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Extending the Cass Trick

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Extending the Cass Trick

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Abstract

In a celebrated 1984 paper, David Cass provided an existence theorem for financial equilibria in incomplete markets with exogenous yields. The theorem showed that, when agents had symmetric information and ordered preferences, equilibria existed on purely financial markets and could be supported by any collection of state prices. This theorem built on the so-called "Cass trick", along which one agent in the economy had an Arrow-Debreu budget set, with one single budget constraint, while all other agents were constrained a la Radner (1972), that is, in every state of nature, given the financial transfers that the asset market permitted. The current paper extends Cass' theorem and the Cass trick to asymmetric information and non-ordered preferences. It shows that any collection of individual state prices under asymmetric information supports an equilibrium, provided one agent had full information. If the latter condition fails, the Cass trick cannot apply. A weaker result holds, namely, equilibrium exists under the no-arbitrage condition.

Key words: sequential equilibrium, perfect foresight, existence of equilibrium, rational expectations, incomplete markets, asymmetric information, arbitrage.

JEL Classification: D52

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1 Introduction

When agents have incomplete or asymmetric information, they seek to infer information from observing markets. A traditional response to that problem is given by the REE (rational expectations equilibrium) model by assuming, quoting Radner (1979), that "*agents have a 'model' or 'expectations' of how equilibrium prices are determined*". Under this assumption, agents know the map between private information signals and equilibrium prices, along a so-called "*forecast function*".

In the simplest setting with two periods, no production and an uncertainty over future states to prevail, Cornet-De Boisdeffre (2002) suggests an alternative approach to the REE, where asymmetric information is represented by private signals, informing each agent that tomorrow's true state will be in a subset of the state space. The latter paper generalizes the classical definitions of equilibrium, no-arbitrage prices and the no-arbitrage condition to asymmetric information. In this model, De Boisdeffre (2007) shows that equilibria exist on purely financial markets if they preclude arbitrage. That no-arbitrage condition, which typically holds under asymmetric information, may always be reached by agents observing asset prices or available financial transfers. Along Cornet-De Boisdeffre (2009), or De Boisdeffre (2016), that learning process requires no price model. Such results differ from Radner's (1979) generic existence of a fully revealing REE.

In this setting, which drops rational expectations, the current paper provides new insights on the existence issue. It examines whether the so-called "*Cass trick*" applies and yields the same results as in the symmetric information case. The answer is negative, in general. However, in a setting where one agent detains all the information of the other agents, Cass' theorem and the Cass trick apply. The Cass trick is a device introduced in Radner's (1972) budget sets and equilibria, which consists in replacing the budget constraints of one agent by a single Arrow-Debreu constraint at the first period, and let the other agents' budget sets unchanged, that is, with one constraint in each state.

This device enables to define asset prices relative to individual state prices, as the weighted sum of payoffs across states. It permits to show that any collection of state prices supports an equilibrium. The current paper extends this result to asymmetric information and to non-ordered preferences, whenever one agent is fully informed. In other cases, the Cass trick cannot apply, and the existence of equilibrium is simply characterized by the no-arbitrage condition, along De Boisdeffre (2007). Equilibria are then supported by some, out of typically many, collections of state prices.

The paper is so organized: Section 2 presents the model. Section 3 states the existence Theorem. Section 4 proves the Theorem. An Appendix proves Lemmas.

2 The model

We consider a pure-exchange financial economy with two periods, $t \in \{0, 1\}$, and an uncertainty, at $t = 0$, upon which state of nature will randomly prevail at $t = 1$. The economy is finite in the sense that the sets, I, S, L and J, respectively, of consumers, states of nature, consumption goods and assets are all finite. The observed state at $t = 0$ is denoted by $s = 0$ and we let $\Sigma' := \{0\} \cup \Sigma$, whenever $\Sigma \subset S$.

2.1 Markets and information

Agents consume or exchange the consumption goods, $l \in L$, on both periods' spot markets. At $t = 0$, each agent, $i \in I$, receives privately the correct information that tomorrow's true state will be in a subset, S_i, of S. We assume costlessly that $S = \cup_{i \in I} S_i$. Thus, the pooled information set, $\underline{\mathbf{S}} := \cap_{i \in I} S_i$, contains the true state, and the relation $\underline{\mathbf{S}} = S$ characterizes symmetric information.

