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Abstract

We study the interaction between multiple information designers who try to influence the behavior of a set of agents. When the set of messages available to each designer is finite, such games always admit subgame perfect equilibria. When designers produce public information about independent pieces of information, every equilibrium of the direct game (in which the set of messages coincides with the set of states) is an equilibrium with larger (possibly infinite) message sets. The converse is true for a class of Markovian equilibria only. When designers produce information for their own corporation of agents, pure strategy equilibria exist and are characterized via an auxiliary normal form game. In an infinite-horizon multi-period extension of information design games, a feasible outcome which Pareto dominates a more informative equilibrium of the one-period game is supported by an equilibrium of the multi-period game.

Keywords: Bayesian persuasion; information design; sharing rules; splitting games; statistical experiments.

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1 Introduction

Decision-makers often receive information from various interested parties who communicate about diverse pieces of information. For instance, consider competing pharmaceutical firms who aim at getting the approval to release new drugs on the market. Each firm aims at persuading the FDA that its product is effective and would prefer products of other firms not to be approved. As other instances, stores conduct advertising campaigns in order to convince consumers to buy from their shops; divisions within an organization or university try to persuade the head of the organization to allocate a position to their department; managers design the distribution of information in organizations in order to improve and coordinate agents' efforts. In all of these examples, interested parties design information in order to influence the behavior of decision-makers. The aim of this paper is to provide a general theoretical framework to analyze such situations.

Our modeling setup is the following. There are n *information designers* and k *agents*, there is an unknown n-dimensional state parameter, the set of possible states is finite. Each designer i controls agents' information about component i of the state by choosing a *statistical experiment* which draws messages received by the agents, as a function of the i-th component. Agents observe the chosen statistical experiments, the realized messages, and the outcome of a public randomization device. Finally, each agent chooses an action from some finite set. The payoffs for the agents and the designers depend on the realized state and on the action of every agent.

In such games of information design, designers' expected utilities are typically discontinuous (and generally not even upper-semi continuous) in the agents' beliefs and hence in the profile of statistical experiments. However, we show that the $(n+k)-$player game between the designers and the agents admits subgame perfect equilibria. We apply the existence result of Simon and Zame (1990) to show that there exists a selection from continuation equilibria induced by statistical experiments such that, by backward induction, the induced n-player game played between the designers has a Nash equilibrium.[1] Contrary to the single-designer case, this existence result requires designers to be able to randomize over statistical experiments. It also requires the agents' strategies to depend not only on their beliefs, but also on the specific statistical experiments used by the designers. We provide a simple strictly competitive example with binary states and a single agent (Example 5, adapted from Sion and Wolfe, 1957) which shows that mixed (ε-)equilibria may fail to exist for some optimal continuation strategy for the agent.

We study two broad classes of information design games for which we investigate equilibrium properties.

First, we consider information design games in which designers control *independent* pieces of information and send *public* messages to the agents. Each profile of experiments chosen by designers induces a public information structure and the set of continuation equilibria depends only on agents' first-order

[1] This methodology is similar to the one used in the literature on existence of subgame perfect equilibria in continuous games. Of particular interest here is the paper of Harris, Reny, and Robson (1995) (see also Mariotti, 2000) who prove the existence of subgame perfect equilibria in continuous games with almost perfect information, assuming as we do that there is a public randomization device. However, our general model does not have almost perfect information since agents may receive private messages from the designers and these results does not apply.

beliefs over each component of the state. For each designer, choosing a statistical experiment is therefore equivalent to choosing a *splitting* of the common prior belief into common posterior beliefs. This class is thus referred to as *splitting games*. Concavification methods can be used to characterize designers' best responses. We study "direct" information design games in which the set of messages available to designer i is equal to the number of states in dimension i. We show that the set of subgame perfect equilibrium outcomes of the direct game is included in the set of subgame perfect equilibrium outcomes of all games with more messages than states (possibly infinitely many messages). The inclusion can be strict even with a single designer and a single agent. However, for the class of *Markovian* equilibria in which agents' strategies depend on their posterior beliefs only, the direct game is "canonical" in that, the set of Markovian subgame perfect equilibrium outcomes of all games with more messages than states, coincides with the set of subgame perfect equilibrium outcomes of the direct game. When there is a continuum of messages, we show that it is without loss of generality to focus on pure strategies for the designers.

Second, we consider *rectangular corporation games* in which each designer i controls the (public and private) information of a given corporation of agents. Each agent belongs to just one corporation. The utility functions of agents in corporation i only depend on the actions taken by agents in corporation i and on the information controlled by designer i. Hence, from the point of view of agents in corporation i, only the messages received from their own designer matter. Designers however interact through how they influence agents' behavior in *all* corporations. In such games, we show that there exists an equilibrium in pure strategies with finite sets of messages. We provide a simple characterization of equilibrium outcomes via an auxiliary $n-$player normal form game. In this game the set of pure strategies of player i is the set of action profiles (or state-dependent action profiles if the designers' preferences are state-dependent) of the agents in corporation i. We show that the set of equilibria of a mixed extension of this auxiliary game with convex constraints on strategies is included in the set of subgame perfect equilibrium outcomes of the original information design game. If each corporation is composed by a single agent then, under a regularity condition, the two sets coincide.

Finally, we consider a multi-period extension of rectangular corporation games in which in every period, designers choose statistical experiments simultaneously. They also choose to continue to the next period or to quit the game. Quitting is a commitment to reveal no further information. These choices are publicly observed and the game continues until either all designers have chosen a non-informative experiment or all have chosen to quit. When this process terminates, all experiments are drawn and agents choose actions. We show that every outcome that Pareto-dominates a *more informative* equilibrium outcome of the one-period game is supported by a subgame perfect equilibrium of the multi-period game.

Related Literature

The characterization of optimal information structures with a single designer has been studied and applied in many articles in the economics literature. See, among others, Lewis and Sappington (1994), Johnson and Myatt (2006), Angeletos and Pavan (2007), Eső and Szentes (2007), Kamenica and

Gentzkow (2011), Eliaz and Serrano (2014), Jehiel (2014), Eliaz and Forges (2015), Taneva (2016), Bergemann and Morris (2016b) and Mathevet, Perego, and Taneva (2017). When there is a single designer and a single agent, our model of information design exactly corresponds to the model of Kamenica and Gentzkow (2011). In such a setting, a subgame perfect equilibrium is obtained by concavifying the designer's indirect utility function, which represents his expected utility as a function of the agent's belief. This characterization is analogous to the one obtained in the literatures on repeated games with incomplete information and on splitting games (Aumann and Maschler, 1967; Aumann, Maschler, and Stearns, 1995; Laraki, 2001a,b; Sorin, 2002; Oliu-Barton, 2017; Mertens, Sorin, and Zamir, 2015).

In Kamenica and Gentzkow (2011), the characterization of optimal information structures is obtained by assuming that when the agent is indifferent between several actions, he chooses an optimal action that favors the designer. Under this assumption, the indirect utility function of the designer is upper-semi continuous and thus an optimal solution exists. With multiple designers who do not have the same preferences, there is no analogue to the designer-preferred tie-breaking rule, so the existence of exact best-responses is not guaranteed. If the strategy of the agent depends on his belief only (and not on the precise statistical experiments used by the designers), an ε-best response of a designer can be obtained by taking the concave hull of his payoff function (with the strategies of the other designers fixed). But this does not guarantee the existence of an ε-equilibrium between the designers. Another important difference between the one-designer and the multiple-designer cases concerns the required richness of the message space (or the ability of the designers to use mixed strategies, *i.e.*, randomizations over statistical experiments). In the single-designer case, the required number of messages is bounded above by the number of actions (by the revelation principle) or the number of states (by a suitable version of Carathéodory's theorem applied to the concave hull of the payoff function). As we illustrate, neither of these results apply with multiple designers, even if there is a single agent.

A few recent articles consider the case of multiple information designers in specific environments. Gentzkow and Kamenica (2017) and Li and Norman (2017a) consider the situation in which the designers can produce the same information for a single agent. This corresponds to our model when the state spaces of all the designers are the same and the states are perfectly correlated. In this case, there is a trivial equilibrium in which designers fully disclose the state. Gentzkow and Kamenica (2017) provide conditions under which all pure strategy equilibria are more informative than the collusive outcome. Li and Norman (2017b) consider the sequential version of the game. Albrecht (2017), Au and Kawai (2017b,a) and Boleslavsky and Cotton (2015, 2017) consider, as in our model, distinct state spaces, and provide equilibrium characterizations in special classes of examples. Albrecht (2017) studies a strictly competitive example (which is also our Example 4). By fixing a symmetric tie-breaking rule for the agent, he shows that the unique equilibrium consists for each designer in generating a continuum of uniformly distributed posteriors.

Our multi-period information design game is related to zero-sum splitting games and acyclic gambling games (Laraki, 2001a; Laraki, 2001b; Sorin, 2002; Oliu-Barton, 2017; Mertens et al., 2015; Laraki and Renault, 2017) with four notable differences: in our model, (i) players observe the experiments chosen in the past but not the message realizations from those experiments; (ii) in each period designers can quit the game, *i.e.*, commit not to reveal any further information; (iii) the analysis is not restricted to 2-player zero-sum games; (iv) the utility functions depend on the state and on agents' actions, so the indirect utilities induced by agents' strategies are typically discontinuous in the posteriors. The first two features guarantee that an equilibrium outcome of the one-period game remains an equilibrium of the multi-period game, a property which is not true even in zero-sum splitting games. Our multi-period analysis is also related to commitment games (Bade, Haeringer, and Renou, 2009; Renou, 2009; Dutta and Ishii, 2016) in the sense that revealing information is irreversible and allows designers to commit to subsets of continuation outcomes.

The next section presents general information design games and an equilibrium existence result. In Section 3, we consider splitting games, *i.e.*, information design games in which designers produce independent pieces of information and disclose information publicly. Section 4 considers rectangular corporations problems. A multi-period extension is studied in Section 5.

2 General Model and Equilibrium Existence

The environment. There is a set $N = \{1, \dots, n\}$ of information *designers* and a set $K = \{1, \dots, k\}$ of *agents*. The set of *states* $\Theta = \prod_{i \in N} \Theta_i$ is endowed with a common prior probability distribution $p^0 \in \Delta(\Theta)$.[2] All players are uninformed about the state. Designer $i \in N$ discloses information about a parameter $\theta_i \in \Theta_i$ and agent $j \in K$ chooses an action $a_j \in A_j$, where the sets Θ_i and A_j are non-empty and finite.[3]

The set of action profiles is $A = \prod_{j \in N} A_j$. The payoff of each player depends on the state and on the action profile. The payoff of designer i (resp., agent j) is denoted $u_i(a; \theta)$ (resp., $v_j(a; \theta)$).

Information structures and statistical experiments. At the ex-ante stage, before states are drawn, each designer i chooses how to disclose information about dimension i of the state to the agents by choosing a mapping that specifies the distributions of messages to the agents conditional on the states in Θ_i. This mapping will be referred to as a *statistical experiment*.

Definition 1. *Given a profile of finite message sets* $(M_i^j)_{j \in K}$*, a* statistical experiment *for designer i is a mapping* $x_i : \Theta_i \to \Delta(M_i)$*, from* Θ_i *to the probability distributions over the set of profiles of messages* $M_i = \prod_{j \in K} M_i^j$.

A statistical experiment induces an information structure for the agents: it selects a profile of messages $m_i = (m_i^j)_{j \in K}$ with probability $x_i(m_i \mid \theta_i)$ and delivers privately the message m_i^j to agent j.

Information design game. Let $M = \prod_i M_i$ be a profile of finite message sets for all designers. Our main object of interest is the $(n + k)-$player game G_M, whose timing is as follows:

1. Each designer i chooses a statistical experiment $x_i \in \Delta(M_i)^{\Theta_i}$; these choices are simultaneous;

[2] In all the paper, for every compact set S, $\Delta(S)$ denotes the set of Borel probability measures over S.

[3] We consider finite actions sets mainly for simplicity. Most of our results generalize to infinite action sets provided that for any information structure, the Bayesian game between agents admits equilibria. E.g., we could assume compact action sets and payoffs continuous w.r.t. actions.

2. Agents publicly observe $(x_1, \ldots, x_n)$;

3. Nature draws the state $\theta = (\theta_1, \ldots, \theta_n)$ according to $p^0 \in \Delta(\Theta)$ and a uniformly distributed public signal $\omega \in [0, 1]$ (called public correlation device hereafter);

4. For each designer i, a profile of messages $m_i = (m_i^1, \ldots, m_i^k)$ is drawn with probability $x_i(m_i \mid \theta_i)$;

5. Each agent j observes the public signal ω and his profile of *private* messages $m^j = (m_i^j)_{i \in N} \in M^j := \prod_{i \in N} M_i^j$, then chooses an action $a_j \in A_j$.

Subgame perfect equilibrium. We will consider subgame perfect equilibria of the game G_M, simply referred to as "equilibria" in the rest of the paper. For each profile of experiments $x = (x_i)_{i \in N} \in \prod_{i \in N} \Delta(M_i)^{\Theta_i}$, let $G_M(x)$ be the k-player Bayesian induced by x, it is the subgame starting from stage 2 in the above timeline.

Let $\mathcal{E}_M(x) \subseteq \Delta(A)^M$ be the set of Bayes Nash equilibrium *outcomes* of $G_M(x)$, *i.e.*, mappings from message profiles to distributions of action profiles. Since the game $G_M(x)$ is a finite game extended by a public randomization device, the set of Bayes Nash equilibrium outcomes $\mathcal{E}_M(x)$ is non-empty and convex, it coincides with the set of public correlated equilibrium outcomes of $G_M(x)$.

In equilibrium, the agents play a Bayes Nash equilibrium of $G_M(x)$ for each x. Consider a Borel measurable mapping $\tau : \prod_{i \in N} \Delta(M_i)^{\Theta_i} \to \Delta(A)^M$ such that for each x, $\tau(x) \in \mathcal{E}_M(x)$, namely a Borel measurable selection from the equilibrium correspondence $\mathcal{E}_M(\cdot)$. Such a selection induces a simultaneous game between the designers with payoff function:

$$U_i^\tau(x_i, x_{-i}) = \sum_{\theta \in \Theta} \sum_{m \in M} \sum_{a \in A} p^0(\theta) \prod_i x_i(m_i|\theta_i) \tau(x)[a|m] u_i(a; \theta),$$

where $\tau(x)[a|m]$ is the probability of a conditional on m under $\tau(x)$.

It is important to note that the function $x \mapsto U_i^\tau(x)$ is possibly discontinuous because $x \mapsto \tau(x)$ may be discontinuous. This is true even with a single agent: when the agent is indifferent between several actions at some belief, his optimal action may switch from one action to another in the neighborhood of his belief (see Example 3).

