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WP 1822 – October 2018

Team production

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Network games, centrality measures, spillovers, bipartite network

JEL codes:
C72, D85, L14

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Abstract

This paper investigates games played on bipartite networks by introducing a team production function allowing for any pattern of cross effects between projects and cross effects between agents. By using a new representation of a bipartite network through a multilayers network, we are able to characterize the interior equilibrium efforts as a function of agents centralities in the multilayers network.

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1. Introduction

Since the work of Ballester et al. (2006) hereafter BCZ, network games with single effort choices and linear best replies have been widely studied. A recent work by Hsieh et al. (2018) introduces a bipartite network game where agents choose multiple efforts across different projects. Their framework can cover a wide range of topics from economics of innovation to economics of crime.
Games on bipartite network should be transformed in order to be solved. This transformation induces a loss of information. Hsieh et al. (2018) propose the use of line graphs (Whitney, 1932) as a solution to minimize this loss. This allows them to examine situations where there is either complementarity or substitutability between efforts made by different agents in a same project, and either congestion or complementarity between efforts of a same agent in various projects with respect to cost. However, in some cases both complementarity and substitutability across agents efforts, as well as both congestion and complementarity across projects in terms of cost reduction, occur.
Our model allows to find the Nash equilibrium efforts levels in these situations as a function of agents centrality. It is built on three pillars: the first one is a team production function inspired from the work of Hsieh et al. (2018), the second one is a new representation of the bipartite network through two others networks, and the last one is a two BCZ-like decomposition of cross-effects, i.e effects across agent's efforts on projects value and effects across agent's efforts on total cost incurred by this agent.

2. Model setup and definitions

Similarly to Hsieh et al. (2018), we consider a fixed and finite set of agents $\mathcal{N}=\{1,2,\ldots,n\}$ and a finite set of projects $\mathcal{P}=\{1,2,\ldots,p\}$. Denote a bipartite network by $\mathcal{B}=(\mathcal{N}\cup\mathcal{P},\mathcal{E})$, where $\mathcal{E}\subseteq\mathcal{N}\times\mathcal{P}$ (see Figure 1 for an example). Each agent i involved in a project s chooses an effort $e_{is}\in\mathbb{R}_+$. The production associated to a project s is a function of efforts made in this project and in other projects. Finally the production associated to the whole network is the sum of the production carried out on all projects.

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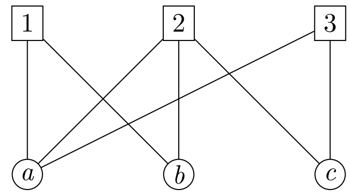


Figure 1: The bipartite network $\mathcal{B}$ of agents (round circles) and projects (squares).

2.1. The production function

To model the value produced in a project $s \in \mathcal{P}$, we adapt the team production function introduced by Hsieh et al. (2018), adding concavity of efforts and a term which represents spillovers across projects (inter-team production effect). The team production function for a project $s \in \mathcal{P}$ is given by

$$Y_s(\mathcal{B}, \mathbf{e}) = (\sum_{i\in\mathcal{N}_s} \alpha_i^s e_{is} + \frac{1}{2}\sum_{i\in\mathcal{N}_s} \sigma_{is,is} e_{is}^2 + \frac{1}{2}\sum_{i\in\mathcal{N}_s}\sum_{j\in\mathcal{N}_s\setminus\{i\}} \sigma_{is,js} e_{is} e_{js}) + \sum_{i\in\mathcal{N}_s}(e_{is}\sum_{t\neq s}\sigma_{is,it} e_{it}) \tag{1}$$

where $\mathcal{N}_s \subseteq \mathcal{N}$ is the set of agents participating in project s, $\alpha_i^s > 0$ is the ability/skill of agent i in project s, $e_{is} \geq 0$ is the effort level of agent i in project s, $\sigma_{is,is}$ <0 captures agent i's concavity of efforts in project s, $\sigma_{is,js}$ (unrestricted) is the complementarity parameter between efforts of agent i and agent j in project s, $\sigma_{is,it}$ (unrestricted) is the complementarity parameter between the effort of agent i made in project t and the effort in project s. Let $\Sigma = [\sigma_{is,jt}]$ be the square matrix of knowledge cross effect. The first part of (1) captures knowledge/information spillovers within a project across agents (intra-team production effect). In addition to Hsieh et al. (2018), we introduce a second part in (1) which captures knowledge/information spillovers across projects (inter-team effects) due to the participation of agents in other projects.

