Equilibrium of a production economy with non-compact attainable allocations set

Senda Ounaies, Jean-Marc Bonnisseau, Souhail Chebbi

▶ To cite this version:

Senda Ounaies, Jean-Marc Bonnisseau, Souhail Chebbi. Equilibrium of a production economy with non-compact attainable allocations set. Advances in Nonlinear Analysis, 2019, 8 (1), pp.979-994. 10.1515/anona-2017-0234 . halshs-01859163

HAL Id: halshs-01859163
https://shs.hal.science/halshs-01859163

Submitted on 3 Apr 2019

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

DE GRUYTER Adv. Nonlinear Anal. 2019; 8: 979–994

Research Article

Senda Ounaies, Jean-Marc Bonnisseau* and Souhail Chebbi

Equilibrium of a production economy with non-compact attainable allocations set

https://doi.org/10.1515/anona-2017-0234
Received October 19, 2017; accepted November 7, 2017

Abstract: In this paper, we consider a production economy with an unbounded attainable set where the consumers may have non-complete non-transitive preferences. To get the existence of an equilibrium, we provide an asymptotic property on preferences for the attainable consumptions and we use a combination of the nonlinear optimization and fixed point theorems on truncated economies together with an asymptotic argument. We show that this condition holds true if the set of attainable allocations is compact or, when the preferences are representable by utility functions, if the set of attainable individually rational utility levels is compact. This assumption generalizes the CPP condition of [N. Allouch, An equilibrium existence result with short selling, *J. Math. Econom.* **37** (2002), no. 2, 81–94] and covers the example of [F. H. Page, Jr., M. H. Wooders and P. K. Monteiro, Inconsequential arbitrage, *J. Math. Econom.* **34** (2000), no. 4, 439–469] when the attainable utility levels set is not compact. So we extend the previous existence results with non-compact attainable sets in two ways by adding a production sector and considering general preferences.

Keywords: Production economy, non-compact attainable allocations, quasi-equilibrium, nonlinear optimization

MSC 2010: 49M37, 91B50, 58C06

1 Introduction

Since the seventies, with the exception of the seminal paper of Mas-Colell [14] and a first paper of Shafer and Sonnenschein [18], equilibrium for a finite-dimensional standard economy is commonly proved using explicitly or implicitly equilibrium existence for the associated abstract economy (see [3, 8, 9, 12, 17, 19]) in which agents are the consumers, the producers and an hypothetical additional agent, the Walrasian auctioneer. Moreover, in exchange economies it is well known that the existence of equilibrium with consumption sets that are not bounded from below requires some non-arbitrage conditions (see [2, 4–7, 13, 20]). In [7], it is shown that these conditions imply the compactness of the individually rational utility level set, which is

Senda Ounaies, Centre d'économie de la Sorbonne, Université Paris 1, Panthéon Sorbonne, 106–112 Boulevard de l'Hôpital, 75647 Paris Cedex 13, France; and Department of Mathematics, College of Science, University El-Manar, Tunis, Tunisia, e-mail: ou-cynda@hotmail.com
***Corresponding author: Jean-Marc Bonnisseau,** Centre d'économie de la Sorbonne, Université Paris 1, Panthéon Sorbonne, 106–112 Boulevard de l'Hôpital, 75647 Paris Cedex 13, France, e-mail: jean-marc.bonnisseau@univ-paris1.fr. https://orcid.org/0000-0002-8156-2229
Souhail Chebbi, Department of Mathematics, College of Science, King Saud University, Box 2455, Riyadh 11451, Saudi Arabia, e-mail: schebbi@ksu.edu.sa

Open Access. © 2019 Walter de Gruyter GmbH, Berlin/Boston.
This work is licensed under the Creative Commons Attribution 4.0 Public License.

Brought to you by | Université Paris 1 Panthéon-Sorbonne/Centre d'Economie de la Sorbonne
Authenticated
Download Date | 3/20/19 5:38 PM

clearly weaker than assuming the compactness of the attainable allocation, and Dana, Le Van and Magnien prove an existence result of an equilibrium under this last condition.

The purpose of our paper is to extend this result to finite-dimensional production economies with non-complete, non-transitive preferences, which may not be representable by a utility function. Furthermore, we also allow the preferences to be other regarding in the sense that the preferred set of an agent depends on the consumption of the other consumers. We posit the standard assumptions about the closedness, the convexity and the continuity on the consumption side as well as on the production side of the economy like in [9], and a survival assumption. We only consider quasi-equilibrium and we refer to the usual interiority of initial endowments or the irreducibility condition to get an equilibrium from a quasi-equilibrium (see, for example, [9, Section 3.2]).

The non-compactness of the attainable sets appears naturally in an economy with financial markets and short-selling. Using Hart's trick [13], we can reduce the problem to a standard exchange economy when the financial markets are frictionless. But if there are some transaction costs, intermediaries like clearing house mechanisms or other kind of frictions, this method is no more working and we then need to introduce a production sector to encompass these frictions. That is why we add in this paper a production sector, which is also justified if we want to analyze a stock market where the payments of an asset depend on the production plan of a firm.

Considering non-complete, non-transitive preferences allows us to deal with Bewley preferences where the agents have several criterions and a consumption is preferred to another one only if all criterions are improved. Such preferences are not representable by utility functions. They appear naturally in financial models where the objective is to minimize the risk according to some consistent measures.

Our main contribution is to provide a sufficient condition (H3) to replace the standard compactness of the attainable allocation set, which is suitably written to deal with general preferences. More precisely, we assume that for each sequence of attainable consumptions there exists an attainable consumption where the preferred consumptions can be approximated by preferred consumptions of the elements of the sequence. Actually, we also restrict our attention to the attainable allocation, which are individually rational, in a sense adapted to the fact that preferences may not be transitive. The formulation of our assumption is in the same spirit as the CPP condition of Allouch [1].

We prove that our condition is satisfied when the attainable set is compact and when preferences are represented by utility functions and the set of attainable individually rational utility levels is compact. So, our result extends the previous ones in the literature. Our asymptotic assumption is weaker than the CPP condition within the framework considered by Allouch where preferences are supposed to be transitive with open lower-sections.

To compare our work with the contribution of Won and Yannelis [21], we provide an asymmetric assumption (EWH3) for exchange economies which is less demanding for one particular consumer. We are not pleased with this assumption since the fundamentals of the economy are symmetric and there is no reason to treat a consumer differently from the others. Won and Yannelis' condition and (EWH3) are not comparable and both of them cover the example of Page, Wooders and Monteiro [15]. Nevertheless, neither of these conditions covers [21, Example 3.1.2]. So, there is room for further works to provide a symmetric assumption covering both examples.

We also remark that our condition deals only with feasible consumptions and not with the associated productions. So, our condition can be identically stated for an exchange economy or for a production economy. This means that even if there exist non-compact feasible productions, an equilibrium still exists if the attainable consumption set remains compact. In other words, the key problem comes from the behavior of the preferences for large consumptions and not from the geometry of the production sets at infinity.

To prove the existence of a quasi-equilibrium, we use several tricks borrowed from various authors. Using a truncated economy in order to apply a fixed point theorem to an artificial compact economy is an old trick as in the first equilibrium proofs. We apply our assumption on the asymptotic behavior of preferences to a sequence of quasi-equilibrium allocations in growing associated truncated economies. We prove that the attainable consumption given by Assumption (H3) is a quasi-equilibrium consumption of the original economy. The originality of the proof is mainly contained in Section 4.

Brought to you by | Université Paris 1 Panthéon-Sorbonne/Centre d'Economie de la Sorbonne
Authenticated
Download Date | 3/20/19 5:38 PM

2 The model

In this paper, we consider the private ownership economy:

$$\mathcal{E} = (\mathbb{R}^L, (X_i, P_i, \omega_i)_{i\in I}, (Y_j)_{j\in J}, (\theta_{ij})_{(i,j)})$$

where L is a finite set of goods so that $\mathbb{R}^L$ is the commodity space and the price space, I is a finite set of consumers, and each consumer i has a consumption set $X_i \subset \mathbb{R}^L$ and an initial endowment $\omega_i \in \mathbb{R}^L$. The tastes of this consumer are described by a preference correspondence $P_i : \prod_{k\in I} X_k \to X_i$, where $P_i(x)$ represents the set of strictly preferred consumption to $x_i \in X_i$ given the consumption $(x_k)_{k\neq i}$ of the other consumers. Furthermore, J is a finite set of producers and $Y_j \subset \mathbb{R}^L$ is the set of possible productions of firm $j \in J$. For each i and j, the portfolio of shares of the consumer i on the profit of the producer j is denoted by θ_{ij}. The θ_{ij} are nonnegative and for every $j \in J$, one has $\sum_{i\in I} \theta_{ij} = 1$. These shares together with their initial endowment determine the wealth of each consumer.

Definition 2.1. An allocation $(x, y) \in \prod_{i\in I} X_i \times \prod_{j\in J} Y_j$ is called attainable if

$$\sum_{i\in I} x_i = \sum_{j\in J} y_j + \sum_{i\in I} \omega_i.$$

We denote by $\mathcal{A}(\mathcal{E})$ the set of attainable allocations.

In this paper, we are only dealing with the existence of quasi-equilibrium. We refer to the large literature on irreducibility, which provides sufficient conditions for a quasi-equilibrium to be an equilibrium. The simplest one is the interiority of the initial endowments linked with the possibility of inaction for the producers.

Definition 2.2. A quasi-equilibrium of the private ownership economy is a pair of an allocation

$$((\bar{x}_i)_{i\in I}, (\bar{y}_j)_{j\in J}) \in \prod_{i\in I} X_i \times \prod_{j\in J} Y_j$$

and a non-zero price vector $\bar{p} \neq 0$, such that the following conditions hold:

(a) (Profit maximization:) For every $j \in J$ and every $y_j \in Y_j$ one has $\bar{p} \cdot y_j \le \bar{p} \cdot \bar{y}_j$.

(b) (Quasi-demand:) For each $i \in I$, one has that

$$\bar{p} \cdot \bar{x}_i \le \bar{p} \cdot \omega_i + \bar{p} \cdot \Big(\sum_{j\in J} \theta_{ij}\bar{y}_j\Big)$$

and $x_i \in P_i(\bar{x})$ implies $\bar{p} \cdot x_i \ge \bar{p} \cdot \bar{x}_i$.

