The Creating Subject, the Brouwer-Kripke Schema, and infinite proofs

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Abstract

Kripke's Schema (better the Brouwer-Kripke Schema) and the Kreisel-Troelstra Theory of the Creating Subject were introduced around the same time for the same purpose, that of analysing Brouwer's 'Creating Subject arguments'; other applications have been found since. I first look in detail at a representative choice of Brouwer's arguments. Then I discuss the original use of the Schema and the Theory, their justification from a Brouwerian perspective, and instances of the Schema that can in fact be found in Brouwer's own writings. Finally, I defend the Schema and the Theory against a number of objections that have been made.

Brouwer's views may be wrong or crazy (e.g. self-contradictory), but one will never find out without looking at their more dubious aspects.
Kreisel [98]

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1 Notation

Unless noticed otherwise, the following notation is used:

$\langle a_n \rangle_n$	a sequence of elements each indexed by n
i, j, k, m, n, p, v, w	variables ranging over the natural numbers
f, g, h	variables ranging over functions
$j(x, y)$	a pairing function $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$
r, s, t	variables ranging over real numbers
x, y, z	variables whose range depends on the context
A, B, C	variables ranging over propositions
P, Q, R	variables ranging over predicates
X, Y, Z	variables ranging over species
$\alpha, \beta, \gamma, \xi$	variables ranging over choice sequences
$\bar{\alpha}m$	the initial segment of α $\langle \alpha(1), \alpha(2), \ldots, \alpha(m) \rangle$

In quotations, notation has been left unchanged.

2 Introduction

Can mathematical arguments depend not only on mathematical objects and their properties, but also on the temporal order in which some ideal mathematician establishes theorems about them? Brouwer affirmed this, and it led him to devise a number of reasonings now known as 'Creating Subject arguments'. The first known example occurs in the 1927 Berlin lectures [47].[1] Let $\mathbb{R}$ be understood as the species of intuitionistic real numbers (convergent choice sequences). Brouwer considers the possibility of an order relation on $\mathbb{R}$ based on 'the naive "before" and "after" ' according to which $r < s$ if and only if on the intuitive continuum (seen with the mind's eye), r appears to the left of s.[2]

Weak counterexample 1 ([47, p.31–32])**.** Let $<$ be the naive order relation. Then there is no hope of showing that

$$\forall x \in \mathbb{R} \, \forall y \in \mathbb{R}(x \neq y \to x < y \lor y < x) \tag{1}$$

[1]Brouwer says in the opening of the 1948 paper 'Essentially negative properties' that he had given an example of such reasoning 'now and then in courses and lectures since 1927' [34, p.963; trl. 45, p.478]. Known places are the second Vienna lecture (see further on in the text), the 1933 Groningen lectures, and the 1934 Geneva lectures. The latter two have remained unpublished, but Niekus [127, section 7; 128, section 10] makes some comments on them. There is no Creating Subject argument in the 1932 lecture 'Will, knowledge, and speech' [30].

[2]In the second Vienna lecture, this is called the 'natural order' [27, p.8] and in the Cambridge lectures the 'intuitive order' [46, p.43].

Plausibility argument 2. Let P be a unary predicate and e a mathematical object in its domain, such that at present neither $\neg P(e)$ nor $\neg\neg P(e)$ have become evident.

Define a real number r as a convergent choice sequence of rationals $r(n)$:

- As long as, by the choice of $r(n)$, one has obtained evidence neither of $P(e)$ nor of $\neg P(e)$, $r(n)$ is chosen to be 0.
- If between the choice of $r(m-1)$ and $r(m)$, one has obtained evidence of $P(e)$, $r(n)$ for all $n \geqslant m$ is chosen to be 2^{-m}.
- If between the choice of $r(m-1)$ and $r(m)$, one has obtained evidence of $\neg P(e)$, $r(n)$ for all $n \geqslant m$ is chosen to be -2^{-m}.

Then it is true that $r \neq 0$ but, as long as neither $\neg P(e)$ nor $\neg\neg P(e)$ has become evident, neither $r < 0$ nor $r > 0$ is true. As one may always expect to be able to find such P and e,[3] there is no hope of showing $\forall x \in \mathbb{R}\, \forall y \in \mathbb{R}(x \neq y \to x < y \vee y < x)$.

The justification is a plausibility argument (for the conclusion that the given proposition will never be demonstrated), not a proof, because it depends on an expectation, albeit one that is highly likely to be fulfillable. Thus, the justification contains not only mathematical reasoning, but also a value judgement. This is the defining characteristic of weak counterexamples.[4]

In the Berlin lectures, this weak counterexample serves as a motivation for introducing the notion of *virtual order*, and it is shown that that is the closest one can come to an order on $\mathbb{R}$; we will come back to that in subsections 3.1 and 3.3. For here, the salient feature of the argument is that the choices in the sequence r depend not only on properties of P and e, but also on *the moment* at which a certain proof about them becomes available to the mathematician constructing the sequence. It will be clear that this sequence r is therefore not determined by a law, to the extent that one accepts the idea that the mathematician's activity depends on its free choices in directing its efforts. The role of freedom in intuitionistic mathematics will be discussed further in subsections 7.1 and 7.3.

'The mathematician' here is evidently an idealised one, who in particular is always there to make the n-th choice, however large n may become. Brouwer baptised this idealised mathematician 'het scheppende subject' in Dutch in 1948 [34, p.963] and 'the Creating Subject' in English in 1949 [37, p.1246].[5] This idealisation will be further explained in

[3] See on this point also the remark on the relation between completely open problems and untested propositions on p.70 below.

[4] As the referee emphasised, Brouwer never calls his weak counterexamples 'theorems'.

[5] The basic meaning of 'scheppend(e)', the participial adjective of the verb 'scheppen', here is 'bringing something into existence'. According to the historical dictionary *Woordenboek der Nederlandsche Taal*, the alternative 'creatief' had already been introduced in Dutch when Brouwer chose 'scheppend', but often

subsection 5.3, and an objection to the claim that Brouwer is making it will be discussed in subsection 7.10.

Brouwer had given a weak counterexample with a very similar conclusion a few years before, in 1924:

Weak counterexample 3 ([20, p.3])**.** There is no hope of showing that

$$\forall x \in \mathbb{R}\, \forall y \in \mathbb{R}(x = y \vee x < y \vee y < x) \tag{2}$$

But the way he had argued for it then had been very different.

Plausibility argument 4.