2.2. Payoffs

An agent i's cost is given by the following quadratic form inspired by Hsieh et al. (2018)

$$c_i(\mathcal{B}, \mathbf{e}) = \frac{1}{2}\sum_{s\in\mathcal{P}} e_{is}(\phi_{is,is} e_{is} - \sum_{t\in\mathcal{P}\setminus\{s\}} \phi_{is,it} e_{it}), \tag{2}$$

where $\phi_{is,is} < 0$ is a convexity parameter possibly heterogeneous across agent/project pairs, $\phi_{is,it}$ (unrestricted) is the complementarity parameter between project s and project t for agent i, that can vary accross agents and pairs of projects.

Following a broad range of work in cooperative game theory we introduce an alternative payoffs specification in terms of surplus sharing. We propose the use of the Myerson's value (Myerson, 1977) to introduce a sharing based on the marginal contribution to the coalition (the set of inventors involved in a project can be viewed as a coalition). For example in our bipartite graph exemple, the set $\{a_1, b_1\}$ is the coalition generated by Project 1. As pointed out by Myerson (Myerson, 1977) in case of complete graph (i.e. in case of clique, every player of a coalition is connected to all other players of the coalition) the Myerson's value is equal to the Shapley's value (Shapley, 1953). The Shapley's value is a well-known solution concept in cooperative game theory, which provides an appealing means of sharing the total value of the cooperative game (in our case the innovation production) among individual players with respect to their marginal contribution to the coalition (Shapley, 1953). Formally the Shapley's value is defined

$$\phi_i(s) = \frac{1}{|\mathcal{N}_s|!}\sum_{s}\Big(v(\pi_i^{\mathcal{N}_s} \cup \{i\}) - v(\pi_i^{\mathcal{N}_s})\Big) \tag{3}$$

where $\mathcal{N}_s$ is the set of players involved in project s, π a permutation on $\mathcal{N}_s$, $\pi_i^{\mathcal{N}_s}$ is the set of players appearing before player i in permutation, $|\mathcal{N}_s|!$ is the number of different permutations of $|\mathcal{N}_s|$ players. Hence, Shapley's value $\phi_i(s)$ can be interpreed as the average marginal contribution of player i to the coalition s assuming all orderings are equally likely. Based on this observation, $\phi_i(s)$ can be viewed as a good means to measure the contribution of player i to the coalition s. Applying these rule of sharing yields the following payoffs formulation

$$\pi_i(\mathcal{B},\mathbf{e}) = \sum_{s=1}^{p}(\alpha e_{js} + \frac{1}{2}\sigma_{js,js}e_{js}^2 + \frac{1}{2}\sum_{k\in\mathcal{N}_s\backslash j}\sigma_{js,ks}e_{ks}e_{js} + e_{js}\sum_{s\neq t}\sigma_{js,jt}e_{jt})\delta_{is} - (\frac{1}{2}\sum_{s\in\mathcal{P}} e_{is}(\phi_{is,is}e_{is} + \sum_{t\in\mathcal{P}\backslash s}\phi_{is,it}e_{it})) \tag{4}$$

where $\delta_{is} \in \{0, 1\}$ indicates whether agent i participates in project s. Note that this profit function allows for any pattern of cross effects between agents and projects with respect to project value, but also any pattern of cross effects across projects in terms of cost reduction.

3. Equilibrium effort of the team production game

In order to isolate knowledge cross effects from cost cross effects, we build two networks from the initial bipartite network $\mathcal{B}$ to which we add weights. This set of weights is built thanks to two BCZ like decomposition of the cross effects.