(c) (Attainability:)

$$\sum_{i\in I} \bar{x}_i = \sum_{i\in I} \omega_i + \sum_{j\in J} \bar{y}_j.$$

Notice that, in view of condition (c), condition (b) can be rephrased as

$$\text{for every } i \in I \text{ one has that } \bar{p} \cdot \bar{x}_i = \bar{p} \cdot \omega_i + \bar{p} \cdot \Big(\sum_{j\in J} \theta_{ij}\bar{y}_j\Big) \text{ and } x_i \in P_i(\bar{x}) \text{ implies } \bar{p} \cdot x_i \ge \bar{p} \cdot \bar{x}_i$$

Before stating the assumptions considered on $\mathcal{E}$, let us introduce some notations:

- $\omega = \sum_{i\in I} \omega_i$ is the total initial endowment.
- $Y = \sum_{j\in J} Y_j$ is the total production set.
- $\hat{X} = \{x \in \prod_{i\in I} X_i$: there exists $y \in Y$ such that $\sum_{i\in I} x_i = \omega + y\}$ is the set of all attainable consumption allocations.
- $\hat{Y} = \{y \in Y$: there exists $x \in \prod_{i\in I} X_i$ such that $\sum_{i\in I} x_i = \omega + y\}$ is the attainable total production set.

In this paper, we consider the following hypothesis.

Assumption (H1). For every $i \in I$, the following conditions hold:

(a) X_i is a non-empty, closed, convex subset of $\mathbb{R}^L$.

(b) (Irreflexivity:) For all $x \in \prod_{i\in I} X_i$, one has $x_i \notin \operatorname{co} P_i(x)$ (the convex hull of $P_i(x)$).

Brought to you by | Université Paris 1 Panthéon-Sorbonne/Centre d'Economie de la Sorbonne
Authenticated
Download Date | 3/20/19 5:38 PM

(c) (Lower semicontinuous:) $P_i : \prod_{k\in I} X_k \to X_i$ is lower semicontinuous.
(d) $\omega_i \in X_i - \sum_{j\in J} \theta_{i,j} Y_j$, i.e. there exists

$$(\underline{x}_i, (\underline{y}_{i,j})) \in X_i \times \prod_{j\in J} Y_j$$

such that

$$\underline{x}_i = \omega_i + \sum_{j\in J} \theta_{i,j} \underline{y}_{i,j}.$$

(e) For each $x \in \hat{X}$, one has $P_i(x) \neq \emptyset$.

Assumption (H2). The set Y is a non-empty, closed and convex subset of $\mathbb{R}^L$.

To overcome the fact that we do not assume local non-satiation, but only non-satiation, we introduce the definition of "augmented preferences" as in [10, 11]. We can avoid the use of augmented preferences if Assumption (H1) (e) is replaced by the assumption that x_i belongs to the closure of $P_i(x)$:

$$\hat{P}_i(x) = \{x_i' \in X_i \mid x_i' = \lambda x_i + (1-\lambda)x_i'',\ 0 \leq \lambda < 1,\ x_i'' \in \operatorname{co} P_i(x)\},$$

Assumption (H3). For all sequences $((x_i^\nu))$ of $\hat{X}$ such that for all i, one has

$$\underline{x}_i \in \overline{\hat{P}_i(x^\nu)^c},$$

there exists a subsequence $((x_i^{\varphi(\nu)})) \in \hat{X}$ and $(\bar{x}_i) \in \hat{X}$ such that for all i and all $\xi_i \in \hat{P}_i(\bar{x})$ there exist an integer ν_1 and a sequence $(\xi_i^{\varphi(\nu)})_{\nu\geq\nu_1}$ convergent to (ξ_i) such that for all $\nu \geq \nu_1$ and all $i \in I$, one has $\xi_i^{\varphi(\nu)} \in \hat{P}_i(x^{\varphi(\nu)})$.

Closedness and convexity are standard assumptions on consumption and production sets. They imply in particular that commodities are perfectly divisible. Assumption (H1) (c) is a weak continuity assumption on preferences. Assumption (H1) (b), i.e. the irreflexivity, is made on the sets $\operatorname{co} P_i(x)$ to avoid to assume the convexity of the preference correspondences P_i. Assumption (H1) (d) implies that by using his own shares in the productive system, consumer i can survive without participating in any exchange. This implies that no trader will be allowed to starve no matter what the prices are. It also insures that the set $\mathcal{A}(\mathcal{E})$ is non-empty. Usually, in exchange economies, this assumption is merely written as $\omega_i \in X_i$, which corresponds to $\omega_i = \underline{x}_i$ and $\underline{y}_{i,j} = 0$ for all j. Assumption (H1) (e) assumes, for every i, the insatiability of the i-th consumer at any point of his attainable consumption set.

Assumption (H3) is an attempt to weaken the compactness assumption on the global attainable set $\mathcal{A}(\mathcal{E})$. A large literature tackles this question by considering what is called a non-arbitrage condition (see, for example, [2, 4, 6, 7]). Our work is much in the spirit of Dana, Le Van and Magnien [6, 7] considering a compact set of attainable utility levels as generalized by Allouch [1]. But we remove the transitivity assumption on preferences like in [21]. We discuss in detail the relationships with these contributions in Section 5.

We assume that for each sequence of attainable consumptions there exists an attainable consumption where the preferred consumptions can be approximated by preferred consumptions of the elements of the sequence. Indeed, the element $\bar{x}$ of $\hat{X}$ is not necessarily a cluster point of the sequence (x^ν), but any element strictly preferred to $\bar{x}$ by any agent is approachable by a sequence of elements strictly preferred to $(x^{\varphi(\nu)})$. This condition imposes some restriction on the asymptotic behavior of the preferences for attainable allocations in the sense that some preferred elements remain at a finite distance of the origin even if the allocation is very far.

Note that the productions are not considered in Assumption (H3). So, only the total production set matters since it determines the attainable consumptions. The fact that some unbounded sequences of individual productions can be attainable does not prevent the existence of an equilibrium as long as the total production set is not modified.

Example 2.3. We present an example of an exchange economy where Assumption (H3) is satisfied while the attainable set is not bounded and the preference correspondences are not representable by utility functions. Then we extend it to a production economy with a class of production sets. Let us consider an exchange economy with two commodities A and B and two consumers.

Brought to you by | Université Paris 1 Panthéon-Sorbonne/Centre d'Economie de la Sorbonne
Authenticated
Download Date | 3/20/19 5:38 PM

The consumption sets are given by

$$X_1 = X_2 = \{(a, b) \in \mathbb{R}^2 \mid a + b \geq 0\}.$$

The attainable allocations set $\mathcal{A}(\mathcal{E})$ of the economy is then

$$\mathcal{A}(\mathcal{E}) = \{((a, b), (\omega_A - a, \omega_B - b)) \mid 0 \leq a + b \leq \omega_A + \omega_B\},$$

where (ω_A, ω_B) with $\omega_A + \omega_B > 0$ denotes the global endowment. The set $\mathcal{A}(\mathcal{E})$ is clearly non-compact.

We consider the continuous function $\Pi : X_i \to \mathbb{R}^2$ defined by

$$\Pi(a, b) = \Big(\frac{1}{2} + \frac{a - b}{(|a - b| + 1)(a^2 + b^2 + 2)}, \frac{1}{2} + \frac{b - a}{(|a - b| + 1)(a^2 + b^2 + 2)}\Big).$$

The preference correspondence is the same for the two consumers and it is defined by $P_i : X_1 \times X_2 \to X_i$ with

$$P_i((a_1, b_1), (a_2, b_2)) = \{(\alpha, \beta) \in X_i \mid \Pi(a_i, b_i) \cdot (\alpha, \beta) > \Pi(a_i, b_i) \cdot (a_i, b_i)\}.$$

One easily checks that Assumption (H1) is satisfied by the preference relations since Π is continuous, so P_i has an open graph and $\Pi(a, b) \gg (0, 0)$, and thus the local non-satiation holds true everywhere.

We remark that if (a_i^ν, b_i^ν) is a sequence of X_i such that $\|(a_i^\nu, b_i^\nu)\|$ converges to $+\infty$ and $a_i^\nu + b_i^\nu$ converges to a finite limit c, then $\Pi(a_i^\nu, b_i^\nu)$ converges to $(\frac{1}{2}, \frac{1}{2})$ and $\Pi(a_i^\nu, b_i^\nu) \cdot (a_i^\nu, b_i^\nu)$ converges to $\lim_\nu \frac{1}{2}(a_i^\nu + b_i^\nu) = \frac{c}{2}$.

Let $((a_1^\nu, b_1^\nu), (a_2^\nu, b_2^\nu))$ be a sequence of $\mathcal{A}(\mathcal{E})$. If it has a bounded subsequence, then this subsequence has a cluster point $((\bar{a}_1, \bar{b}_1), (\bar{a}_2, \bar{b}_2))$. Then the desired property of Assumption (H3) holds true thanks to the fact that the preference correspondences have an open graph. See the proof of Proposition 5.1 (i).