Definition 2. *An equilibrium of G_M is a Borel measurable selection τ from the correspondence $\mathcal{E}_M(\cdot)$ and a mixed strategy Nash equilibrium of the game $((U_i^\tau)_i, (\Delta(M_i)^{\Theta_i})_i)$.*

Notice that we allow for mixed strategies that is, such that each designer i chooses a Borel probability measure over $\Delta(M_i)^{\Theta_i}$. The main result in this section is that, for every profile of finite message sets M, equilibria exist for the information design game G_M.

Theorem 1. *For every profile of finite message sets, the game G_M admits an equilibrium.*

Proof. For all $x \in \prod_{i \in N} \Delta(M_i)^{\Theta_i}$ and $y \in \Delta(A)^M$, let

$$U_i(x, y) = \sum_{\theta \in \Theta} \sum_{m \in M} \sum_{a \in A} p^0(\theta) \prod_i x_i(m_i \mid \theta_i) y(a \mid m) u_i(a; \theta),$$

be the expected utility of designer i given the profile of statistical experiments x and the outcome y of $G_M(x)$. Let $U(x,y) = (U_i(x,y))_{i\in N}$ and define correspondences $\mathcal{U} : \prod_{i\in N} \Delta(M_i)^{\Theta_i} \to \mathbb{R}^N$ by

$$\mathcal{U}(x) = \{U(x,y) : y \in \mathcal{E}_M(x)\}.$$

The equilibrium correspondence $\mathcal{E}_M(\cdot)$ has compact convex values and is upper hemicontinuous. By continuity of the payoff function $U(x,y)$, $\mathcal{U}(\cdot)$ also has compact convex values and is upper hemicontinuous. From the main theorem of Simon and Zame (1990), there exists a Borel measurable selection $U^*(x) \in \mathcal{U}(x)$ from $\mathcal{U}$ such that the normal form game $((U_i^*)_i, (\Delta(M_i)^{\Theta_i})_i)$ admits a Nash equilibrium in mixed strategies. Hence, there exists a Borel measurable selection τ from the correspondence $\mathcal{E}_M(\cdot)$ such that $(U_i^\tau)_i = (U_i^*)_i$ and $((U_i^\tau)_i, (\Delta(M_i)^{\Theta_i})_i)$ admits a Nash equilibrium in mixed strategies. ■

Remark 1. Several comments are in order.

1. The correspondance $\mathcal{U}(\cdot)$ does not generally admit continuous selections, see Example 3.
2. An (ε-)equilibrium may not exist for some selections $\tau(\cdot) \in \mathcal{E}_M(\cdot)$. This is shown in Example 5 with two designers, one agent, and binary states.
3. The theorem extends directly to the case where each designer i is constrained to choose a statistical experiment x_i from a compact subset of statistical experiments $X_i \subseteq \Delta(M_i)^{\Theta_i}$. For example, X_i can be the set of all public statistical experiments, *i.e.*, statistical experiments in which the messages received by all agents from designer i are perfectly correlated (as in splitting games; see Section 3). This can also be used to model verifiable information setups where the designer can choose a partition of Θ_i for each agent and the element of the partition containing the true state is announced. This is a particular subset of experiments.
4. If there is a single designer i and if $|M_i^j| \geq |A_j|$ for every $j \in K$, then the set $\bigcup_x \mathcal{E}_M(x)$ is the set of *Bayes correlated equilibria* (BCE) (Bergemann and Morris, 2016a), which is closed and compact. Thus *one* (but not all) equilibrium is found by maximizing the designer's payoff over the set of BCE. In such an equilibrium, the public randomization device is not needed and the designer selects his favorite continuation Bayes Nash equilibrium along the equilibrium path. Of course, with multiple designers, such a selection does not exist if designers have misaligned preferences.
5. Even in the case of one designer, it is not clear how the *set* of equilibrium outcomes varies with the size of the message spaces. It can be shown that information structures with higher-order beliefs that appear in global games (Rubinstein, 1989, Carlson and van Damme, 1993) can be used by the designer to induce a unique continuation equilibrium outcome in $\mathcal{E}_M(x)$. Such structures typically use message sets which are larger than the action or the state space.[4] A fortiori, the set of equilibrium outcomes with multiple designers also depends on the message spaces, even if $|M_i^j| \geq |A_j|$ and $|M_i^j| \geq |\Theta_i|$ for every i, j. We will provide sharper equilibrium characterizations

[4]See Moriya and Yamashita (2017) for an explicit example involving an infinite number of messages with one designer, two agents, two states and two actions.

with respect to the message spaces in splitting games (Section 3) and rectangular corporation games (Section 4).

6. Relatedly, an equilibrium outcome of G_M may not be an equilibrium outcome of an alternative game in which the designers can also choose the message spaces strategically. Again, our equilibrium characterizations in the special classes of splitting games and rectangular corporation games allow to conclude that equilibrium outcomes in these classes of information design games are robust to such deviations.

7. A public correlation device is not needed in the case of one agent because the set of optimal continuation strategies of a single agent is convex. Alternatively, if there is more than one agent, and at least two players are able to send a public message in $[0, 1]$ between stage 1 and stage 4 of the information design game, jointly controlled lotteries (see e.g. Aumann et al., 1995) can replace the public correlation device. Finally, if each designer controls the information of separate corporations of agents (as in Section 4), we can characterize equilibria without a public correlation device.

3 Splitting Games: Public and Independent Information

In this section we assume that the prior probability distribution is the product of its marginal distributions: $p^0 = \otimes_{i\in N} p_i^0$, with $p_i^0 \in \Delta(\Theta_i)$. That is, designers produce and disclose independent pieces of information to the agents. In addition, we assume that each designer i sends public messages to all agents. With some abuse of notation with respect to the previous section, we denote by $m_i \in M_i$ the public message received from designer i. Hence, each designer i chooses a statistical experiment $x_i : \Theta_i \to \Delta(M_i)$, from Θ_i to the set of probability distributions over the set of public messages M_i.

This special class of information design games is referred to as *splitting games* because the choice of a statistical experiment $x_i : \Theta_i \to \Delta(M_i)$ by designer i will be reduced to a choice of a *splitting* of the common prior p_i^0 into $|M_i|$ common posteriors for the agents. Notice that every information design game with a single agent is a splitting game. When in addition there is a single designer, such games are called "Bayesian persuasion" games (see Section 3.4.1).

3.1 From Statistical Experiments to Splittings

Given a statistical experiment $x_i \in \Delta(M_i)^{\Theta_i}$, a message $m_i \in M_i$ is publicly observed by the agents with total probability

$$\lambda_i(m_i) = \sum_{\theta_i \in \Theta_i} x_i(m_i \mid \theta_i) p_i^0(\theta_i).$$

If $\lambda_i(m_i) > 0$, the agents' posterior beliefs conditional on m_i are derived from Bayes' rule as follows:

$$\mathbf{p}_i(\theta_i \mid m_i) = \frac{x_i(m_i \mid \theta_i) p_i^0(\theta_i)}{\lambda_i(m_i)}, \quad \text{for every } \theta_i \in \Theta_i.$$

Posterior beliefs satisfy $\sum_m \lambda_i(m_i)\mathbf{p}_i(m_i) = p_i^0$, where $\mathbf{p}_i(m_i) = (\mathbf{p}_i(\theta_i|m_i))_{\theta_i\in\Theta_i}$. The set of distributions of posteriors which average on $p_i^0 \in \Delta(\Theta_i)$ will be referred to as the set of *splittings* of p_i^0:

$$\mathcal{S}_i(p_i^0) = \left\{\mu_i \in \Delta(\Delta(\Theta_i)) : \int_{p_i\in\Delta(\Theta_i)} p_i \, d\mu_i(p_i) = p_i^0\right\},$$

where $\Delta(\Delta(\Theta_i))$ is the set of Borel probability measures over the compact set $\Delta(\Theta_i)$.

We denote by $\mathcal{S}_i^{M_i}(p_i^0)$ the set of splittings with finite support of cardinality at most $|M_i|$. Any such splitting can be represented by a pair $\mu_i = (\lambda_i, \mathbf{p}_i)$ with $\lambda_i \in \Delta(M_i)$, $\mathbf{p}_i : M_i \to \Delta(\Theta_i)$ and $\sum_m \lambda_i(m_i)\mathbf{p}_i(m_i) = p_i^0$.

Every statistical experiment $x_i \in \Delta(M_i)^{\Theta_i}$ induces a splitting in $\mathcal{S}_i^{M_i}(p_i^0)$ given by:

$$p_i^0(\theta_i) = \sum_{m_i\in M_i} \lambda_i(m_i)\mathbf{p}_i(\theta_i \mid m_i).$$

Conversely, from the splitting lemma (Aumann et al., 1995), every splitting $(\lambda_i, \mathbf{p}_i)$ with finite support is generated by the statistical experiment given by $x_i(m_i \mid \theta_i) = \frac{\lambda_i(m_i)\mathbf{p}_i(\theta_i|m_i)}{p_i^0(\theta_i)}$ for θ_i in the support of p_i^0.

Example 1 (Binary states and messages)**.** Take a single designer ($n = 1$) who designs information on $\Theta = \{\theta^1, \theta^2\}$ using messages in $M = \{m^1, m^2\}$, the prior probability of θ^1 is $p^0 \in [0, 1]$. Then, a statistical experiment $x : \Theta \to \Delta(M)$ induces the following splitting $(\lambda, \mathbf{p})$ of p^0:

$$\lambda(m^1) = x(m^1|\theta^1)p^0 + x(m^1|\theta^2)(1 - p^0),$$

$$\mathbf{p}(\theta^1|m^1) = \frac{x(m^1|\theta^1)p^0}{\lambda(m^1)},$$
$$\mathbf{p}(\theta^1|m^2) = \frac{x(m^2|\theta^1)p^0}{\lambda(m^2)}.$$

Conversely, every splitting $(\lambda, \mathbf{p})$ can be induced by a statistical experiment $x : \Theta \to \Delta(M)$ which solves the three equations just stated above.

Since in splitting games all messages are public, beliefs are common among agents and the set $\mathcal{E}_M(x) \subseteq \Delta(A)^M$ of Bayes Nash equilibrium outcomes of $G_M(x)$ only depends on the distributions of posteriors (the splittings) and on the realized posteriors induced by x. Hence, the information design game in which each designer i chooses a statistical experiment $x_i \in \Delta(M_i)^{\Theta_i}$ is equivalent to the game in which each designer i chooses a splitting $\mu_i \in \mathcal{S}_i^{M_i}(p_i^0)$ and agents publicly observe $\mu = (\mu_i)_{i\in N}$ and $\mathbf{p}(m) = (\mathbf{p}_i(m_i))_{i\in N}$. We denote this game by SG_M. Similarly, we let SG_∞ be the game where each designer i chooses any splitting $\mu_i \in \mathcal{S}_i(p_i^0)$ with possibly infinite support.

Remark 2. While every statistical experiment in $\Delta(M_i)^{\Theta_i}$ induces a unique splitting, a splitting in $\mathcal{S}_i^{M_i}(p_i^0)$ can be induced by many statistical experiments in $\Delta(M_i)^{\Theta_i}$. Hence, strictly speaking, the game in which designers choose statistical experiments has more strategies than the game in which they choose splittings. However, it is readily observed that equilibrium outcomes of the game in which every designer i chooses statistical experiments in $\Delta(M_i)^{\Theta_i}$ coincide with the equilibrium outcomes of the game in which he chooses splittings in $\mathcal{S}_i^{M_i}(p_i^0)$. Indeed, if a statistical experiment of designer i uses several messages inducing the same posterior p_i on Θ_i for some message profile m_{-i} of the other designers, then in the corresponding game in which designer i uses a splitting, it suffices to replace the continuation equilibrium played by the agents at p_i (given m_{-i}) by the average continuation equilibrium of the agents when p_i has been induced in the game in which designer i chooses the statistical experiment. Notice that this equivalence between splittings and statistical experiments does not hold in the more general information design games of the previous section. Indeed, with multiple agents and private messages, statistical experiments can induce higher-order beliefs that cannot be captured by first-order beliefs over Θ.

With some abuse of notations, we extend the utility function of each designer $i \in N$ and agent $j \in K$ as follows. For every distribution of action profiles $y \in \Delta(A)$, let $v_j(y;\theta) = \sum_{a\in A} y(a)v_j(a;\theta)$ and $u_i(y;\theta) = \sum_{a\in A} y(a)u_i(a;\theta)$ for $i \in N$. For every $p \in \prod_{i\in N}\Delta(\Theta_i)$ we also denote

$$v_j(y;p) = \sum_{\theta\in\Theta}\prod_{j\in N} p_j(\theta_j)v_j(y;\theta) \text{ and } u_i(y;p) = \sum_{\theta\in\Theta}\prod_{j\in N} p_j(\theta_j)u_i(y;\theta).$$

For each $p \in \prod_{i\in N}\Delta(\Theta_i)$, let $Y(p) \subseteq \Delta(A)$ be the set of public correlated equilibrium outcomes (*i.e.*, distributions of action profiles) of the game played between the agents when their common belief is given by p. That is, $Y(p)$ is the convex hull of the set of (mixed) Nash equilibria of the game $((v_j(\cdot,p))_j,(A_j)_j)$. The set $Y(p)$ is non-empty, compact and convex.

In the splitting game SG_M, agents' strategies induce a continuation outcome:

$$\tau : \prod_{i\in N}\mathcal{S}_i^{M_i}(p_i^0)\times\Delta(\Theta_i)\to\Delta(A).$$

For every profile of splittings $\mu = (\mu_i)_{i\in N} \in \prod_{i\in N}\mathcal{S}_i^{M_i}(p_i^0)$ and every profile of posteriors $p \in \prod_{i\in N}\Delta(\Theta_i)$, $\tau(\mu,p)$ is the distribution of actions played by the agents in the continuation game followed by μ and p.

In an equilibrium of SG_M, it must be that $\tau(\mu,p)\in Y(p)$ for every $\mu \in \prod_{i\in N}\mathcal{S}_i^{M_i}(p_i^0)$ and every p in the support of μ. Any such Borel measurable selection τ induces a game between the designers with payoffs:

$$U_i^\tau(\mu) = \int_p u_i(\tau(\mu,p);p)\,d\mu(p), \quad i = 1\ldots,n.$$

An equilibrium of SG_M is a continuation equilibrium outcome τ, with $\tau(\mu,p)\in Y(p)$ for every $\mu\in\prod_{i\in N}\mathcal{S}_i^{M_i}(p_i^0)$ and p in the support of μ, and a Nash equilibrium of the game $((U_i^\tau)_i,(\mathcal{S}_i^{M_i}(p_i^0))_i)$. We consider both pure strategy equilibrium for the designers $\mu\in\prod_i\mathcal{S}_i^{M_i}(p_i^0)$ and mixed strategy equilibrium for the designers, *i.e.*, a profile of Borel probability distributions $\zeta\in\prod_i\Delta(\mathcal{S}_i^{M_i}(p_i^0))$ over the compact sets of splittings. A mixed strategy $\zeta_i\in\Delta(\mathcal{S}_i^{M_i}(p_i^0))$ for designer i is called a *mixed splitting*.