3.1. Creation of the two networks

Given a network $\mathcal{B}$, its line graph $\mathcal{L}(\mathcal{B})$ is a graph such that each node of $\mathcal{L}(\mathcal{B})$ represents an edge of $\mathcal{B}$, and two nodes of $\mathcal{L}(\mathcal{B})$ are connected if and only if their corresponding edges share a common endpoint in $\mathcal{B}$ (see West et al. (2001)). This network is represented in Figure 2a. By abuse of notation $\mathcal{L}(\mathcal{B})$ stands also for the adjacency matrix of the line graph. The elements of the adjacency matrix of the line graph of $\mathcal{B}$ are given by

$$\mathcal{L}_{(i,s),(j,t)}(\mathcal{B}) = \begin{cases} 1 & i=j,\ s=t \\ 1 & i=j,\ s\neq t \\ 1 & i\neq j,\ s=t \\ 0 & otherwise \end{cases} \tag{5}$$

Observe that an agent i is possibly represented by multiple nodes, one node by project in which the agent is involved. The second network $\mathcal{P}(\mathcal{B})$ is built by removing each link of $\mathcal{L}(\mathcal{B})$ which links two different agents. This network is represented in Figure 2b. By abuse of notation $\mathcal{P}(\mathcal{B})$ stands also for the adjacency matrix of this network. The elements of the adjacency matrix of $\mathcal{P}(\mathcal{B})$ are given by

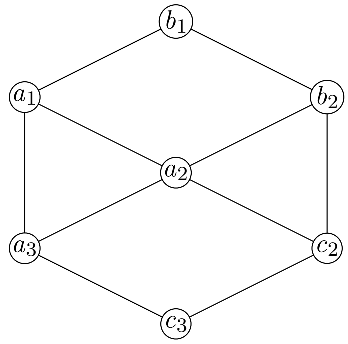
$$\mathcal{P}_{(i,s),(j,t)}(\mathcal{B}) = \begin{cases} 1 & i=j,\ s\neq t \\ 0 & otherwise \end{cases} \tag{6}$$

3.2. Decomposition of the cross effects: weighting of the two networks

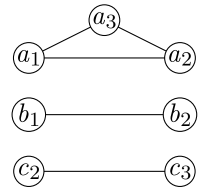
Now we turn to the decomposition of cross effects which will allow us to weight the two networks $\mathcal{L}(\mathcal{B})$ and $\mathcal{P}(\mathcal{B})$. Let $\mathbf{I}$ denote the $|\mathcal{E}|$-square identity matrix and $\mathbf{U}$ denote the $|\mathcal{E}|$-square matrix of ones, where $|\mathcal{E}|$ is the number of links in $\mathcal{B}$.

3.2.1. The Σ matrix as a weighting of matrix $\mathcal{L}(\mathcal{B})$

We decompose the Σ matrix into an idiosyncratic concavity component, a global subtituability component, and a local complementarity component, in the following way.
Let $\underline{\sigma} = \min\{\sigma_{is,jt} | i\neq j \; and \; s=t \; or \; i=j \; and \; s\neq t\}$ and $\bar{\sigma} = \max\{\sigma_{is,jt} | i\neq j \; and \; s=t \; or \; i=j \; and \; s\neq t\}$. Let $\sigma < \min\{\underline{\sigma}, 0\}$ and $\gamma = -\min\{\underline{\sigma}, 0\} \geq 0$. If there exists some subtituability between efforts, then $\underline{\sigma} < 0$ and $\gamma > 0$; otherwise $\underline{\sigma} \geq 0$ and $\gamma = 0$. Let $\lambda = \bar{\sigma} + \gamma \geq 0$. We assume that $\lambda > 0$. Let $g_{is,js} = \frac{\sigma_{is,js}+\gamma}{\lambda}$ for



(a) The $\mathcal{L}(\mathcal{B})$ network.



(b) The $\mathcal{P}(\mathcal{B})$ network.

Figure 2: The two created networks.

$i \neq j$, $g_{is,it} = \frac{\sigma_{is,it}+\gamma}{\lambda}$ for $s \neq t$ and set $g_{is,is} = 0$. By construction $0 \leq g_{is,js} \leq 1$, and $0 \leq g_{is,it} \leq 1$. Let $\mathbf{G} = [g_{is,jt}] = \mathbf{G}_1 + \mathbf{G}_2$ denote the weighted adjacency matrix of network $\mathcal{L}(\mathcal{B})$, a zero diagonal non-negative symmetric square matrix. Each element $g_{is,js}$ in $\mathbf{G}_1$ measures the relative complementarity between efforts of agent i and j in project s with respect to the benchmark value γ and is expressed as a fraction of λ. Likewise each element $g_{is,it}$ in $\mathbf{G}_2$ measures the relative complementarity between efforts made in project s and t for agent i with respect to the benchmark value γ and is expressed as a fraction of λ . Finally let $\sigma = -\beta - \gamma$, where $\beta > 0$. As in BCZ, the $\mathbf{\Sigma}$ matrix is decomposed additively as:

$$\mathbf{\Sigma} = -\beta\mathbf{I} - \gamma\mathbf{U} + \lambda\mathbf{G}_1 + \lambda\mathbf{G}_2, \tag{7}$$

where β and γ are positive parameters. The idiosyncratic effect, $-\beta\mathbf{I}$, shows the concavity of the payoffs in terms of own efforts. The global interaction effect, $-\gamma\mathbf{U}$, gives a uniform subtituability in efforts across all pairs of nodes. The local interaction effect, $\lambda(\mathbf{G}_1 + \mathbf{G}_2)$, shows the complementarity in players' efforts, potentially heterogeneous across different pairs of "nodes".

3.2.2. The $\mathbf{\Phi}$ matrix as a weighting of matrix $\mathcal{P}(\mathcal{B})$

We decompose the $\mathbf{\Phi}$ matrix into an idiosyncratic convexity component, a global congestion component, and a local complementarity component, in the following way.
Let $\underline{\phi} = \max\{\phi_{is,it} | s \neq t\}$ and $\bar{\phi} = \min\{\phi_{is,it} | s \neq t\}$. This unusual notation is due to a mirror effect caused by the presence of a minus before the cost in (**??**). Let $\phi > \max\{\underline{\phi}, 0\}$ and $\theta = -\max\{\underline{\phi}, 0\} \leq 0$. If there exists some congestion effects between two projects, then $\underline{\phi} > 0$ and $\theta < 0$; otherwise $\underline{\phi} \leq 0$ and $\theta = 0$. Let $\psi = \bar{\phi} + \theta \leq 0$. We assume that $\psi < 0$. Let $p_{is,it} = \frac{\phi_{is,it}+\theta}{\psi}$ for $s \neq t$ and set $p_{is,is} = 0$. By construction $0 \leq p_{is,it} \leq 1$. Let $\mathbf{P} = [p_{is,it}]$ denotes the weighted adjacency matrix of network $\mathcal{P}(\mathcal{B})$, a zero diagonal non-negative symmetric square matrix. Each element $p_{is,it}$ in $\mathbf{P}$ measures the relative complementarity between projects s and t for agent i with respect to the benchmark value θ and is expressed as a fraction of ψ. Finally let $\phi = -w - \theta$, where $w < 0$. We have

$$\mathbf{\Phi} = -w\mathbf{I} - \theta\mathbf{U} + \psi\mathbf{P}, \tag{8}$$

where, $-w\mathbf{I}$ reflects the convexity of costs, $-\theta\mathbf{U}$ the uniform congestion effect, and $\psi\mathbf{P}$ the complementarity between efforts.

Using the decomposition of the $\mathbf{\Sigma}$ matrix (7) and the decomposition of the $\mathbf{\Phi}$ matrix (8), the payoff (4) can be written as

$$\pi_i(\mathcal{B},\mathbf{e}) = \sum_{s=1}^{p}(\alpha\ e_{js} - \frac{1}{2}(\beta+\gamma)\sum_{j\in\mathcal{N}_s} e_{js}^2 - \gamma\sum_{js\neq ks} e_{js}e_{ks} + \frac{\lambda}{2}\sum_{j\neq k\in\mathcal{N}_s\setminus\{j\}} g_{js,ks}e_{js}e_{ks} + \lambda e_{js}\sum_{s\neq t} g_{js,jt}e_{jt})\delta_{is}$$
$$+\frac{1}{2}\sum_{s\in\mathcal{P}}(w+\theta)e_{is}^2 + \theta\sum_{is\neq it} e_{is}e_{it} - \frac{\psi}{2}\sum_{s\in\mathcal{P}}\sum_{t\in\mathcal{P}\setminus\{s\}} p_{is,it}e_{is}e_{it} \tag{9}$$

3.3. Equilibrium of the game

We now present our result. The proof is given in the Appendix.
Let $\mu_1(\frac{1}{2}\mathbf{G}_1+\mathbf{G}_2)$ and $\mu_1(\mathbf{P})$ denote respectively the largest eigenvalue of $(\frac{1}{2}\mathbf{G}_1+\mathbf{G}_2)$ and $\mathbf{P}$. Define $\boldsymbol{\alpha}$ $= [\alpha,\ldots,\alpha]^\top$, $\psi^*=\frac{\psi}{\beta-w}$, $\lambda^*=\frac{\lambda}{\beta-w}$, $\mathbf{a}=[\mathbf{I}\text{-}\lambda^*(\frac{1}{2}\mathbf{G}_1+\mathbf{G}_2)+\psi^*\mathbf{P}]^{-1}\mathbb{1}$ and $\mathbb{1}$ is the $(|\mathcal{E}|,1)$ vector of ones. Finally denotes by $\mathbf{e^*}$ the $(|\mathcal{E}|,1)$ vector of equilibrium efforts.