We let $P := \{p \in \mathbb{R}^{L \times \underline{\mathbf{S}}'} : \|p\| \leqslant 1\}$ be the set of admissible commodity prices, which each agent is assumed to observe, or anticipate perfectly, a la Radner (1972). Moreover, each agent with an incomplete information forms her private forecasts in the unrealizable states. Such forecasts, (s, p_s^i), are pairs of a state, $s \in S_i \backslash \underline{\mathbf{S}}$, and a price, $p_s^i \in \mathbb{R}^L_{++}$, that the generic i^{th} agent believes to be the spot price in state s.

Agents may operate financial transfers across states in S' (actually in $\underline{\mathbf{S}}'$) by exchanging, at $t = 0$, finitely many nominal assets, $j \in J$, which pay off, at $t = 1$, conditionally on the realization of the state. We assume that $\#J \leqslant \#\underline{\mathbf{S}}$. Payoffs define a $S \times J$ matrix, V, whose generic row in state $s \in S$, denoted by $V(s) \in \mathbb{R}^J$, does not depend on prices. Thus, at asset price, $q \in \mathbb{R}^J$, agents may buy or sell unrestrictively portfolios of assets, $z = (z_j) \in \mathbb{R}^J$, for $q \cdot z$ units of account at $t = 0$, against the promise of delivery of a flow, $V(s) \cdot z$, of conditional payoffs across states.

2.2 The consumer's behaviour and concept of equilibrium

Each agent, $i \in I$, receives an endowment, $e_i := (e_{is})$, granting the commodity bundles, $e_{i0} \in \mathbb{R}^L_+$ at $t = 0$, and $e_{is} \in \mathbb{R}^L_+$, in each expected state, $s \in S_i$, if it prevails. Given the market prices, $p := (p_s) \in P$ and $q \in \mathbb{R}^J$, and her forecasts, the generic i^{th} agent's consumption set is $X_i := \mathbb{R}^{L \times S_i'}_+$ and her budget set is defined as follows:

$$B_i(p,q) := \{ \ (x,z) \in X_i \times \mathbb{R}^J : p_0 \cdot (x_0 - e_{i0}) \leqslant -q \cdot z \ \ and \ \ p_s \cdot (x_s - e_{is}) \leqslant V(s) \cdot z, \ \forall s \in \underline{\mathbf{S}}$$
$$and \ \ p_s^i \cdot (x_s - e_{is}) \leqslant V(s) \cdot z, \ \forall s \in S_i \backslash \underline{\mathbf{S}} \ \}.$$

Each consumer, $i \in I$, is endowed with a complete preordering, $\precsim_i$, over her consumption set, representing her preferences. Her strict preferences, $\prec_i$, are represented, for each $x \in X_i$, by the set, $P_i(x) := \{ y \in X_i : x \prec_i y \}$, of consumptions which are strictly preferred to x. In the above economy, denoted by $\mathcal{E} = \{(I, S, L, J), V, (S_i)_{i\in I}, (p^i_s)_{(i,s)\in I\times S_i\setminus\underline{\mathbf{S}}}, (e_i)_{i\in I}, (\precsim_i)_{i\in I}\}$, agents optimise their consumptions in the budget sets. This yields the following concept of equilibrium:

Definition 1 *A collection of prices,* $p = (p_s) \in P$, $q \in \mathbb{R}^J$, *& decisions,* $(x_i, z_i) \in B_i(p, q)$, *for each* $i \in I$, *is an equilibrium of the economy,* $\mathcal{E}$, *if the following conditions hold:*

(a) $\forall i \in I,\ (x_i, z_i) \in B_i(p, q)$ *and* $P_i(x_i) \times \mathbb{R}^J \cap B_i(p, q) = \varnothing$;

(b) $\sum_{i\in I} (x_{is} - e_{is}) = 0,\ \forall s \in \underline{\mathbf{S}}'$;

(c) $\sum_{i\in I} z_i = 0$.