When the continuation outcome does not depend on μ, we write $y(p) = \tau(\mu, p)$ and

$$U_i^y(p) = u_i(y(p); p), \;\; U_i^y(\mu) = \int_p u_i(y(p); p)\, d\mu(p).$$

3.2 Robustness to Message Sets

The first result of this section is that the set of equilibrium outcomes expands with the number of messages when the sets of messages are large enough ($|M_i| \geq |\Theta_i|$ for every $i \in N$). This implies that every equilibrium outcome of the *direct game* SG_Θ, where the message space M_i of each designer i has the same cardinality as Θ_i, is also an equilibrium outcome of every game with larger (possibly infinite) message spaces. It also implies that such an equilibrium is "robust" in the sense that it would still be an equilibrium in the broader game where each designer i is allowed to choose the set of messages M_i strategically.

Theorem 2. *Let M and M' be two profiles of message sets such that $|M_i'| \geq |M_i| \geq |\Theta_i|$ for every $i \in N$. The set of equilibrium outcomes of SG_M is included in the set of equilibrium outcomes of $SG_{M'}$ as well as in the set of equilibrium outcomes of SG_∞.*

The intuition is as follows. Take an equilibrium profile of mixture of splittings ζ^* of SG_Θ, and let designers play the same mixed splittings in SG_M, this is possible because $S_i^{\Theta_i}(p_i^0) \subseteq S_i^{M_i}(p_i^0)$ whenever $|M_i| \geq |\Theta_i|$. Along the equilibrium path in SG_M, consider the same continuation equilibria as in SG_Θ. Off the equilibrium path, if a designer i deviates unilaterally from ζ_i^* to a splitting $\mu_i \in S_i^{M_i}(p_i^0)$, consider the *worst* continuation equilibrium for him, independently of μ_i. If this deviation is profitable in SG_M, then it is also profitable in SG_Θ because the best response of designer i is obtained as a concavification with $|\Theta_i|$ posteriors, a contradiction. Here is the formal proof.

Proof. Consider an equilibrium $((\zeta_i^*)_{i\in N}, \tau)$ of SG_M, where each ζ_i^* is a mixed strategy in $\Delta(\mathcal{S}^{M_i}(p_i^0))$ and $\tau : \prod_i \mathcal{S}^{M_i}(p_i^0) \times \Delta(\Theta_i) \to \Delta(A)$ is a continuation equilibrium outcome for the agents. We will prove the result by extending τ to all splittings in such a way that no designer can profitably deviate to any splitting, with either finite or infinite support.

Recall that for all (μ, p), $\tau(\mu, p) \in Y(p)$ is a continuation equilibrium outcome given the public belief p. For each designer i and belief p, fix $y^i(p) \in \arg\min_{y \in Y(p)} u_i(y, p)$. Define the extension $\tilde{\tau} : \prod_i \mathcal{S}_i(p_i^0) \times \Delta(\Theta_i) \to \Delta(A)$ as follows.

- If $\mu_i \in \mathcal{S}^{M_i}(p_i^0)$ for all i, then $\tilde{\tau}(\mu, p) = \tau(\mu, p)$ for all p.
- If there is a single designer i such that $\mu_i \notin \mathcal{S}^{M_i}(p_i^0)$, choose $\tilde{\tau}(\mu_i, \mu_{-i}, p) = y^i(p)$.
- For all other profiles μ, let $\tilde{\tau}(\mu, p)$ be arbitrary in $Y(p)$.

We claim that $((\zeta_i^*)_{i\in N}, \tilde{\tau})$ is an equilibrium of SG_∞. Clearly, no designer i can profitably deviate to $\tilde{\mu}_i \in \mathcal{S}^{M_i}(p_i^0)$. Suppose that designer i deviates to an arbitrary $\tilde{\mu}_i \in \mathcal{S}(p_i^0)$. His expected payoff is,

$$\begin{aligned} U_i^{\tilde{\tau}}(\tilde{\mu}_i, \zeta_{-i}^*) &= \int_{\mu_{-i}} \left\{ \int_{p_i, p_{-i}} u_i(y^i(p_i, p_{-i}); p_i, p_{-i}), p_i, p_{-i}) d\tilde{\mu}_i(p_i) d\mu_{-i}(p_{-i}) \right\} d\zeta_{-i}(\mu_{-i}) \\ &= \int_{p_i} \left\{ \int_{\mu_{-i}} \int_{p_{-i}} u_i(y^i(p_i, p_{-i}); p_i, p_{-i}), p_i, p_{-i}) d\mu_{-i}(p_{-i}) d\zeta_{-i}(\mu_{-i}) \right\} d\tilde{\mu}_i(p_i). \end{aligned}$$

Denoting $U_i^{\tilde{\tau}}(p_i, \zeta_{-i}^*) = \int_{\mu_{-i}} \int_{p_{-i}} u_i(y^i(p_i, p_{-i}); p_i, p_{-i}), p_i, p_{-i}) d\mu_{-i}(p_{-i}) d\zeta_{-i}(\mu_{-i})$, we have

$$U_i^{\tilde{\tau}}(\tilde{\mu}_i, \zeta_{-i}^*) = \int_{p_i} U_i^{\tilde{\tau}}(p_i, \zeta_{-i}^*) d\tilde{\mu}_i(p_i) \le \sup_{\mu_i \in \mathcal{S}_i(p_i^0)} \int_{p_i} U_i^{\tilde{\tau}}(p_i, \zeta_{-i}^*) d\mu_i(p_i) = \sup_{\mu_i \in \mathcal{S}_i^{\Theta_i}(p_i^0)} \int_{p_i} U_i^{\tilde{\tau}}(p_i, \zeta_{-i}^*) d\mu_i(p_i).$$

Indeed, the right-hand-side is the concave hull $\mathrm{cav}_{p_i} U_i^{\tilde{\tau}}(p_i, \zeta_{-i}^*)$ of $U_i^{\tilde{\tau}}(p_i, \zeta_{-i}^*)$ with respect to p_i and the supremum is achieved with $|\Theta_i|$ posteriors.[5]

If the deviation $\tilde{\mu}_i$ is profitable, then there exists $\mu_i^* \in S^{\Theta_i}(p_i^0) \subseteq S^{M_i}(p_i^0)$ such that

$$U_i^{\tau}(\zeta_i^*, \zeta_{-i}^*) = U_i^{\tilde{\tau}}(\zeta_i^*, \zeta_{-i}^*) < U_i^{\tilde{\tau}}(\tilde{\mu}_i, \zeta_{-i}^*) \le U_i^{\tilde{\tau}}(\mu_i^*, \zeta_{-i}^*) \le U_i^{\tau}(\mu_i^*, \zeta_{-i}^*),$$

where the last inequality follows from the fact that $\tilde{\tau}(\cdot, p)$ is the least preferred continuation equilibrium of designer i. This contradicts that ζ^* is an equilibrium of SG_M.

Therefore, if ζ^* is an equilibrium of SG_M, it is also an equilibrium in the splitting games with more messages, as desired. ■

Theorem 2 shows that the set of equilibrium outcomes increases with the number of messages, *i.e.*, with the size of the supports of splittings. In general the inclusions can be strict as illustrated in the next example.

Example 2. Consider one designer, one agent, and two states $\Theta = \{0, 1\}$. The prior probability of state $\theta = 1$ is $p^0 = 1/2$. The set of actions of the agent is $A = A^1 \times A^2$ where $\{0, 1/2, 1\} \subseteq A^1 \subseteq [0, 1]$ and $A^2 = \{0, 1\}$. We choose a utility function $v(a^1, a^2; p)$ of the agent that depends only on $a^1 \in A^1$ and which is such that $a^1 = 0$ is optimal *only* at belief $p = 0$, $a^1 = 1/2$ is optimal *only* at belief $p = 1/2$, and $a^1 = 1$ is optimal *only* at belief $p = 1$. The utility of the designer depends only on the agent's action and is given by $u(a^1, a^2) = \frac{1}{4} - (a^1 - 1/2)^2 - a^2$. Note that a^2 is here only for punishing or rewarding the designer.

Consider the splitting μ^3 of SG_{M^3}, with $|M^3| = 3$, which induces the distribution $(1/4, 1/2, 1/4)$ over the posteriors $\{0, 1/2, 1\}$. Consider the continuation equilibrium (*i.e.*, optimal strategy for the agent) τ which consists in playing the pair of actions $\tau(\mu, p) = (p, 0) \in A$ if $\mu = \mu^3$, and $\tau(\mu, p) = (a^1(p), 1) \in A$ if $\mu \ne \mu^3$, where $a^1(p)$ maximizes $v(a^1, a^2; p)$. That is, the agent plays optimally given his belief, but he punishes with $a^2 = 1$ when the designer deviates from μ^3 (it is optimal for the agent to do so).

[5] For any function $\varphi : \Delta(\Theta_i) \to \mathbb{R}$, $\sup_{\mu_i \in \mathcal{S}_i(p_i^0)} \int_{p_i} \varphi(p_i)\, d\mu_i(p_i)$ is the (pointwise) smallest concave function $\mathrm{cav}\, \varphi(p_i)$ on $\Delta(\Theta_i)$ satisfying $\mathrm{cav}\, \varphi(p_i) \ge \varphi(p_i)$ for all $p_i \in \Delta(\Theta_i)$, and the supremum is achieved with $|\Theta_i|$ posteriors. See, *e.g.*, Aumann et al., 1995, or Rockafellar, 1970, Corollary 17.1.5, p. 157 for more technical details.

Clearly, (μ^3, τ) is an equilibrium of SG_{M^3} because the designer gets a positive payoff, while he gets a strictly negative payoff if he deviates. As observed above, τ is also a continuation equilibrium outcome. This equilibrium in SG_{M^3} induces the outcome $\rho \in \Delta(A)^{\Theta}$ which is such that

$$\rho(0,0 \mid \theta = 0) = \rho(1/2, 0 \mid \theta = 0) = \rho(1, 0 \mid \theta = 1) = \rho(1/2, 0 \mid \theta = 1) = 1/2.$$

In particular, it induces the ex-ante distribution $(1/4, 1/2, 1/4)$ over the actions $\{(0,0), (1/2, 0), (1, 0)\}$.

Consider now the game SG_{M^2}, with $|M^2| = 2$. To induce the same outcome ρ as above, the strategy profile must induce the ex-ante distribution $(1/4, 1/2, 1/4)$ over the actions $\{(0,0), (1/2, 0), (1, 0)\}$ and thus, the designer must induce the distribution $(1/4, 1/2, 1/4)$ over the posteriors $\{0, 1/2, 1\}$. The only way to achieve this with splittings with two posteriors is to use a mixed splitting $\zeta \in \Delta(S^{M^2}(p^0))$ which randomizes over the non-revealing splitting μ^2_{NR} and the fully revealing splitting μ^2_{FR} with equal probabilities. In addition, along the equilibrium path, the agent must choose $a^2 = 0$. However, the designer has a profitable deviation from ζ which consists in playing the splitting μ^2_{NR} with probability 1. Indeed, playing μ^2_{NR} gives him a payoff of $1/4$, while playing μ^2_{FR} gives him a payoff of 0.

This example shows that there exists an equilibrium outcome of SG_{M^3} which is not an equilibrium outcome of the direct game SG_{M^2}. This example can be generalized to show that for every $M' > M \geq 2$, there is an equilibrium outcome of $SG_{M'}$ which is not an equilibrium outcome of SG_M.

The second result in this section states that in the game SG_∞, it is without loss of generality to focus on pure strategies for the designers.

Theorem 3. *The set of equilibrium outcomes of SG_∞ coincides with the set of equilibrium outcomes of SG_∞ in which designers use pure strategies.*

The proof goes as follows. For each profile of mixed splittings in $\prod_i \Delta(\mathcal{S}_i(p_i^0))$, we construct a profile of pure splittings in $\prod_i \mathcal{S}_i(p_i^0)$ which induces the same distributions of posterior beliefs. Then, it suffices to construct a continuation outcome after the profile of pure splittings in $\prod_i \mathcal{S}_i(p_i^0)$ which induces the same expected payoffs for the designers.

Combined with the previous theorem, this result implies that every equilibrium outcome of the direct game SG_Θ is an equilibrium outcome of SG_∞ in which designers use pure strategies. Together with Theorem 1, this also implies that the game SG_∞ admits an equilibrium in pure strategies for the designers.

Proof. Consider an equilibrium $((\zeta_i, \zeta_{-i}), \tau)$ of SG_∞ in mixed strategies. The mixed splitting $\zeta_i \in \Delta(\mathcal{S}_i(p_i^0))$ for designer i induces a distribution $F_i \in \Delta(\mathcal{S}_i(p_i^0) \times \Delta(\Theta_i))$ defined by $dF_i(\mu_i, p_i) = d\mu_i(p_i) d\zeta_i(\mu_i)$. To be precise, for any a Borel set $B \subseteq \mathcal{S}_i(p_i^0) \times \Delta(\Theta_i)$ denote $B_{\mu_i} = \{p_i : (\mu_i, p_i) \in B\}$ and let

$$F_i(B) = \int_{\mu_i, p_i} \mathbf{1}_B dF_i(\mu_i, p_i) = \int_{\mu_i} \left(\int_{p_i} \mathbf{1}_{B_{\mu_i}} d\mu_i(p_i) \right) d\zeta_i(\mu_i) = \int_{\mu_i} \mu_i(B_{\mu_i}) d\zeta_i(\mu_i).$$

The marginal distribution of p_i under F_i is a splitting (the "expected splitting") denoted by μ_{F_i}.