Proposition 1. *Consider the n-player simultaneous move game, and the payoff function given by* (9). *Assume $\beta > \lambda\mu_1((\frac{1}{2}\boldsymbol{G}_1+\boldsymbol{G}_2))$ and $-w > -\psi\mu_1(\boldsymbol{P})$, then the matrix $[\boldsymbol{I}\text{-}\lambda^*(\frac{1}{2}\boldsymbol{G}_1+\boldsymbol{G}_2)+\psi^*\boldsymbol{P}]$ is well defined and nonnegative, and the game $(\boldsymbol{\Sigma},\boldsymbol{\Phi})$ has a unique Nash equilibrium, which is interior and given by*

$$\boldsymbol{e^*} = \frac{\alpha}{(\gamma-\theta)\mathbb{1}^\intercal\boldsymbol{a} + (\beta-w)}[\boldsymbol{I} - \lambda^*(\frac{1}{2}\boldsymbol{G}_1 + \boldsymbol{G}_2) + \psi^*\boldsymbol{P}]^{-1}\mathbb{1} \tag{10}$$

Let us comment on the conditions for the existence and uniqueness of the equilibrium derived in Proposition 1.[1] The assumption on β and $\mu_1(\frac{1}{2}\mathbf{G}_1+\mathbf{G}_2)$ requires that inter and intra-team effect $(\mu_1(\frac{1}{2}\mathbf{G}_1+\mathbf{G}_2))$ are not too strong with respect to concavity (β). A similar condition is commonly assumed in network games with one layer (König et al., forthcoming). The second condition which is specif to our model, requires that the portfolio effects $(-\psi\mu_1(\mathbf{P}))$ are too strong with respect to convexity of cost $(-w)$.
The second important observation is that the equilibrium can be viewed as a function of centrality in an aggregated network. Observe that $[\mathrm{I}-\lambda(\frac{1}{2}\mathbf{G}_1+\mathbf{G}_2)+\psi\mathbf{P}]^{-1}$ under the following assumption $\beta > \lambda\mu_1(\frac{1}{2}\mathbf{G}_1+\mathbf{G}_2)$ and $-w > -\psi\mu_1(\mathbf{P})$ can be rewritten as $[\mathbf{I}-\mathbf{M}]^{-1}$, where $\mathbf{M} = \lambda^*(\frac{1}{2}\mathbf{G}_1+\mathbf{G}_2) - \psi\mathbf{P}$. The matrix $\mathbf{M}$ can be interpreted as an aggregated adjacency matrix and so $[\mathbf{I}-\mathbf{M}]^{-1}\mathbb{1}$ as representing players centrality in this network.

4. Conclusion

[VERY PRELIMINARY AND INCOMPLETED] We adapt the team production model of Hsieh et al. (2018) to the production of non-public good with the introduction of a marginal sharing of the ouput. The next step of the work will be to bring the model to the data.

Appendix

Before turning to the proof we restate two useful theorems from Debreu and Herstein (1953), and a result from Weyl.

Theorem I from (Debreu and Herstein, 1953)

Let $A \geq 0$ be indecomposable. Then A has a characteristic root $r > 0$ such that to r can be associated an eigen-vector $x_0 > 0$; if α is any characteristic root of A, $|\alpha| \leq r$; r increases when any element of A increases; r is a simple root.

[1] A necessary and sufficient condition for existence and uniqueness of an interior Nash equilibrium can be found in the appendix.

Theorem III* from (Debreu and Herstein, 1953)

$(sI - A)^{-1} \geq 0$ if and only if $s > r$. Where r is the largest eigenvalue of A.

Weyl's inequality

Let A and B two hermitian matrices[2]. Then $\mu_1(A+B) \leq \mu_1(A) + \mu_1(B)$, where μ_1 denotes the largest eigenvalue.