The economy, $\mathcal{E}$, is called standard if it meets the following conditions:

Assumption A1 *(monotonicity):* $\forall (i, x, y) \in I\times(X_i)^2,\ (x \leqslant y,\ x \neq y) \Rightarrow (x \prec_i y)$;

Assumption A2 *(strong survival):* $\forall i \in I,\ e_i \in \mathbb{R}^{L\times S'_i}_{++}$;

Assumption A3*:* $\forall i \in I$, $\prec_i$ *is lower semicontinuous convex-open-valued and such that* $x \prec_i x + \lambda(y - x)$, *whenever* $(x, y, \lambda) \in X_i \times P_i(x)\times\ ]0, 1]$;

Assumption A4*:* $\exists z \in \mathbb{R}^J$, $V(s) \cdot z > 0$, $\forall s \in \underline{\mathbf{S}}$.

We end this Section with a standard Claim, which will serve later.

Claim 1 *Let* $q \in \mathbb{R}^J$ *be given. The following Assertions are equivalent:*

(i) $\nexists (i, z) \in I \times \mathbb{R}^J$, $-q \cdot z \geqslant 0$ *and* $V(s) \cdot z \geqslant 0$, $\forall s \in S_i$, *with one strict inequality;*

(ii) $\forall i \in I$, $\exists \lambda_i := (\lambda_{is}) \in \mathbb{R}^{S_i}_{++}$, $q = \sum_{s\in S_i} \lambda_{is} V(s)$.

Proof See Cornet-De Boisdeffre (2002, Lemma 1, p. 398). □

We henceforth assume, at no cost from Cornet-De Boisdeffre (2009), that the structure, (S_i), is arbitrage-free, that is, admits an asset price, $q \in \mathbb{R}^J$, which meets the conditions of Claim 1. Indeed, the latter paper shows that agents, starting from any information structure, (S_i), may always infer from observing markets, and with no price model, a refined information structure, which is arbitrage-free.

3 The existence Theorem and proof

Theorem 1 *Let* $(\lambda_{is}) \in \times_{i\in I}\mathbb{R}^{S_i}_{++}$ *be a collection of individual state prices, i.e., of scalars which meet the conditions of Claim 1-*(ii) *above, for some* $q \in \mathbb{R}^J$*. Assume that one agent in a standard economy,* $\mathcal{E}$*, say* $i = 1$*, has full information, that is,* $S_1 = \underline{\mathbf{S}}$*. Then, the economy,* $\mathcal{E}$*, admits an equilibrium,* $(p, q, [(x_i, z_i)]) \in P\times\mathbb{R}^J\times(\times_{i\in I}B_i(p, q))$*, such that* $p := (p_s) \in \mathbb{R}^{L\times\underline{\mathbf{S}}'}_{++}$ *and* $q = \sum_{s\in S_i} \lambda_{is}V(s)$*, for every* $i \in I$*.*

To prove Theorem 1, we henceforth set as given arbitrary state prices, $(\lambda_i) \in \times_{i\in I}\mathbb{R}^{S_i}_{++}$, which satisfy the conditions of Claim 1, we assume that agent $i = 1$ has full information and let $q = \sum_{s\in S_1} \lambda_{1s}V(s)$ be given.

The proof's main argument is the Gale-Mas-Colell (1975, 1979) fixed-point-like theorem, henceforth GMC. First, we need define, for each agent, $i \in I$, and for markets, lower semi-continuous reaction correspondences over a convex compact set. Thus, Sub-Section 1 presents an auxiliary compact economy, with slightly modified budget sets. Sub-Section 2 defines the reaction correspondences and applies the GMC theorem to them. This yields a so-called (with slight abuse) "*fixed point*". Sub-Section 3 shows this fixed point defines an equilibrium.

3.1 An auxiliary compact economy with modified budget sets

For every pair $(i, p) \in I\times P$, we let $Z_i := \{z \in \mathbb{R}^J : V(s)\cdot z = 0, \forall s \in S_i\}$, its orthogonal, $Z_i^\perp$, and $Z_i^\perp(p) := \{z \in Z_i^\perp : \exists x \in X_i,\ (x, z) \in B_i(p, q)\}$ meet the following property:

Lemma 1 $\exists r_1 > 0 : \forall p \in P,\ \forall i \in I,\ \forall z \in Z_i^{\perp}(p),\ \|z\|\ < r_1$.

