *A4*), a C.F.E. exists whatever their beliefs. Markets clear ex post at one self-fulfilling common anticipation. We now argue why the set of all equilibrium forecasts may be one of "*minimum uncertainty*" and how it could be assessed.

On the first issue, when today's beliefs are private, no equilibrium price should be ruled out *a priori*, given agents' unknown beliefs and decisions today. Theoretically, this set is of incompressible uncertainty. Practically, it would be so because no agent has structural knowledge, along Kurz (1994). Past price series confirm that erratic fluctuations occur, and not only in periods of enhanced uncertainty.

Yet, if no agent has structural knowledge and access to private data, how can this minimum uncertainty set, or a bigger set, be inferred ? The response may simply be empirical, that is, only require observations.

On this second issue, the model specifies *normalized* spot prices (in a large sense from Remark 1). It is often possible to observe past prices and reckon their *relative* values, in a wide array of situations, or states, which typically replicate over time (hence, embed $\underline{\mathbf{S}}$). Relative prices vary between observable upper and lower bounds. It is sensible to assume that markets are mostly at equilibrium and, with long enough series, that all equilibrium prices lie within the bounds of their convex hulls.

Statistical methods of that kind, and the iterative verification across periods that sets like convex hulls contain the true forthcoming price, require no price model and need not be performed by consumers. They would rather be implemented by a tradehouse or financial or public institution. The latter have greater computational facilities than the former, and obvious applications to infer, e.g., to finance.

On consumer side, if agents should agree on a minimal span of price uncertainty (and would take advice from specialists), they typically keep private their beliefs and have remaining idiosyncratic anticipations, explaining their likely asymmetries.

4 The existence proof

Hereafter, we set as given an arbitrary anticipation structure, $(\Omega_i) \in \mathcal{AS}$, and beliefs, $(\pi_i) \in \Pi_{(\Omega_i)}$, and assume that the economy, $\mathcal{E}_{(\pi_i)}$, meets Assumptions *A1-A2-A3-A4*. The proof proceeds in three steps. Sub-Section 4.1 defines, via finite partitions, a non-decreasing sequence, $\{(\Omega_i^n)\}_{n\in\mathbb{N}}$, of finite refinements of (Ω_i), whose limit is dense in (Ω_i). Sub-Section 4.2 constructs a sequence of finite auxiliary economies, which all admit equilibria along our companion paper [7]. Sub-Section 4.3 derives a CFE of the economy $\mathcal{E}_{(\pi_i)}$ from these auxiliary equilibria.

4.1 Finite partitions of agents' anticipation sets

- Let $(i,n) \in I \times \mathbb{N}$ be given. We define an integer, $K_{(i,n)} \in \mathbb{N}$, and a partition, $\mathcal{P}_i^n = \{\Omega_{(i,n)}^k\}_{1 \leqslant k \leqslant K_{(i,n)}}$, of Ω_i, such that $\pi_i(\Omega_{(i,n)}^k) > 0$, for each $k \in \{1, \ ..., \ K_{(i,n)}\}$.
- In each set $\Omega_{(i,n)}^k$ (for $k \leqslant K_{(i,n)}$), we select exactly one element, $\omega_{(i,n)}^k$, to form the discrete sub-set, $\Omega_i^n := \{\omega_{(i,n)}^k\}_{1 \leqslant k \leqslant K_{(i,n)}}$, of Ω_i.
- We define mappings, $\pi_i^n : \Omega_i^n \to \mathbb{R}_{++}$, by $\pi_i^n(\omega_{(i,n)}^k) = \pi_i(\Omega_{(i,n)}^k)$ and $\Phi_i^n : \Omega_i \to \Omega_i^n$, by its restrictions, $\Phi^n_{i \ / \ \Omega^k_{(i,n)}}(\omega) = \omega_{(i,n)}^k$, for each $k \leqslant K_{(i,n)}$ and every $\omega \in \Omega_{(i,n)}^k$.

And we henceforth assume that the Assertions of the following Lemma hold.

Lemma 2 *For each* $(i,n) \in I \times \mathbb{N}$*, we may choose the above* $\mathcal{P}_i^n$*,* Ω_i^n*,* Φ_i^n*, such that:*

(i) $\Omega_i^n \subset \Omega_i^{n+1}$ *and* $\mathcal{P}_i^{n+1}$ *is finer than* $\mathcal{P}_i^n$*;*

(*ii*) $\{V(\omega)\}_{\omega\in\cap_{i\in I}\Omega_i^n}$ *contains* $\#J$ *independent vectors, whenever* $n \geqslant \#J$*;*

(*iii*) $\cup_{n\in\mathbb{N}}\Omega_i^n$ *is everywhere dense in* Ω_i*;*

(*iv*) *for every* $\omega \in \Omega_i$*,* $\Phi_i^n(\omega)$ *converges uniformly to* ω*.*