Proposition 1

Proof. When $\delta_{is} = 1$, the first order condition of (9) wrt. e_{is} is given by[1]

$$\frac{\partial \pi_i(\mathcal{G}, \mathcal{P}, \mathbf{e})}{\partial e_{is}} = \alpha - (\beta + \gamma)e_{is} - \gamma \sum_{js \neq is} e_{js} + \frac{\lambda}{2} \sum_{j \in \mathcal{N}_s \setminus \{i\}} g_{is,js} e_{js} + \lambda \sum_{s \neq t} g_{is,it} e_{it}$$
$$+ (w + \theta)e_{is} + \theta \sum_{it \neq is} e_{it} - \psi \sum_{t \in \mathcal{P} \setminus \{s\}} p_{is,it} e_{it} = 0$$

An interior Nash equilibrium in pure strategies $\mathbf{e}^* \in \mathbb{R}_+^{|\mathcal{E}|}$ is such that $\partial \pi_i / \partial e_{is}(\mathbf{e}^*) = 0$ and $e_{is}^* > 0$ for all $is \in \mathbb{E}$. If such an equilibrium exists, it then solves

$$\boldsymbol{\alpha} = [(\beta - w)\mathbf{I} + (\gamma - \theta)\mathbf{U} - \lambda(\frac{1}{2}\mathbf{G}_1 + \mathbf{G}_2) + \psi \mathbf{P}]\mathbf{e} \tag{11}$$

Since $\beta > \lambda \mu_1(\frac{1}{2}\mathbf{G}_1 + \mathbf{G}_2)$ and $-w > -\psi \mu_1(\mathbf{P})$, we have

$$\beta - w > \lambda \mu_1(\frac{1}{2}\mathbf{G}_1 + \mathbf{G}_2) - \psi \mu_1(\mathbf{P}) \tag{12}$$

Moreover by Weyl's inequality we have

$$\lambda \mu_1(\frac{1}{2}\mathbf{G}_1 + \mathbf{G}_2) - \psi \mu_1(\mathbf{P}) \geq \mu_1((\frac{1}{2}\mathbf{G}_1 + \mathbf{G}_2) - \psi \mathbf{P}) \tag{13}$$

Plugging (13) into (12), we get

$$\mu_1(\lambda(\frac{1}{2}\mathbf{G}_1 + \mathbf{G}_2) - \psi \mathbf{P}) < \beta - w$$

So, from Theorem III* of Debreu and Herstein (1953) it follows that the necessary and sufficient condition for $[(\beta - w)\mathbf{I} - \lambda(\frac{1}{2}\mathbf{G}_1 + \mathbf{G}_2) + \psi \mathbf{P}]^{-1}$ to be well defined and nonnegative, is satisfied. Since the matrix $[(\beta - w)\mathbf{I} + (\gamma - \theta)\mathbf{U} - \lambda(\frac{1}{2}\mathbf{G}_1 + \mathbf{G}_2) + \psi \mathbf{P}]$ is generically nonsingular[3], (11) has a unique generic solution in $\mathbb{R}^{|\mathcal{E}|}$, denoted by $\mathbf{e}^*$.

Let $\mathbb{E} = \{is \in \mathbf{G}\}$ denote the set of active nodes in network $\mathbf{G}$. Using the fact that $\mathbf{U}\mathbf{e}^{\boldsymbol{*}} = \hat{e}^* \mathbb{1}$, where $\hat{e}^* = \sum_{is \in \mathbb{E}} e_{is}^*$, (11) is equivalent to:

$$(\alpha - (\gamma - \theta)\hat{e}^*)\mathbb{1} = (\beta - w)[\mathbf{I} - \lambda^*(\frac{1}{2}\mathbf{G}_1 + \mathbf{G}_2) + \psi^* \mathbf{P}]\mathbf{e}^{\boldsymbol{*}} \tag{14}$$

$$\Leftrightarrow (\beta - w)\mathbf{e}^{\boldsymbol{*}} = (\alpha - (\gamma - \theta)\hat{e}^*)[\mathbf{I} - \lambda^*(\frac{1}{2}\mathbf{G}_1 + \mathbf{G}_2) + \psi^* \mathbf{P}]^{-1}\mathbb{1} \tag{15}$$

[2] Observe that every real symmetric matrix is an Hermitian matrix.