If the sequence is unbounded, we note that the sequences $(a_1^\nu + b_1^\nu)$ and $(a_2^\nu + b_2^\nu)$ belong to $[0, \omega_A + \omega_B]$, and $a_1^\nu + b_1^\nu + a_2^\nu + b_2^\nu = \omega_A + \omega_B$ for all ν. So, there exists a subsequence $((a_1^{\varphi(\nu)}, b_1^{\varphi(\nu)}), (a_2^{\varphi(\nu)}, b_2^{\varphi(\nu)}))$ such that the sequences $(a_1^{\varphi(\nu)} + b_1^{\varphi(\nu)})$ and $(a_2^{\varphi(\nu)} + b_2^{\varphi(\nu)})$ converge to $c \in [0, \omega_A + \omega_B]$ and $\omega_A + \omega_B - c$, respectively. Let us consider the attainable allocation

$$\Big(\Big(\bar{a}_1 = \frac{c}{2}, \bar{b}_1 = \frac{c}{2}\Big), \Big(\bar{a}_2 = \frac{\omega_A + \omega_B - c}{2}, \bar{b}_2 = \frac{\omega_A + \omega_B - c}{2}\Big)\Big).$$

We remark that

$$\Pi(\bar{a}_1, \bar{b}_1) = \Pi(\bar{a}_2, \bar{b}_2) = \Big(\frac{1}{2}, \frac{1}{2}\Big),$$
$$\Pi(\bar{a}_1, \bar{b}_1) \cdot (\bar{a}_1, \bar{b}_1) = \frac{1}{2}(\bar{a}_1 + \bar{b}_1) = \frac{c}{2},$$
$$\Pi(\bar{a}_2, \bar{b}_2) \cdot (\bar{a}_2, \bar{b}_2) = \frac{1}{2}(\bar{a}_2 + \bar{b}_2) = \frac{\omega_A + \omega_B - c}{2}.$$

Let $i = 1, 2$ and $(a_i, b_i) \in X_i$ be such that $(a_i, b_i) \in P_i((\bar{a}_1, \bar{b}_1), (\bar{a}_2, \bar{b}_2))$. From the definition of P_i one deduces that

$$\frac{1}{2}(a_i + b_i) > \frac{1}{2}(\bar{a}_i + \bar{b}_i) = \frac{1}{2} \lim_{\nu \to \infty} (a_i^{\varphi(\nu)} + b_i^{\varphi(\nu)}) = \lim_{\nu \to \infty} \Pi(a_i^{\varphi(\nu)}, b_i^{\varphi(\nu)}) \cdot (a_i^{\varphi(\nu)}, b_i^{\varphi(\nu)}).$$

Furthermore, since $\Pi(a_i^{\varphi(\nu)}, b_i^{\varphi(\nu)})$ converges to $(\frac{1}{2}, \frac{1}{2})$, we have

$$\frac{1}{2}(a_i + b_i) = \lim_{\nu \to \infty} \Pi(a_i^{\varphi(\nu)}, b_i^{\varphi(\nu)}) \cdot (a_i, b_i).$$

Consequently, for ν large enough,

$$\Pi(a_i^{\varphi(\nu)}, b_i^{\varphi(\nu)}) \cdot (a_i, b_i) > \lim_{\nu \to \infty} \Pi(a_i^{\varphi(\nu)}, b_i^{\varphi(\nu)}) \cdot (a_i^{\varphi(\nu)}, b_i^{\varphi(\nu)}),$$

which means that

$$(a_i, b_i) \in P_i((a_1^{\varphi(\nu)}, b_1^{\varphi(\nu)}), (a_2^{\varphi(\nu)}, b_2^{\varphi(\nu)})),$$

so the desired property in Assumption (H3) holds true.

We now consider a finite collection of production sets $(Y_j)_{j \in J}$ of $\mathbb{R}^2$ such that $Y = \sum_{j \in J} Y_j$ is closed, convex, contains 0 and $y_A + y_B \leq 0$ for all $(y_A, y_B) \in Y$. Let us consider the production economy where the consumption sector is as above, the production sector is described by $(Y_j)_{j \in J}$ and the portfolio shares (θ_{ij}) are any ones satisfying the standard conditions. One easily checks that Assumption (H3) is satisfied by this production economy since the attainable consumption set is smaller or equal to the one of the exchange economy.

Brought to you by | Université Paris 1 Panthéon-Sorbonne/Centre d'Economie de la Sorbonne
Authenticated
Download Date | 3/20/19 5:38 PM

The main result of this paper is the following existence theorem of a quasi-equilibrium for a production economy.

Theorem 2.4. *Under Assumptions* (H1), (H2) *and* (H3), *there exists a quasi-equilibrium of the economy* $\mathcal{E}$.

3 Preliminary results

First, we show that some properties of the preference correspondences P_i are still true for $\hat{P}_i$.

Proposition 3.1. *Assume that X_i is convex for all i.*
(i) *If P_i is lower semicontinuous on $\prod_{i\in I} X_i$, then the same is true for $\hat{P}_i$.*
(ii) *$\hat{P}_i(x)$ has convex values. Furthermore, if $x_i \notin \operatorname{co} P_i(x)$ for all $x_i \in X_i$, then $x_i \notin \hat{P}_i(x)$.*

Proof. (i) Let $x \in \prod_{i\in I} X_i$ and let V be an open subset of X_i such that

$$V \cap \hat{P}_i(x) \neq \emptyset.$$

Then there exists $\xi_i \in \hat{P}_i(x) \cap V$, which means that $\xi_i = \lambda x_i + (1-\lambda)\zeta_i$ for some $\lambda \in [0,1[$, $\zeta_i \in \operatorname{co} P_i(x)$. Let $\epsilon > 0$ be such that $B(\xi_i,\epsilon) \subset V$. Since the correspondence P_i is lower semicontinuous, $\operatorname{co} P_i$ is lower semicontinuous (see [9, p. 154]). Consequently, there exists a neighborhood W of x in $\prod_{i\in I} X_i$ such that

$$x' \in W \quad \text{implies} \quad \operatorname{co} P_i(x') \cap B(\zeta_i,\epsilon) \neq \emptyset.$$

Thus, for all $x' \in W$, there exists $\zeta_i' \in \operatorname{co} P_i(x') \cap B(\zeta_i,\epsilon)$. Let W' be such that

$$W' = \{x' \in W \mid \|x_i' - x_i\| < \epsilon\}.$$

Let $x' \in W'$ and $\xi_i' = \lambda x_i' + (1-\lambda)\zeta_i'$. Then $\xi_i' \in \hat{P}_i(x')$ such that

$$\|\xi_i' - \xi_i\| \leq \lambda\|x_i' - x_i\| + (1-\lambda)\|\zeta_i' - \zeta_i\| < \epsilon.$$

Then one gets $\xi_i' \in B(\xi_i,\epsilon) \subset V$. Hence, $\xi_i' \in \hat{P}_i(x') \cap V$, which proves the lower semicontinuity of $\hat{P}_i$.

(ii) Let $x \in \prod_{i\in I} X_i$ and $z_i, z_i' \in \hat{P}_i(x)$ be such that $z_i = x_i + \lambda(\xi_i - x_i)$ and $z_i' = x_i + \beta(\xi_i' - x_i)$ for some $\lambda, \beta \in]0,1]$ and $\xi_i, \xi_i' \in \operatorname{co} P_i(x)$. For $\alpha \in]0,1[$, we have

$$\begin{aligned}\alpha z_i + (1-\alpha)z_i' &= x_i + \alpha\lambda\xi_i + (1-\alpha)\beta\xi_i' - [\alpha\lambda x_i + (1-\alpha)\beta x_i]\\ &= x_i + \alpha\lambda\xi_i + (1-\alpha)\beta\xi_i' - [\alpha\lambda + (1-\alpha)\beta]x_i\\ &= x_i + \gamma(\xi_i'' - x_i),\end{aligned}$$

where

$$\gamma = \alpha\lambda + (1-\alpha)\beta \quad \text{and} \quad \xi_i'' = \frac{\alpha\lambda}{\gamma}\xi_i + \frac{(1-\alpha)\beta}{\gamma}\xi_i'.$$

One easily checks that $\gamma \in]0,1]$ since $\lambda, \beta \in]0,1]$ and $\xi_i'' \in \operatorname{co} P_i(x)$. Then $\alpha z_i + (1-\alpha)z_i' \in \hat{P}_i(x)$, which means that $\hat{P}_i$ has convex values.

Finally, we prove the irreflexivity by contraposition. Let us suppose that $x_i \in \hat{P}_i(x)$ for some i. Then $x_i = \lambda x_i + (1-\lambda)x_i'$ with $\lambda \in [0,1[$ and $x_i' \in \operatorname{co} P_i(x)$. Hence, we have $x_i = x_i' \in \operatorname{co} P_i(x)$, which contradicts Assumption (H1) (b). □

Now, we consider the following economy:

$$\mathcal{E}' = (\mathbb{R}^L, (X_i, \hat{P}_i, \omega_i)_{i\in I}, (Y_j')_{j\in J}, (\theta_{ij})_{(i,j)}),$$

where the preference correspondences are replaced by the augmented preference correspondences and the production sets are replaced by their closed convex hull, that is, $Y_j' = \overline{\operatorname{co}} Y_j$ for each j.

Lemma 3.2. *Under Assumption* (H2), *the economies $\mathcal{E}$ and $\mathcal{E}'$ have the same total production set so the same attainable consumption set $\hat{X}$.*

Brought to you by | Université Paris 1 Panthéon-Sorbonne/Centre d'Economie de la Sorbonne
Authenticated
Download Date | 3/20/19 5:38 PM

Proof. Let $Y' = \sum_{j\in J} Y'_j$.

It is clear that $Y \subset Y'$. Conversely, $Y' = \sum_{j\in J} \overline{\text{co}}Y_j \subset \text{cl}(\sum_{j\in J} \text{co } Y_j)$; see [16, p. 48, Corollary 6.6]. Since the convex hull of a sum is the sum of the convex hulls, one gets

$$Y' = \sum_{j\in J} \overline{\text{co}}Y_j \subset \text{cl}\Big(\sum_{j\in J} \text{co } Y_j\Big) = \text{cl}\Big(\text{co}\Big(\sum_{j\in J} Y_j\Big)\Big) = \overline{\text{co}}Y.$$

Since Y is a non-empty closed, convex subset of $\mathbb{R}^L$, one has $\overline{\text{co}}Y = Y$. Hence, $Y = Y'$. □

Proposition 3.3. *If $((\bar{x}_i), (\bar{\zeta}_j), \bar{p})$ is a quasi-equilibrium of $\mathcal{E}'$, then there exists $\bar{y} \in \prod_{j\in J} Y_j$ such that $((\bar{x}_i), (\bar{y}_j), \bar{p})$ is a quasi-equilibrium of $\mathcal{E}$.*

Proof. Let $((\bar{x}_i), (\bar{\zeta}_j), \bar{p})$ be a quasi-equilibrium of $\mathcal{E}'$. So, $\sum_{j\in J} \bar{\zeta}_j \in \sum_{j\in J} Y'_j$. By Lemma 3.2, $\sum_{j\in J} Y'_j = Y$. Consequently, there exists $\bar{y} \in \prod_{j\in J} Y_j$ such that $\sum_{j\in J} \bar{\zeta}_j = \sum_{j\in J} \bar{y}_j$. Hence,

$$\sum_{i\in I} \bar{x}_i = \omega + \sum_{j\in J} \bar{y}_j.$$

In other words, condition (c) of Definition 2.2 is satisfied.