> Let d_ν be the ν-th digit after the decimal point in the decimal expansion of π and $m = k_n$, when in the ongoing decimal expansion of π at d_m it happens for the n-th time that the part $d_m d_{m+1} \dots d_{m+9}$ of this decimal expansion forms a sequence 0123456789. Let furthermore $c_\nu = (-1/2)^{k_1}$ if $\nu \geqslant k_1$, otherwise $c_\nu = (-1/2)^\nu$; then the infinite sequence $c_1, c_2, c_3, \dots$ defines a real number r, for which neither $r = 0$, nor $r > 0$, nor $r < 0$ holds. [20, p.3]

Note that the choices in the sequence c_ν are not defined in terms of the moment at which the Creating Subject comes to know the truth of a certain proposition, as they would in Plausibility argument 2. As the decimal expansion of π is lawlike, and for every ν it is decidable whether $\nu \geqslant k_1$, r is itself lawlike.

When Brouwer wrote this, no value k_1 was known, but in the meantime it has been found that $k_1 = 17{,}387{,}594{,}880$ [7]; so $r > 0$. But as Brouwer remarks in a footnote [20, p.3n4], r can be defined using any other decidable property of natural numbers of which one knows neither that there is an instance, nor that there cannot be one. He came to call such a property a 'fleeing property'.

Definition 5 ([26, p.161; 46, p.6–7])**.** A *fleeing property* is defined by a decidable predicate P on the natural numbers such that at present there is evidence of neither $\exists n P(n)$ nor $\forall n \neg P(n)$. The *critical number* k_P of P is the as yet hypothetical smallest number k such that $P(k)$.

This notion is used in the second of the known Creating Subject arguments, which is made in the second Vienna lecture from 1928. It was published in 1930 and, as observed by Dirk van Dalen [168], that makes it the first occurrence of a Creating Subject argument in print. Its conclusion is that of Weak counterexample 1:

takes on a restricted sense expressing that this bringing about happens in an imaginative, original, or artistic way. The *Oxford English Dictionary* shows the same relation between 'creating' and 'creative'. To reflect the Subject's ontological responsibility in its full generality, then, Brouwer's chosen terms in Dutch and English are more apt than the often seen 'creatief subject' and 'creative subject'.

Plausibility argument 6.

> That the continuum is *not ordered* by the sequence of its elements as derived from intuition [durch die der Anschauung entnommene Reihenfolge ihrer Elemente] is shown by an element p determined by the convergent sequence $c_1, c_2, \ldots$, for which I[6] choose c_1 to be the zero point and every $c_{\nu+1} = c_\nu$ with one exception: As soon as I find a critical number λ_f of a certain fleeing property f, I choose the next c_ν to be equal to $-2^{-\nu-1}$, and as soon as I find a proof of the absurdity of such a critical number, I choose the next c_ν to be equal to $2^{-\nu-1}$. This element p is different from zero, and yet it is neither smaller nor greater than zero. [26, p.7–8; trl. [109, p.59], modified]

Like the 1924 argument, this one depends on a fleeing property, but here, if a critical number is found, it influences the choices in the sequence not through its value but through the moment at which that value was found. Thus, p is not a lawlike real number.

Neither in the Berlin nor in the Vienna lecture Brouwer paused to isolate the notion of Creating Subject, even though it is clearly operative in the arguments, and the audiences will have easily missed this novelty [168, p.308–310; 173, p.516–517]. It is in a series of publications beginning in 1948 that Creating Subject arguments appear in central position.

Wider attention to that series, and indeed to Creating Subject arguments as such, was first drawn by Heyting's discussion in his book *Intuitionism* from 1956 [70]. In the early comments by Heyting himself, Van Dantzig, and Kleene (who had spent January-June 1950 at the University of Amsterdam) the Creating Subject arguments were considered controversial or not mathematical at all [175; 70, title of chapter 8; 81, p.175], because these arguments were considered to let in an empirical or quasi-empirical element. On the other hand, Kripke and Kreisel in the mid-1960s encouraged further discussion of Brouwer's arguments by presenting explicitations in terms of what have become known as Kripke's Schema and the Theory of the Creative Subject. (For reasons given on p.46 and p.32, it is more appropriate to speak of 'the Brouwer-Kripke Schema' and 'the Theory of the Creating Subject'.) And whereas Brouwer had used Creating Subject arguments only in weak and strong counterexamples to classical principles, other uses of the idea have been found, mostly using the Brouwer-Kripke Schema.

The purpose of this paper is to present Brouwer's Creating Subject arguments in their original context, to give a critical survey of the debate of these arguments, and to argue for their intuitionistic correctness via a defense of the intuitionistic validity of the Brouwer-Kripke Schema and the correctness of the Theory of the Creating Subject. Given the historical and philosophical concern with what was, or might reasonably be inferred

[6] Brouwer's use of 'I' here is conventional, and does not mean that this is no Creating Subject argument. See subsections 5.3 and 7.10, and compare footnote 116.

to have been, Brouwer's thinking when introducing and employing the Creating Subject arguments, more recent alternatives to his proofs and notions, or intended outright replacements thereof, are taken to be of additional, but not of primary interest.

3 Three Creating Subject arguments

This section is an exposition of three of Brouwer's post-war Creating Subject arguments that, together, bring out the mathematical and historical aspects relevant to the later debate. Two more of Brouwer's Creating Subject arguments will be discussed in subsection 6.1.

3.1 Ordering relations, testability, judgeability, decidability

In the counterexamples that will be discussed in detail, the following relations and notions are used.

Definition 7 ([27, p.8–9; 37, p.1246n]). Let x, and y be two real numbers given by convergent choice sequences.[7] Define 'x *coincides* with y' by

$$x = y \equiv \forall n \, \exists m \, \forall i (i \geqslant m \rightarrow |x(i) - y(i)| < 2^{-n}) \tag{3}$$

Define 'x is *measurably smaller* than y'[8] by

$$x < \!\!\circ\, y \equiv \exists m \, \exists n \, \forall i (i \geqslant m \rightarrow y(i) - x(i) > 2^{-n}) \tag{4}$$

Correspondingly, $y \circ\!\!> x$ means that y is '*measurably greater*' than x. Further,

$$x \leqslant y \quad \equiv \quad \neg(x \circ\!\!> \gamma) \tag{5}$$

$$x < y \quad \equiv \quad \neg(x \circ\!\!> y) \wedge \neg(x = y) \tag{6}$$

The relation $x < y$ is negative[9] and $x < \!\!\circ\, y$ positive (an existence statement). Brouwer calls $< \!\!\circ$ the 'constructive' [36] or 'natural measurable' order [39] on the continuum, because $x < \!\!\circ\, y$ exactly if, on the intuitive continuum on which a scale has been placed such that 0 is to the left of 1, x is to the left of y. Thus, where it applies, this order mathematically captures the 'naive' order $<$ that we saw in the introduction.

[7] I read Brouwer's characterisations of choice sequences [21, p.245n3; 32, p.323] in such a way that they include that of a lawlike sequence as a limiting case.