Proof See the Appendix, which provides one example of such sets and maps. □

4.2 The auxiliary economies, $\mathcal{E}^n$

Given $n \in \mathbb{N}$, we define an economy, $\mathcal{E}^n = \{(I, S, L, J), V, (\Omega_i'^n), (e_i), (u_i^n)\}$, with same periods, sets of agents, goods and endowments as above. The realizable states and the generic i^{th} agent's expectations are defined as follows:

- $\Omega_i'^n := \underline{\mathbf{S}} \cup \Omega_i^n$ is the agent's information set, defining the information structure, $(\Omega_i'^n)$, of a formal state space, $\Omega^n := \cup_{i\in I}\Omega_i'^n$, whose set of realizable states is $\underline{\mathbf{S}}$.
- In each state $s \in \underline{\mathbf{S}}$, the i^{th} agent has a perfect foresight of the spot price.
- In each state $(s, p) \in \Omega_i^n$, the i^{th} agent is certain that price $p \in P$ will prevail.

Let $\mathcal{P}^* := \{p \in \mathbb{R}^L_{++} : \|p\| \leqslant 1\} \times \mathcal{P} := \{p \in \mathbb{R}^L_{++} : \|p\| \leqslant 1\} \times \{p \in \mathbb{R}^L_{++} : \|p\| = 1\}^{\underline{\mathbf{S}}}$. By induction on $n \in \mathbb{N}$, we define a sequence of equilibrium prices, $(p^n, q^n) \in \mathcal{P}^* \times Q$ in the following way. For all prices, $(p := (p_s), q) \in \mathcal{P}^* \times Q$, we let the generic i^{th} agent's consumption set, budget set, and utility function in the economy $\mathcal{E}^n$ be:

$$X_i^n := \mathbb{R}_+^{L\times\Omega_i'^n}, \text{ whose generic element is denoted by } x := [(x_s)_{s\in\underline{\mathbf{S}}'}, (x_\omega)_{\omega\in\Omega_i^n}];$$

$$B_i^n(p,q) := \{\ (x,z) \in X_i^n \times \mathbb{R}^J :\ p_0\cdot(x_0 - e_{i0}) \leqslant -q\cdot z, \quad p_s\cdot(x_s - e_{is}) \leqslant V(s, p_s)\cdot z,\ \forall s \in \underline{\mathbf{S}}$$

$$and\ \ \overline{p}_s\cdot(x_\omega - e_{is}) \leqslant V(\omega)\cdot z,\ \forall \omega := (s, \overline{p}_s) \in \Omega_i^n\ \};$$

$$\text{and } x \in X_i^n \mapsto\ u_i^n(x) := \frac{1}{n\#\underline{\mathbf{S}}} \sum_{s\in\underline{\mathbf{S}}} u_i(x_0, x_s)\ + (1 - \tfrac{1}{n}) \sum_{\omega\in\Omega_i^n} u_i(x_0, x_\omega)\pi_i^n(\omega).$$

The above economy, $\mathcal{E}^n$, is formally the same as our companion paper's [7]. From its existence Theorem 1 and proof, and from Lemma 2-(*ii*) above, it admits an equilibrium, for every $n \geqslant \#J$ (with slight abuse, we say for $n \in \mathbb{N}$):

Definition 4 *A collection of prices,* $(p,q) \in \mathcal{P}^* \times Q$*, and decisions,* $(x_i, z_i) \in B_i^n(p,q)$*, for each* $i \in I$*, is an equilibrium of the economy,* $\mathcal{E}^n$*, if the following Conditions hold:*

(a) $\forall i \in I,\ (x_i, z_i) \in \arg\max_{(x,z)\in B_i^n(p,q)}\ u_i^n(x)$;

(b) $\sum_{i\in I}\ (x_{is} - e_{is}) = 0,\ \forall s \in \underline{\mathbf{S}}'$;

(c) $\sum_{i\in I}\ z_i = 0$.

We set as given one such equilibrium, $\mathcal{C}^n := \{p^n, q^n, [(x_i^n, z_i^n)]\}$, in the economy, $\mathcal{E}^n$, for each $n \in \mathbb{N}$. From the proof of Theorem 1 in [7], $\mathcal{C}^n$ satisfies $\|p_0^n\| + \|q^n\| \geqslant 1$, for each $n \in \mathbb{N}$, and the following properties:

Lemma 3 *The following Assertions hold:*

(i) *the price sequences* $\{p^n\}$*,* $\{q^n\}$ *may be assumed to converge, say to* $p^* := (p_s^*) \in \overline{\mathcal{P}}^*$ *and* $q^* \in Q$*, such that* $\omega_s^* := (s, p_s^*) \in \overline{\Delta} \subset \Omega_i$*, for each* $(i,s) \in I \times \underline{\mathbf{S}}$*; we let* $\omega_0^* := (p_0^*, q^*)$*;*