Proof : see the Appendix. □

Along Lemma 1, for each $i \in I$ and all $p := (p_s) \in P$, we let $Z_i^* := \{z \in Z_i^{\perp} : \|z\| \leqslant r_1\}$, and define the following modified budget sets:

$B_1(p) := \{x \in X_1 : p_0 \cdot (x_0 - e_{10}) + \sum_{s \in \underline{\mathbf{S}}} \lambda_{1s} p_s \cdot (x_s - e_{1s}) \leqslant 1\}$, and

$B_i(p) := \{\ (x, z) \in X_i \times Z_i^* : p_0 \cdot (x_0 - e_{i0}) + \sum_{s \in \underline{\mathbf{S}}} \lambda_{1s} p_s \cdot (x_s - e_{is}) \leqslant 1$

$\quad$ *and* $p_s \cdot (x_s - e_{is}) \leqslant V(s) \cdot z,\ \forall s \in \underline{\mathbf{S}}$

$\quad$ *and* $p_s^i \cdot (x_s - e_{is}) \leqslant V(s) \cdot z,\ \forall s \in S_i \backslash \underline{\mathbf{S}}\ \}$, for every $i \in I \backslash \{1\}$;

$\mathcal{A}(p) := \{\ [x_1, (x_i, z_i)] \in \times_{i \in I}\ B_i(p) :\ \sum_{i \in I}(x_{is} - e_{is}) = 0,\ \forall s \in \underline{\mathbf{S}}'\ \}$.

These sets meet the following boundary condition:

Lemma 2 $\exists r_2 > 0 : \forall p \in P,\ \forall\ [x_1, (x_i, z_i)] \in \mathcal{A}(p),\ \sum_{i \in I}\ \|x_i\|\ < r_2$.

Proof : Let $p \in P$ and $[x_1, (x_i, z_i)] \in \mathcal{A}(p)$ be given.

The relations, $x_{is} \in [0, e]^L$, where $e := \sum_{i \in I} \|e_i\|$, hold, for every $(i, s) \in I \times \underline{\mathbf{S}}'$, from the non-negativity and market clearing conditions on $\mathcal{A}(p)$. Then, Lemma 2 stems from the compactness of Z_i^* and the relations $p_s^i \in \mathbb{R}_{++}^L$, for each $(i, s) \in I \times S_i \backslash \underline{\mathbf{S}}$. □

Along Lemma 2, for every $(i, p := (p_s)) \in I \times P$, we let $X_i^* := \{x \in X_i : \|x\| \leqslant r_2\}$ and define the following convex compact sets:

$B_1'(p) := \{x \in X_1^* : p_0 \cdot (x_0 - e_{i0}) + \sum_{s \in \underline{\mathbf{S}}} \lambda_{1s} p_s \cdot (x_s - e_{1s}) \leqslant \gamma_p\}$, and for each $i \in I \backslash \{1\}$,

$B_i'(p) := \{\ (x, z) \in X_i^* \times Z_i^* : p_0 \cdot (x_0 - e_{i0}) + \sum_{s \in \underline{\mathbf{S}}} \lambda_{1s} p_s \cdot (x_s - e_{is}) \leqslant \gamma_p$

$\quad$ *and* $p_s \cdot (x_s - e_{is}) \leqslant V(s) \cdot z,\ \forall s \in \underline{\mathbf{S}}$

$\quad$ *and* $p_s^i \cdot (x_s - e_{is}) \leqslant V(s) \cdot z,\ \forall s \in S_i \backslash \underline{\mathbf{S}}\ \}$,

where $\gamma_p := 1 - \|p\|$, so that $B_i'(p) \subset B_i(p)$, for every $i \in I$.

The auxiliary economy is alike that of Section 2, up to the change in budget sets. We notice that the first period budget constraint is defined with reference to one unique collection of state prices, (λ_{1s}). Agents' behaviours are replaced by reaction correspondences, presented hereafter. Their budget sets satisfy Claim 2.