The expected payoff of designer i under $((\zeta_i, \zeta_{-i}), \tau)$ writes:

$$\begin{aligned}
U_i^\tau(\zeta_i, \zeta_{-i}) &= \int_{\mu_i, \mu_{-i}} \left\{ \int_{p_i, p_{-i}} u_i(\tau(\mu_i, \mu_{-i}, p_i, p_{-i}), p_i, p_{-i}) d\mu_i(p_i) d\mu_{-i}(p_{-i}) \right\} d\zeta_i(\mu_i) d\zeta_{-i}(\mu_{-i}) \\
&= \int_{\mu, p} u_i(\tau(\mu_i, \mu_{-i}, p_i, p_{-i}), p_i, p_{-i}) dF_i(p_i, \mu_i) dF_{-i}(p_{-i}, \mu_{-i}).
\end{aligned}$$

Now, we use Fubini's theorem to exchange integration order. Let's write the distribution $dF_i(p_i, \mu_i)$ as the product of its marginal distribution $dF_i(p_i)$ and of a conditional distribution $dF_i(\mu_i|p_i)$ (these are well defined since the sets $\mathcal{S}_i(p_i^0)$ and $\Delta(\Theta_i)$ are compact metric). We get,

$$\begin{aligned}
U_i^\tau(\zeta_i, \zeta_{-i}) &= \int_{\mu, p} u_i(\tau(\mu_i, \mu_{-i}, p_i, p_{-i}), p_i, p_{-i}) dF_i(\mu_i|p_i) dF_i(p_i) dF_{-i}(\mu_{-i}|p_{-i}) dF_{-i}(p_{-i}) \\
&= \int_p u_i \left(\int_\mu \tau(\mu_i, \mu_{-i}, p_i, p_{-i}) dF_i(\mu_i|p_i) dF_{-i}(\mu_{-i}|p_{-i});\ p_i, p_{-i} \right) dF_i(p_i) dF_{-i}(p_{-i}).
\end{aligned}$$

If player i deviates to ξ_i inducing the distribution G_i, we have $U_i^\tau(\xi_i, \zeta_{-i}) \leq U_i^\tau(\zeta_i, \zeta_{-i})$ and,

$$U_i^\tau(\xi_i, \zeta_{-i}) = \int_p u_i \left(\int_\mu \tau(\mu_i, \mu_{-i}, p_i, p_{-i}) dG_i(\mu_i|p_i) dF_{-i}(\mu_{-i}|p_{-i});\ p_i, p_{-i} \right) dG_i(p_i) dF_{-i}(p_{-i}).$$

Let F_i, F_{-i} be the representations of mixed splittings as joint distributions and let $\mu_i^* = \mu_{F_i}$ be the expected splitting of designer i.

Define $\tau^* : \prod_i \mathcal{S}_i(p_i^0) \times \Delta(\Theta_i) \to \Delta(A)$ as follows:

- $\tau^*(\mu^*, p) = \int_\mu \tau(\mu_i, \mu_{-i}, p_i, p_{-i}) dF_i(\mu_i|p_i) dF_{-i}(\mu_{-i}|p_{-i})$,
- $\tau^*(\mu_i, \mu_{-i}^*, p) = y^i(p) \in \arg\min_{y \in Y(p)} u_i(y, p)$,
- For all other profiles μ, let $\tau^*(\mu, p)$ be arbitrary in $Y(p)$.

By construction, the payoff on the equilibrium path is

$$\begin{aligned}
U_i^\tau(\zeta_i, \zeta_{-i}) &= \int_p u_i \left(\int_\mu \tau(\mu_i, \mu_{-i}, p_i, p_{-i}) dF_i(\mu_i|p_i) dF_{-i}(\mu_{-i}|p_{-i});\ p_i, p_{-i} \right) dF_i(p_i) dF_{-i}(p_{-i}) \\
&= \int_p u_i(\tau^*(\mu^*, p), p) d\mu_i^*(p_i) d\mu_{-i}^*(p_{-i}) = U_i^{\tau^*}(\mu_i^*, \mu_{-i}^*).
\end{aligned}$$

Deviating to μ_i gives a payoff

$$\begin{aligned}
U_i^{\tau^*}(\mu_i, \mu_{-i}^*) &= \int_p u_i(y^i(p), p) d\mu_i(p_i) d\mu_{-i}^*(p_{-i}) \\
&= \int_p u_i(y^i(p), p) dG_i(p_i) dF_{-i}(p_{-i}) \\
&\leq \int_p u_i \left(\int_\mu \tau(\mu_i, \mu_{-i}, p_i, p_{-i}) dG_i(\mu_i|p_i) dF_{-i}(\mu_{-i}|p_{-i});\ p_i, p_{-i} \right) dG_i(p_i) dF_{-i}(p_{-i}) \\
&= U_i^\tau(\xi_i, \zeta_{-i}) \leq U_i^\tau(\zeta_i, \zeta_{-i}) = U_i^{\tau^*}(\mu_i^*, \mu_{-i}^*),
\end{aligned}$$

where the first inequality holds because $y^i(p)$ is the punishing continuation equilibrium outcome. This concludes the proof. ■

3.3 Canonical Games and Markovian Equilibria

In Example 2, more messages yield more equilibrium outcomes. This finding relies on equilibria where the action of the agent depends not only on the posterior belief, but also on the splitting used by the designer. Let's consider equilibria where the actions of agents depend only on the belief.

Definition 3. *An equilibrium $((\zeta_i^*)_{i\in N}, \tau)$ is* Markovian *if $\tau(\mu, p) = y(p) \in Y(p)$ does not depend on μ.*

This equilibrium refinement requires the strategies to be Markovian[6] with respect to common beliefs, a natural state variable for this model. The next result is that the set of outcomes of Markovian equilibria of the direct game coincides with the set of outcomes of Markovian equilibria of games with more messages (possibly infinitely many). Thus, the direct game may be called *canonical* for the set of Markovian equilibria.

Theorem 4. *Let M be a profile of message sets such that $|M_i| \geq |\Theta_i|$ for every $i \in N$. The set of Markovian equilibrium outcomes of SG_M coincides with the set of Markovian equilibrium outcomes of SG_∞. This set also coincides with the set of Markovian equilibrium outcomes of SG_∞ in which designers use pure strategies.*

The proof of this result relies on a lemma which states that every splitting can be replicated by mixing over splittings of the direct game. Let's call $\mathcal{S}_i^{\Theta_i}(p_i^0)$ the set of *canonical splittings* for designer i. A *mixed canonical splitting* of p_i^0 is a Borel probability distribution $\zeta_i \in \Delta(\mathcal{S}_i^{\Theta_i}(p_i^0))$ over the compact set of canonical splittings. A mixed splitting ζ_i induces an "expected" splitting μ_{ζ_i}, defined for each Borel set $B \subseteq \Delta(\Theta_i)$ by,

$$\mu_{\zeta_i}(B) = \int \mu_i(B) d\zeta_i(\mu_i).$$

It is easy to see that $\mu_{\zeta_i} \in \mathcal{S}_i^{\Theta_i}(p_i^0)$. We have the following representation result.

Lemma 1.

$$\mathcal{S}_i(p_i^0) = \overline{\text{co}}\, \mathcal{S}_i^{\Theta_i}(p_i^0) = \{\mu_{\zeta_i} : \zeta_i \in \Delta(\mathcal{S}_i^{\Theta_i}(p_i^0))\},$$

where $\overline{\text{co}}$ denotes the closure of the convex hull.

In words, any splitting is a (limit of) convex combination of canonical splittings. Equivalently, any splitting is the expected splitting induced by randomizing over canonical splittings.

Now, the intuition that every Markovian equilibrium outcome is an equilibrium outcome of the direct game (Theorem 4) is the following. By Lemma 1, every equilibrium splitting in SG_∞ can replaced by

[6] This equilibrium refinement should be more concisely called *public correlated* Markovian equilibrium whenever public correlation is needed to generate $y(p) \in Y(p)$. With a slight abuse of words, we simply call this refinement Markovian equilibrium.

a mixed canonical splitting which induces the same distribution of posteriors in SG_Θ. When the equilibrium is Markovian, continuation equilibrium outcomes depend only on agents' beliefs. Hence, if a deviation from the mixed canonical splitting is profitable in the direct game, it is also profitable in the original game, a contradiction. This argument does not hold if continuation equilibrium outcomes depend also on designers' splittings, as illustrated in Example 2.

Let's turn to the formal proofs.

Proof of Lemma 1. Since $\mathcal{S}_i(p_i^0)$ is convex and compact, $\overline{\mathrm{co}}\,\mathcal{S}_i^{\Theta_i}(p_i^0) \subseteq \mathcal{S}_i(p_i^0)$. Let $\mu_i^* \in \mathcal{S}_i(p_i^0)$, we want to show that $\mu_i^* \in \overline{\mathrm{co}}\,\mathcal{S}_i^{\Theta_i}(p_i^0)$. First observe that $\mathcal{S}_i^{\Theta_i}(p_i^0)$ is compact in $\Delta(\Delta(\Theta_i))$ and therefore the closure of its convex hull $\overline{\mathrm{co}}\,\mathcal{S}_i^{\Theta_i}(p_i^0)$ is also compact (see e.g., Aliprantis and Border, Theorem 5.35, page 185). Suppose that $\mu_i^* \notin \overline{\mathrm{co}}\,\mathcal{S}_i^{\Theta_i}(p_i^0)$. By the separation theorem (Aliprantis and Border, Theorem 5.79, page 207), there exists a continuous function $f : \Delta(\Theta_i) \to \mathbb{R}$ such that,

$$\int_{p_i} f(p_i)d\mu_i^*(p_i) > \sup_{\mu_i \in \overline{\mathrm{co}}\,\mathcal{S}_i^{\Theta_i}(p_i^0)} \int_{p_i} f(p_i)d\mu_i^*(p_i) \geq \sup_{\mu_i^* \in \mathcal{S}_i^{\Theta_i}(p_i^0)} \int_{p_i} f(p_i)d\mu_i^*(p_i).$$

This inequality is impossible. The LHS is smaller than $\mathrm{cav}\, f(p_i^0)$ and the RHS is $\mathrm{cav}\, f(p_i^0)$, since the cav of any function is achieved with $|\Theta_i|$ posteriors (Rockafellar, 1970, Corollary 17.1.5, p. 157). We conclude that $\mu_i^* \in \overline{\mathrm{co}}\,\mathcal{S}_i^{\Theta_i}(p_i^0)$ and therefore, $\mathcal{S}_i(p_i^0) = \overline{\mathrm{co}}\,\mathcal{S}_i^{\Theta_i}(p_i^0)$.

The other set equality in Lemma 1 follows from three observations. First, $\mathrm{co}\,\mathcal{S}_i^{\Theta_i}(p_i^0)$ is the set of all μ_{ζ_i} for ζ_i in $\Delta(\mathcal{S}_i^{\Theta_i}(p_i^0))$ with finite support. Second, the mapping $\zeta_i \mapsto \mu_{\zeta_i}$ is continuous for the weak-* topology. To see this, suppose that ζ_i^n weak-* converges to ζ_i. For every continuous function $f : \Delta(\Theta_i) \to \mathbb{R}$,

$$\int_{p_i} f(p_i)d\mu_{\zeta_i^n}(p_i) = \int_{\mu_i} \left(\int_{p_i} f(p_i)d\mu_i(p_i) \right) d\zeta_i^n(\mu_i).$$

The weak-* convergence of ζ_i^n implies that this converges to

$$\int_{\mu_i} \left(\int_{p_i} f(p_i)d\mu_i(p_i) \right) d\zeta_i(\mu_i) = \int_{p_i} f(p_i)d\mu_{\zeta_i}(p_i).$$

Thus, $\mu_{\zeta_i^n}$ weak-* converges to μ_{ζ_i}. Third, the set of probability measures with finite support is dense in $\Delta(\mathcal{S}_i^{\Theta_i}(p_i^0))$ (Aliprantis and Border, Theorem 15.10, page 513). The closure of the convex hull $\overline{\mathrm{co}}\,\mathcal{S}_i^{\Theta_i}(p_i^0)$ is thus the set of all μ_{ζ_i}'s. ■

Proof of Theorem 4. The argument of the proof of Theorem 2 directly applies to show that a Markovian equilibrium outcome of SG_M is a Markovian equilibrium outcome of SG_∞. Conversely, consider a Markovian equilibrium $((\zeta_i^*)_{i\in N}, y)$ of SG_∞. First, we reduce it to a Markovian equilibrium $((\mu_i^*)_{i\in N}, y)$ of SG_∞ in pure strategies for the designers. As before, a mixed splitting ζ_i^* is represented by a joint distribution $F_i^* \in \Delta(\mathcal{S}_i(p_i^0) \times \Delta(\Theta_i))$. The equilibrium payoff of designer i is:

$$\begin{aligned}
U_i^y(\zeta_i^*, \zeta_{-i}^*) &= \int_\mu \int_p u_i(y(p_i, p_{-i}), p_i, p_{-i}) d\mu_i(p_i) d\mu_{-i}(p_{-i}) d\zeta_i^*(\mu_i) d\zeta_{-i}^*(\mu_{-i}) \\
&= \int_{\mu,p} u_i(y(p_i, p_{-i}), p_i, p_{-i}) dF_i^*(\mu_i, p_i) dF_{-i}^*(\mu_{-i}, p_{-i}) \\
&= \int_p u_i(y(p_i, p_{-i}), p_i, p_{-i}) \int_\mu dF_i^*(\mu_i, p_i) dF_{-i}^*(\mu_{-i}, p_{-i}) \\
&= \int_p u_i(y(p_i, p_{-i}), p_i, p_{-i}) d\mu_{F_i^*}(p_i) d\mu_{F_{-i}^*}(p_{-i}) = U_i^y(\mu_{F_i^*}, \mu_{F_{-i}^*}).
\end{aligned}$$

If designer i deviates from the mixed equilibrium to a pure $\tilde{\mu}_i$, $U_i^y(\tilde{\mu}_i, \zeta_{-i}^*) \leq U_i^y(\zeta_i^*, \zeta_{-i}^*)$ and,

$$U_i^y(\tilde{\mu}_i, \zeta_{-i}^*) = \int_p u_i(y(p_i, p_{-i}), p_i, p_{-i}) d\tilde{\mu}_i(p_i) d\mu_{F_{-i}^*}(p_{-i}) = U_i^y(\tilde{\mu}_i, \mu_{F_{-i}^*}),$$

which shows that $((\mu_{F_i^*}, \mu_{F_{-i}^*}), y)$ is an equilibrium.

Now, let (μ^*, y) be a Markovian equilibrium of SG_∞ in pure strategies for the designers. From Lemma 1, there exists $\zeta_i^* \in \Delta(\mathcal{S}^{\Theta_i}(p^0))$ such that $\mu_i^* = \mu_{\zeta_i^*}$. We claim that $((\zeta_i^*)_i, y)$ is a mixed equilibrium of SG_Θ. We have,

$$\begin{aligned}
U_i^y(\mu_i^*, \mu_{-i}^*) &= \int_\mu \int_p u_i(y(p_i, p_{-i}), p_i, p_{-i}) d\mu_i(p_i) d\mu_{-i}(p_{-i}) d\zeta_i^*(\mu_i) d\zeta_{-i}^*(\mu_{-i}) \\
&= U_i^y(\zeta_i^*, \zeta_{-i}^*).
\end{aligned}$$

Suppose that designer i deviates to some pure $\tilde{\mu}_i \in \mathcal{S}^{\Theta_i}(p^0)$. The expected payoff is,

$$\begin{aligned}
U_i^y(\tilde{\mu}_i, \zeta_{-i}^*) &= \int_{\mu_{-i}} \int_p u_i(y(p_i, p_{-i}), p_i, p_{-i}) d\tilde{\mu}_i(p_i) d\mu_{-i}(p_{-i}) d\zeta_{-i}^*(\mu_{-i}) \\
&= U_i^y(\tilde{\mu}_i, \mu_{-i}^*) \leq U_i^y(\mu_i^*, \mu_{-i}^*) = U_i^y(\zeta_i^*, \zeta_{-i}^*).
\end{aligned}$$

This completes the proof of the theorem. ■

Remark 3. It can be noted that those results extend directly to models with continuous actions. The only important assumption in the model is that the correspondence $p \mapsto Y(p)$ be non-empty convex compact valued and upper hemicontinuous. In particular, when $Y(p)$ is a singleton for each p, then all equilibria are Markovian. This is the case, for instance, if there is a single agent with strictly concave preferences, but also for many games with quadratic preferences such as beauty contests, Cournot competition, network games. For such environments, the game SG_Θ is canonical for *all* equilibria.