[8] The English terminology is Brouwer's own [39, p.3]. In Dutch, Brouwer used 'tastbaar kleiner' [29, lecture 11], literally 'tangibly smaller'. (Vesley [81, p.143] uses '$\geqslant 2^{-n}$'.)

[9] Brouwer [34, p.963] defines 'a simply negative property' as 'the absurdity of a constructive property'; here we have a negative property that is a conjunction of two simple ones.

Brouwer calls $<$ as defined in (6) the ‘virtual’ [e.g. 39] or ‘negative’ order [36] on the continuum. It is different from the order $<$ in the Creating Subject argument in the introduction.[10] Brouwer’s recasting of that argument based on the new definition is discussed in subsection 3.2 below.

From the natural measurable order, Brouwer also defined ‘apartness’:

$$x \# y \equiv x \lessdot \gamma \vee x \gtrdot y \tag{7}$$

where the octothorpe is pronounced ‘is apart from’.[11] We have [22, p.254]

$$\neg(x \# y) \leftrightarrow x = y \tag{8}$$

and thereby

$$x \neq y \leftrightarrow \neg\neg x \# y \tag{9}$$

Properties of $<$ and $\lessdot$ that are immediate from the definitions are

$$x \gtrdot 0 \to x > 0 \tag{10}$$

and also [37, p.1245 and 1248]

$$\neg\neg x > 0 \leftrightarrow x > 0 \tag{11}$$

$$\neg\neg x > 0 \leftrightarrow \neg\neg x \gtrdot 0 \tag{12}$$

and hence

$$x > 0 \leftrightarrow \neg\neg x \gtrdot 0 \tag{13}$$

But, as shown by a Creating Subject argument that will be discussed in subsection 3.2, we do not have

$$x > 0 \to x \gtrdot 0 \tag{14}$$

and therefore neither

$$\neg\neg x \gtrdot 0 \to x \gtrdot 0 \tag{15}$$

This means that $<$ is the weaker order relation. As we will see, Brouwer used that Creating Subject argument also to show that $<$ is not a complete order.

[10] There are more places where the order $<$ is not the virtual one [27, p.8; 29, section 8; 31, lecture 2; 46, p.40-41].

[11] The term ‘apart’ (Dutch ‘verwijderd’, ‘plaatselijk verschillend’ [17, section 2]; German ‘entfernt’ [22, p.254], ‘örtlich verschieden’ [14, p.3]) and the definition are Brouwer’s [37, p.1246], the notation was introduced by Heyting [69, p.20], who earlier had used ω [68, section 6].

Definition 8. A proposition A is *decidable* if a method is known to prove $A \vee \neg A$. Instead of 'decidable', Brouwer wrote 'judgeable' [e.g. 44, p.114]. Testability of A is decidability of $\neg A$: a proposition A is *testable* if a method is known to prove $\neg A \vee \neg\neg A$.[12]

While decidability implies testability, if only a proof of $\neg\neg A$ is known then A has been tested without having been decided. No proposition can be untestable or undecidable in an absolute sense, as both $\neg(\neg A \vee \neg\neg A)$ and $\neg(A \vee \neg A)$ are contradictory.

3.2 An argument from 1948

In 'Essentially negative properties', Brouwer presented

Weak counterexample 9 ([34])**.** There is no hope of showing that

$$\forall x \in \mathbb{R}(x \neq 0 \to x < 0 \vee x > 0) \tag{16}$$

let alone of

$$\forall x \in \mathbb{R}(x \neq 0 \to x \lessdot 0 \vee x \gtrdot 0) \tag{17}$$

with the common notational variant

$$\forall x \in \mathbb{R}(x \neq 0 \to x \# 0) \tag{18}$$

Plausibility argument 10. Let A be a proposition that is at present not testable. The Creating Subject constructs a choice sequence r of rational numbers $r(n)$:

- As long as, by the choice of $r(n)$, the Creating Subject has obtained evidence neither of A nor of $\neg A$, $r(n)$ is chosen to be 0.
- If between the choice of $r(m-1)$ and $r(m)$, the Creating Subject has obtained evidence of A, $r(n)$ for all $n \geqslant m$ is chosen to be 2^{-m}.
- If between the choice of $r(m-1)$ and $r(m)$, the Creating Subject has obtained evidence of $\neg A$, $r(n)$ for all $n \geqslant m$ is chosen to be -2^{-m}.

The choice sequence r converges,[13] hence r is a real number. By its definition,

$$r = 0 \leftrightarrow \neg A \wedge \neg\neg A \tag{19}$$

[12] As a schema, $\neg A \vee \neg\neg A$ has been called 'weak excluded middle'. The intermediate logic obtained by adding it to intuitionistic logic was first studied by Yankov [194].

[13] Let ϵ be given, and determine an n such that $2^{-n} < \epsilon$. Construct the sequence r up to $r(n)$, which can be done as each choice is decidable. If $r(n) = 0$, all further choices will be in the interval $[-2^{-(n+1)}, 2^{-(n+1)}]$ and hence within ϵ from one another. If $r(n) \neq 0$, then the choices in r have already been fixed, and hence within ϵ from one another.

and so the Creating Subject knows that $r \neq 0$. But it also reasons

$$\begin{array}{lr} r < 0 & \text{(assumption)} \\ \neg(r > 0) & \text{(def. } <) \\ \text{There never is evidence of } A & \text{(def. } r) \\ \neg A & \text{(meaning of } \neg) \\ A \text{ has been tested} & \end{array} \tag{20}$$

and, symmetrically,

$$\begin{array}{lr} r > 0 & \text{(assumption)} \\ \neg(r < 0) & \text{(def. } >) \\ \text{There never is evidence of } \neg A & \text{(def. } r) \\ \neg\neg A & \text{(meaning of } \neg) \\ A \text{ has been tested} & \end{array} \tag{21}$$

In both cases, the Creating Subject arrives at a conclusion that contradicts the hypothesis that A cannot yet be tested. Hence, as long as A cannot be tested, $r < 0 \vee r > 0$ cannot be proved, and, by implication, neither can $r \lessdot 0 \vee r \gtrdot 0$.

This reasoning could not be reproduced starting from the hypothesis that A is at present undecidable, as that would allow for A having been tested (the Creating Subject may have a proof of $\neg\neg A$), in which case no contradiction arises. On the other hand, it can be reproduced starting from the hypothesis that A is untestable but replacing $>$ by $\gtrdot$; this strengthens the assumptions towards a contradiction in (20) and (21), but yields a correspondingly weaker result, namely, a weak counterexample to (17), without implying one to (16).