[1] Observe that the second order condition $\frac{\partial^2 \pi_i(\mathcal{G}, \mathbf{e})}{\partial e_{is}^2} = (w + \theta) - (\beta + \gamma) < 0$ is satisfied, since $w < 0$, $\theta < 0$, $\beta > 0$ and $\gamma > 0$.

[3] The set of parameters β, γ, λ, w, θ, ψ for which det $((\beta - w)\mathbf{I} + (\gamma - \theta)\mathbf{U} - \lambda(\frac{1}{2}\mathbf{G}_1 + \mathbf{G}_2) + \psi \mathbf{P}) = 0$ has a Lebesgue measure zero in $\mathbb{R}^6$.

Multiplying to the left by $\mathbb{1}^\intercal$ and solving for $\hat{e}^*$, we get

$$\hat{e}^* = \frac{\alpha \mathbb{1}^\intercal \mathbf{a}}{(\beta - w) + (\gamma - \theta)\mathbb{1}^\intercal \mathbf{a}} \tag{16}$$

Plugging (16) into (15) we get

$$\mathbf{e^*} = \frac{\alpha}{(\gamma - \theta)\mathbb{1}^\intercal \mathbf{a} + (\beta - w)}[\mathbf{I} - \lambda^*(\frac{1}{2}\mathbf{G}_1 + \mathbf{G}_2) + \psi^* \mathbf{P}]^{-1}\mathbb{1}$$

Given that $\alpha > 0$ and $([\mathbf{I} - \lambda^* \mathbf{G} + \psi^* \mathbf{P}]^{-1}\mathbb{1})_{is} \geq 1$[4] for all is, it follows that $\mathbf{e}^*$ is interior.
We now establish uniqueness. First the previous argument shows that $\mathbf{e}^*$ is a unique interior equilibrium. We now deal with corner solutions.
Let $\beta(\Sigma)$, $\gamma(\Sigma)$, $\lambda(\Sigma)$, $w(\Phi)$, $\theta(\Phi)$, $\psi(\Phi)$ be the elements of the decompositions of $\mathbf{\Sigma}$ and $\mathbf{\Phi}$. In what follows, we omit the dependence in $\mathbf{\Sigma}$ and $\mathbf{\Phi}$ when there is no confusion. For all matrices $\mathbf{Y}$, vector $\mathbf{y}$ and set $\mathbb{S} \subset \mathbb{E}$, let $\mathbf{Y}_{\mathbb{S}}$ be a submatrix of $\mathbf{Y}$ with s rows and columns, and $\mathbf{y}_{\mathbb{S}}$ be the subvector with rows in $\mathbb{S}$. It can be shown[5] that $\gamma(\Sigma_S) \leq \gamma(\Sigma)$, $\beta(\Sigma_S) \geq \beta(\Sigma)$, $\lambda(\Sigma_S) \leq \lambda(\Sigma)$, $\theta(\mathbf{\Phi}_S) \leq \theta(\Phi)$, $-w(\mathbf{\Phi}_S) \leq -w(\mathbf{\Phi})$ and $-\psi(\mathbf{\Phi}_S) \leq -\psi(\mathbf{\Phi})$. Since $\lambda(\mathbf{G}_1 + \mathbf{G}_2) = \mathbf{\Sigma} + \gamma(\mathbf{U} - \mathbf{I}) - \sigma \mathbf{I}$, it follows that the coefficients in the S rows and columns of $\lambda(\mathbf{G}_1 + \mathbf{G}_2)$ are at least as high as the corresponding coefficients in $\lambda(\Sigma_S)(\mathbf{G}_1 + \mathbf{G}_2)_S$. Since $\psi \mathbf{P} = \mathbf{\Phi} + \theta(\mathbf{U} - \mathbf{I}) - \phi \mathbf{I}$, it follows that the coefficients in the S rows and columns of $\psi \mathbf{P}$ are at most as high as the corresponding coefficients in $\psi(\mathbf{\Phi}_S)\mathbf{P}_S$.
It follows from Theorem 1 in Debreu and Herstein (1953) that we have, $\mu_1(\lambda(\Sigma_S)\mathbf{G}_S) \leq \mu_1(\lambda(\Sigma)\mathbf{G})$ and $\mu_1(-\psi(\mathbf{\Phi}_S)\mathbf{P}_S) \geq \mu_1(-\psi(\mathbf{\Phi})\mathbf{P})$. Therefore $\beta(\Sigma) > \lambda(\Sigma)\mu_1(\mathbf{G})$ implies $\beta(\Sigma_S) > \lambda(\Sigma_S)\mu_1((\mathbf{G}_1 + \mathbf{G}_2)_S)$ and $-w(\mathbf{\Phi}) > -\psi\mu_1(\mathrm{P})$ implies $-w(\mathbf{\Phi}_S) > -\psi\mu_1(P_S)$.
To introduce a contradiction suppose that $\mathbf{e^*}$ is a non-interior Nash equilibrium of the network game. Let $\mathbb{S} = \{is \in \mathbb{E} : e^*_{is} > 0\}$. Since for $\mathbf{e}^* = \mathbf{0}$, we have $\frac{\partial \Pi_i(\mathcal{G},\mathcal{P},\mathbf{e})}{\partial e_{is}} = \alpha > 0$, $\mathbf{0}$ cannot be a Nash equilibrium. So we know that $\mathbb{S} \neq \emptyset$. Then for all $is \in \mathbb{S}$, the equilibrium solves