Moreover, one can remark that $\bar{y}_\in Y'_j$ for every j. Consequently, $\bar{p} \cdot \bar{y}_j \le \bar{p} \cdot \bar{\zeta}_j$. But since $\sum_{j\in J} \bar{\zeta}_j = \sum_{j\in J} \bar{y}_j$, one gets $\bar{p} \cdot \bar{y}_j = \bar{p} \cdot \bar{\zeta}_j$.

We now show that condition (a) is satisfied. Let $j \in J$ and $y_j \in Y_j$. Then $y_j \in Y'_j$, so $\bar{p} \cdot y_j \le \bar{p} \cdot \bar{\zeta}_j = \bar{p} \cdot \bar{y}_j$. Hence, $\bar{p} \cdot y_j \le \bar{p} \cdot \bar{y}_j$ and condition (a) of Definition 2.2 is satisfied.

Lastly, we show that condition (b) is satisfied. Since $\bar{p} \cdot \bar{\zeta}_j = \bar{p} \cdot \bar{y}_j$ for all $j \in J$, we have

$$\bar{p} \cdot \bar{x}_i \le \bar{p} \cdot \omega_i + \sum_{j\in J} \theta_{i,j}\bar{p} \cdot \bar{y}_j$$

for all i. Now, let $i \in I$ and $x_i \in X_i$ be such that $x_i \in P_i(\bar{x})$. Since $P_i(\bar{x}) \subset \hat{P}_i(\bar{x})$, we obtain $\bar{p} \cdot x_i \ge \bar{p} \cdot \bar{x}_i$. □

4 Existence of quasi-equilibria

In this section, we consider the economy $\mathcal{E}'$ as defined above. We have seen in the previous section that we can deduce the existence of a quasi-equilibrium of $\mathcal{E}$ from a quasi-equilibrium of $\mathcal{E}'$.

In what follows, we will consider Assumptions (H1'), (H2') which correspond to (H1), (H2), but are adapted to $\mathcal{E}'$ and the asymptotic assumption (WH3). In the previous section, we have shown that (H1') and (H2') are satisfied by $\mathcal{E}'$ if Assumptions (H1), (H2) are satisfied by $\mathcal{E}$ and (WH3) is weaker than (H3).

Assumption (H1'). For every $i \in I$, the following conditions hold:

(a) X_i is a non-empty closed, convex subset of $\mathbb{R}^L$.

(b) (Irreflexivity:) For all $x \in \prod_{i\in I} X_i$, one has $x_i \notin \hat{P}_i(x)$.

(c) (Lower semicontinuous:) $\hat{P}_i : \prod_{k\in I} X_k \to X_i$ is lower semicontinuous and convex-valued.

(d) $\omega_i \in X_i - \sum_{j\in J} \theta_{i,j} Y'_j$, i.e. there exists

$$(\underline{x}_i, (\underline{y}_{i,j})) \in X_i \times \prod_{j\in J} Y'_j$$

such that

$$\underline{x}_i = \omega_i + \sum_{j\in J} \theta_{i,j}\underline{y}_{i,j}.$$

(e) For each $x \in \hat{X}$, one has $\hat{P}_i(x) \ne \emptyset$, and for all $\xi_i \in \hat{P}_i(x)$ and all $t \in]0, 1]$, one has $t\xi_i + (1-t)x_i \in \hat{P}_i(x)$.

Assumption (H2'). The set Y'_j is a closed, convex subset of $\mathbb{R}^L$ for each $j \in J$.

To prepare the discussion on the relationships with the paper of Won and Yannelis, we consider the following weakening of Assumption (H3). If A is a subset of $\mathbb{R}^L$, cone A is the cone spanned by A.

Brought to you by | Université Paris 1 Panthéon-Sorbonne/Centre d'Economie de la Sorbonne
Authenticated
Download Date | 3/20/19 5:38 PM

Assumption (WH3). There exists a consumer i_0 such that for all sequences $((x_i^\nu))$ of $\hat{X}$ with

$$\underline{x}_i \in \overline{\hat{P}_i(x^\nu)^c}$$

for all i, there exist a subsequence $((x_i^{\varphi(\nu)})) \in \hat{X}$ and $(\bar{x}_i) \in \hat{X}$ with the condition that for all i and all $\xi_i \in \hat{P}_i(\bar{x})$ there exist an integer ν_1 and a sequence $(\xi_i^{\varphi(\nu)})_{\nu \geq \nu_1}$ convergent to (ξ_i) such that for all $\nu \geq \nu_1$,

$$\xi_{i_0}^{\varphi(\nu)} \in \operatorname{cone}[\hat{P}_{i_0}(x^{\varphi(\nu)}) - \bar{x}_{i_0}^{\varphi(\nu)}] + \bar{x}_{i_0}^{\varphi(\nu)},$$

and for all $i \neq i_0$,

$$\xi_i^{\varphi(\nu)} \in \hat{P}_i(x^{\varphi(\nu)}).$$

Assumption (WH3) is clearly weaker than (H3) since

$$\hat{P}_{i_0}(x^{\varphi(\nu)}) \subset \operatorname{cone}[\hat{P}_{i_0}(x^{\varphi(\nu)}) - \bar{x}_{i_0}^{\varphi(\nu)}] + \bar{x}_{i_0}^{\varphi(\nu)}.$$

But this assumption exhibits the drawback of being asymmetric. That is why we did not emphasize it before since we think that further works should provide an even weaker but symmetric assumption. We provide more comments in Section 5 when we discuss the link to the work of Won and Yannelis.

We now state the existence result of a quasi-equilibrium for a finite private ownership economy satisfying Assumptions (H1'), (H2') and (WH3).

Theorem 4.1. *If Assumptions* (H1'), (H2') *and* (WH3) *are satisfied, then there exists a quasi-equilibrium of the economy* $\mathcal{E}'$.

The idea of the proof is as follows: We first truncate consumption and production sets with a closed ball with a radius large enough. Following an idea of Bergstrom [3], we modify the budget sets in such a way that they coincide with the original ones when the price belongs to the unit sphere. Then, by applying the well-known result of Gale and Mas-Colell [11] and Bergstrom [3] about the existence of maximal elements to a suitable family of lower semicontinuous correspondences, we obtain a sequence $((x^\nu), (y^\nu), p^\nu)$ such that $((x^\nu), (y^\nu))$ is an attainable allocation of the economy $\mathcal{A}(\mathcal{E}')$, p^ν belongs to the unit ball of $\mathbb{R}^L$, the domain of admissible prices, the producers maximize the profit over the truncated production sets, and the consumptions are maximal elements of the preferences on the truncated consumption sets, but with a relaxed budget constraint. From Assumption (WH3) and the compactness of the price set, we obtain a subsequence $(x^{\varphi(\nu)}, y^{\varphi(\nu)}, p^{\varphi(\nu)})$ and an element $(\bar{x}, \bar{y}, \bar{p})$ such that the preferences at this point are close to the preferences at $x^{\varphi(\nu)}$ for ν large enough and $p^{\varphi(\nu)}$ converges to $\bar{p}$. Finally, we prove that $(\bar{x}, \bar{y}, \bar{p})$ is a quasi-equilibrium of $\mathcal{E}'$. Note that the difficulty of the limit argument comes from the fact that $(\bar{x}, \bar{y})$ is not necessarily the limit of $(x^{\varphi(\nu)}, y^{\varphi(\nu)})$.

4.1 The fixed-point argument

From Assumption (H1') (d), let us fix $\underline{x}_i \in X_i$ and $\underline{y}_{i,j} \in Y'_j$ such that

$$\underline{x}_i = \omega_i + \sum_{j \in J} \theta_{i,j} \underline{y}_{i,j}$$

for every $i \in I$. Let $\bar{B}^\nu$ be the closed ball with center 0 and radius ν with ν large enough so that ω, $\underline{x}_i$, $\underline{y}_{i,j}$ and ω_i belong to B^ν, the interior of $\bar{B}^\nu$, for all i, j. We consider the truncated economy obtained by replacing agent's consumption sets by $X_i^\nu = X_i \cap \bar{B}^\nu$ for all $i \neq i_0$, and

$$X_{i_0}^\nu = X_{i_0} \cap \bar{B}^{(\sharp I + \sharp J)\nu}.$$

The production set becomes $Y_j^\nu = Y'_j \cap \bar{B}^\nu$ and the augmented preference correspondences are $\hat{P}_i^\nu = \hat{P}_i \cap B^\nu$ for $i \neq i_0$ and

$$\hat{P}_{i_0}^\nu = \hat{P}_{i_0} \cap B^{(\sharp I + \sharp J)\nu}.$$

Brought to you by | Université Paris 1 Panthéon-Sorbonne/Centre d'Economie de la Sorbonne
Authenticated
Download Date | 3/20/19 5:38 PM

The closed unit ball $\bar{B} = \{x \in \mathbb{R}^L : \|x\| \leq 1\}$ will be the price set. The truncation of X_{i_0} is chosen in such a way that if $(x, y) \in \prod_{i\in I} X_i^\nu \times \prod_{j\in J} Y_j^\nu$ is feasible, that is, $\sum_{i\in I} x_i = \omega + \sum_{j\in J} y_j$, then x_{i_0} belongs to the open ball $B^{(\sharp I+\sharp J)\nu}$.

We now consider the economy

$$\mathcal{E}^\nu = (\mathbb{R}^L, (X_i^\nu, \hat{P}_i^\nu, \omega_i)_{i\in I}, (Y_j^\nu)_{j\in J}, (\theta_{i,j})_{(i\in I, j\in J)}),$$

where the consumption and production sets are compact.

Remark 4.2. For all i, the correspondence $\hat{P}_i^\nu$ is lower semicontinuous. Indeed, $\hat{P}_i^\nu$ is the intersection of the lower semicontinuous correspondence $\hat{P}_i$ and the constant correspondence B^ν (or $B^{(\sharp I+\sharp J)\nu}$), which has an open graph.