Claim 2 *For every $i \in I$, B'_i is upper semicontinuous.*

Proof Let $i \in I$ be given. The correspondence B'_i is, as standard, upper semicontinuous, for having a closed graph in a compact set. □

3.2 The fixed-point-like argument

Budget sets were modified in sub-section 3.1, so that their interiors be non-empty. This was required to prove the lower semi-continuity of the reaction correspondences under Lemma 3, below. For every $p \in P$, these interior budget sets are as follows:

$$B''_1(p) := \{x \in X^*_1 : p_0 \cdot (x_0 - e_{10}) + \textstyle\sum_{s \in \underline{\mathbf{S}}} \lambda_{1s} p_s \cdot (x_s - e_{1s}) < \gamma_p\} \quad \text{and}$$

$$B''_i(p) := \{\ (x, z) \in X^*_i \times Z^*_i : p_0 \cdot (x_0 - e_{i0}) + \textstyle\sum_{s \in \underline{\mathbf{S}}} \lambda_{1s} p_s \cdot (x_s - e_{is}) < \gamma_p$$
$$\text{and} \ \ p_s \cdot (x_s - e_{is}) < V(s) \cdot z, \ \forall s \in \underline{\mathbf{S}}$$
$$\text{and} \ \ p^i_s \cdot (x_s - e_{is}) < V(s) \cdot z, \ \forall s \in S_i \backslash \underline{\mathbf{S}}\ \}, \quad \text{for every } i \in I \backslash \{1\}.$$

Claim 3 *The following Assertions hold, for each $i \in I$:*

(i) $\forall p \in P$, $B''_i(p) \neq \varnothing$;

(ii) *the correspondence B''_i is lower semicontinuous.*

Proof Let $p \in P$ and $i \in I \backslash \{1\}$ be given. Assertion (i) The non vacuity of $B''_1(p)$ is obvious from Assumption *A2* and the definition of $B''_1(p)$. From Assumption *A2* and the definition of forecasts (i.e., $(p^i_s) \in \mathbb{R}^{L \times S_i \backslash \underline{\mathbf{S}}}_{++}$) and of $B''_i(p)$, we may choose $x \in X^*_i$, such that $(x, 0)$ meets the (weak) budget constraints of $B'_i(p)$ in all states $s \in S'_i$, and with a strict inequality in every state, $s \in \underline{\mathbf{S}}$, such that $p_s \neq 0$ or $s \notin \underline{\mathbf{S}}$. Then, from Assumption *A4* and the definition, there exists $z \in \mathbb{R}^J$, such that $(x, z) \in B''_i(p)$. □

Assertion (ii) The convexity of $B''_i(p)$ (for $i \in I$) holds and yields, from Assertion (i), $B'_i(p) = \overline{B''_i(p)}$. From the continuity of the scalar product, the correspondences B''_i and B''_1 are lower semicontinuous for having an open graph in a compact set. □

We now introduce an agent representing markets $(i = 0)$ and a reaction correspondence, for each agent, on the convex compact set, $\Theta := P \times X_1^* \times (\times_{i \in I \backslash \{1\}} X_i^* \times Z_i^*)$. Thus, we let, for each $i \in I \backslash \{1\}$ and every $\theta := (p, [x_1, (x_i, z_i)]) \in \Theta$:

$$\Psi_0(\theta) := \{p' \in P : \textstyle\sum_{s \in \underline{\mathbf{S}}'} [(p'_s - p_s) \cdot \sum_{i \in I} (x_{is} - e_{is})] > 0\};$$

$$\Psi_1(\theta) := \left\{ \begin{array}{lll} B'_1(p) & if & x_1 \notin B'_1(p) \\ B''_1(p) \cap P_1(x_1) & if & x_1 \in B'_1(p) \end{array} \right\};$$

$$\Psi_i(\theta) := \left\{ \begin{array}{lll} B'_i(p) & if & (x_i, z_i) \notin B'_i(p) \\ B''_i(p) \cap P_i(x_i) \times Z_i^* & if & (x_i, z_i) \in B'_i(p) \end{array} \right\};$$

Lemma 3 *For each $i \in I \cup \{0\}$, Ψ_i is lower semicontinuous.*

Proof See the Appendix. □

We can now apply a fixed-point argument to the above reaction correspondences:

Claim 4 *There exists $\theta^* := (p^*, [x_1^*, (x_i^*, z_i^*)]) \in \Theta$, such that:*

(i) $\forall p \in P$, $\sum_{s \in \underline{\mathbf{S}}'} [(p_s^* - p_s) \cdot \sum_{i \in I} (x_{is}^* - e_{is})] \geqslant 0$;

(ii) $x_1^* \in B'_1(p^*)$ *and* $B''_1(p^*) \cap P_1(x_1^*) = \varnothing$;

(iii) $\forall i \in I \backslash \{1\}$, $(x_i^*, z_i^*) \in B'_i(p^*)$ *and* $B''_i(p^*) \cap P_i(x_i^*) \times Z_i^* = \varnothing$.