For a weak counterexample only to (17), from either an untestable or an undecidable proposition A, there is an alternative argument (not presented by Brouwer):

Plausibility argument 11. As above, but instead of (20) and (21), the reasoning is

$$\begin{array}{lr} r \lessdot 0 & \text{(assumption)} \\ \exists n(r(n) < 0) & \text{(def. } \lessdot) \\ \text{By the choice of } r(n) \text{, the Creating Subject} & \text{(def. } r) \\ \quad \text{has obtained evidence of } \neg A & \\ A \text{ has been decided (and hence tested)} & \end{array} \tag{22}$$

and

$$
\begin{array}{ll}
r \mathrel{\circ\!\!>} 0 & \text{(assumption)} \\
\exists n(r(n) > 0) & \text{(def. } \mathrel{\circ\!\!>}\text{)} \\
\text{By the choice of } r(n)\text{, the Creating Subject} & \text{(def. } r\text{)} \\
\quad \text{has obtained evidence of } A & \\
A \text{ has been decided (and hence tested)} &
\end{array}
\tag{23}
$$

The reason why this shorter argument does not also provide a weak counterexample to (16) if $\mathrel{<\!\!\circ}$ is replaced by $<$, which would weaken the assumption in (22) and (23) from which to work towards a contradiction, is that although by definition of $<$ we have

$$r < 0 \rightarrow \neg\neg\exists n(r(n) < 0) \tag{24}$$

we do not have

$$r < 0 \rightarrow \exists n(r(n) < 0) \tag{25}$$

To get from the former to the latter, Markov's Principle (MP)[14] should hold for non-recursive sequences. MP comes in several forms [136, section 4; 141; 148, section 4.5]; the one relevant here is

$$\neg\neg\exists n(\alpha(n) = 1) \rightarrow \exists n(\alpha(n) = 1) \tag{26}$$

and was originally formulated and defended for recursive sequences. I will not here go into the question whether for such sequences it is valid.[15] But it would certainly not be correct to apply MP to a sequence such as r. Briefly, the Creating Subject is free to go about its activity, and hence its knowing, from the hypothesis $r < 0$, that it is not the case that there is no stage at which it establishes $\neg A$, does not suffice for knowing a stage at which it does. For more on the non-recursive nature of the Subject's activity, see subsection 7.1.

Brouwer (in effect) once used a form of MP himself, in a paper from 1918; this is discussed below in Appendix A.

In the same paper from 1948 as that in which Weak counterexample 9 appears, Brouwer also observes that a slight modification to the definition of r yields the following weak counterexample:

[14]Markov introduced this principle in lectures 1952–1953 [55, p.44] and called it 'the Leningrad Principle' [111], after the place where he worked before moving to Moscow in 1955, and continued to use it much later [110, p.284n20]. Another name he used was 'the principle of constructive glean' [149, p.137]. Finally, it is also known as 'the Principle of Constructive Choice' [55, p.44]. Philosophical and historical aspects of Markov's engagement with the work of Brouwer and Heyting have been analysed by Vandoulakis [181].

[15]It seems very likely to me that it is; I give a plausibility argument elsewhere [153, section 2].

Weak counterexample 12 ([34])**.** There is no hope of showing that

$$\forall x \in \mathbb{R}(x > 0 \to x \gtrdot 0) \tag{27}$$

Plausibility argument 13. As Plausibility argument 10, modifying the clause in the definition of r for the case that evidence has been obtained of $\neg A$ by changing the number chosen from -2^{-n} to 2^{-n}. By the argument above, $r \neq 0$ and now also $r > 0$, but not, as long as the proposition A has not been tested, $r \gtrdot 0$.

This also presents a weak counterexample to MP ((mis)applied to non-recursive sequences) that predates the introduction of the latter.[16]

As Brouwer mentions at the beginning of 'Essentially negative properties' [34], he was moved to publish these weak counterexamples by Griss' and Van Dantzig's contention that negation is not acceptable in intuitionistic mathematics [65, 66; 174; 160, section 6.2.2]. The philosophical conclusion that Brouwer draws from his counterexamples is that $\neq$ and $>$ are *essentially* negative in the sense that there is no hope of ever proving that they are equivalent to positive relations, for he believes that there will always be untestable propositions. Krivtsov [103, p.167] and also Veldman (in conversation) have pointed out that that conclusion is of a greater generality than is supported by Brouwer's actual arguments. While Brouwer shows that $\neq$ does not coincide with $\#$, nor $>$ with $\gtrdot$, he does not address the question whether there may not be other possibilities. That question, while crucial to Brouwer's discussion with Griss and Van Dantzig, goes beyond that of the acceptability of Brouwer's counterexamples themselves, and will be of no further concern here.

3.3 A refinement by Heyting

Because the Creating Subject does not know in advance by which choice $r(n)$ it will have obtained evidence of A or, as the case may be, of $\neg A$, it can not know in advance any properties of n to know which would require knowing its exact value,. In other words, if the Subject does know such a property of n, it must have decided A already. The definition of a sequence r may be made to depend on such properties.

This idea was used by Heyting in 1956 to argue that we cannot expect to prove that Brouwer's order on the continuum $<$, which is a virtual order, is also a pseudo-order. The notion of a virtual order was introduced by Brouwer in print in 1926 [23, p.453], that of a pseudo-order by Heyting in his dissertation [68, p.8]. These order concepts have in

[16]See footnote 14.

common the axioms for partial order:

$$x \sqsubset y \to \neg(y \sqsubset x) \wedge \neg(x = y) \tag{28}$$
$$x = y \wedge y \sqsubset z \to x \sqsubset z \tag{29}$$
$$x \sqsubset y \wedge y = z \to x \sqsubset z \tag{30}$$
$$x \sqsubset y \wedge y \sqsubset z \to x \sqsubset z \tag{31}$$

A pseudo-order satisfies, in addition,

$$\neg(x \sqsubset y) \wedge \neg(y \sqsubset x) \to x = y \tag{32}$$
$$x \sqsubset y \to \forall z(x \sqsubset z \vee z \sqsubset y) \tag{33}$$

whereas a virtual order satisfies, in addition to (28)–(31) and (32),

$$\neg x \sqsubset y \wedge \neg(x = y) \to y \sqsubset x \tag{34}$$

Both notions are weaker than that of a complete order, which satisfies

$$x = y \vee x \sqsubset y \vee y \sqsubset x \tag{35}$$

As we saw in subsection 3.2, Brouwer had shown that his order on the continuum $<$, which is a virtual order, is weaker than his order on the continuum $<\!\!\circ$, which is an example of Heyting's pseudo-orders. Heyting's argument then makes it plausible that no other choice of pseudo-order would have yielded an equivalence:

Weak counterexample 14 ([70, p.117])**.** There is no hope of showing that Brouwer's virtual order on the continuum is a pseudo-order.[17]

Plausibility argument 15. Let A be a proposition that has not been tested. The Creating Subject constructs two real numbers, r and s:

- As long as, by the choice of $r(n)$, the Creating Subject has not tested A, $r(n)$ is chosen to be 2^{-n}.
- If between the choice of $r(m-1)$ and $r(m)$, the Creating Subject has tested A, $r(n)$ for all $n \geqslant m$ is chosen to be 2^{-m}.