Remark 4.3. With the above remark and since $\bar{B}^\nu$ is convex and closed, note that the compact economy $\mathcal{E}^\nu$ satisfies Assumption (H2') and Assumption (H1'), but not the non-satiation of preferences at attainable allocations. Furthermore, Y_j^ν is now compact.

Since each Y_j^ν is compact, we can define for every $p \in \bar{B}$ the profit function

$$\pi_j^\nu(p) = \sup p.Y_j^\nu = \sup\{p.y_j : y_j \in Y_j^\nu\},$$

and the wealth of consumer i is defined by

$$\gamma_i^\nu(p) = p.\omega_i + \sum_{j\in J} \theta_{ij}\pi_j^\nu(p).$$

Note that the function $\pi_j^\nu : \bar{B} \to \mathbb{R}$ is continuous since it is finite and convex.

In what follows, we will use the following notations for simplicity

$$\begin{aligned} Z^\nu &= \prod_{i\in I} X_i^\nu \times \prod_{j\in J} Y_j^\nu \times \bar{B} \quad \text{and } z = (x, y, p) \text{ denotes a typical element of } Z^\nu, \\ \hat{\gamma}_i^\nu(z) &= \gamma_i^\nu(p) + \frac{1 - \|p\|}{\sharp I}, \\ \widetilde{\gamma}_i^\nu(z) &= \max\Big\{\hat{\gamma}_i^\nu(z), \frac{1}{2}[\hat{\gamma}_i^\nu(p) + p \cdot x_i]\Big\}. \end{aligned}$$

Remark 4.4. Note that $p \cdot x_i > \widetilde{\gamma}_i^\nu(z) > \hat{\gamma}_i^\nu(z)$ when $p \cdot x_i > \hat{\gamma}_i^\nu(z)$, and $\widetilde{\gamma}_i^\nu(z) = \hat{\gamma}_i^\nu(z)$ when $p \cdot x_i \leq \hat{\gamma}_i^\nu(z)$.

Let now $N = I \cup J \cup \{0\}$ be the union of the set of consumers I indexed by i, the set of producers J indexed by j, and an additional agent 0 whose function is to react with prices to a given total excess demand.

For all $i \in I$, we define the correspondences $\alpha_i^\nu : Z^\nu \to X_i^\nu$ and $\widetilde{\beta}_i^\nu : Z^\nu \to X_i^\nu$ as follows:

$$\begin{aligned} \alpha_i^\nu(z) &= \{\xi_i \in X_i^\nu : p \cdot \xi_i \leq \hat{\gamma}_i^\nu(z)\}, \\ \widetilde{\beta_i}^\nu(z) &= \{\xi_i \in X_i^\nu : p \cdot \xi_i < \widetilde{\gamma}_i^\nu(z)\}. \end{aligned}$$

From the construction of the extended budget set, one checks that for all i the consumption $\underline{x}_i$ belongs to $\widetilde{\beta_i}^\nu(z)$ if $x_i \notin \alpha_i^\nu(z)$. Indeed, from (H1') (d),

$$\underline{x}_i = \omega_i + \sum_{j\in J} \theta_{i,j}\underline{y}_{i,j}$$

since $x_i \notin \alpha_i{}^\nu(z)$, $p \cdot x_i > \hat{\gamma}_i^\nu(z)$ and $\widetilde{\gamma}_i^\nu(z) > \hat{\gamma}_i^\nu(z)$. Furthermore,

$$p \cdot \underline{x}_i = p \cdot \omega_i + \sum_{j\in J} \theta_{i,j} p \cdot \underline{y}_{i,j} \leq p \cdot \omega_i + \sum_{j\in J} \theta_{i,j}\pi_j^\nu(p) \leq \hat{\gamma}_i^\nu(z) < \widetilde{\gamma}_i^\nu(z),$$

which means that $\underline{x}_i$ belongs to $\widetilde{\beta_i}^\nu(z)$. Moreover, since $\widetilde{\gamma}_i^\nu$ is continuous, the correspondence $\widetilde{\beta_i}^\nu$ has an open graph in $Z^\nu \times X_i^\nu$.

Brought to you by | Université Paris 1 Panthéon-Sorbonne/Centre d'Economie de la Sorbonne
Authenticated
Download Date | 3/20/19 5:38 PM

Now, for $i \in I$, we consider the mapping ϕ_i^ν defined from Z^ν to X_i^ν by

$$\phi_i^\nu(z) = \begin{cases} \widetilde{\beta}_i^\nu(z) & \text{if } x_i \notin \alpha_i^\nu(z), \\ \widetilde{\beta}_i^\nu(z) \cap \hat{P}_i^\nu(x) & \text{if } x_i \in \alpha_i^\nu(z). \end{cases}$$

For $j \in J$, we define ϕ_j^ν from Z^ν to Y_j^ν by

$$\phi_j^\nu(z) = \{y_j' \in Y_j^\nu \mid p \cdot y_j < p \cdot y_j'\},$$

and the mapping ϕ_0^ν from Z^ν to $\bar{B}$ is defined by

$$\phi_0^\nu(z) = \Big\{q \in \bar{B} \;\Big|\; (q - p) \cdot \Big(\sum_{i \in I} x_i - \omega - \sum_{j \in J} y_j \Big) > 0 \Big\}.$$

Now we will apply to Z^ν and the correspondences $(\phi_i)_{i \in I}^\nu$, $(\phi_j)_{j \in J}^\nu$ and ϕ_0^ν the well-known theorem of Gale and Mas-Colell [11]. We will actually use the Bergstrom version of this theorem in [3], which is more adapted to our setting.

Theorem 4.5 ([3, 11]). *For each $k = 1, \dots, \bar{k}$, let Z_k be a non-empty, compact, convex subset of some finite-dimensional Euclidean space. Given $Z = \prod_{k=1}^{\bar{k}} Z_k$, for each k, let $\phi_k : Z \to Z_k$ be a lower semicontinuous correspondence satisfying $z_k \notin \operatorname{co} \phi_k(z)$ for all $z \in Z$. Then there exists $\bar{z} \in Z$ such that for each $k = 1, \dots, \bar{k}$, one has*

$$\phi_k(\bar{z}) = \emptyset.$$

For the correspondences $(\phi_j^\nu)_{j \in J}$ and ϕ_0^ν, one easily checks that they are convex-valued, irreflexive and lower semicontinuous since they have an open graph.

We now check that for all $i \in I$ the correspondence ϕ_i^ν satisfies the assumptions of Theorem 4.5. We first remark that ϕ_i^ν is convex-valued since $\widetilde{\beta}_i^\nu$ and $\hat{P}_i$ are convex. We now check the irreflexivity. If $x_i \in \alpha_i^\nu(z)$, then, from Assumption (H1') (b), $x_i \notin \hat{P}_i(x)$, so $x_i \notin \phi_i^\nu(x)$ since $\phi_i^\nu(x) \subset \hat{P}_i(x)$. If $x_i \notin \alpha_i^\nu(z)$, then from Remark 4.4, we obtain $p \cdot x_i > \widetilde{\gamma}_i^\nu(z)$, so $x_i \notin \widetilde{\beta}_i^\nu(z) = \phi_i^\nu(z)$.

For the lower semicontinuity, let V be an open set and let z be such that $\phi_i^\nu(z) \cap V \neq \emptyset$. If $x_i \notin \alpha_i^\nu(z)$, then $p \cdot x_i > \hat{\gamma}_i^\nu(z)$. Since $\hat{\gamma}_i^\nu$ is continuous, there exists a neighborhood W of z such that $p' \cdot x_i' > \hat{\gamma}_i^\nu(z')$ for all $z' \in W$. Since $\widetilde{\beta}_i^\nu$ has an open graph, there exists a neighborhood W' of z such that $\widetilde{\beta}_i^\nu(z') \cap V \neq \emptyset$ for all $z' \in W'$. So, $\phi_i^\nu(z') \cap V \neq \emptyset$ for all $z' \in W \cap W'$, and consequently ϕ_i^ν is lower semicontinuous at z. If $x_i \in \alpha_i^\nu(z)$, we first remark that $\widetilde{\beta_i}^\nu \cap \hat{P}_i^\nu$ is lower semicontinuous as an intersection of a lower semicontinuous correspondence with an open graph correspondence. So, there exists a neighborhood W of z such that $\widetilde{\beta_i}^\nu(z') \cap \hat{P}_i^\nu(x') \cap V \neq \emptyset$ for all $z' \in W$. This implies that $\widetilde{\beta_i}^\nu(z') \cap V \neq \emptyset$. Hence, in both cases $x_i' \in \alpha_i^\nu(z')$ or $x_i' \notin \alpha_i^\nu(z')$, we obtain $\phi_i^\nu(z') \cap V \neq \emptyset$ from the definition of ϕ_i^ν. Thus ϕ_i^ν is also lower semicontinuous at z in this case.

From Theorem 4.5 there exists $\bar{z}^\nu = (\bar{x}^\nu, \bar{y}^\nu, \bar{p}^\nu) \in Z^\nu$ such that, for all $k \in N$,

$$\phi_k^\nu(\bar{z}^\nu) = \emptyset.$$

As already noticed, since $\underline{x}_i \in \widetilde{\beta}_i^\nu(\bar{z}^\nu)$ and $\phi_i^\nu(\bar{z}^\nu) = \emptyset$ for all $i \in I$, we conclude from the definition of ϕ_i^ν that

$$\begin{cases} \bar{p}^\nu \cdot \bar{x}_i^\nu \leq \hat{\gamma}_i^\nu(\bar{z}^\nu), \\ \widetilde{\beta}_i^\nu(\bar{z}^\nu) \cap \hat{P}_i^\nu(\bar{x}^\nu) = \emptyset. \end{cases} \tag{4.1}$$

Furthermore, from Remark 4.4 one deduces that $\widetilde{\gamma}_i^\nu(\bar{z}^\nu) = \hat{\gamma}_i^\nu(\bar{z}^\nu)$.