(ii) *the sequences* $\{(x_{is}^n)_{s\in\underline{\mathbf{S}}'}\}$ *and* $\{z_i^n\}$*, for each* $i \in I$*, may be assumed to converge, to* $(x_{i\omega_s^*}^*) := (x_{is}^*)_{s\in\underline{\mathbf{S}}'} \in \mathbb{R}_+^{L\times\underline{\mathbf{S}}'}$ *and* $z_i^* \in \mathbb{R}^J$*, s. t.* $\sum_{i\in I}\ (x_{is}^* - e_{is})_{s\in\underline{\mathbf{S}}'} = 0$ *and* $\sum_{i\in I}\ z_i^* = 0$.

Lemma 4 *Let* $B_i(\omega, z) = \{x \in \mathbb{R}_+^L : p\cdot(x - e_{is}) \leqslant V(\omega)\cdot z\}$*, be given sets, for every* $z \in \mathbb{R}^J$ *and all* $\omega := (s,p) \in \Omega_i$*. Along Lemma 3, the following Assertions hold for each* $i \in I$:

(i) *the correspondence* $\omega \in \Omega_i \mapsto \arg\max\ u_i(x_{i0}^*, x)$*, for* $x \in B_i(\omega, z_i^*)$*, is a continuous mapping, whose embedding,* $x_i^* : \omega \in \Omega_i' \mapsto x_{i\omega}^*$*, is a consumption, that is,* $x_i^* \in X_i$*;*

(ii) $U_i^{\pi_i}(x_i^*) = \lim_{n\to\infty} u_i^n(x_i^n)$.

Proof of the Lemmas See the Appendix. □

4.3 An equilibrium of the initial economy

We now prove Theorem 1, via the following Claim.

Claim 1 *The collection,* $\{p^*, q^*, (\Omega_i), (\pi_i), [(x_i^*, z_i^*)]\}$*, of prices, anticipation sets, beliefs, and decisions of Lemmas 3-4, defines a CFE of the economy* $\mathcal{E}_{(\pi_i)}$.

Proof We let $\mathcal{C}^* := \{p^*, q^*, (\Omega_i), (\pi_i), [(x_i^*, z_i^*)]\}$ be defined as in Claim 1. From Lemma 3, $\mathcal{C}^*$ meets Conditions (a)-(c)-(d) of Definition 2 of equilibrium, above. We now show that $\mathcal{C}^*$ meets Condition (b) of the same Definition 2.

From the definition of $\mathcal{C}^n$, the relations $p_0^n \cdot (x_{i0}^n - e_{i0}) \leqslant -q^n \cdot z_i^n$ hold, for each $(i, n) \in I \times \mathbb{N}$, and yield $p_0^* \cdot (x_{i0}^* - e_{i0}) \leqslant -q^* \cdot z_i^*$, for each $i \in I$, in the limit $(n \to \infty)$. We let $\omega_0^* := (p_0^*, q^*)$. From Lemma 4-$(i)$, the relations $x_i^* \in X_i$ and $p_s \cdot (x_{i\omega}^* - e_{is}) \leqslant V(\omega) \cdot z_i^*$ also hold, for every $i \in I$ and every $\omega = (s, p_s) \in \Omega_i$, and imply: $[(x_i^*, z_i^*)]_{i \in I} \in \times_{i \in I} B_i(\omega_0^*)$.

Next, we assume, by contraposition, that $\mathcal{C}^*$ fails to meet Condition (b) of Definition 2, that is, there exist $i \in I$, $(x, z) \in B_i(\omega_0^*)$ and $\varepsilon \in \mathbb{R}_{++}$, such that:

$$(I) \qquad \varepsilon + U_i^{\pi_i}(x_i^*) < U_i^{\pi_i}(x).$$

We may, moreover, assume that $(x, z) \in B_i(\omega_0^*)$ is such that:

$$(II) \qquad \exists\ (\delta, M) \in \mathbb{R}_{++}^2 \colon x_\omega \in [\delta, M]^L,\ \forall \omega \in \Omega_i.$$

The existence of an upper bound to consumptions x_ω (for $\omega \in \Omega_i$) results from the relation $(x, z) \in B_i(\omega_0^*)$, which implies a bound to financial transfers and from the fact that Ω_i is closed in $S_i \times P$. Moreover, for $\alpha \in]0, 1]$ small enough, the decision $(x^\alpha, z^\alpha) := ((1 - \alpha)x + \alpha e_i, (1 - \alpha)z) \in B_i(\omega_0^*)$ meets both relations (I) and (II), from Assumption *A1* and from the uniform continuity (on a compact set) of the mapping $(\alpha, \omega) \in [0, 1] \times \Omega_i \mapsto (x_\omega^\alpha,\ u_i(x_0^\alpha, x_\omega^\alpha))$. So, relations (II) may indeed be assumed.