3.4 Examples

3.4.1 The One-Designer, One-Agent Case

Splitting games subsume the model of Bayesian persuasion (Kamenica and Gentzkow, 2011) where there is a single designer and a single agent. When there is a single agent, the set $Y(p)$ of continuation equilibrium outcomes is simply the set of optimal mixed actions of the agent:

$$Y(p) = \Delta(A(p)), \text{ where } A(p) = \arg\max_{a \in A} v(a; p).$$

An equilibrium is a Borel measurable selection $\tau(\mu, p) \in Y(p)$ for each μ and p and an optimal strategy for the designer that solves $\sup_{\mu \in \mathcal{S}(p^0)} U^{\tau}(\mu)$. The best equilibrium for the designer is Markovian and is obtained by selecting the designer-preferred optimal strategy τ^* of the agent,

$$\tau^*(\mu, p) = y^*(p) \in \arg\max_{y \in Y(p)} u(y, p).$$

Then, the induced utility for the designer, denoted by $U^*(p) = u(y^*(p); p)$ is upper semi-continuous and the designer has a best-response inducing the payoff:

$$\max_{\mu \in \mathcal{S}(p^0)} U^{\tau^*}(\mu) = \operatorname*{cav}_{p} U^*(p^0).$$

Example 3 (Binary states and messages, cont)**.** As in Example 1, the designer can design information on $\Theta = \{\theta^1, \theta^2\}$, and $|M| = 2$. The agent has three possible actions, $A = \{a^1, a^2, a^3\}$, and his utility function is given by:

$$v(a, \theta) = \begin{array}{c|c|c|c|} \multicolumn{1}{c}{} & \multicolumn{1}{c}{a^1} & \multicolumn{1}{c}{a^2} & \multicolumn{1}{c}{a^3} \\ \cline{2-4} \theta^1 & 2 & 3 & 3 \\ \cline{2-4} \theta^2 & 2 & 1 & 0 \\ \cline{2-4} \end{array}$$

Denote by p the probability of θ^1. The optimal action of the agent is a^1 when $p < \frac{1}{2}$ and a^2 when $p \in (\frac{1}{2}, 1)$. Every randomization between a_1 and a_2 is optimal at $p = \frac{1}{2}$ and every randomization between a_2 and a_3 is optimal at $p = 1$. Let $y(\frac{1}{2}) \in Y(\frac{1}{2})$ be identified with the probability that the agent plays a^2 at $p = \frac{1}{2}$, and $y(1) \in Y(1)$ be identified with the probability that the agent plays a^3 at $p = 1$. Consider a designer's utility function which depends only on the action and is given by:

$$u(a, \theta) = \begin{cases} 0 & \text{if } a = a^1, \\ 1 & \text{if } a = a^2, \\ 3 & \text{if } a = a^3. \end{cases}$$

Given the strategy $y(p)$ of the agent, the induced utility of the designer as a function of the posterior p, is given by the function $u(y(p); p)$ represented in Figure 1. The concavification is given by the dotted curve. Observe that the concavification depends on the tie-breaking rule at $p = 1$, but not at $p = 1/2$.

The designer-preferred optimal strategy y^* is such that $y^*(\frac{1}{2}) = y^*(1) = 1$, which yields

$$\operatorname*{cav}_p u(y^*(p^0); p^0) = \operatorname*{cav}_p U^*(p^0) = 3p^0.$$

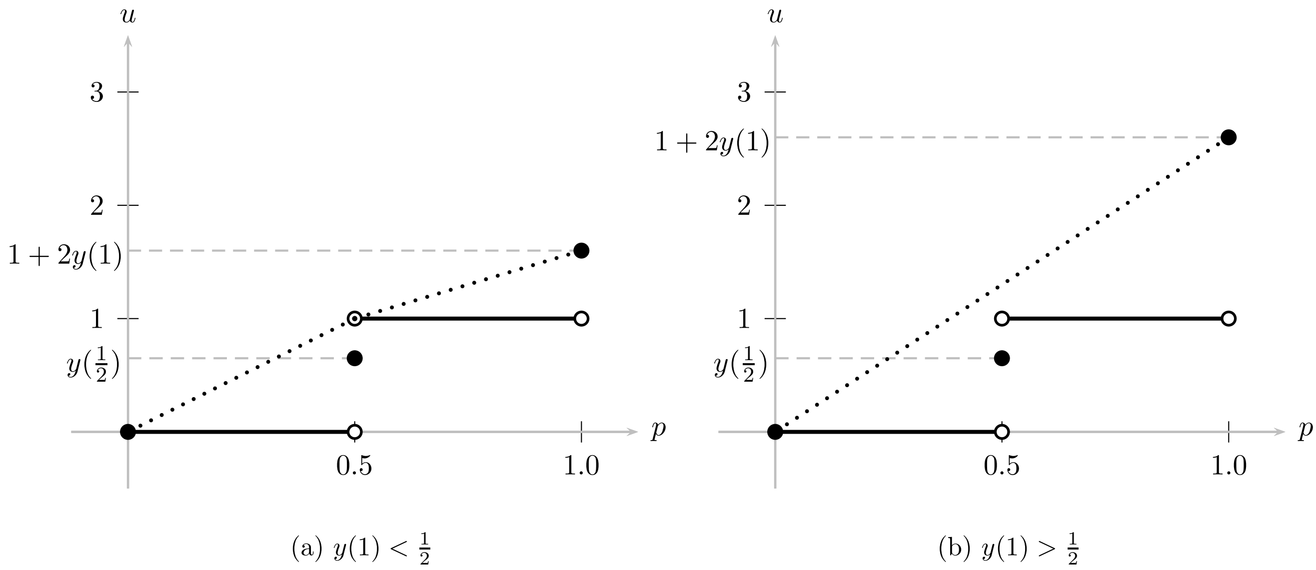

Figure 1: Concavification in Example 3. $u(y(p); p)$ in solid lines and $\operatorname{cav}_p u(y(p); p)$ in dotted lines.

3.4.2 Two Strictly Competitive Examples

In the following example we show that with a finite set of messages, if we fix a continuation equilibrium arbitrarily, then an equilibrium (and even an ε-equilibrium) in pure strategies may not exist. An equilibrium in pure strategies with a finite set of messages exists for *some* continuation equilibrium which depends both on the posteriors *and* on the splittings used by the designers.

Example 4 (Being perceived as better)**.** The agent (a buyer) decides to buy either from firm 1 (action 1) or from firm 2 (action 2). Each designer (firm) wants to maximize the probability of trade. The state (valuations of the buyer) is $\theta = (\theta_1, \theta_2) \in \{0, 1\} \times \{0, 1\}$, and the prior probability that $\theta_i = 1$ is $p_i^0 \in (0, 1)$. The agent's payoff is $v(i, \theta) = \theta_i$, $i = 1, 2$, so the set of optimal actions is given by:

$$A(p_1, p_2) = \begin{cases} \{1\} & \text{if } p_1 > p_2 \\ \{2\} & \text{if } p_1 < p_2 \\ \{1, 2\} & \text{if } p_1 = p_2, \end{cases}$$

where action $a = i$ means buying from i. Notice that for every strategy of the agent, the game between the designers is a zero-sum game.

A constant tie-breaking rule is given by a fixed $\alpha \in [0, 1]$ such that the probability to choose action

$a = 1$ is α whenever $p_1 = p_2$. After normalization of the utility functions we have (see Figure 2):

$$U_1(p_1, p_2) = -U_2(p_1, p_2) = U(p_1, p_2) = \begin{cases} 1 & \text{if } p_1 > p_2, \\ -1 & \text{if } p_2 > p_1, \\ 2\alpha - 1 & \text{if } p_1 = p_2. \end{cases}$$

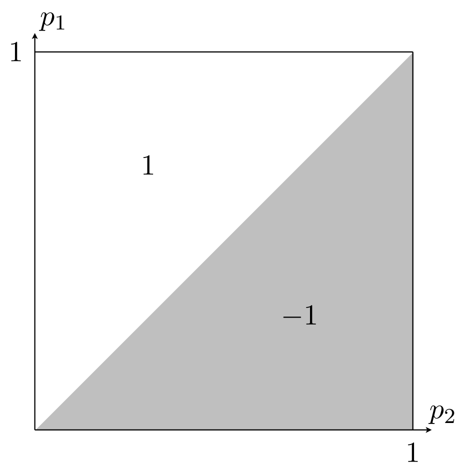

Figure 2: Payoff of designer 1 in Example 4 as a function of the agent's beliefs (p_1, p_2).

In the game SG_M with $|M_1| = |M_2| = 2$, consider a splitting μ_2 of designer 2 given by the convex combination $\frac{1}{2} = \lambda p_2 + (1-\lambda)p_2'$ with $p_2 \leq \frac{1}{2} \leq p_2'$. The best-reply payoff of designer 1 is $\text{cav}_{p_1} U(p_1, \mu_2)$, where cav_{p_1} denotes the concavification with respect to the first variable, the second one being fixed. We have,

$$U(p_1, \mu_2) = \begin{cases} -1 & \text{if } p_1 < p_2, \\ \lambda(2\alpha - 1) - (1 - \lambda) = 2\alpha\lambda - 1 & \text{if } p_1 = p_2, \\ \lambda - (1 - \lambda) = 2\lambda - 1 & \text{if } p_2 < p_1 < p_2', \\ \lambda + (1 - \lambda)(2\alpha - 1) = 2\lambda + 2\alpha(1 - \lambda) - 1 & \text{if } p_1 = p_2', \\ 1 & \text{if } p_1 > p_2'. \end{cases}$$

It is easy to verify that $\text{cav}_{p_1} U(\frac{1}{2}, \mu_2) \geq \frac{\alpha}{2}$ for every μ_2. In addition, if designer 2 plays fully revealing (*i.e.*, $p_2 = 0$, $p_2' = 1$), then $\text{cav}_{p_1} U(\frac{1}{2}, \mu_2) = \frac{\alpha}{2}$. Hence,

$$\min_{\mu_2 \in \mathcal{S}_2(\frac{1}{2})} \sup_{\mu_1 \in \mathcal{S}_2(\frac{1}{2})} U(\mu_1, \mu_2) = \frac{\alpha}{2}.$$

The reasoning is symmetric for designer 2 and we get:

$$\max_{\mu_1 \in \mathcal{S}_2(\frac{1}{2})} \inf_{\mu_2 \in \mathcal{S}_2(\frac{1}{2})} U(\mu_1, \mu_2) = -\frac{(1 - \alpha)}{2}.$$

Therefore for every fixed $\alpha \in [0,1]$, the induced 2-player game has no value in pure strategies and thus no equilibrium in which designers use pure strategies.[7]

The game has an equilibrium in pure strategies if the agent's strategy depends on the splittings (hence, this equilibrium is not Markovian according to Definition 3). Indeed, suppose that both designers completely reveal the state, in which case the agent randomizes equally when he is indifferent ($\alpha = \frac{1}{2}$). This yields an expected payoff equal to zero for both designers. If designer 1 deviates, then the agent chooses designer 2's preferred action ($\alpha = 0$), and if designer 2 deviates then the agent chooses designer 1's preferred action ($\alpha = 1$). It is easy to see that with this strategy, no designer can achieve an expected probability of trade higher than $\frac{1}{2}$, so this constitutes an equilibrium in which designers use pure strategies.

Now suppose that designers can use mixed strategies and consider any optimal strategy for the agent. Suppose that designer 2 plays the "uniform strategy": he randomizes uniformly over all possible symmetric splittings $\frac{1}{2} = \frac{1}{2}p_2 + \frac{1}{2}(1-p_2)$, with $p_2 \in [0,1]$ (uniformly distributed).[8] Then, the payoff of designer 1 is:

$$U(p_1, \mu_2) = \int_0^1 U(p_1, p_2)dp_2 = \int_0^{p_1} 1dp_2 + \int_{p_1}^1 -1dp_2 = 2p_1 - 1.$$

This is linear (thus concave) and the value at $\frac{1}{2}$ is 0. Thus, against the uniform strategy, 0 is the best payoff that designer 1 can achieve. Hence, the game has a value (equal to 0) and each designer has an optimal strategy (the uniform one), whatever the optimal strategy of the agent is.

In the previous example, there exists a mixed equilibrium between the designers for every optimal strategy of the agent. The next example adapted from Sion and Wolfe (1957) shows that, for some optimal strategy of the agent, an ε-equilibrium (in pure or mixed strategies) may not exist, regardless of the message spaces of the designers. This justifies our approach to consider the $(n+k)-$player game involving the agents, instead of fixing continuation equilibrium outcomes for the agents and analyzing the induced $n-$player game between the designers.

Example 5. As in Example 4, the state is $\theta = (\theta_1, \theta_2) \in \{0,1\} \times \{0,1\}$, and the prior probability that $\theta_i = 1$ is $p_i^0 \in (0,1)$. The space of posteriors $[0,1]\times[0,1]$ is partitioned into nine regions depending on the optimal action of the agent, who has thus at least nine possible actions. It is easy to construct payoffs for the agent which generate those regions.[9] His optimal action leads to payoffs $U(p_1,p_2) = U_1(p_1,p_2) = -U_2(p_1,p_2) = 1$ when his belief belongs to the white area of Figure 3, and $U_1(p_1,p_2) = -U_2(p_1,p_2) = -1$ when it belongs to the gray area. The agent is indifferent between several actions at all points of discontinuities, in which case $U_1(p_1,p_2) = -U_2(p_1,p_2)$ can take any value in $[-1,1]$.

Notice first that $\operatorname{cav}_{p_1} U(\frac{1}{2}, p_2) = 1$ for every $p_2 \in [0,1]$. Therefore $\operatorname{vex}_{p_2} \operatorname{cav}_{p_1} U(\frac{1}{2}, \frac{1}{2}) = 1$, where $\operatorname{vex}_{p_2}(f) = -\operatorname{cav}_{p_2}(-f)$ is the *convexification* of f with respect to the second variable. This means that, for every optimal strategy of the agent, if designer 1 *could observe* the posterior realized by designer 2,

[7] Assuming $\alpha = 1/2$, Albrecht (2017) shows that there is no pure strategy equilibrium for any finite number of messages.