In addition, for all $j \in J$, since $\phi_j^\nu(\bar{z}^\nu) = \emptyset$, we deduce that

$$\bar{p}^\nu \cdot y_j \leq \bar{p}^\nu \cdot \bar{y}_j^\nu = \pi_j^\nu(\bar{p}^\nu) \quad \text{for all } y_j \in Y_j^\nu, \tag{4.2}$$

and since $\phi_0^\nu(\bar{z}^\nu) = \emptyset$, we have

$$p \cdot \Big(\sum_{i \in I} \bar{x}_i^\nu - \omega - \sum_{j \in J} \bar{y}_j^\nu \Big) \leq \bar{p}^\nu \cdot \Big(\sum_{i \in I} \bar{x}_i^\nu - \omega - \sum_{j \in J} \bar{y}_j^\nu \Big) \quad \text{for all } p \in \bar{B}. \tag{4.3}$$

Brought to you by | Université Paris 1 Panthéon-Sorbonne/Centre d'Economie de la Sorbonne
Authenticated
Download Date | 3/20/19 5:38 PM

We now prove that $(\sum_{i\in I}\bar{x}_i^\nu - \omega - \sum_{j\in J}\bar{y}_j^\nu) = 0$. Indeed, if not, it follows from (4.3) that $\bar{p}^\nu$ belongs to the boundary of $\bar{B}$, that is,

$$\|\bar{p}^\nu\| = 1 \quad \text{and} \quad \bar{p}^\nu \cdot \Big(\sum_{i\in I}\bar{x}_i^\nu - \omega - \sum_{j\in J}\bar{y}_j^\nu\Big) > 0.$$

Now, by (4.1) and (4.2), $\bar{p}^\nu \cdot \bar{x}_i^\nu \le \hat{\gamma}_i^\nu(\bar{z}^\nu) = \gamma_i^\nu(\bar{z}^\nu) = \bar{p}^\nu \cdot \omega_i + \sum_{j\in J}\theta_{i,j}\bar{p}^\nu \cdot \bar{y}_j^\nu$ for all i. Summing these inequalities over $i \in I$, we get

$$\bar{p}^\nu \cdot \Big(\sum_{i\in I}\bar{x}_i^\nu - \omega - \sum_{j\in J}\bar{y}_j^\nu\Big) \le 0,$$

which yields a contradiction. We thus have proved that $(\bar{x}^\nu, \bar{y}^\nu) \in \mathcal{A}(\mathcal{E}^\nu)$.

Remark 4.6. Since $(\bar{x}^\nu, \bar{y}^\nu)$ is feasible, we deduce that $\bar{x}_{i_0}^\nu$ belongs to the open ball $B^{(\sharp I+\sharp J)\nu}$. By Assumption (H1') (e), $\hat{P}_{i_0}(\bar{x}^\nu)$ is non-empty and for all $\xi_{i_0} \in \hat{P}_{i_0}(\bar{x}^\nu)$ and all $t \in\,]0,1]$ one has $t\xi_{i_0} + (1-t)\bar{x}_{i_0}^\nu \in \hat{P}_{i_0}(\bar{x}^\nu)$. For t small enough, $t\xi_{i_0} + (1-t)\bar{x}_{i_0}^\nu$ belongs to $B^{(\sharp I+\sharp J)\nu}$, so also to $\hat{P}_{i_0}^\nu(\bar{x}^\nu)$. From (4.1),

$$\bar{p}^\nu \cdot (t\xi_{i_0} + (1-t)\bar{x}_{i_0}^\nu) \ge \hat{\gamma}_{i_0}^\nu(\bar{z}^\nu).$$

At the limit when t tends to 1, knowing from (4.1) that $\bar{p}^\nu \cdot \bar{x}_{i_0}^\nu \le \hat{\gamma}_{i_0}^\nu(\bar{z}^\nu)$, we get

$$\bar{p}^\nu \cdot \bar{x}_{i_0}^\nu = \hat{\gamma}_{i_0}^\nu(\bar{z}^\nu), \tag{4.4}$$

from which one deduces that

$$\bar{p}^\nu \cdot \xi_{i_0} \ge \hat{\gamma}_{i_0}^\nu(\bar{z}^\nu) \quad \text{for all } \xi_{i_0} \in \hat{P}_{i_0}(\bar{x}^\nu). \tag{4.5}$$

4.2 The limit argument

We first show that we can apply Assumption (WH3) to the sequence $((\bar{x}_i^\nu))$ built in the previous subsection. We have already proved that $\bar{x}^\nu$ is attainable in the truncated economy $\mathcal{E}^\nu$, so it is also attainable in the economy $\mathcal{E}'$. It remains to show that

$$\underline{x}_i \in \overline{\hat{P}_i(\bar{x}^\nu)^c}$$

for all i.

There are two cases. First, if $\bar{p}^\nu \cdot \underline{x}_i < \hat{\gamma}_i^\nu(\bar{z}^\nu)$, which means that $\underline{x}_i \in \widetilde{\beta}_i^\nu(\bar{z}^\nu)$, then, from (4.1),

$$\underline{x}_i \notin \hat{P}_i^\nu(\bar{x}^\nu) = \hat{P}_i(\bar{x}^\nu) \cap B^\nu.$$

Since $\underline{x}_i \in B^\nu$ as ν has been chosen large enough, one deduces that $\underline{x}_i \notin \hat{P}_i(\bar{x}^\nu)$, and therefore

$$\underline{x}_i \in \overline{\hat{P}_i(\bar{x}^\nu)^c}.$$

If $\bar{p}^\nu \cdot \underline{x}_i \ge \hat{\gamma}_i^\nu(\bar{z}^\nu)$, as $\underline{x}_i \in \widetilde{\beta}_i^\nu(\bar{z}^\nu)$ and $\hat{\gamma}_i^\nu(\bar{z}^\nu) = \widetilde{\gamma}_i^\nu(\bar{z}^\nu)$, we actually have the equality $\bar{p}^\nu \cdot \underline{x}_i = \hat{\gamma}_i^\nu(\bar{z}^\nu)$. We remark that

$$\hat{\gamma}_i^\nu(\bar{z}^\nu) = \gamma_i^\nu(\bar{z}^\nu) + \frac{1 - \|\bar{p}^\nu\|}{\sharp I} = \bar{p}^\nu \cdot \underline{x}_i = \bar{p}^\nu \cdot \Big(\omega_i + \sum_{j\in J}\theta_{i,j}\underline{y}_{i,j}\Big) \le \gamma_i^\nu(\bar{z}^\nu).$$

So, $\|\bar{p}^\nu\| = 1$. By contradiction, we prove that

$$\underline{x}_i \in \overline{\hat{P}_i(\bar{x}^\nu)^c}.$$

Indeed, if not, then $\underline{x}_i \in \operatorname{int}\hat{P}_i(\bar{x}^\nu)$ and there exists $\rho > 0$ such that $B(\underline{x}_i, \rho) \subset \hat{P}_i(\bar{x}^\nu)$ and $B(\underline{x}_i, \rho) \subset B^\nu$. Since $\bar{p}^\nu \neq 0$, there exists $\xi_i^\nu \in B(\underline{x}_i, \rho)$ such that $\bar{p}^\nu \cdot \xi_i^\nu < \bar{p}^\nu \cdot \underline{x}_i = \hat{\gamma}_i^\nu(\bar{z}^\nu)$, and this contradicts (4.1) since $\xi_i^\nu \in B(\underline{x}_i, \rho) \subset \hat{P}_i^\nu(\bar{x}^\nu)$.

We now consider the subsequence $((\bar{x}_i^{\varphi(\nu)}))$ of $\hat{X}$ and $((\bar{x}_i)) \in \hat{X}$ as given by Assumption (WH3). From the definition of $\hat{X}$ there exists $(\bar{y}_j) \in \prod_{j\in J} Y_j'$ such that

$$\sum_{i\in I}\bar{x}_i = \sum_{i\in I}\omega_i + \sum_{j\in J}\bar{y}_j.$$

Since $\bar{B}$ is compact, we can assume without any loss of generality that the sequence $(\bar{p}^{\varphi(\nu)})$ converges to $\bar{p} \in \bar{B}$.

Brought to you by | Université Paris 1 Panthéon-Sorbonne/Centre d'Economie de la Sorbonne
Authenticated
Download Date | 3/20/19 5:38 PM

Now let $(y_j) \in \prod_{j\in J} Y'_j$, $(\xi_i) \in \prod_{i\in I} \hat{P}_i(\bar{x})$ and $\lambda \in [0,1[$. Such (ξ_i) exists from Assumption (H1') (e). Furthermore, from the definition of the extended preferences, note that $\xi_i^\lambda = \lambda\bar{x}_i + (1-\lambda)\xi_i \in \hat{P}_i(\bar{x})$.

By (WH3), there exists an integer ν_1 and a sequence $(\xi_i^{\varphi(\nu)})_{\nu\geq\nu_1}$ convergent to ξ_i^λ such that for all $\nu \geq \nu_1$,

$$\xi_{i_0}^{\varphi(\nu)} \in \operatorname{cone}\{\hat{P}_{i_0}(\bar{x}^{\varphi(\nu)}) - \bar{x}_{i_0}^{\varphi(\nu)}\} + \bar{x}_{i_0}^{\varphi(\nu)}$$

and $\xi_i^{\varphi(\nu)} \in \hat{P}_i(\bar{x}^{\varphi(\nu)})$ for all $i \neq i_0$. Since the sequence $(\xi_i^{\varphi(\nu)})_{\nu\geq\nu_1}$ is convergent, it is bounded and, for ν large enough, $\xi_i^{\varphi(\nu)}$ belongs to B^ν for all $i \neq i_0$, so $\xi_i^{\varphi(\nu)} \in \hat{P}_i^\nu(\bar{x}^{\varphi(\nu)})$ for all $\nu \geq \nu_1$. We deduce from (4.1) that $\xi_i^{\varphi(\nu)} \notin \widetilde{\beta}_i^\nu(\bar{z}^{\varphi(\nu)})$, that is, $\bar{p}^\nu \cdot \xi_i^{\varphi(\nu)} \geq \widetilde{\gamma}_i^\nu(\bar{z}^\nu) = \hat{\gamma}_i^\nu(\bar{z}^\nu)$. Using the same argument as in Remark 4.6, we deduce that $\bar{p}^\nu \cdot \bar{x}_i^{\varphi(\nu)} = \hat{\gamma}_i^\nu(\bar{z}^\nu)$. So, from Remark 4.6, for all $i \in I$,

$$\bar{p}^\nu \cdot \bar{x}_i^{\varphi(\nu)} = \bar{p}^{\varphi(\nu)} \cdot \omega_i + \sum_{j\in J} \theta_{i,j}\bar{p}^{\varphi(\nu)} \cdot \bar{y}_j^{\varphi(\nu)} + \frac{1 - \|\bar{p}^{\varphi(\nu)}\|}{\sharp I}.$$

Summing these inequalities over i and knowing that $(\bar{x}^{\varphi(\nu)}, \bar{y}^{\varphi(\nu)})$ is a feasible allocation, we conclude that $\|\bar{p}^{\varphi(\nu)}\| = 1$ and $\|\bar{p}\| = 1$ at the limit.