Proof Quoting Gale-Mas-Colell (1975, 1979): "*Given $X = \times_{i=1}^m Xi$, where X_i is a non-empty compact convex subset of $\mathbb{R}^n$, let $\varphi_i : X \to X_i$ be m convex (possibly empty) valued correspondences, which are lower semicontinuous. Then, there exists*

x in X such that for each i either $x_i \in \varphi_i(x)$ or $\varphi_i(x) = \varnothing$". The correspondences Ψ_i, for each $i \in I \cup \{0\}$, meet all conditions of the above theorem and yield Claim 4. $\square$

3.3 An equilibrium of the economy $\mathcal{E}$

The above fixed point, θ^*, meets the following properties, proving Theorem 1:

Claim 5 *Let $\theta^* := (p^*, [x_1^*, (x_i^*, z_i^*)]) \in \Theta$, along Claim 3, and $z_1^* := -\sum_{i \in I \backslash \{1\}} z_i^*$ be given. The following Assertions hold:*

(i) $[x_1^*, (x_i^*, z_i^*)] \in \mathcal{A}(p^*)$, hence, $\sum_{i \in I} \|x_i^*\| < r_2$;

(ii) $x_1^* \in B_1'(p^*)$ and $B_1'(p^*) \cap P_1(x_1^*) = \varnothing$;

(iii) for every $i \in I \backslash \{1\}$, $(x_i^*, z_i^*) \in B_i'(p^*)$ and $B_i'(p^*) \cap P_i(x_i^*) \times Z_i^* = \varnothing$;

(iv) $(p^*, q, [(x_i^*, z_i^*)_{i \in I}])$ is an equilibrium of the economy $\mathcal{E}$, such that $p^* \in \mathbb{R}_{++}^{L \times \underline{\mathbf{S}}'}$.

Proof Assertion (i) For every $s \in \underline{\mathbf{S}}'$, the relation $p_s^* \cdot \sum_{i \in I} (x_{is}^* - e_{is}) \geqslant 0$ holds from Claim 4-(i), and with strict inequality whenever $\sum_{i \in I} (x_{is}^* - e_{is}) \neq 0$.

Assume, by contraposition, that $\sum_{i \in I} (x_{is}^* - e_{is}) \neq 0$, for some $s \in \underline{\mathbf{S}}'$. Then, from Claim 4, $\gamma_{p^*} = 0$, and the budget constraints, $p_0 \cdot (x_{i0}^* - e_{i0}) + \sum_{s \in \underline{\mathbf{S}}} \lambda_{1s} p_s \cdot (x_{is}^* - e_{is}) \leqslant 0$, hold for every $i \in I$. Summing them up (for $i \in I$), yields, from above:

$$0 < p_0 \cdot \sum_{i \in I} (x_{i0}^* - e_{i0}) + \sum_{s \in \underline{\mathbf{S}}} \lambda_{1s} p_s \cdot \sum_{i \in I} (x_{is}^* - e_{is}) \leqslant 0.$$

This contradiction proves that $\sum_{i \in I} (x_{is}^* - e_{is}) = 0$, for every $s \in \underline{\mathbf{S}}'$. Then, it follows from Claim 4-(ii) and Lemma 2 that $[x_1^*, (x_i^*, z_i^*)] \in \mathcal{A}(p^*)$ and $\sum_{i \in I} \|x_i^*\| < r_2$. $\square$

Assertion (iii) (and Assertion (ii) alike) Let $i \in I \backslash \{1\}$ be given. From Claim 4-(ii), we need only show that $B_i'(p^*) \cap P_i(x_i^*) \times Z_i^* = \varnothing$. Assume, by contraposition, that there exists $(x_i, z_i) \in B_i'(p^*) \cap P_i(x_i^*) \times Z_i^*$.