The sequence s depends on r:

[17] In fact, the Continuity Principle (Principle 24 below) can be used to obtain the stronger result that the proposition stating the equivalence of these two orders is absurd. (I thank the referee for pointing this out.)

- As long as, by the choice of $s(n)$, the Creating Subject has not tested A, $s(n)$ is chosen to be 2^{-n}.
- If between the choice of $r(m-1)$ and $r(m)$, the Creating Subject has tested A and m is odd, $s(n)$ for all $n \geqslant m$ is chosen to be 2^{-n}.
- If between the choice of $r(m-1)$ and $r(m)$, the Creating Subject has tested A and m is even, $s(n)$ for all $n \geqslant m$ is chosen to be 2^{-m}.

Then $0 < r$. By definition, $0 < s \to s \neq 0$; and if $s \# 0$, the third clause in the definition of s must have applied, so $s \# 0 \to s = r$, and therefore, by $s < r \to s \neq r$ and (8), $s < r \to s = 0$. Combining the two gives $0 < s \vee s < r \to s \neq 0 \vee s = 0$. Truth of $s \neq 0$ means that it is not possible that A will not be tested between the choices $r(m-1)$ and $r(m)$ for some even m, and truth of $s = 0$ means that A will not be tested between the choices $r(m-1)$ and $r(m)$ for some even m. Either alternative cannot be known before A has, in fact, been tested. On the hypothesis that A has not yet been tested, the Creating Subject knows that $0 < r$, but cannot yet know that $0 < s \vee s < r$. Therefore, axiom (33) does not hold for $<$.

Brouwer proved that any virtual order is an inextensible order (and vice versa), in the sense that any consistent addition to it is already contained in it [47, ch.3; 38].[18] Van Dalen [47, p.12] observes that for Brouwer, what with his drive for generality, this seems to have resulted in a fondness for $<$; that would explain why Brouwer, unlike Heyting, never isolated the notion of pseudo-order.[19] But Heyting and later constructivists have valued the pseudo-order $\lessdot$ because its positivity makes it more practical, and for (notational) simplicity write it as $<$ [70, p.107; 47, p.29n1].

3.4 Drifts and checking numbers

To be able to discuss weak counterexamples in general terms, Brouwer introduces the following definitions [35; 36; 37, p.1246].

Definition 16. A *drift* γ is the union[20] of the real numbers in a converging sequence $\langle c_n(\gamma)\rangle_n$ and the real number $c(\gamma)$ it converges to, where the $c_n(\gamma)$ are all apart from

[18] Martino [114] analyses Brouwer's (changing) argument.

[19] In the 1934 Geneva lectures, Brouwer precedes the definition of $\lessdot$ with the remark 'This will be just an auxiliary relation for the virtual order' [31, lecture 2, trl. MvA].

[20] The term is of course to be taken in Brouwer's species-theoretical sense, not the set-theoretical sense: the union of two species M and N is the species of objects that have either been proved to be an element of the species M, or been proved to be an element of the species N. [13, p.4]. Thinking set-theoretically, one would construe a drift as an ordered pair.

one another and from $c(\gamma)$.[21] The $c_n(\gamma)$ are the *counting numbers* of the drift and $c(\gamma)$ its *kernel*. A counting number $c_n(\gamma) \lessdot c(\gamma)$ is a *left counting number*, and a counting number $c_n(\gamma) \gtrdot c(\gamma)$ a *right counting number*. If all counting numbers of a drift are left counting numbers, the drift is *left-winged*; if all are right counting numbers, *right-winged*. If the sequence $\langle c_n(\gamma)\rangle_n$ is the union of an infinite sequence $\langle l_n(\gamma)\rangle_n$ of left counting numbers and an infinite sequence $\langle d_n(\gamma)\rangle_n$ of right counting numbers, the drift is *two-winged*.

Definition 17. Let A be a proposition and γ a drift. The *direct checking number of γ through A* is the real number $D(\gamma, A)$, constructed as follows.

The Creating Subject constructs a choice sequence $R(\gamma, A)$ of real numbers $c_1(\gamma, A)$, $c_2(\gamma, A), \ldots$:

- As long as, by the choice of $c_n(\gamma, A)$, the Creating Subject has obtained evidence neither of A nor of $\neg A$, $c_n(\gamma, A)$ is chosen to be $c(\gamma)$.

- If $n = 1$ and the Creating Subject has obtained evidence either of A or of $\neg A$ before making the choice of $c_1(\gamma, A)$,
  or $n > 1$ and the Creating Subject has obtained evidence either of A or of $\neg A$ between the choice of $c_{m-1}(\gamma, A)$ and that of $c_m(\gamma, A)$ for some $m \leqslant n$,
  then $c_n(\gamma, A)$ is chosen to be $c_m(\gamma)$.

$D(\gamma, A)$ is the real number to which the sequence $R(\gamma, A)$ converges.[22]

From this definition, it follows that $D(\gamma, A) \neq c(\gamma)$, that $\neg\neg\exists n(D(\gamma, A) = c_n(\gamma))$, and that $\exists n(D(\gamma, A) = c_n(\gamma))$ cannot be proved until A has been decided.

[21] The metaphor Brouwer has in mind is that of a sand drift; 'drift' translates 'aanstuiving' [35, p.1239; 36, p.122] which originally refers to the wind's blowing sand to a given place, thereby forming a dune. The related term 'checking number' in Definition 17 is his translation of 'dempingsgetal', derived from the verb 'dempen', one of whose meanings is 'to check', as in 'to check a sand drift'. (The historical dictionary *Woordenboek der Nederlandsche Taal* gives the example phrase 'Het dempen der zandverstuivingen onder Eerbeek wordt op kleine schaal voortgezet'.) In the inland area where Brouwer lived, sand drifts and the resulting dunes were, and are, well known.