For all $i \neq i_0$,

$$\bar{p}^{\varphi(\nu)} \cdot \xi_i^{\varphi(\nu)} \geq \gamma_i^\nu(\bar{z}^\nu) = \bar{p}^{\varphi(\nu)} \cdot \omega_i + \sum_{j\in J} \theta_{i,j}\bar{p}^{\varphi(\nu)} \cdot \bar{y}_j^{\varphi(\nu)},$$

and for i_0 there exist $\alpha \geq 0$ and $\zeta_{i_0}^{\varphi(\nu)} \in \hat{P}_{i_0}(\bar{x}^{\varphi(\nu)})$ such that

$$\bar{p}^{\varphi(\nu)} \cdot \xi_{i_0}^{\varphi(\nu)} = \bar{p}^{\varphi(\nu)} \cdot [\bar{x}_{i_0}^{\varphi(\nu)} + \alpha(\zeta_{i_0}^{\varphi(\nu)} - \bar{x}_{i_0}^{\varphi(\nu)})].$$

From (4.4) and (4.5),

$$\bar{p}^{\varphi(\nu)} \cdot \bar{x}_{i_0}^{\varphi(\nu)} = \hat{\gamma}_{i_0}^\nu(\bar{z}^\nu) = \gamma_{i_0}^\nu(\bar{z}^\nu) \quad \text{and} \quad \bar{p}^{\varphi(\nu)} \cdot \zeta_{i_0}^{\varphi(\nu)} \geq \hat{\gamma}_{i_0}^\nu(\bar{z}^\nu) = \gamma_{i_0}^\nu(\bar{z}^\nu),$$

so, since $\alpha \geq 0$, one concludes that

$$\bar{p}^{\varphi(\nu)} \cdot \xi_{i_0}^{\varphi(\nu)} \geq \gamma_{i_0}^\nu(\bar{z}^\nu) = \bar{p}^{\varphi(\nu)} \cdot \omega_{i_0} + \sum_{j\in J} \theta_{i_0,j}\bar{p}^{\varphi(\nu)} \cdot \bar{y}_j^{\varphi(\nu)}.$$

For ν large enough, $y_j \in \bar{B}^\nu$ for all $j \in J$. So, $(y_j) \in \prod_{j\in J} Y_j^\nu$, and from (4.2) one gets

$$\bar{p}^{\varphi(\nu)} \cdot \xi_i^{\varphi(\nu)} \geq \bar{p}^{\varphi(\nu)} \cdot \omega_i + \sum_{j\in J} \theta_{i,j}\bar{p}^{\varphi(\nu)} \cdot y_j. \tag{4.6}$$

In particular, for $(\bar{y}_j) \in \prod_{j\in J} Y'_j$, one gets

$$\bar{p}^{\varphi(\nu)} \cdot \xi_i^{\varphi(\nu)} \geq \bar{p}^{\varphi(\nu)} \cdot \omega_i + \sum_{j\in J} \theta_{i,j}\bar{p}^{\varphi(\nu)} \cdot \bar{y}_j. \tag{4.7}$$

Passing to the limit in (4.6) and (4.7), we obtain

$$\bar{p} \cdot \xi_i^\lambda \geq \bar{p} \cdot \omega_i + \sum_{j\in J} \theta_{i,j}\bar{p} \cdot y_j \tag{4.8}$$

and

$$\bar{p} \cdot \xi_i^\lambda \geq \bar{p} \cdot \omega_i + \sum_{j\in J} \theta_{i,j}\bar{p} \cdot \bar{y}_j. \tag{4.9}$$

Inequalities (4.8) and (4.9) hold true for any $i \in I$, $\xi_i \in \hat{P}_i(\bar{x})$, $\lambda \in [0,1[$ and $(y_j) \in \prod_{j\in J} Y'_j$. Knowing that $(\bar{x}, \bar{y})$ is an attainable allocation, we will show that $(\bar{x}, \bar{y}, \bar{p})$ is a quasi-equilibrium of the economy $\mathcal{E}'$, which completes the proof.

When λ goes to 1 in (4.8) and (4.9), one gets

$$\bar{p} \cdot \bar{x}_i \geq \bar{p} \cdot \omega_i + \sum_{j\in J} \theta_{i,j}\bar{p} \cdot y_j \tag{4.10}$$

Brought to you by | Université Paris 1 Panthéon-Sorbonne/Centre d'Economie de la Sorbonne
Authenticated
Download Date | 3/20/19 5:38 PM

and

$$\bar{p} \cdot \bar{x}_i \geq \bar{p} \cdot \omega_i + \sum_{j \in J} \theta_{i,j} \bar{p} \cdot \bar{y}_j. \tag{4.11}$$

Since $(\bar{x}, \bar{y})$ is a feasible allocation, summing the inequalities over i in (4.11), we deduce that

$$\bar{p} \cdot \bar{x}_i = \bar{p} \cdot \omega_i + \sum_{j \in J} \theta_{i,j} \bar{p} \cdot \bar{y}_j.$$

Taking $\lambda = 0$ in (4.9), we obtain for all $i \in I$ and all $\xi_i \in \hat{P}_i(\bar{x})$,

$$\bar{p} \cdot \xi_i \geq \bar{p} \cdot \omega_i + \sum_{j \in J} \theta_{i,j} \bar{p} \cdot \bar{y}_j.$$

So, the quasi-demand condition (b) of Definition 2.2 is satisfied.

Finally, from (4.10) and (4.11), for all $(y_j) \in \prod_{j \in J} Y'_j$, one gets

$$\bar{p} \cdot \omega_i + \sum_{j \in J} \theta_{i,j} \bar{p} \cdot y_j \leq \bar{p} \cdot \omega_i + \sum_{j \in J} \theta_{i,j} \bar{p} \cdot \bar{y}_j.$$

Summing over i, we get

$$\sum_{j \in J} \bar{p} \cdot y_j \leq \sum_{j \in J} \bar{p} \cdot \bar{y}_j.$$

For any $j \in J$, by applying this inequality to $y' \in \prod_{j \in J} Y'_j$ defined by

$$y'_{j'} = \begin{cases} y_j & \text{if } j' = j, \\ \bar{y}'_j & \text{if } j' \neq j, \end{cases}$$

it readily follows that

$$\bar{p} \cdot y_j \leq \bar{p} \cdot \bar{y}_j,$$

which means that the profit maximization condition (a) of Definition 2.2 is also satisfied.

5 Relationship with the literature

In this section, we compare Assumption (H3) with other conditions in the literature on the existence of equilibrium with non-compact attainable sets. We show that Assumption (H3) is weaker than the compactness of the set of individually rational and attainable allocations or utility levels and the CPP condition of Allouch. We also explain the relationships with the condition of Won and Yannelis.

5.1 Compactness of the attainable utility set

The following proposition shows that Assumption (H3) is weaker than the compactness of $\mathcal{A}(\mathcal{E})$ or the attainable utility set U. We use the following assumption on preferences as in Allouch.

Assumption (H4). The utility function u_i is lower semicontinuous and strictly quasi-concave, that is, for all $(x_i, z_i) \in X_i \times X_i$ with $u_i(z_i) > u_i(x_i)$, one has $u_i(\lambda x_i + (1 - \lambda) z_i) > u_i(x_i)$ for all $\lambda \in [0, 1[$.

If P_i is represented by a utility function u_i satisfying Assumption (H4), i.e.

$$P_i(x) = \{x'_i \in X_i \mid u_i(x'_i) > u_i(x_i)\},$$

then $P_i(x) = \hat{P}_i(x)$ for all $x \in \prod_{i \in I} X_i$. If the preferences of all consumers are represented by a utility function, the set of attainable utility level U is defined by

$$U = \{(v_1, v_2, \dots, v_m) \in \mathbb{R}^I_+ \mid \text{there exists } x \in \hat{X} \text{ such that } u_i(\underline{x}_i) \leq v_i \leq u_i(x_i)\}.$$

Brought to you by | Université Paris 1 Panthéon-Sorbonne/Centre d'Economie de la Sorbonne
Authenticated
Download Date | 3/20/19 5:38 PM

In an exchange economy with the survival assumption $\omega_i \in X_i$ for all i, the set U is just the set of individually rational attainable consumptions.

Proposition 5.1. *Under Assumption* (H1), *the following assertions hold:*
(i) *If* $\mathcal{A}(\mathcal{E})$ *is compact, then* (H3) *is satisfied.*
(ii) *If the preferences of all consumers are represented by a utility function satisfying Assumption* (H4) *and if* U *is compact, then Assumption* (H3) *is satisfied.*

The proof of this proposition as a consequence of the lower semicontinuity of preferences for (i) and the utility representation for (ii) is left to the reader.

5.2 Comparison with the CPP condition of Allouch

We recall the following definition of the CPP condition considered by Allouch [1].

Definition 5.2. The economy $\mathcal{E}$ satisfies the CPP condition if for every sequence $((x_i^\nu))$ of $\hat{X}$ there exists a subsequence $((x_i^{\varphi(\nu)})) \in \hat{X}$, an element $(\xi_i) \in \hat{X}$ and a sequence $(\xi_i^{\varphi(\nu)})_{\nu \geq \nu_1}$ convergent to ξ_i with $\xi_i^{\varphi(\nu)} \in \hat{P}_i(x^{\varphi(\nu)})$ for all ν.