From Lemmas 1-3, $p^* \in \mathbb{R}_+^L \times [\delta, 1]^{L \times \mathbf{\underline{S}}}$. Then, from the relations (I)-(II) and $(x, z) \in B_i(\omega_0^*)$, the definition of Ω_i, Assumptions *A1-A2* and uniform continuity arguments, we may also assume there exists $\gamma \in \mathbb{R}_{++}$, such that:

$$(III) \quad p_0^* \cdot (x_0 - e_{i0}) \leqslant -q^* \cdot z \text{ and } p_s \cdot (x_\omega - e_{is}) < -\gamma + V(\omega) \cdot z,\ \forall \omega := (s, p_s) \in \Omega_i.$$

From relations (I)-(II)-(III), we may also assume there exists $\gamma' \in]0, \gamma[$, such that:

$$(IV) \quad p_0^* \cdot (x_0 - e_{i0}) \leqslant -\gamma' - q^* \cdot z \text{ and } p_s \cdot (x_\omega - e_{is}) \leqslant -\gamma' + V(\omega) \cdot z, \ \forall \omega := (s, p_s) \in \Omega_i.$$

We recall from above that $\|p_0^*\| + \|q^*\| \geqslant 1$. The above assertion is obvious, from relations (III), if $p_0^* \cdot (x_0 - e_{i0}) < -q^* \cdot z$. Assume that $p_0^* \cdot (x_0 - e_{i0}) = -q^* \cdot z$. If $p_0^* = 0$, then, $q^* \neq 0$, and relations (IV) hold if we replace z by $z - q^*/N$, for $N \in \mathbb{N}$ big enough. If $p_0^* \neq 0$ and $x_0 \neq 0$, the desired assertion results from Assumption *A1-A2* and above. Else, $-q^* \cdot z = -p_0^* \cdot e_{i0} < 0$, and a slight change in portfolio insures relations (IV).

From relations (IV), the continuity of the scalar product and Lemmas 1-2-3, there exists $N_1 \in \mathbb{N}$, such that, for every $n \geqslant N_1$:

$$(V) \quad \begin{cases} p_0^n \cdot (x_0 - e_{i0}) \leqslant -q^n \cdot z \\ p_s^n \cdot (x_{(s, p_s^*)} - e_{is}) \leqslant V(s, p_s^n) \cdot z, \ \forall s \in \underline{\mathbf{S}} \\ p_s \cdot (x_\omega - e_{is}) \leqslant V(s, p_s) \cdot z, \ \forall \omega := (s, p_s) \in \Omega_i^n \end{cases}$$

Along relations (V), for each $n \geqslant N_1$, we define, in $\mathcal{E}^n$, the decision $(x^n, z) \in B_i^n(p^n, q^n)$ by $x_0^n := x_0$, $x_s^n := x_{(s, p_s^*)}$, $x_\omega^n := x_\omega$, for $(s, \omega) \in \underline{\mathbf{S}} \times \Omega_i^n$, and recall that:

- $U_i^{\pi_i}(x) := \int_{\omega \in \Omega_i} u_i(x_0, x_\omega) d\pi_i(\omega)$;
- $u_i^n(x^n) := \frac{1}{n \# \underline{\mathbf{S}}} \sum_{s \in \underline{\mathbf{S}}} u_i(x_0, x_s^n) + (1 - \frac{1}{n}) \sum_{\omega \in \Omega_i^n} u_i(x_0, x_\omega) \pi_i^n(\omega)$.

Then, from above, from relation (II), Lemma 2, and the uniform continuity of $x \in X_i$ and u_i on compact sets, there exists $N_2 \geqslant N_1$ such that:

$$(VI) \quad |U_i^{\pi_i}(x) \text{-} u_i^n(x^n)| < \int_{\omega \in \Omega_i} |u_i(x_0, x_\omega) \text{-} u_i(x_0, x_{\Phi_i^n(\omega)})| d\pi_i(\omega) + \tfrac{\varepsilon}{4} < \tfrac{\varepsilon}{2}, \text{ for every } n \geqslant N_2.$$

From equilibrium conditions and Lemma 4-(ii), there exists $N_3 \geqslant N_2$, such that:

$$(VII) \qquad u_i^n(x^n) \leqslant u_i^n(x_i^n) < \tfrac{\varepsilon}{2} + U_i^{\pi_i}(x_i^*), \text{ for every } n \geqslant N_3.$$

Let $n \geqslant N_3$ be given. The above Conditions (I)-(VI)-(VII) yield, jointly:

$U_i^{\pi_i}(x) < \frac{\varepsilon}{2} + u_i^n(x^n) \leqslant \frac{\varepsilon}{2} + u_i^n(x_i^n) < \varepsilon + U_i^{\pi_i}(x_i^*) < U_i^{\pi_i}(x)$.