[8] A corresponding infinite statistical experiment $x_2 : \Theta_2 \to [0,1]$ draws a message in $[0,1]$ from the conditional density functions $f_{x_2}(m \mid \theta_2 = 0) = 2m$ and $f_{x_2}(m \mid \theta_2 = 1) = 2(1-m)$.

[9] Given any straight line in $[0,1] \times [0,1]$, one can find a corresponding decision problem with two actions. It is then possible to combine such problems with additively separable payoffs to generate any polyhedral region.

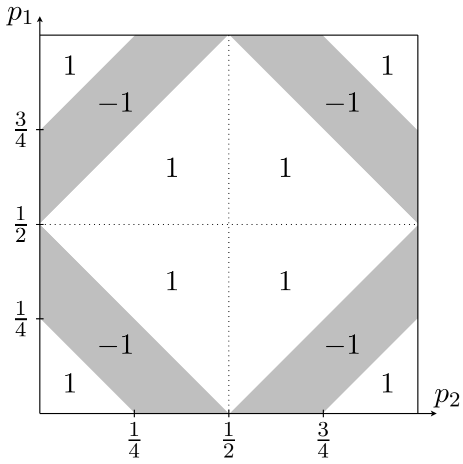

Figure 3: Payoff of designer 1 in Example 5 as a function of the agent's beliefs (p_1, p_2).

then he would achieve a payoff of 1: if the posterior induced by designer 2 is such that $p_2 \in (0,1)$, then designer 1 does not reveal any information; otherwise, designer 1 fully discloses θ_1. In other words, $\text{vex}_{p_2} \text{cav}_{p_1} U(\frac{1}{2}, \frac{1}{2})$ is the value of the zero-sum game where designer 2 plays first and designer 1 observes the realized posterior before choosing his splitting. This is an upper bound of the sup inf value of the simultaneous move game.

The value of $\text{vex}_{p_2} U(p_1, \frac{1}{2})$ depends on the strategy of the agent at posteriors $(p_1, p_2) = (\frac{1}{2}, 0)$ and $(p_1, p_2) = (\frac{1}{2}, 1)$. For example, if $U(\frac{1}{2}, 0) = U(\frac{1}{2}, 1) = \bar{u} \in [-1, 1]$, then

$$\operatorname*{vex}_{p_2} U\left(p_1, \frac{1}{2}\right) = \begin{cases} -1 & \text{if } p_1 \neq \frac{1}{2} \\ \bar{u} & \text{if } p_1 = \frac{1}{2}, \end{cases}$$

which implies $\text{cav}_{p_1} \text{vex}_{p_2} U(\frac{1}{2}, \frac{1}{2}) = \bar{u}$. We have:

$$\operatorname*{cav}_{p_1} \operatorname*{vex}_{p_2} U\left(\frac{1}{2}, \frac{1}{2}\right) \leq \max_{\mu_1 \in \mathcal{S}(\frac{1}{2})} \inf_{\mu_2 \in \mathcal{S}(\frac{1}{2})} U(\mu_1, \mu_2) \leq \min_{\mu_1 \in \mathcal{S}(\frac{1}{2})} \sup_{\mu_2 \in \mathcal{S}(\frac{1}{2})} U(\mu_1, \mu_2) \leq \operatorname*{vex}_{p_2} \operatorname*{cav}_{p_1} U\left(\frac{1}{2}, \frac{1}{2}\right).$$

Thus, for the strategy always favoring designer 1 ($\bar{u} = 1$), the information design game has an equilibrium value:

$$\max_{\mu_1 \in \mathcal{S}(\frac{1}{2})} \inf_{\mu_2 \in \mathcal{S}(\frac{1}{2})} U(\mu_1, \mu_2) = \min_{\mu_1 \in \mathcal{S}(\frac{1}{2})} \sup_{\mu_2 \in \mathcal{S}(\frac{1}{2})} U(\mu_1, \mu_2) = 1.$$

This equilibrium is very simple: designer 1 does not reveal any information and he gets a payoff of 1 whatever the strategy of designer 2.

Consider now the "symmetric" optimal strategy for the agent: he randomizes in such a way that the payoff is equal to 0 at all points of discontinuities of $U(p_1, p_2)$ of Figure 3. If the induced game between the designers has a value (sup inf = inf sup), from the previous discussion it should belong to $[\bar{u}, 1] = [0, 1]$ (with the symmetric strategy, $\bar{u} = 0$). However, we can now follow Sion and Wolfe (1957) to show that this game has no value. The example of Sion and Wolfe (1957) is a normal form game in which player 1 chooses an action $x \in [0, \frac{1}{2}]$, player 2 chooses an action $y \in [\frac{1}{2}, 1]$, and where the payoff is given by our function $U(p_1, p_2)$ with $p_1 = x$, $p_2 = y$ and the symmetric optimal strategy for the agent. It is easy to see that the mixed extension of this game is equivalent to the information design game induced by the symmetric optimal strategy for the agent. Indeed, for every mixed strategy in their example, there exists a symmetric splitting in the information design game inducing the same payoff. Conversely, for every (symmetric or asymmetric) splitting, there exists an equivalent mixed strategy in their game.

Sion and Wolfe (1957) show that the maxmin payoff is $\frac{1}{3}$ and that the minmax payoff is $\frac{3}{7}$. A maxmin strategy for designer 1 is the splitting $\frac{1}{2} = (1/6) \times 0 + (1/6) \times 1 + (2/3) \times \frac{1}{2}$. A minmax strategy for designer 2 is the splitting $\frac{1}{2} = (2/7) \times 0 + (2/7) \times 1 + (1/7) \times \frac{1}{4} + (1/7) \times \frac{3}{4} + (1/14) \times \frac{3}{8} + (1/14) \times \frac{5}{8}$. Hence,

$$\max_{\mu_1 \in \mathcal{S}(\frac{1}{2})} \inf_{\mu_2 \in \mathcal{S}(\frac{1}{2})} U(\mu_1, \mu_2) = \frac{1}{3} < \min_{\mu_1 \in \mathcal{S}(\frac{1}{2})} \sup_{\mu_2 \in \mathcal{S}(\frac{1}{2})} U(\mu_1, \mu_2) = \frac{3}{7}.$$

This shows that with the symmetric optimal strategy for the agent, the induced information design game between the designers has no value and therefore no ε-equilibrium.

4 Rectangular Corporation Problems

As in the previous section we assume that the prior probability distribution $p^0 = \otimes_{i \in N} p_i^0$ is the product of its marginal distributions $p_i^0 \in \Delta(\Theta_i)$. We do not restrict attention to public messages anymore, but we assume that each designer provides information that is only relevant for a group of agents, that we call his *corporation*. Precisely, for each designer i, there is a group of n_i agents numbered ij, $j = 1, \ldots, n_i$. Each agent ij has a finite set of actions A_i^j and we let $A_i = \prod_{j=1}^{n_i} A_i^j$ denote the set of action profiles for corporation i.

For simplicity we consider state-independent preferences for the designers, but our results extend to state-dependent preferences (see Remark 4). The payoff $u_i(a)$ of each designer i depends on $a = (a_i)_{i \in N} \in \prod_{i \in N} A_i$, the profile of actions of all agents (from all corporations). The payoff $v_i^j(\theta_i, a_i)$ of each agent ij in corporation i depends on θ_i and a_i, that is on the state and actions of corporation i. The set of messages from designer i to agent ij is a finite set M_i^j and we assume that $|M_i^j| \geq |A_i^j|$. Denote $M_i = \prod_{j=1}^{n_i} M_i^j$ and $M = (M_i)_{i \in N}$.

We focus on information design games satisfying the assumptions above and in which (i) there is no public correlation device and (ii) agents in corporation i only observe the statistical experiment chosen by designer i. We call such games *rectangular corporation games* and denote them by RG_M. Since agents' payoffs in corporation i do not depend on the statistical experiments of the designers other than i and since public correlation does not affect payoffs directly, equilibrium outcomes of RG_M are also equilibrium outcomes of the corresponding information design game as defined in our general model.

Rectangular corporation games fit well the example of competing pharmaceutical companies mentioned in the introduction, where the agents (*e.g.*, members of the Center for Drug Evaluation and Research of the FDA) review the new drug application of each company to find out if the drug is safe and effective. See also Example 6 below where an agent is a patent office which reviews applications and decides whether patents are granted to complementary inventions.

Notice that every information design game with a single designer is a rectangular corporation game. Notice also that Examples 4 and 5 in the previous section are *not* rectangular corporation games.

Now, when designer i chooses the information structure for his corporation, he induces a Bayesian game for it. The set of equilibria induced for the corporation is naturally captured by the Bayes Correlated Equilibrium concept of Bergemann and Morris (2016a). A statistical experiment x_i for corporation i is called *direct* if $x_i \in \Delta(A_i)^{\Theta_i}$.

Definition 4. *A direct statistical experiment x_i^* is a* Bayes Correlated Equilibrium *(BCE) for corporation i, if for each agent ij and each pair of actions $a_i^j, b_i^j \in A_i^j$,*

$$\sum_{\theta_i \in \Theta_i} \sum_{a_i^{-j}} p_i^0(\theta_i) x_i^*(a_i^j, a_i^{-j} | \theta_i) v_i^j(a_i^j, a_i^{-j}; \theta_i) \geq \sum_{\theta_i \in \Theta_i} \sum_{a_i^{-j}} p_i^0(\theta_i) x_i^*(a_i^j, a_i^{-j} | \theta_i) v_i^j(b_i^j, a_i^{-j}; \theta_i).$$

In words, a BCE is a statistical experiment where each agent's message consists of a privately recommended action, with the property that each agent has an incentive to play it if others do so. Denote by $C_i(p_i^0)$ the set of distributions of action profiles of corporation i induced by BCEs. That is, the set of $y_i \in \Delta(A^i)$ for which there exists a BCE x_i^* such that $y_i(a_i) = \sum_{\theta_i \in \Theta_i} p_i^0(\theta_i) x_i^*(a_i | \theta_i)$.

Consider the auxiliary n-player normal form game $(C_i(p_i^0), u_i)_{i \in N}$.

Theorem 5. *The normal form game $(C_i(p_i^0), u_i)_{i \in N}$ has a non-empty compact set of equilibria $E(p^0)$. Every distribution of actions in $E(p^0)$ is an equilibrium outcome of the rectangular corporation game RG_M.*

Proof. The non-empty set of BCEs is described by finitely many linear inequalities, it is therefore convex and compact and so is $C_i(p_i^0)$. The game $(C_i(p_i^0), u_i)_{i \in N}$ is obtained from the mixed extension of a finite game, by considering a non-empty, convex and compact subset of feasible mixed strategies. Therefore it has an equilibrium by Nash's theorem.

Given an equilibrium of $(C_i(p_i^0), u_i)_{i \in N}$, consider the corresponding experiments x_i^*. Since these are BCEs, each agent for each corporation has an incentive to play the recommended action.

Suppose that designer i deviates to experiment x_i. In every equilibrium, agents in corporation i play an equilibrium of the continuation Bayesian game with information structure x_i. A simplified version of the revelation principle in Myerson (1982, Proposition 2) shows that this continuation equilibrium is a Bayes correlated equilibrium. Therefore, it induces some distribution in $C_i(p_i^0)$ which implies that the deviation is not profitable for designer i. ■

Remark 4. We make several comments.

1. Every distribution of actions in $E(p^0)$ can be induced by finite statistical experiments with $|A_i|$ messages for each designer and pure strategies for each designer i and each agent ij. This implies that all rectangular corporation games RG_M with $|M_i^j| \geq |A_i^j|$ for every i, j, have a common equilibrium in pure strategies.

2. Theorem 5 easily extends to the case where the utilities of the designers depend on the state. Suppose that the utility function of designer i is:

$$u_i(a_1, \ldots, a_n; \theta_1, \ldots, \theta_n).$$

Denote by $\mathcal{B}_i(p_i^0)$ the set of BCEs for corporation i given the prior p_i^0 and consider the game where each designer i chooses a BCE $b_i \in \mathcal{B}_i(p_i^0)$. Given a strategy profile $(b_1, \ldots, b_n)$ in this game, the expected payoff of designer i is:

$$U_i(b_1, \ldots, b_n) = \sum_{\theta \in \Theta} p^0(\theta) u_i(b_1(\theta_1), \ldots, b_n(\theta_n); \theta).$$

The normal form game $(\mathcal{B}_i(p_i^0), U_i)_i$ admits a Nash equilibrium. Indeed, the set $\mathcal{B}_i(p_i^0)$ is non-empty, convex and compact and the mapping $b_i \mapsto U_i(b_i, b_{-i})$ is linear, so Nash's theorem applies. The rest of the proof of Theorem 5 extends directly.

3. In general, $E(p^0)$ is a strict subset of equilibrium outcomes of RG_M, even for generic games. To see this, consider one designer with a corporation of two agents. The two agents play a game with complete information given by the following matrix in all states:

	L	R
T	1,1	0,0
B	0,0	1,1

Suppose that the utility of the designer is 1 if (T, L) is played and 0 otherwise. In this game, $E(p^0)$ is a singleton obtained by selecting the correlated equilibrium that the designer prefers, namely (T, L). However, (B, R) is a Bayes Nash equilibrium of the matrix game for every statistical experiment (here, simply a correlation device). Thus, there is an equilibrium of the game of information design where the agents choose (B, R) irrespective of experiments and messages. $E(p^0)$ is thus a strict subset of the equilibrium outcomes of information design game. This latter property also holds for a neighborhood of this game. Indeed, for payoff functions of the agents close enough to the matrix above, (T, L) and (B, R) are strict Bayes Nash equilibria in all states, so the same logic applies.

4. An equilibrium outcome of the information design game may not be an equilibrium outcome of $(C_i(p_i^0), u_i)_{i \in N}$ even with a single agent. To see this, consider the single-designer game of Example 3 with $p^0 = \frac{1}{2}$. There is an equilibrium with no information disclosure in which the agent plays action a^1 if $p < \frac{1}{2}$ and action a^2 if $p \geq \frac{1}{2}$, so action a^2 is played with probability one on the equilibrium path. However, the game $(C(\frac{1}{2}), u)$ has a unique equilibrium outcome: $E(\frac{1}{2}) = \left\{\frac{1}{2}a^1 + \frac{1}{2}a^3\right\}$. This example is however non-generic in the sense that it has an action (namely a^3) which is not essential: there exists no belief under which a^3 is the unique optimal action. Similarly, in Example 2, no action is essential for the agent, the unique equilibrium of the

auxiliary game $(C(\frac{1}{2}), u)$ is the first best for the designer (the non-revealing splitting), but the information design game has many other equilibrium outcomes.