Beside this assumption, Allouch also assumes that the preference relations are transitive, have open lower-section and that the augmented preferences are equal to the preferences. Assumption (H3) and the CPP condition have the same flavor, but the transitivity allows us to consider a unique sequence $(\xi_i^{\varphi(\nu)})$ whereas Assumption (H3) needs a sequence for each preferred element.

Proposition 5.3. *Let us assume that the preference relations are transitive, have open lower-section and are equal to the augmented preferences. Then if the CPP condition is satisfied, Assumption* (H3) *holds true.*

The proof is a direct consequence of the transitivity of preferences and the open lower-section, which allow us to get the desired property under the CPP condition.

5.3 Comparison with the work of Won and Yannelis

To compare our contribution to the one of Won and Yannelis [21], we restrict our attention to an exchange economy. Indeed, the initial endowments ω_i are used as a reference point on the budget line and there is no equivalent consumption in a production economy. The frameworks and the basic assumptions are quite similar and we focused our attention on the asymptotic condition corresponding to our Assumption (H3). To state it, we borrow the following notations from [21]: for $x \in \prod_{i \in I} X_i$ and all $i \in I$, we set $r_i(x) = \max\{\|x_k\| \mid k \neq i\}$, and $\bar{B}(0, r)$ denotes the closed ball of center 0 and radius r. We now state Assumption (B7a) of Won and Yannelis.

Assumption (B7a). There exists a consumer $i_0 \in I$ such that for all sequences $((x_i^\nu))$ of $\hat{X}$ with

$$\omega_i \in \overline{\hat{P}_i(x^\nu)^c}$$

for all i and all ν, there exist a subsequence $((x_i^{\varphi(\nu)}))$ and a sequence $(y^{\varphi(\nu)})$ convergent to a point $y \in \hat{X}$, such that, for all ν,

$$P_{i_0}(y^{\varphi(\nu)}) \subset \operatorname{cone}[P_{i_0}(x^{\varphi(\nu)}) - \{\omega_{i_0}\}] + \{\omega_{i_0}\},$$

and for all $i \neq i_0$,

$$P_i(y^{\varphi(\nu)}) \cap \bar{B}(0, r_{i_0}(x^{\varphi(\nu)})) \subset \operatorname{cone}[P_i(x^{\varphi(\nu)}) \cap \bar{B}(0, r_{i_0}(x^{\varphi(\nu)})) - \{\omega_i\}] + \{\omega_i\}.$$

We first remark that Assumption (H3) does not require the sequence $(y^{\varphi(\nu)})$ and the inclusion of the associated preferred set, or a truncation of it, in a set generated by the preferred set of $x^{\varphi(\nu)}$. Indeed, our assumption has the flavor of the CPP condition of Allouch.

Brought to you by | Université Paris 1 Panthéon-Sorbonne/Centre d'Economie de la Sorbonne
Authenticated
Download Date | 3/20/19 5:38 PM

Note that the use of the cone operator enlarges the set

$$[P_{i_0}(x^{\varphi(\nu)}) - \{\omega_{i_0}\}] \quad \text{or} \quad [P_i(x\{\omega_{i_0}\}) \cap \bar{B}(0, r_{i_0}(x\{\omega_{i_0}\})) - \{\omega_i\}],$$

so the condition is weaker than assuming

$$P_{i_0}(y^{\varphi(\nu)}) \subset P_{i_0}(x^{\varphi(\nu)}) \quad \text{and} \quad P_i(y^{\varphi(\nu)}) \cap \bar{B}(0, r_{i_0}(x^{\varphi(\nu)})) \subset P_i(y^{\varphi(\nu)}) \cap \bar{B}(0, r_i(x^{\varphi(\nu)}))$$

for all $i \neq i_0$. Note that, thanks to the lower semicontinuity of the preferences, Assumption (H3) is weaker than assuming the existence of the convergent sequence $(y^{\varphi(\nu)})$ and the inclusion $P_i(y^{\varphi(\nu)}) \subset P_i(x^{\varphi(\nu)})$. So, at this stage, the two assumptions are not comparable.

The major advantage of Assumption (B7a) comes from the fact that it is satisfied by an example by Page, Wooders and Monteiro [15] where an equilibrium exists with a non-compact set of attainable individually rational utility level. We easily check that this example satisfies the following asymmetric weakening of Assumption (H3) in the framework of an exchange economy.

Assumption (EWH3). There exists a consumer $i_0 \in I$ such that for all sequences $((x_i^\nu))$ of $\hat{X}$ with

$$\omega_i \in \overline{\hat{P}_i(x^\nu)^c}$$

for all i, there exist a subsequence $((x_i^{\varphi(\nu)})) \in \hat{X}$ and $(\bar{x}_i) \in \hat{X}$ with the condition that for all i and all $\xi_i \in \hat{P}_i(\bar{x})$, there exist an integer ν_1 and a sequence $(\xi_i^{\varphi(\nu)})_{\nu \geq \nu_1}$ convergent to ξ_i with, for all $\nu \geq \nu_1$,

$$\xi_{i_0}^{\varphi(\nu)} - \omega_{i_0} \in \operatorname{cone}[\hat{P}_{i_0}(x^{\varphi(\nu)}) - \omega_{i_0}],$$

and for all $i \neq i_0$,

$$\xi_i^{\varphi(\nu)} \in \hat{P}_i(x^{\varphi(\nu)}).$$

We did not consider and emphasize this assumption previously since its asymmetry is an hint that there is still room for improvements to get a still weaker and symmetric assumption. We can easily adapt the proof of Section 4 to check that Assumption (EWH3) is sufficient for the existence of quasi-equilibrium in exchange economies.

Finally, we discuss [21, Example 3.1.2]. Clearly, Assumption (EWH3) does not cover this example. Won and Yannelis claim that this example satisfies their weaker assumption (B7). The argument is based on the fact that there is no equilibrium in the truncated economy except the no-trade one with the two consumptions equal to 0 and any positive price. Actually, it seems to us that the price $p = (0, 1)$ associated to the consumptions $x_1 = (r, 0)$ and $x_2 = (-r, 0)$ is an equilibrium when the first agent has a truncated budget set $\bar{B}(0, r)$. In that case, the set $P_1(x) \cap \bar{B}(0, r_2(x))$ is empty, and so is the set $G_2(x)$ with the notation of that paper. Consequently, finding an assumption covering [21, Example 3.1.2] is still an open challenge.

Funding: This project was funded by the National Plan for Science Technology and Innovation (MAARIFAH), King Abdulaziz City for Science and Technology, Kingdom of Saudi Arabia, award number (12-MAT2703-02).

References

[1] N. Allouch, An equilibrium existence result with short selling, *J. Math. Econom.* **37** (2002), no. 2, 81–94.
[2] N. Allouch and M. Florenzano, Edgeworth and Walras equilibria of an arbitrage-free exchange economy, *Econom. Theory* **23** (2004), no. 2, 353–370.
[3] T. C. Bergstrom, The existence of maximal elements and equilibria in the absence of transitivity, Technical Report, University of Michigan, 1975.
[4] D. J. Brown and J. Werner, Arbitrage and existence of equilibrium in infinite asset markets, *Rev. Econ. Stud.* **62** (1995), 101–114.
[5] H. H. C. Cheng, Asset market equilibrium in infinite-dimensional complete markets, *J. Math. Econom.* **20** (1991), no. 1, 137–152.

Brought to you by | Université Paris 1 Panthéon-Sorbonne/Centre d'Economie de la Sorbonne
Authenticated
Download Date | 3/20/19 5:38 PM

[6] R.-A. Dana, C. Le Van and F. Magnien, General equilibrium in asset markets with or without short-selling, *J. Math. Anal. Appl.* **206** (1997), no. 2, 567–588.
[7] R.-A. Dana, C. Le Van and F. Magnien, On the different notions of arbitrage and existence of equilibrium, *J. Econom. Theory* **87** (1999), no. 1, 169–193.
[8] M. Florenzano, *L'équilibre économique général transitif et intransitif: problème d'existence*, Éditions CNRS, Paris, 1981.
[9] M. Florenzano, *General Equilibrium Analysis Existence and Optimality Properties of Equilibria*, Kluwer Academic, Boston, 2003.
[10] D. Gale and A. Mas-Colell, An equilibrium existence theorem for a general model without ordered preferences, *J. Math. Econom.* **2** (1975), no. 1, 9–15.
[11] D. Gale and A. Mas-Colell, Corrections to: "An equilibrium existence theorem for a general model without ordered preferences" [J. Math. Econom. 2 (1975), 9–15; MR 52 2542], *J. Math. Econom.* **6** (1979), no. 3, 297–298.
[12] J. Greenberg, Quasi-equilibrium in abstract economies without ordered preferences, *J. Math. Econom.* **4** (1977), no. 2, 163–165.
[13] O. D. Hart, On the existence of equilibrium in a securities model, *J. Econom. Theory* **9** (1974), no. 3, 293–311.
[14] A. Mas-Colell, An equilibrium existence theorem without complete or transitive preferences, *J. Math. Econom.* **1** (1974), no. 3, 237–246.
[15] F. H. Page, Jr., M. H. Wooders and P. K. Monteiro, Inconsequential arbitrage, *J. Math. Econom.* **34** (2000), no. 4, 439–469.
[16] R. T. Rockafellar, *Convex Analysis*, Princeton Math. Ser. 28, Princeton University Press, Princeton, 1970.
[17] W. Shafer and H. Sonnenschein, Equilibrium in abstract economies without ordered preferences, *J. Math. Econom.* **2** (1975), no. 3, 345–348.
[18] W. J. Shafer, The nontransitive consumer, *Econometrica* **42** (1974), 913–919.
[19] W. J. Shafer, Equilibrium in economies without ordered preferences or free disposal, *J. Math. Econom.* **3** (1976), no. 2, 135–137.
[20] J. Werner, Arbitrage and the existence of competitive equilibrium, *Econometrica* **55** (1987), no. 6, 1403–1418.
[21] D. C. Won and N. C. Yannelis, Equilibrium theory with unbounded consumption sets and non-ordered preferences. I. Non-satiation, *J. Math. Econom.* **44** (2008), no. 11, 1266–1283.

Brought to you by | Université Paris 1 Panthéon-Sorbonne/Centre d'Economie de la Sorbonne
Authenticated
Download Date | 3/20/19 5:38 PM