[22] Let ϵ be given. First consider the sequence $c_n(\gamma)$. By hypothesis, it converges to $c(\gamma)$, so a number n_0 can be constructed such that all values from place n_0 onward are within $\epsilon/2$ from one another and from $c(\gamma)$. With this n_0 in hand, turn to the sequence $R(\gamma, A)$ and assume that two values $c_k(\gamma, A)$ and $c_m(\gamma, A)$ have been constructed, $n_0 \leqslant k < m$. Then it is decidable whether, by the time of constructing $c_k(\gamma, A)$, A has been proved. If it has, then $R(\gamma, A)$ has become constant, and $c_k(\gamma, A)$ and $c_m(\gamma, A)$ certainly lie within ϵ from one another. If A has not been proved by the time of constructing $c_k(\gamma, A)$, but it has been proved by the time of constructing $c_m(\gamma, A)$, then the values at places k to m in $R(\gamma, A)$ are all within $\epsilon/2$ from $c(\gamma)$, and hence within ϵ from one another. Finally, if A has not yet been proved by the time of constructing $c_m(\gamma, A)$, then the values at places k to m in $R(\gamma, A)$ are just $c(\gamma)$, and hence within ϵ from one another. So in the sequence $R(\gamma, A)$, all values from place n_0 onward lie within ϵ from one another.

For the discussion of BKS further on, it should be noted that in Definition 17 (and Definitions 18 and 19) Brouwer poses no requirement that A be untestable or undecidable. Indeed, the possibility that A has been decided before the construction of the direct checking number begins is explicitly taken into account, and in a footnote Brouwer mentions that untested A form (implicitly, not the general but) a special case.

Definition 18. Let A be a proposition and γ a drift. The *conditional checking number of* γ *through* A is the real number $C(\gamma, A)$ constructed as follows.

The Creating Subject constructs a choice sequence $Q(\gamma, p)$ of real numbers $c_1(\gamma, A)$, $c_2(\gamma, A), \ldots$:

- As long as, by the choice of $c_n(\gamma, A)$, the Creating Subject has not obtained evidence of A, $c_n(\gamma, A)$ is chosen to be $c(\gamma)$.

- If $n = 1$ and the Creating Subject has obtained evidence of A before making the choice of $c_1(\gamma, A)$,
  or $n > 1$ and the Creating Subject has obtained evidence of A between the choice of $c_{m-1}(\gamma, A)$ and that of $c_m(\gamma, A)$ for some $m \leqslant n$,
  then $c_n(\gamma, A)$ is chosen to be $c_m(\gamma)$.

$C(\gamma, A)$ is the real number to which the sequence $Q(\gamma, A)$ converges.

Definition 19. Let A be a proposition and γ a two-winged drift with $\langle c_n(\gamma)\rangle_n$ the union of $\langle l_n(\gamma)\rangle_n$ and $\langle d_n(\gamma)\rangle_n$. The *two-sided checking number of* γ *through* A is the real number $E(\gamma, A)$ constructed as follows.

The Creating Subject constructs a choice sequence $S(\gamma, A)$ of real numbers $c_1(\gamma, A)$, $c_2(\gamma, A), \ldots$:

- As long as, by the choice of $c_n(\gamma, A)$, the Creating Subject has obtained evidence neither of A nor of $\neg A$, $c_n(\gamma, A)$ is chosen to be $c(\gamma)$.

- If $n = 1$ and the Creating Subject has obtained evidence of A before making the choice of $c_1(\gamma, A)$,
  or $n > 1$ and the Creating Subject has obtained evidence of A between the choice of $c_{m-1}(\gamma, A)$ and that of $c_m(\gamma, A)$ for some $m \leqslant n$,
  then $c_n(\gamma, A)$ is chosen to be $d_m(\gamma)$.

- If $n = 1$ and the Creating Subject has obtained evidence of $\neg A$ before making the choice of $c_1(\gamma, A)$,
  or $n > 1$ and the Creating Subject has obtained evidence of $\neg A$ between the choice of $c_{m-1}(\gamma, A)$ and that of $c_m(\gamma, A)$ for some $m \leqslant n$,
  then $c_n(\gamma, A)$ is chosen to be $l_m(\gamma)$.

$E(\gamma, A)$ is the real number to which the sequence $S(\gamma, A)$ converges.

Plausibility argument 10 in subsection 3.2 can be rephrased in terms of a two-sided checking number, based on a two-winged drift γ with kernel 0 and counting numbers $\langle l_n(\gamma)\rangle_n = \langle -2^{-1}, -2^{-2}, \ldots\rangle$ and $\langle d_n(\gamma)\rangle_n = \langle 2^{-1}, 2^{-2}, \ldots\rangle$ An argument involving a conditional checking number is discussed in the next subsection.

3.5 An argument from 1949

In 1949 Brouwer improved on the 1948 result and devised a strong counterexample:

Theorem 20 ([36])**.**

$$\neg\forall x \in \mathbb{R}(x > 0 \to x \mathrel{\circ\!\!>} 0) \tag{36}$$

Corollary 21. Not stated by Brouwer, but immediate from his definitions:

$$\neg\forall x \in \mathbb{R}(x \neq 0 \to x \# 0) \tag{37}$$

Corollary 22 ([148, p.205–206])**.** Not stated by Brouwer, who never isolated MP:

$$\neg\mathrm{MP} \tag{38}$$

Proof 23 (of Corollary 22)**.** Assume MP. Let $r \in \mathbb{R}$ be an arbitrary real number, given as a choice sequence of rationals, for which $r > 0$. Without loss of generality, we may assume that $\forall n(|r(n) - r| < 2^{-n})$. The assumption $r > 0$ implies $\neg\forall n(|r(n)| < 2^{-n})$ and hence $\neg\neg\exists n(|r(n)|) \geqslant 2^{-n})$. Define α by $\alpha(n) = 0$ if $|r(n)| < 2^{-n}$ and $\alpha(n) = 1$ if $|r(n)| \geqslant 2^{-n}$. Then $\neg\neg\exists n(\alpha(n) = 1)$ and, by MP, $\exists n(\alpha(n) = 1)$. Hence for some n, $|r(n)| \geqslant 2^{-n}$. As, by assumption, $|r(n) - r| < 2^{-n}$, also, for some m, $|r(n) - r| < 2^{-n} - 2^{-m}$, and therefore $r > 2^{-m}$. By arbitrariness of r, we conclude $\forall x \in \mathbb{R}(x > 0 \to x \mathrel{\circ\!\!>} 0)$, which contradicts Theorem 20.

Brouwer's proof of Theorem 20 contains a mistake, as was pointed out by Myhill, who also showed how to repair it. We will look at Brouwer's argument in this subsection, and at Myhill's reaction in the next.

It is not possible to establish Theorem 20 by first proving

$$\exists x \in \mathbb{R} \neg (x > 0 \to x \gtrdot 0) \tag{39}$$

for as $\neg\neg x > 0 \leftrightarrow x > 0$ and $\neg\neg x > 0 \leftrightarrow \neg\neg x \gtrdot 0$, the formula between the brackets is equivalent to $\neg\neg x \gtrdot 0 \to x \gtrdot 0$, which is equivalent to an instance of PEM [22, p.252] which is consistent [11], and so (39) is contradictory.[23] A strong counterexample, if there is one, should therefore involve a property specific to universal quantification over the real numbers. For this, Brouwer had the principle of Weak Continuity for Numbers[24] [13, p.13; 18, p.189; 21, p.253; 24, p.63; 41, p.15] and the Fan Theorem.