This contradiction proves that $\mathcal{C}^*$ meets Condition (b) of Definition 2, hence, from above, is a C.F.E. of the economy $\mathcal{E}_{(\pi_i)}$. This completes the proof of Theorem 1. □

Appendix

Lemma 1 *Under Assumptions A1-A2, the following Assertion holds:*

(i) $\exists \delta > 0 : \Delta \subset \underline{\mathbf{S}} \times [\delta, 1]^L$.

Proof We introduce new notations and let, for all $(i, s, x := (x_0, x_s)) \in I \times \underline{\mathbf{S}} \times \mathbb{R}_+^{L \times L}$:

- $y \succ_s^i x$ denote a vector, $y := (y_0, y_s) \in \mathbb{R}_+^{L \times L}$, s.t. $u_i(y_0, y_s) > u_i(x_0, x_s)$ and $y_0 = x_0$;
- $\mathcal{A}_s := \{(x_i) := ((x_{i0}, x_{is})) \in \mathbb{R}_+^{L \times L \times I} : \sum_{i \in I} x_i = \sum_{i \in I} (e_{i0}, e_{is})\}$;
- $P_s := \{p \in \overline{\mathcal{P}}^* : \exists j \in I,\ \exists (x_i) \in \mathcal{A}_s,$ *such that* $(y \succ_s^j x_j) \Rightarrow (p_s \cdot y_s \geqslant p_s \cdot x_{js} \geqslant p_s \cdot e_{js})\}$.

The proof of Lemma 1 relies on the following Lemmata:

Lemmata 1 *The following Assertions hold:*

(i) $\forall s \in \underline{\mathbf{S}}$, P_s *is a closed, hence, compact set;*

(ii) $\exists\ \delta > 0 : \forall (s, l) \in \underline{\mathbf{S}} \times L,\ \forall p := (p_{s'}^l) \in P_s,\ p_s^l \geqslant \delta$.

Proof of Lemmata 1 Assertion (i) From the definition, for each $(n, s) \in \mathbb{N} \times \underline{\mathbf{S}}$, the set P_s contains $p^n \in \mathcal{P}^*$. Let $s \in \underline{\mathbf{S}}$ and a converging sequence $\{p^k\}_{k \in \mathbb{N}}$ of elements of P_s be given. Its limit, p, is in $\overline{\mathcal{P}}^*$, a closed set. We may assume there exist (a same) $j \in I$ and a sequence, $\{x^k\}_{k \in \mathbb{N}} := \{(x_i^k)\}_{k \in \mathbb{N}}$, of elements of $\mathcal{A}_s$, which meet the condition of the definition of P_s, and which converges to some $x := (x_i) \in \mathcal{A}_s$. Indeed, the sequence $\{x^k\}_{k\in\mathbb{N}}$ is bounded from the non-negativity and market clearing conditions in $\mathcal{A}_s$.

Then, the relations $p^k_s \cdot (x^k_{js} - e_{js}) \geqslant 0$, which hold from the definition, for each $k \in \mathbb{N}$, yield, in the limit, $p_s \cdot (x_{js} - e_{js}) \geqslant 0$. We show that (p, j, x) satisfies the conditions of the definition of P_s (hence, $p = \lim p^k \in P_s$ and P_s is closed).

By contraposition, assume that this is not the case, i.e., there exists $y \in \mathbb{R}^{L\times L}_+$, such that $y_0 = x_{j0}$, $u_j(x_{j0}, y_s) > u_j(x_{j0}, x_{js})$ and $p_s \cdot (y_s - x_{js}) < 0$. Then, we show:

(I) $\quad \forall K \in \mathbb{N}, \exists k > K, \ u_j(x^k_{j0}, y_s) > u_j(x^k_{j0}, x^k_{js})$.

If not, one has $u_j(x^k_{j0}, y_s) \leqslant u_j(x^k_{j0}, x^k_{js})$, for k big enough, which implies, in the limit $(k \to \infty)$, $u_j(x_{j0}, y_s) \leqslant u_j(x_{j0}, x_{js})$, in contradiction with the above opposite relation. Hence, relations (I) hold. From the definition of the sequence $\{x^k\}$, relations (I) imply $p^k_s \cdot (y_s - x^k_{js}) \geqslant 0$ (for each $k \in \mathbb{N}$ such that $u_j(x^k_{j0}, y_s) > u_j(x^k_{j0}, x^k_{js})$) and, in the limit $(k \to \infty)$, $p_s \cdot (y_s - x_{js}) \geqslant 0$, in contradiction with the above opposite inequality. This contradiction proves that $p := \lim p^k \in P_s$, hence, all P_s are compact. □