Now, we consider games with a single agent in each corporation and give conditions under which the set $E(p^0)$ coincides with the set of equilibrium outcomes of RG_M.

Definition 5. *Consider a rectangular corporation game with one agent in each corporation, and let $A_i(p_i) = \arg\max_{a_i \in A_i} v_i(a_i; p_i)$ be the set of optimal actions of agent i in corporation i. The game is* regular *is for each i and for each a_i there exists $p_i \in \Delta(\Theta_i)$ such that $A_i(p_i) = \{a_i\}$.*

Theorem 6. *Consider a regular rectangular corporation game RG_M with one agent in each corporation. Then, $E(p^0)$ coincides with the set of equilibrium outcomes of RG_M.*

Proof. The regularity assumption implies that for every $p_i \in \Delta(\Theta_i)$, every $a_i \in A_i(p_i)$, and every $\varepsilon > 0$, there exists $\tilde{p}_i$ with $\|p_i - \tilde{p}_i\| < \varepsilon$, such that $A_i(\tilde{p}_i) = \{a_i\}$ (choose any norm on the finite dimensional set $\Delta(\Theta_i)$). Indeed, there exists q_i such that $A_i(q_i) = \{a_i\}$, so for each $\varepsilon > 0$, a_i is the only optimal action for $\tilde{p}_i = (1-\varepsilon)p_i + \varepsilon q_i$.

Assume by contradiction that there is an equilibrium outcome of RG_M that is not in $E(p^0)$. This means that some designer i has a deviation $\bar{y}_i \in C_i(p_i^0)$ which is profitable by more than some $\varepsilon > 0$. Consider the splitting which induces $\bar{y}_i$. From the regularity assumption, it is possible to perturb the posteriors slightly in such a way that they are still induced by a splitting of p^0 and at each posterior, there is a single optimal action for the agent. Moreover, we can do it by selecting the optimal action that designer i prefers. For small perturbations, this would be a deviation profitable by $\varepsilon/2$ in the information design game, a contradiction. ■

The single-designer/single-agent problem is a particular case of a rectangular corporation problem. Hence, the previous theorem implies that, under regularity, the information design game (with at least as many messages as actions) has a unique equilibrium payoff for the designer, given by $\text{cav}_p\, U^*(p^0)$ and which is achievable by pure strategies.

The following example illustrates our characterization in a simple patent game with two designers.

Example 6. Consider two designers who represent entrepreneurs who own projects for complementary new technologies (for example, new hardware and software, new battery and microprocessor). For each entrepreneur, investing in the new technology is valuable only if he is granted an enforceable patent and if the other entrepreneur also invests in the new technology and gets an enforceable patent. The patentability characteristics of the technology of entrepreneur i are uncertain and denoted by $\theta_i \in \Theta_i = \{\underline{\theta}, \bar{\theta}\}$. The characteristics are good ($\theta_i = \bar{\theta}$) with probability $p_i^0 \in (0,1)$ and bad ($\theta_i = \underline{\theta}$) with probability $1 - p_i^0$. Assume without loss of generality that $p_2^0 \leq p_1^0$.

The agent for each entrepreneur i is a patent office who examines the application and decides whether an enforceable patent is granted to entrepreneur i. The agent grants the patent if his belief p_i that $\theta_i = \bar{\theta}$ is greater than or equal to some threshold $\bar{p} \in (p_1^0, 1)$. An entrepreneur's payoff is normalized to 0 if he is not granted a patent. If a single entrepreneur is granted a patent, he incurs a cost equal to $c > 0$. If both entrepreneurs obtain an enforceable patent, each gets a net benefit of 1.

The designers' payoffs as a function of the agent's beliefs $(p_1, p_2) \in [0,1]^2$ are given by Figure 4. Denote by y_i the probability that designer i gets an enforceable patent. The set of such probabilities induced by all statistical experiments of designer i and optimal strategies of the agent is given by:

$$C_i(p_i^0) = [0, \frac{p_i^0}{\bar{p}}].$$

Designer i strictly prefers to get the patent *iff* the other designer gets it with probability $y_{-i} > \frac{c}{1+c}$. Hence from Theorem 6, if $\frac{c}{1+c} > \frac{p_2^0}{\bar{p}}$ the unique equilibrium outcome of the information design game is $y_1 = y_2 = 0$, entrepreneurs never invest in the new technology. If $\frac{c}{1+c} < \frac{p_2^0}{\bar{p}}$, the information design game has three equilibrium outcomes: $y_1 = y_2 = 0$, $y_1 = y_2 = \frac{c}{1+c}$, and $y_1 = \frac{p_1^0}{\bar{p}}$, $y_2 = \frac{p_2^0}{\bar{p}}$.

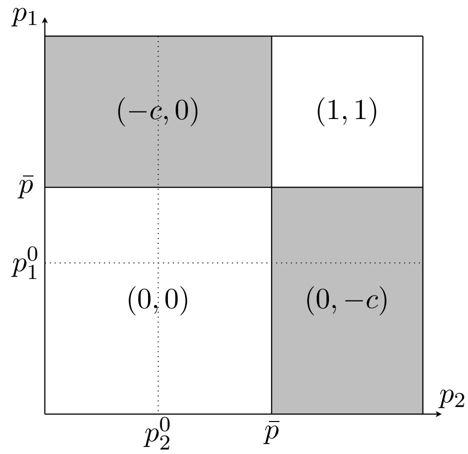

Figure 4: Payoff of the designers in Example 6 as a function of agents' beliefs (p_1, p_2).

5 A Multi-Period Model of Information Design

In this section, we study multi-period information design games which extend simple rectangular corporation games. Consider a rectangular problem as described in Section 4, assume that there is a single agent in each corporation and fix message sets $M = (M_1, \ldots, M_n)$ with $|M_i| \geq |A_i|$ for each i.

In the multi-period game, designers choose experiments simultaneously in each period and observe those choices. In the next period, each designer has the possibility to choose another statistical experiment. This process repeats and experiments pile up as long as at least one designer continues to produce informative experiments. Then, the states are drawn, each agent receives messages from all experiments chosen by his designer and chooses an action. More precisely, the multi-period game, denoted by MRG_M, unfolds as follows.

Statistical experiments and decisions. In each period $t \geq 1$, each designer i chooses both a statistical experiment $x_i^t : \Theta_i \to \Delta(M_i)$ and a decision $d_i^t \in \{C, Q\}$, where C and Q are respectively interpreted as *Continue* and *Quit*. Statistical experiments and decisions are chosen simultaneously in each period and are observed by all designers.

Stopping rule. The game continues until a stopping condition is fulfilled. A first rule for stopping is that when player i chooses to quit at period t, then he commits not to reveal any further information. When all players have chosen to quit, the game terminates. Second, if all players who did not quit yet choose simultaneously a non-informative experiment, the game also terminates.

Denote $NR_i = \{x_i \in \Delta(M_i)^{\Theta_i} : \forall\, \theta_i, \theta_i',\ x_i(\theta_i) = x_i(\theta_i')\}$ the set of non-revealing experiments of designer i and (x_i^t, d_i^t) the experiment and decision of designer i at period t. The stopping rules are the following.

(i) If $d_i^t = Q$, then $d_i^s = Q$ and $x_i^s \in NR_i$ for each $s > t$.

(ii) If $x_i^t \in NR_i$ for each designer i, then $x_i^s \in NR_i$ for each $s > t$ and each $i \in N$.

The game terminates at the first period T such that $x_i^T \in NR_i$ for each designer i.

Outcomes and payoffs. Suppose that the game terminates in period T. Then there is a sequence of statistical experiments $(x_i^t)_{i,t=1,\ldots,T}$. For each i, a state θ_i and a sequence of messages $(m_i^t)_{t=1}^T$ are drawn with probability:

$$p_i^0(\theta_i) \times \prod_{t=1}^{T} x_i^t(m_i^t|\theta_i).$$

Each agent observes the sequence of messages $(m_i^t)_{t\leq T}$ and chooses an action a_i. The payoff of designer i is $\delta^{T-1}u_i(a)$ and the payoff of agent i is $\delta^{T-1}v_i(a_i;\theta_i)$, where $\delta \in [0,1)$ is a discount factor.

If the game never terminates, then all players receives a payoff of 0.

Remark 5. This game resembles dynamic commitments models, in particular the one of Dutta and Ishii (2016). Choosing a statistical experiment is a commitment to reveal some amount of information. In our multi-period version, the designers commit to reveal more and more as time goes by. Similarly to Dutta and Ishii (2016), we allow for the possibility of committing to reveal no further information, that is, to quit the game.

Strategies. We denote by $h^t = (x_i^s, d_i^s)_{i\in N, s\leq t}$ the history of designers up to period t. A terminal history is such that $x_i^t \in NR_i$ for all i. A behavior strategy σ_i for designer i is a mapping from non-terminal histories to distributions of experiments and decisions:

$$\sigma_i(h^t) \in \Delta\Big(\Delta(M_i)^{\Theta_i} \times \{C, Q\}\Big) \text{ for each non-terminal } h^t.$$

At a non-terminal history, each designer who did not quit yet can choose a new experiment and whether to quit or to continue. At a terminal history, there is a chance move disclosing information to the agents who choose actions.

Equilibrium. An equilibrium specifies for each agent an action $\tau_i((x_i^t, m_i^t)_{t\leq T})$ which is optimal given his belief induced by the sequence of experiments and messages. Given a Borel measurable choice τ of actions, the strategy profile of designers form an equilibrium of the multi-period game on experiments induced by τ.

Theorem 7. *The game MRG_M has an equilibrium.*

Proof. The proof is constructive. Consider an equilibrium of the 1-period game RG_M as constructed in Theorem 5 which describes the statistical experiments chosen by the designers and the choice of actions for the agents. Now, let a strategy profile be such that the designers play this equilibrium in period 1 and quit. It is easily seen that it is a Nash equilibrium of MRG_M. Suppose that all designers but i play their one-period equilibrium strategy and quit immediately. The best-response of designer i in the multi-period game is thus the best-response in the one-period game, since other players will not react at future periods.

To make this profile an equilibrium, we define it such that after every public non-terminal history, designers play a Nash equilibrium of the corresponding one-period game and quit immediately. This way, from the same logic as above, the best-response of each designer i in the multi-period subgame is the same as in the one-period subgame.

The task is thus to prove that the one-period game $RG_M(h)$ which follows a history h has a Nash equilibrium. Let us first define this game precisely.

Let $h = (x_i^s, d_i^s)_{i\in N, s\leq t}$ be a non-terminal history and let $I(h) = \{i \in N, d_i^t = C\}$ be the set of *active* designers. In the game $RG_M(h)$, each designer $i \in I(h)$ chooses $x_i^{t+1} \in \Delta(M_i)^{\Theta_i}$, choices being simultaneous. For each $i \in N \setminus I(h)$, let $x_i^{t+1} \in NR_i$. Then, for each $i \in N$, a state θ_i and a sequence of messages $(m_i^s)_{s=1}^{t+1}$ are drawn with probability:

$$p_i^0(\theta_i) \times \prod_{s=1}^{t} x_i^s(m_i^s|\theta_i) \times x_i^{t+1}(m_i^{t+1}|\theta_i).$$

Each agent i observes his sequence of messages and chooses action a_i. The payoff of designer i is $u_i(a)$ and the payoff of agent i is $v_i(a_i;\theta_i)$.

For each i, define the experiment $X_i^h : \Theta_i \to \Delta((M_i)^t)$ given by the sequence $(x_i^s)_{s=1}^t$. That is $X_i^h((m_i^s)_{s=1}^t|\theta_i) = \prod_{s=1}^t x_i^s(m_i^s|\theta_i)$ and consider the induced splitting $\mu_i^h = (\lambda_i(\ell_i); \mathbf{p}_i(\ell_i))_{\ell_i}$ of p_i^0,

$$p_i^0 = \sum_{\ell_i} \lambda_i(\ell_i)\mathbf{p}_i(\ell_i),$$

where $\ell_i = (m_i^s)_{s=1}^t$ is the generic element of $(M_i)^t$. After receiving message ℓ_i, agent i has belief $\mathbf{p}_i(\ell_i)$ and receives information from x_i^{t+1}. It should be noticed that x_i^{t+1} does not depend on the realized value of ℓ_i. Let's characterize the distributions over A_i generated by some $x_i^{t+1} = x_i$ and some optimal strategy of agent i.

Fix i and a belief $\mathbf{p}_i(\ell_i)$. For each experiment $x_i : \Theta_i \to \Delta(M_i)$, agent i solves a decision problem.

His strategy is a mapping $\tau_i^{\ell_i} : M_i \to \Delta(A_i)$ and it is optimal if:

$$\sum_{\theta_i, m_i, a_i} \mathbf{p}_i(\theta_i|\ell_i) x_i(m_i|\theta_i) \tau_i^{\ell_i}(a_i|m_i) v_i(a_i;\theta_i) \geq \sum_{\theta_i, m_i} \mathbf{p}_i(\theta_i|\ell_i) x_i(m_i|\theta_i) v_i(b_i;\theta_i), \; \forall b_i \in A_i. \tag{1}$$

Let then $C_i(\mu_i^h)$ be the set of distributions over A_i given the splitting μ_i^h and obtained by letting x_i and $(\tau_i(\ell_i))_{\ell_i}$ vary. That is, $y_i \in \Delta(A_i)$ belongs to $C_i(\mu_i^h)$ if there exists $x_i : \Theta_i \to \Delta(M_i)$ and $(\tau_i(\ell_i))_{\ell_i}$ such that Equation (1) holds and,

$$y_i(a_i) = \sum_{\ell_i, \theta_i, m_i} \lambda_i(\ell_i) \mathbf{p}_i(\theta_i|\ell_i) x_i(m_i|\theta_i) \tau_i^{\ell_i}(a_i|m_i), \;\; \forall\, a_i.$$

Claim 1. *The set $C_i(\mu_i^h)$ is convex and compact.*

Define $Q_i^{\ell_i}(m_i, a_i|\theta_i) = x_i(m_i|\theta_i)\tau_i^{\ell_i}(a_i|m_i)$ the joint distribution of the message and action conditional on ℓ_i, θ_i. With this change of variables, we see that $y_i \in C_i(\mu_i^h)$ if and only if there exists a family of distributions $Q_i^{\ell_i}(\cdot|\theta_i) \in \Delta(M_i \times A_i)$ such that:

1. $\forall \ell_i, \ell_i'$, $\sum_{a_i} Q_i^{\ell_i}(m_i, a_i|\theta_i) = \sum_{a_i} Q_i^{\ell_i'}(m_i, a_i|\theta_i)$,
2. $\forall \ell_i$, $\forall a_i, b_i$, $\sum_{\theta_i} \mathbf{p}_i(\theta_i|\ell_i) \sum_{m_i, a_i} Q_i^{\ell_i}(m_i, a_i|\theta_i)(v_i(a_i;\theta_i) - v_i(b_i;\theta_i)) \geq 0$,

and for each a_i,

$$y_i(a_i) = \sum_{\ell_i} \lambda_i(\ell_i) \sum_{\theta_i} \mathbf{p}_i(\theta_i|\ell_i) \sum_{m_i} Q_i^{\ell_i}(m_i, a_i|\theta_i).$$

Condition 1 expresses that the probability of m_i conditional on θ_i does not depend on ℓ_i. Condition 2 expresses that the agent randomizes over optimal actions. Now, the set of families $Q_i^{\ell_i}(\cdot|\theta_i)$ satisfying these two conditions is given by linear equalities and weak inequalities, therefore it is convex and compact. The claim follows since y_i is a linear function of $Q_i^{\ell_i}(\cdot|\theta_i)$.