Principle 24 (Weak Continuity for Numbers)**.**

$$\forall \alpha \, \exists x \in \mathbb{N} A(\alpha, x) \to \forall \alpha \, \exists m \in \mathbb{N} \, \exists x \in \mathbb{N} \, \forall \beta (\bar{\alpha} m = \bar{\beta} m \to A(\beta, x)) \tag{WC-N}$$

where $\bar{\alpha} m$ stands for the initial segment $\langle \alpha(1), \alpha(2), \ldots, \alpha(m) \rangle$.

Brouwer never gave an explicit justification of WC-N, which he must have had; I refer to [162] and the later [157, ch.7] for discussion, a justification, and further references.

WC-N can be strengthened to

Principle 25 (Continuity for Numbers)**.**

$$\forall \alpha \, \exists x \in \mathbb{N} A(\alpha, x) \to \exists F \in K_0 \, \forall \alpha A(\alpha, F(\alpha)) \tag{C-N}$$

An informal justification runs as follows.[25] The quantifier combination $\forall \alpha \, \exists x$ entails that x can be constructed from α by a method, whose existence can be made explicit. By WC-N, for a given α a value for x can be constructed from an initial segment of α. It can be assumed that the method will assign the same x to extensions of that segment, as additional information should not change its outcome; and the constructivity of the method should entail that it is decidable whether a given initial segment is long enough for the method to produce its output. The methods for which these assumptions hold can be represented by a class of continuous functionals K_0 (which can be defined inductively).

[23] Likewise, $\exists x \in \mathbb{R} \neg (x \neq 0 \to x \# 0)$ is contradictory, because of (9) and the consistency of $\neg\neg \beta \# \gamma \to \beta \# \gamma$.

[24] Thus named by Troelstra and Van Dalen [148, p.208–209]; in Brouwer's writings it remained nameless.

[25] For details, see Kleene and Vesley [81, p.71–73], whose term is 'Brouwer's Principle for Numbers', and Troelstra and Van Dalen [148, p.211–212], who coined the term '(strong) continuity for numbers'. One also finds the principle referred to as '$\forall \alpha \, \exists x$-continuity' (e.g., by Myhill in 98, p.175), which is confusing (to some) because that is the quantifier combination in the antecedent of WC-N as well. But C-N may be held to lay greater claim to the term, because it gives fuller expression to the intuitionistic meaning of that combination.

Alternatively, C-N can be proved from WC-N, monotone bar induction, and AC-NF [56, p.65–66].[26]

WC-N is used in Brouwer's proofs of the Fan Theorem from 1927.[27]

Definition 26 ([16, p.4; 40, p.143])**.** A *fan* is a finitely branching tree. An infinite path through a fan is called an *element* of it.

When considered as a spread, the unit continuum [0,1] is itself constructed as an infinitely branching tree, the paths through which are choice sequences. The following theorem shows that it can in a sense be represented by a fan:

Theorem 27 ([18, p.192])**.** There is a fan J that coincides with the unit continuum $[0,1]$ in that every real number $r \in J$ coincides with a real number $s \in [0,1]$, and every $s \in [0,1]$ coincides with a real number $r \in J$.

Theorem 28 (Fan Theorem [18, p.192; 24, p.66])**.** Let F be a method that assigns to every element e of a fan a (not necessarily unique) number $F(e) \in \mathbb{N}$. Then there exists an $m \in \mathbb{N}$, dependent on F but not on e, such that $F(e)$ depends on only the first m chosen values in e.[28]

The conclusion of the Fan Theorem should be seen not only in light of the contrast between the presence and absence of a (uniform) bound on the determination of $F(e)$, but also of the contrast between a determination of $F(e)$ from only values in e and a

[26] Thanks to Joan Moschovakis for reminding me of this fact and for the reference to Dummett. See also [95, p.333–334, 5.6.3(ii)]. For AC-NF, see Principle 32 below. For a statement of monotone bar induction, see, e.g., [56, p.63]. The intuitions that justify any of the forms of bar induction are not simpler than those appealed to in the intuitive justification of C-N from WC-N.

[27] In 'Über Definitionsbereiche von Funktionen' from 1927 [24], Brouwer presents two proofs, in both of which the Fan Theorem is a corollary of the Bar Theorem. A *bar* is a set of nodes in a tree such that every path through the tree intersects it. The Bar Theorem states that if a tree contains a bar, then it contains a bar that admits of a well-ordering. One proof of the Bar Theorem is based on a general induction principle indicated in footnote 7 of Brouwer's paper, the other, in section 2 of the main text, on the insight that proofs of the hypothesis of the Bar Theorem, when considered as mental objects, can be put into a canonical form. In both the bar is defined by an application of WC-N. Brouwer should have included that in the statement of the theorem (or decidability, uniqueness, or monotonicity of the bar); for otherwise, as Kleene has shown, the theorem is false [81, p.87–88]. (On this account, in Brouwer's presentation of the Bar Theorem in 1954 [41], where the bars are not taken to arise by applications of WC-N, there is a gap.) Kleene gives various correct formulations [81, p.54–55]. There is ample discussion of Brouwer's argument for the Bar Theorem based on canonical proofs [129; 115; 57, section 3.2; 156, ch.4; 186].

[28] Kleene [81, p.59] observes that classically this theorem is false, because of its dependence on WC-N. He states a version that is also classically true: if there is a decidable bar in a fan, then there is a uniform bound on the depth of the paths to the bar (contraposition of König's Lemma, which itself is intuitionistically incorrect). Current discussion of the Fan Theorem in non-intuitionistic constructivism is concerned with variants of this latter version [56, section 3.2; 8, section 4.1; 187].

determination of $F(e)$ from, or also from, restrictions that the Creating Subject may have imposed on how these values are chosen.

Proof 29 (of Theorem 20)**.** This is Brouwer's incorrect proof of 1949; in this presentation we will use mostly Brouwer's own notation. Let γ be a drift with kernel 0 and counting numbers $2^{-1}, 2^{-2}, 2^{-3}, \ldots$, and let f be an arbitrary point of the fan J of Theorem 27. A general condition imposed on the choices in f is that f_n must be chosen after the choice of $c_n(\gamma)$ but before that of $c_{n+1}(\gamma)$. This is to ensure that the Creating Subject continues the construction of the direct checking number as it continues constructing f;[29] it also ensures that the choice of $c_{n+1}(\gamma)$ is always informed by an initial segment of f of length exactly n, which means that, in appropriate cases, from a hypothesis about $c_{n+1}(\gamma)$ we may infer a property of that initial segment.