Assertion (ii) Let $(s, l) \in \underline{\mathbf{S}} \times L$ and $p := (p^{l'}_{s'}) \in P_s$ be given. Let $e \in \mathbb{R}^L$ have zero components but the l^{th}, equal to 1. We prove that $p^l_s = p_s \cdot e > 0$. Indeed, let $(p, j, (x_i)) \in P_s \times I \times \mathcal{A}_s$ meet the conditions of the definition of P_s. For every $n > 1$, we let $x^n_j \in \mathbb{R}^{L\times L}_+$ be such that $x^n_{js} := (1 - \frac{1}{n})x_{js}$ and $x^n_{j0} := x_{j0}$. It satisfies $p_s \cdot (x^n_{js} - x_{js}) < 0$ (since $p_s \cdot x_{js} \geqslant p_s \cdot e_{js} > 0$, from Assumption *A1* and the definition of $\mathcal{P}^*$).

Let $E := (0, e) \in \mathbb{R}^{L\times L}_+$. Along Assumption *A2*, there exists $n \in \mathbb{N}$, such that $y := (x^n_j + (1 - \frac{1}{n})E)$ satisfies $u_j(y_0, y_s) = u_j(x_{j0}, y_s) > u_j(x_{j0}, x_{js})$, implying $p_s \cdot x_{js} \leqslant p_s \cdot y_s = p_s \cdot (x^n_{js} + (1 - \frac{1}{n})e) < p_s \cdot x_{js} + (1 - \frac{1}{n})p_s \cdot e$. Hence, $p^l_s = p_s \cdot e > 0$. The mapping $\varphi_{(s,l)} : P_s \to \mathbb{R}_{++}$, defined by $\varphi_{(s,l)}(p) := p_s \cdot e$ is continuous and attains its minimum on the compact set P_s, say $\delta_{(s,l)} > 0$. Then, Assertion (ii) holds for $\delta := \min_{(s,l)\in\underline{\mathbf{S}}\times L} \ \delta_{(s,l)}$. □

Proof of Lemma 1 Referring to Definition 3, Lemma 1 results from the relation $\Lambda \subset \cap_{s\in\underline{\mathbf{S}}} P_s$, which holds from the definitions, and from Lemmata 1-(ii) above. $\square$

Lemma 2 *For each* $(i,n) \in I \times \mathbb{N}$*, we may choose the above* $\mathcal{P}_i^n$*,* Ω_i^n*,* Φ_i^n*, such that:*

(i) $\Omega_i^n \subset \Omega_i^{n+1}$ *and* $\mathcal{P}_i^{n+1}$ *is finer than* $\mathcal{P}_i^n$*;*

(ii) $\{V(\omega)\}_{\omega\in\cap_{i\in I}\Omega_i^n}$ *contains* $\#J$ *independent vectors, whenever* $n \geqslant \#J$*;*

(iii) $\cup_{n\in\mathbb{N}} \Omega_i^n$ *is everywhere dense in* Ω_i*;*

(iv) *for every* $\omega \in \Omega_i$*,* $\Phi_i^n(\omega)$ *converges uniformly to* ω*.*

Proof Let $i \in I$, $n \in \mathbb{N}$ and $K^n := \{1, ..., 2^{n-1}\}^L$ be given (letting $\mathbb{N}$ start from $n = 1$).

From the definition, $\Omega_i := \cup_{s\in S_i}\{s\} \times P_s^i \subset S \times P$. For each pair $(s, k := (k^l)) \in S_i \times K^n$, we define the (possibly empty) subset, $\Omega_{(i,n)}^{(s,k)} := \{s\} \times (P_s^i \cap \times_{l\in L}]\frac{k^l-1}{2^{n-1}}, \frac{k^l}{2^{n-1}}])$, of Ω_i. To simplify notations, we let $K_{(i,n)} := \#\ \{(s,k) \in S_i \times K^n : \pi_i(\Omega_{(i,n)}^{(s,k)}) > 0\}$ and identify the latter set, $\{(s,k) \in S_i \times K^n : \pi_i(\Omega_{(i,n)}^{(s,k)}) > 0\}$, to the subset, $\{1, ..., K_{(i,n)}\}$, of $\mathbb{N}$. Then, the partitions, $\mathcal{P}_i^n := \{\Omega_{(i,n)}^k\}_{1\leqslant k\leqslant K_{(i,n)}}$, of Ω_i are ever finer as $n \in \mathbb{N}$ increases.