The meaning of this claim is that, for any given history of experiments, the set of distributions of actions induced by choosing an experiment at the next period (and quitting) is convex and compact. Note that the claim also holds if the designer is inactive, *i.e.*, did quit earlier in the history.

As in the previous section, the game $(C_i(\mu_i^h), u_i)_{i \in N}$ is the mixed extension of a finite game, where each player is restricted to a convex and compact subset of mixed strategies. There is an equilibrium by Nash's theorem. This concludes the proof of the theorem ■

A direct implication of the construction is the following.

Corollary 1. *An equilibrium outcome of the one-period game RG_M is an equilibrium outcome of the multi-period game MRG_M.*

Remark 6. Again, some comments are in order.

1. For simplicity, we have assumed that the preferences of the designers depend only on actions, but Theorem 7 extends to state-dependent preferences in the same way as in Remark 4.

2. Still for simplicity, we assume that there is a single agent per corporation. To extend the results to more agents, one should consider all the Bayes Nash equilibria of the game induced on a corporation by a sequence of experiments and allow for public randomization devices within corporations to make the set convex.

3. The splitting games of Section 3 could similarly be extended to multi-period splitting games. With infinite sets of messages, the proof of Theorem 7 can be extended to show the existence of an equilibrium in multi-period splitting games with infinite sets of messages. Indeed, Theorem 3 implies the existence of equilibria in *pure* strategies in SG_∞ and the same reasoning as in Theorem 7 applies.

Prisoners' dilemma for information design We consider a game between two designers which has the same structure as the prisoners' dilemma. States are binary $\Theta_i = \{0,1\}$ and let p_i be the probability that $\theta_i = 1$. The actions sets are $A_i = \{a_i^1, a_i^2, a_i^3\}$. Without specifying it, we assume that the utility function of the agent gives rise to the following optimal actions:

$$A_i(p_i) = \begin{cases} \{a_i^1\} & \text{if } p_i < 1/3 \\ \{a_i^2\} & \text{if } 1/3 < p_i < 2/3 \\ \{a_i^3\} & \text{if } p_i > 2/3. \end{cases}$$

Thus, for each $i = 1,2$, the set of posteriors is identified with the interval $[0,1]$ and is partitioned into three sub-intervals depending on which action is played. Now, we specify the payoffs (u_1, u_2) of the designers: the game given by Figure 5 is a version of the prisoners' dilemma.

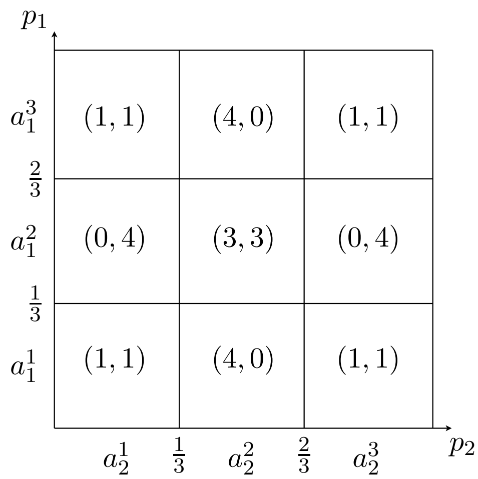

Figure 5: A rectangular 3×3 prisoners' dilemma.

The one-period game has a unique equilibrium outcome given by full revelation. Indeed, inducing the extreme actions a_i^1, a_i^3 strictly dominates inducing the middle action a_i^2. The unique one-period equilibrium payoff is $(1, 1)$.

However, in a Folk Theorem manner, the cooperative payoff $(3, 3)$ is an equilibrium payoff in the multi-period game. The supporting equilibrium is as follows. In the first period, each designer chooses a non-informative experiment and continues. If both players do this, the game ends and the payoff is $(3, 3)$. If one player deviates, the other player fully reveals the state in the next period and quits (he does so after any other history). Thus, deviating yields a payoff less than or equal to 1.

Cooperation is sustained by a trigger strategy. This is possible only because the punishment, *i.e.* the dominant strategy, is *more informative* than the cooperative one. Thus, it is possible to react to the deviation of the opponent. That would not be the case in the alternative version of the prisoners' dilemma represented by Figure 6.

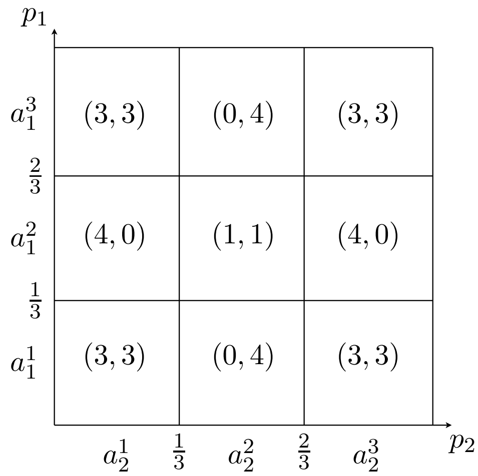

Figure 6: Alternative rectangular 3×3 prisoners' dilemma.

In this game, the dominant strategy is non-revealing, while the cooperative one is fully revealing. The one-period equilibrium outcome is $(1, 1)$ and is also the unique equilibrium outcome of the multi-period game. Each player can play non-revealing in the first period and quit immediately, securing a payoff greater than or equal to 1. Therefore, 1 is a lower bound for equilibrium payoffs. Consider an equilibrium where some player i induces a_i^1 or a_i^3 with positive probability after some history and consider the first period where this happens. At this point, the opponent plays non-revealing and quits, reaping the payoff 4 with positive probability. Thus, player i would get a payoff strictly less than 1, a contradiction.

The logic of these examples can be extended as a simple Folk Theorem result.

Definition 6.

1. *A profile of statistical experiments* $(x_i)_i$ induces *a payoff vector* $\mathbf{u} \in \mathbb{R}^{\mathbf{N}}$ *if there exist optimal continuation strategies for the agents in the one-period game, such that the vector of expected payoffs of the designers in the one-period game given* $(x_i)_i$ *and these continuation strategies, is equal to* $\mathbf{u}$.

2. *A payoff vector* $\mathbf{u}$ *is* more informative *than a payoff vector* $\tilde{\mathbf{u}}$ *if there exist profiles of experiments*

$(x_i)_i$ and $(\tilde{x}_i)_i$ inducing $\mathbf{u}$ and $\tilde{\mathbf{u}}$ respectively such that for each i, x_i is Blackwell more informative than $\tilde{x}_i$.

Experiments can be replaced by splittings: μ_i is more informative than $\tilde{\mu}_i$ if splitting the posteriors of μ_i gives $\tilde{\mu}_i$, *i.e.*, $\tilde{\mu}_i$ is a mean-preserving spread of μ_i.

Theorem 8. *Let $\mathbf{u}, \tilde{\mathbf{u}} \in \mathbb{R}^N$ be two payoff vectors for the designers. Assume that $\tilde{\mathbf{u}}$ Pareto-dominates $\mathbf{u}$, that $\mathbf{u}$ is an equilibrium payoff of the one-period game and that $\mathbf{u}$ is more informative than $\tilde{\mathbf{u}}$. Then $\delta\tilde{\mathbf{u}}$ is an equilibrium payoff of the multi-period game with discount factor δ.*

Notice that, formally, this multi-period game is not a repeated game, since payoffs are obtained in the last period only. Two comments follow. First, Theorem 8 is valid for every fixed discount factor δ, contrary to usual folk theorems in the literature. Second, strict Pareto dominance is not necessary: in our setup, even if some player deviates, he will not get his period payoff during the deviation, but only his punishment payoff in the end.

Proof. The equilibrium is as follows. In the first period players should choose experiments $(\tilde{x}_i)_i$ implementing $\tilde{\mathbf{u}}$ and continue. If they all play these actions in the first period, each player plays non-revealing and continues. If there is a unilateral deviation, players play the one-period equilibrium leading to $\mathbf{u}$ and quit. After every other history, players play an equilibrium of the next one-period game (see the proof of Theorem 7) and quit.

If all players adhere to this profile, $(\tilde{x}_i)_i$ is played in the first period followed by no revelation, the game terminates and $\tilde{\mathbf{u}}$ is implemented. The payoff is thus $\delta\tilde{\mathbf{u}}_i$ for player i. If some player deviates, since the one-period equilibrium is more informative than $(\tilde{x}_i)_i$, it can be implemented by other players. Thus the payoff of the deviating player i is no more than $\delta\mathbf{u_i} \leq \delta\tilde{\mathbf{u}}_\mathbf{i}$. By construction, there is no profitable unilateral deviation at other histories.

Notice that it is important that players *do not* quit on the equilibrium path, the game terminates because they collude on no revelation at the second step. This ensures that each player retains his punishment ability. ■

References

Albrecht, B. C. (2017): "Political Persuasion," *mimeo.*

Aliprantis, C. and K. Border (2006): *Infinite dimensional analysis*, Springer, Berlin.

Angeletos, G.-M. and A. Pavan (2007): "Efficient Use of Information and Social Value of Information," *Econometrica*, 75, 1103–1142.

Au, P. H. and K. Kawai (2017a): "Competitive Disclosure of Correlated Information," *mimeo.*

——— (2017b): "Competitive Information Disclosure by Multiple Senders," *mimeo.*

Aumann, R. J. and M. B. Maschler (1967): "Repeated Games with Incomplete Information: A Survey of Recent Results," Report of the U.S. Arms Control and Disarmament Agency, ST-116, Chapter III, pp. 287–403.

Aumann, R. J., M. B. Maschler, and R. Stearns (1995): *Repeated Games of Incomplete Information*, Cambridge, Massachusetts: MIT Press.

Bade, S., G. Haeringer, and L. Renou (2009): "Bilateral commitment," *Journal of Economic Theory*, 144, 1817–1831.

Bergemann, D. and S. Morris (2016a): "Bayes correlated equilibrium and the comparison of information structures in games," *Theoretical Economics*, 11, 487–522.

——— (2016b): "Information Design, Bayesian Persuasion, and Bayes Correlated Equilibrium," *The American Economic Review*, 106, 586–591.

Boleslavsky, R. and C. Cotton (2015): "Grading standards and education quality," *American Economic Journal: Microeconomics*, 7, 248–279.

——— (2017): "Limited capacity in project selection: Competition through evidence production," *Economic Theory*, forthcoming.

Carlson, H. and E. van Damme (1993): "Global Games and Equilibrium Selection," *Econometrica*, 61, 989–1018.

Dutta, R. and R. Ishii (2016): "Dynamic commitment games, efficiency and coordination," *Journal of Economic Theory*, 163, 699–727.

Eliaz, K. and F. Forges (2015): "Information disclosure to Cournot duopolists," *Economics Letters*.

Eliaz, K. and R. Serrano (2014): "Sending information to interactive receivers playing a generalized prisoners' dilemma," *International Journal of Game Theory*, 43, 245–267.

Eső, P. and B. Szentes (2007): "Optimal information disclosure in auctions and the handicap auction," *The Review of Economic Studies*, 74, 705.

Gentzkow, M. and E. Kamenica (2017): "Competition in persuasion," *The Review of Economic Studies*, 84, 300–322.

Harris, C., P. Reny, and A. Robson (1995): "The existence of subgame-perfect equilibrium in continuous games with almost perfect information: A case for public randomization," *Econometrica: Journal of the Econometric Society*, 507–544.

Jehiel, P. (2014): "On transparency in organizations," *The Review of Economic Studies*, 82, 736–761.

Johnson, J. P. and D. P. Myatt (2006): "On the Simple Economics of Advertising, Marketing, and Product Design," *American Economic Review*, 96, 756–784.

Kamenica, E. and M. Gentzkow (2011): "Bayesian Persuasion," *American Economic Review*, 101, 2590–2615.

Laraki, R. (2001a): "The splitting game and applications," *International Journal of Game Theory*, 30, 359–376.

——— (2001b): "Variational inequalities, system of functional equations and incomplete information repeated games," *SIAM Journal on Control and Optimization*, 40, 516–524.

Laraki, R. and J. Renault (2017): "Acyclic Gambling Games," *mimeo.*

Lewis, T. R. and D. E. M. Sappington (1994): "Supplying Information to Facilitate Price Discrimination," *International Economic Review*, 35, 309–327.

Li, F. and P. Norman (2017a): "On Bayesian persuasion with multiple senders," *mimeo.*

——— (2017b): "Sequential Persuasion," *mimeo.*

Mariotti, T. (2000): "Subgame-perfect equilibrium outcomes in continuous games of almost perfect information1," *Journal of Mathematical Economics*, 34, 99–128.

Mathevet, L., J. Perego, and I. Taneva (2017): "On Information Design in Games," *mimeo.*

Mertens, J.-F., S. Sorin, and S. Zamir (2015): *Repeated games*, Cambridge University Press.

Moriya, F. and T. Yamashita (2017): "Asymmetric information allocation to avoid coordination failure," *mimeo.*

Myerson, R. B. (1982): "Optimal Coordination Mechanisms in Generalized Principal-Agent Problems," *Journal of Mathematical Economics*, 10, 67–81.

Oliu-Barton, M. (2017): "The Splitting Game: Uniform Value and Optimal Strategies," *Dynamic Games and Applications*, 1–23.

Renou, L. (2009): "Commitment games," *Games and Economic Behavior*, 66, 488–505.

Rockafellar, R. (1970): *Convex Analysis*, Princeton University Press.

Rubinstein, A. (1989): "The Electronic Mail Game: Strategic Behavior Under 'Almost Common Knowledge'," *American Economic Review*, 79, 385–391.

Simon, L. K. and W. R. Zame (1990): "Discontinuous games and endogenous sharing rules," *Econometrica*, 861–872.

Sion, M. and P. Wolfe (1957): "On a game without a value," *Contributions to the theory of games*, 3, 299–306.

Sorin, S. (2002): *A First Course on Zero-Sum Repeated Games*, Springer.

Taneva, I. (2016): "Information Design," *mimeo.*