$$\boldsymbol{\alpha}_S = [(\beta - w)\mathbf{I}_S + (\gamma - \theta)\mathbf{U}_S - \lambda(\frac{1}{2}\mathbf{G}_{S1} + \mathbf{G}_{S2}) + \psi \mathbf{P}_S]\mathbf{e}^*_S$$

Given that $\beta > \mu_1(\frac{1}{2}\mathbf{G}_{S1} + \mathbf{G}_{S2})$ and $-w > -\psi\mu_1(\mathbf{P}_S)$, from Theorem III* of Debreu and Herstein (1953) it follows that $[\mathbf{I}_S - \lambda^*(\frac{1}{2}\mathbf{G}_{S1} + \mathbf{G}_{S2}) + \psi^* \mathbf{P}_S]^{-1}$ is well defined and nonnegative and we have

$$\mathbf{e}^*_S = \frac{(\alpha - (\gamma - \theta)\hat{e}^*_S)}{\beta - w}[\mathbf{I}_S - \lambda^*(\frac{1}{2}\mathbf{G}_{S1} + \mathbf{G}_{S2}) + \psi^* \mathbf{P}_S]^{-1}\mathbb{1}_S \tag{17}$$

Since for all $js \in \mathbb{E} \setminus \mathbb{S}$, we have $e^*_{js} = 0$, we should have

$$\frac{\partial \Pi_i}{\partial e_{js}}(\mathbf{e}^*) = \alpha + \frac{\lambda}{2}\sum_{i \in \mathcal{N}_s \setminus \{j\}} \sigma_{is,js} e^*_{is} + \lambda \sum_{jt \in \mathbb{S}} \sigma_{js,jt} e^*_{jt} - \psi \sum_{jt \in \mathbb{S}} \phi_{js,jt} e^*_{jt} \leq 0 \tag{18}$$

Using (17) we rewrite this inequality as:

$$(\alpha - (\gamma - \theta)\hat{e}^*_S)[1 + \frac{\lambda}{2}\sum_{i \in \mathcal{N}_s \setminus \{j\}} g_{is,js}\mathbf{a}_{is} + \lambda \sum_{s \neq t, jt \in \mathbb{S}} g_{js,jt}\mathbf{a}_{jt} - \psi \sum_{jt \in \mathbb{S}} p_{js,jt}\mathbf{a}_{jt}] \leq 0,$$

where $\mathbf{a}_{is}$ stands for the entry in the vector $\mathbf{a}$ associate to agent i in project s. This inequality implies that $(\alpha - (\gamma - \theta)\hat{e}^*_S) \leq 0$. However if $(\alpha - (\gamma - \theta)\hat{e}^*_S) \leq 0$, then it follows by (17) that $e^*_{is} \leq 0$ for all $is \in \mathbb{S}$, which is a contradiction. □

[4] Given the assumptions on λ and ψ, this expression can be rewritten as a Neumann series, and the inequality follows.

[5] To show the set of inequalities that follows, it is sufficient to study the two polar cases, namely the case where the higher (respectively the lower) σ^{st}_{ij} (ϕ^i_{st}) is removed from the game.

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