For every integer, $k \leqslant K_{(i,n)}$, we choose one element, $\omega_{(i,n)}^k \in \Omega_{(i,n)}^k$, and just one. We may always construct the sets, $\Omega_i^n := \{\omega_{(i,n)}^k\}_{1\leqslant k\leqslant K_{(i,n)}}$, such that $\Omega_i^n \subset \Omega_i^{n+1}$, for every $n \in \mathbb{N}$, and, from Assumption *A3*, such that Assertion (ii) holds. Then, we define the mappings, Φ_i^n and π_i^n, as in sub-Section 4.1 and Lemma 2 holds. $\square$

Lemma 3 *The following Assertions hold:*

(i) *the price sequences* $\{p^n\}$*,* $\{q^n\}$ *may be assumed to converge, say to* $p^* := (p_s^*) \in \overline{\mathcal{P}}^*$ *and* $q^* \in Q$*, such that* $\omega_s^* := (s, p_s^*) \in \overline{\Delta} \subset \Omega_i$*, for each* $(i,s) \in I \times \underline{\mathbf{S}}$*; we let* $\omega_0^* := (p_0^*, q^*)$*;*

(ii) *the sequences* $\{(x_{is}^n)_{s\in\underline{\mathbf{S}}'}\}$ *and* $\{z_i^n\}$*, for each* $i \in I$*, may be assumed to converge, to* $(x_{i\omega_s^*}^*) := (x_{is}^*)_{s\in\underline{\mathbf{S}}'} \in \mathbb{R}_+^{L\times\underline{\mathbf{S}}'}$ *and* $z_i^* \in \mathbb{R}^J$*, s. t.* $\sum_{i\in I} (x_{is}^* - e_{is})_{s\in\underline{\mathbf{S}}'} = 0$ *and* $\sum_{i\in I}\ z_i^* = 0$*.*

Proof Assertion (i) is obvious from the definitions, Lemma 1, the relations $p^n \in \Lambda$ (see sub-Section 4.2) for each $n \in \mathbb{N}$, Assumption *A4* and compactness arguments.$\square$

Assertion (ii) The non-negativity and market clearance conditions over auxiliary equilibrium allocations imply that $\{(x_{is}^n)_{s\in\underline{\mathbf{S}}'}\}$ is bounded, hence, may be assumed to converge, for each $i \in I$. The market clearance conditions of equilibrium, $\sum_{i\in I}(x_{is}^n - e_{is})_{s\in\underline{\mathbf{S}}'} = 0$, which hold for each $n \in \mathbb{N}$, yield the limit: $\sum_{i\in I}(x_{is}^* - e_{is})_{s\in\underline{\mathbf{S}}'} = 0$.

By contraposition, assume that there exists an extracted sequence, $\{(z_i^{\varphi(n)})\}$, such that $\lim_{n\to\infty} k_n := \|(z_i^{\varphi(n)})\| = \infty$. We let $(z_i'^n) := \frac{(z_i^{\varphi(n)})}{\|(z_i^{\varphi(n)})\|}$ be such that $\|(z_i'^n)\| = 1$, for every $n \in \mathbb{N}$ and denote $\alpha := 1 + \|(e_i)\| > 0$. The bounded sequence $\{(z_i'^n)\}$ admits a cluster point, (z_i), such that $\|(z_i)\| = 1$. The definition of auxiliary equilibria yields:

$\sum_{i\in I} z_i'^n = 0$ and $V(\omega)\cdot z_i'^n \geqslant -\alpha/k_n,\ \forall (i,n,\omega) \in I \times \mathbb{N} \times \cap_{i\in I}\Omega_i^n$, and

$\sum_{i\in I} z_i = 0$ and $V(\omega)\cdot z_i \geqslant 0,\ \forall (i,\omega) \in I \times \cap_{i\in I}\Omega_i$, when passing to the limit.

The latter relations imply $V(\omega)\cdot z_i = 0$ for every $(i,\omega) \in I \times \cap_{i\in I}\Omega_i$, and, from Assertion *A3*, $(z_i) = 0 \in \mathbb{R}^{J\times I}$. This contradicts the above relation $\|(z_i)\| = 1$. Hence, the sequence $\{(z_i^n)_{i\in I}\}$ is bounded, and may be assumed to converge, say to $(z_i^*) \in \mathbb{R}^{J\times I}$. The relations $\sum_{i\in I} z_i^n = 0$, for each $n \in \mathbb{N}$, yield, in the limit, $\sum_{i\in I} z_i^* = 0$. $\square$

Lemma 4 *Let* $B_i(\omega,z) = \{x \in \mathbb{R}_+^L : p\cdot(x - e_{is}) \leqslant V(\omega)\cdot z\}$, *be given sets, for every* $z \in \mathbb{R}^J$ *and all* $\omega := (s,p) \in \Omega_i$. *Along Lemma 3, the following Assertions hold for each* $i \in I$:
(i) *the correspondence* $\omega \in \Omega_i \mapsto \arg\max\ u_i(x_{i0}^*, x)$, *for* $x \in B_i(\omega, z_i^*)$, *is a continuous mapping, whose embedding,* $x_i^* : \omega \in \Omega_i' \mapsto x_{i\omega}^*$, *is a consumption, that is,* $x_i^* \in X_i$*;*
(ii) $U_i^{\pi_i}(x_i^*) = \lim_{n\to\infty} u_i^n(x_i^n)$.