From Claim 3, there exists $(x_i', z_i') \in B_i''(p^*) \subset B_i'(p^*)$. By construction, the relations $(x_i^n, z_i^n) := [\frac{1}{n}(x_i', z_i') + (1 - \frac{1}{n})(x_i, z_i)] \in B_i''(p^*)$ hold, for every $n \in \mathbb{N}$. From Assumption *A3*, the relation $(x_i^N, z_i^N) \in P_i(x_i^*) \times Z_i^*$ also holds, for $N \in \mathbb{N}$ big enough, which implies the relation $(x_i^N, z_i^N) \in B_i''(p^*, q^*) \cap P_i(x_i^*) \times Z_i^*$, in contradiction with Claim 4-(ii). □

Assertion (iv) The relation $p_s^* \in \mathbb{R}_{++}^{L \times \mathbf{S}'}$ is standard from Assertions (i)-(ii)-(iii) and Assumptions *A1-A2*. From Assertions (i)-(ii)-(iii) and Assumption *A1*, agents' budget constraints hold with equality. Then, from Assertion (i), the relations $\sum_{i \in I} (x_{is}^* - e_{is}) = 0$, for $s \in \underline{\mathbf{S}}$, yield: $p_s^* \cdot (x_{1s}^* - e_{1s}) = -\sum_{i \in I \backslash \{1\}} p_s^* \cdot (x_{is}^* - e_{is}) = -\sum_{i \in I \backslash \{1\}} V(s) \cdot z_i^* = V(s) \cdot z_1^*$.

Summing the above relations with state prices yields:

$$p_0^* \cdot (x_{10}^* - e_{10}) + \sum_{s \in \underline{\mathbf{S}}} \lambda_{1s}\, p_s^* \cdot (x_{1s}^* - e_{1s}) = p_0^* \cdot (x_{10}^* - e_{10}) + \sum_{s \in \underline{\mathbf{S}}} \lambda_{1s} V(s) \cdot z_1^* = p_0^* \cdot (x_{10}^* - e_{10}) + q \cdot z_1^*.$$

Similarly, for every $i \in I \backslash \{1\}$, the following relations stem from Assertion (i)-(iii):

$$p_0^* \cdot (x_{i0}^* - e_{i0}) + \sum_{s \in \underline{\mathbf{S}}} \lambda_{1s}\, p_s^* \cdot (x_{is}^* - e_{is}) = p_0^* \cdot (x_{i0}^* - e_{i0}) + \sum_{s \in \underline{\mathbf{S}}} \lambda_{1s} V(s) \cdot z_i^* = p_0^* \cdot (x_{i0}^* - e_{i0}) + q \cdot z_i^*.$$

It follows from above that the saturated budget constraints, $p_0^* \cdot (x_{i0}^* - e_{i0}) + q \cdot z_i^* = \gamma_{p^*}$, hold, for every $i \in I$. Summing them up yields, from Assertion (i):

$$p_0^* \cdot \sum_{i \in I} (x_{i0}^* - e_{i0}) + q \cdot \sum_{i \in I} z_i^* = 0 = \#I\ \gamma_{p^*}.$$

Hence, $\gamma_{p^*} = 0$. It results from above that $(x_i^*, z_i^*) \in B_i(p^*, q)$, for every $i \in I$, hence, $\|z_i^*\| < r_1$, fom Lemma 1. Then, from Assertions (i)-(ii)-(iii), Lemmas 1-2 and above, (x_i^*, z_i^*) is optimal in $B_i(p^*, q)$, for every $i \in I \backslash \{1\}$, and x_1^* is optimal in $B_1'(p^*)$, which is (with slight abuse) a bigger budget set than $B_1(p^*, q)$. Since $(x_1^*, z_1^*) \in B_1(p^*, q)$, the decision (x_1^*, z_1^*) is also optimal in $B_1(p^*, q)$. Hence, the collection, $\mathcal{C} := (p^*, q, [(x_i^*, z_i^*)_{i \in I}])$, meets Condition (a) of Definition 1 of equilibrium. From Assertion (i) and the definition of $z_1^* := -\sum_{i \in I \backslash \{1\}} z_i^*$, $\mathcal{C}$ also meets Conditions (b)-(c) of Definition 1. □

The proof of Claim 5 shows why a fully informed agent is needed to apply the Cass trick. It is because budget constraints are saturated and commodity markets clear (in realizable states) that the optimal decision (x_1^*, z_1^*) belongs to the budget set $B_1(p^*, q)$. Nothing guarantees this outcome otherwise. If no agent is fully informed, Cass' existence result needs be replaced by De Boisdeffre's (2007) weaker one, which characterizes the existence of equilibrium by the no-arbitrage condition.