Proof Assertion (i) Let $i \in I$ be given and $\mathcal{C}^n := \{p^n, q^n, [(x_i^n, z_i^n)]\}$, for each $n \in \mathbb{N}$, be the auxiliary equilibrium chosen in sub-Section 4.2.

From Definition 4, Lemma 2 and Assumption $A2$, the relations $\{(x^n_{i\Phi^n_i(\omega)})\} = \arg\max_{x\in B_i(\Phi^n_i(\omega),z^n_i)} u_i(x^n_{i0}, x)$ hold for each $n \in \mathbb{N}$ and every $\omega \in \Omega_i$.

Let R be the subset of $\Omega_i \times \mathbb{R}^J$ upon which the correspondence B_i of Lemma 4 has non-empty values. These values are convex from the definitionof B_i, and compact from the definition of Ω_i. As standard from Berge Theorem (see, e.g., Debreu, 1959, p. 19) the correspondence (which is a mapping from Assumption $A2$), $(x_0, \omega, z) \in \mathbb{R}^L_+ \times R \mapsto \arg\max_{x\in B_i(\omega,z)}\ u_i(x_0, x)$, is continuous, since u_i is continuous, and, from the definition of Ω_i, B_i is also continuous (that is, upper and lower semicontinuous).

From Lemmas 2 and 3, the relations $(x^*_{i0}, \omega, z^*_i) = \lim_{n\to\infty}(x^n_{i0}, \Phi^n_i(\omega), z^n_i)$ hold for every $(i,\omega) \in I \times \Omega_i$. Hence, from Berge's theorem, the above relations, $\{x^n_{i\Phi^n_i(\omega)}\} = \arg\max_{x\in B_i(\Phi^n_i(\omega),z^n_i)} u_i(x^n_{i0}, x)$, for all $n \in \mathbb{N}$ and $\omega \in \Omega_i$, pass to the limit and yield a continuous mapping, $\omega \in \Omega_i \mapsto x^*_{i\omega} := \arg\max_{x\in B_i(\omega,z^*_i)} u_i(x^*_{i0}, x)$, whose embedding, $x^*_i : \omega \in \Omega'_i := \{0\} \cup \Omega_i \mapsto x^*_{i\omega}$, is a consumption of the economy $\mathcal{E}_{(\pi_i)}$, i.e., $x^*_i \in X_i$. □

Assertion (ii) Let $i \in I$ be given and $x^*_i \in X_i$ be defined from above. By the same token (with same notations as above), we let $\varphi_i : (x_0, \omega, z) \in \mathbb{R}^L_+ \times R \mapsto \arg\max_{x\in B_i(\omega,z)}\ u_i(x_0, x)$ be defined continuous on its domain. The continuity of u_i implies that of $U_i : (x_0, \omega, z) \in \mathbb{R}^L_+ \times R \mapsto u_i(x_0, \varphi_i(x_0, \omega, z))$. Moreover, the relations $(x^*_{i0}, \omega, z^*_i) = \lim_{n\to\infty}(x^n_{i0}, \Phi^n_i(\omega), z^n_i)$, $u_i(x^*_{i0}, x^*_{i\omega}) = U_i(x^*_{i0}, \omega, z^*_i)$ and $u_i(x^n_{i0}, x^n_{i\Phi^n_i(\omega)}) = U_i(x^n_{i0}, \Phi^n_i(\omega), z^n_i)$ hold, from Lemmas 2-3 and above, for every $(\omega, n) \in \Omega_i \times \mathbb{N}$. Hence, Lemma 2, the uniform continuity of u_i and U_i on compact sets, and the definitions of utilities, yield, for each $(i, n) \in I \times \mathbb{N}$:

(I) $\quad \forall\varepsilon > 0,\ \exists N \in \mathbb{N} : \forall n > N,\ \forall\omega \in \Omega_i,\ |\ u_i(x^*_{i0}, x^*_{i\omega}) - u_i(x^n_{i0}, x^n_{i\Phi^n_i(\omega)})\ |\ < \varepsilon.$

(II) $\quad U^{\pi_i}_i(x^*_i) := \int_{\omega\in\Omega_i} u_i(x^*_{i0}, x^*_{i\omega})d\pi_i(\omega);$

(III) $\quad u^n_i(x^n) := \frac{1}{n\#\underline{\mathbf{S}}} \sum_{s\in\underline{\mathbf{S}}} u_i(x^n_0, x^n_s) + (1 - \frac{1}{n}) \sum_{\omega\in\Omega^n_i} u_i(x^n_0, x^n_\omega)\pi^n_i(\omega).$

Then, Assertion (ii) results immediately from the relations (I)-(II)-(III) above. □

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