Appendix

Lemma 1 $\exists r_1 > 0 : \forall p \in P,\ \forall i \in I,\ \forall z \in Z_i^{\perp}(p),\ \|z\|\ < r_1$

Proof Let $\delta = 1 + \sum_{i \in I} \|e_i\|$ and $i \in I$ be given. Assume, by contraposition, that, for every $k \in \mathbb{N}$, there exist $p^k \in P$ and $z^k \in Z_i^{\perp}(p^k)$, such that $\|z_i^k\| = \alpha_k > k$. For every $k \in \mathbb{N}$, let $z_i'^k := z_i^k/\alpha_k$. The bounded sequence, $\{z_i'^k\}$, admits a cluster point, $z_i \in Z_i^{\perp}$, such that $\|z_i\| = 1$. For each $k \in \mathbb{N}$, the relations $z^k \in Z_i^{\perp}(p^k)$ hold and imply:

$V(s) \cdot z'^k \geqslant -\delta/k$, for every $s \in S_i$, and $-q \cdot z'^k \geqslant -\delta/k$, which yields, in the limit,

$V(s) \cdot z \geqslant 0$, for every $s \in S_i$, and $-q \cdot z \geqslant 0$.

Since $q = \sum_{s \in S_i}\ \lambda_{is} V(s)$, the latter relations imply $z \in Z_i \cap Z_i^{\perp} = \{0\}$, from Claim 1 and above, which contradicts the above relation $\|z\| = 1$ and proves Lemma 1. □

Lemma 3 *For each* $i \in I \cup \{0\}$, Ψ_i *is lower semicontinuous.*

Proof The correspondences Ψ_0 is lower semicontinuous for having an open graph.

We now set $i \in I \backslash \{1\}$ and $\theta \in \Theta$ as given (the proof is similar for $i = 1$).

- Assume that $(x_i, z_i) \notin B_i'(p)$. Then, $\Psi_i(\theta) = B_i'(p)$.

Let V be an open set in $X_i^* \times Z_i^*$, such that $V \cap B_i'(p) \neq \varnothing$. It follows from the convexity of $B_i'(p)$ and the non-emptyness of the open set $B_i''(p)$ that $V \cap B_i''(p) \neq \varnothing$. From Claim 3, there exists a neighborhood U of p, such that $V \cap B_i'(p') \supset V \cap B_i''(p') \neq \varnothing$, for every $p' \in U$.

Since $B_i'(p)$ is nonempty, closed, convex in the compact set $X_i^* \times Z_i^*$, there exist open sets V_1 and V_2 in $X_i^* \times Z_i^*$, such that $(x_i, z_i) \in V_1$, $B_i'(p) \subset V_2$ and $V_1 \cap V_2 = \varnothing$. From Claim 2, there exists a neighborhood $U_1 \subset U$ of p, such that $B_i'(p') \subset V_2$, for every $p' \in U_1$. Let $W = U_1 \times (\times_{j \in I} W_j)$, where $W_i := V_1$, $W_1 := X_1^*$ and $W_j := X_j^* \times Z_j^*$, for each $j \in I \backslash \{i, 1\}$. Then, W is a neighborhood of θ, such that $\Psi_i(\theta') = B_i'(p')$, and $V \cap \Psi_i(\theta') \neq \varnothing$, for every $\theta' \in W$. Thus, Ψ_i is lower semicontinuous at θ. □

- Assume that $(x_i, z_i) \in B_i'(p)$, i.e., $\Psi_i(\theta) = B_i''(p) \cap P_i(x) \times Z_i^*$.

Lower semicontinuity results fromthe definition if $\Psi_i(\theta) = \varnothing$. Assume $\Psi_i(\theta) \neq \varnothing$. We recall that P_i (from Assumption *A3*) is lower semicontinuous with open values and that B_i'' has an open graph. As corollary, the correspondence $(p', [x_1', (x_i', z_i')]) \in \Theta \mapsto B_i''(p') \cap P_i(x_i') \times Z_i^* \subset B_i'(p')$ is lower semicontinuous at θ. Then, from the latter inclusions, Ψ_i is lower semicontinuous at θ. □

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