Let α_f be the proposition $f \in \mathbb{Q}$, ρ the species of the sequences $R(\gamma, \alpha_f)$ and δ the species of the direct checking numbers of γ through α_f, $D(\gamma, \alpha_f)$.

Assume, towards a contradiction, that $x > y \leftrightarrow x \circ\!\!> y$.

It follows from $\forall f(\neg(\neg\alpha_f \wedge \neg\neg\alpha_f))$ and the definition of a direct checking number that $\forall f(D(\gamma, \alpha_f) > 0)$. By the assumption, then also $\forall f(D(\gamma, \alpha_f) \circ\!\!> 0)$, i.e., $\forall e \in \delta\, \exists \nu \in \mathbb{N}(e \circ\!\!> 2^{-\nu})$. Call such a ν depending on e, which will not be unique, $n(e)$. It follows that for each f, $D(\gamma, \alpha_f) \in \mathbb{Q}$ can be proved before $n(e)$ choices have been made in $D(\gamma, \alpha_f)$, hence the proposition $f \in \mathbb{Q}$ can be decided before $n(e)$ choices have been made in $R(\gamma, \alpha_f)$, and hence, by the general condition on the choices in f, the proposition $f \in \mathbb{Q}$ can be decided before $n(e)$ choices have been made in f.

This situation is reflected in the fan J. From every $p = R(\gamma, \alpha_f) \in \rho$ a real number $D(\gamma, \alpha_f) \in [0,1]$ can be obtained; by Theorem 27, an element $f(p) \in J$ can then be found that coincides with the latter. By the above, the proposition $D(\gamma, \alpha_f) \in \mathbb{Q}$ can be proved before $n(e)$ choices have been made in p, so there correspondingly is an $n(p) \in \mathbb{N}$ such that the proposition $f(p) \in \mathbb{Q}$ can be proved before $n(p)$ choices have been made in $f(p)$. (While coinciding, p and $f(p)$ may converge in different manners, so that if for some predicate P one can prove $P(p)$ as soon as $\langle p_1, \ldots, p_k \rangle$ have been chosen, the number of choices in $f(p)$ needed to prove $P(f(p))$ may be different.) The number $n(p)$ depends, through $n(e)$, on f, and the proposition $f(p) \in \mathbb{Q}$ is equivalent to $f \in \mathbb{Q} \vee f \notin \mathbb{Q}$. Thus, for each $f \in J$, an $n(f) \in \mathbb{N}$ can be found such that the proposition $f \in \mathbb{Q} \vee f \notin \mathbb{Q}$ can be proved before $n(f)$ choices have been made in f.

By the Fan Theorem, the species $\{n(f) \mid f \in J\}$ has a maximum element m. Among the sequences admitted to the fan J are sequences g which up to choice m have been chosen freely, subject to the general restriction on elements of J and no others. Therefore, $g \in \mathbb{Q} \vee g \notin \mathbb{Q}$ cannot be proved by stage m. But the conclusion of the previous paragraph,

[29] Without such a condition the Subject would not be obliged to do so, on account of its creative freedom.

which depends on the assumption that $x > y \rightarrow x \gtrdot y$, implies that it can, and hence this assumption has led to a contradiction.

3.6 Myhill's objection to, and repair of, the argument from 1949

In July 1966, John Myhill wrote a letter to Brouwer claiming that the appeal to the Fan Theorem in Proof 29 is not correct [172, p.465–466].[30] No (draft) reply from Brouwer is known.[31] Myhill remarked on the problem in print in 1967 [123, p.296].

> In your version, if I understand it correctly, the proof runs as follows. For each real number $\alpha \in [0,1]$, the real number $\phi(\alpha)$ is defined as follows: as long as the creating subject has not judged the proposition 'α is rational', let $\phi(\alpha)(n) = 1/2^n$; if at the kth step (after k choices for α) he decides that α is rational or irrational, let $\phi(\alpha)(k+q) = 1/2^k$ for all q. Then $\phi(\alpha)$ cannot be 0, for if it were α could be neither rational nor not rational. All this is quite clear. The difficulty lies in the second half of the proof.
>
> [...]
>
> It is the application of the fan theorem which I question here. As I understood it, the fan theorem applies only to those cases in which to every free choice sequence α belonging to the finitary spread F we can assign a natural number n_α *using only the values* $\alpha(0), \alpha(1), \alpha(2), \ldots$. The proof of the fan theorem, it seems to me, depends essentially on this condition (which is met in the usual mathematical cases: for instance in the theorem, that I used above, that [0,1] has no detachable subspecies.) But it is not met in the situation to which you apply the fan-theorem here, because in computing the n from the α one is allowed to use also the values of $\phi(\alpha)$, which may depend not only on α but also on what restrictions have been placed on α, and on what properties of $\phi(\alpha)$ the creating subject may have inferred from these.

In terms of Brouwer's proof, the objection amounts to this. In the construction of $n(f)$ from $f(p)$, and hence from f, the Creating Subject may use, besides the values in f, the values in $D(\gamma, \alpha_f)$, and the latter may depend on restrictions that it has imposed on f; whereas application of the Fan Theorem would require that $n(f)$ can be calculated from only values in $f(p)$, and hence from only values in f.

To my mind, Myhill is right about this. Under the assumption of $x > y \leftrightarrow x \gtrdot y$, the inference from $\forall f(D(\gamma, \alpha_f) > 0)$ to $\forall f(D(\gamma, \alpha_f) \gtrdot 0)$, that is, to the existence of a method

[30] This objection was not yet made in Myhill's contribution to the discussion printed after Kreisel's seminal paper [98]; the preface to the volume in which it appeared [105] is dated July 1966 as well.

[31] Brouwer in his note to that letter errs in taking it to refer to 'Points and spaces', which contains the argument discussed in subsection 3.7 below, but not one using the Fan Theorem.

that transforms any f into a proof of $D(\gamma, \alpha_f) \gg 0$, entails no condition on the information about f that this method may depend on to arrive at that proof; but it is from that proof that, via $n(e)$ and $n(p)$, $n(f)$ is constructed. On the other hand, use of the Fan Theorem does impose the condition Myhill states, because it is required by WC-N, on which the proof of that theorem depends.

Brouwer and Myhill both appeal to the impossibility of deciding the question of rationality of a real number from a freely chosen initial segment of a given length without restrictions; but Brouwer in his reductio argument applies the Fan Theorem to steer toward it, whereas, as Myhill observes, it should have kept him from applying the Fan Theorem in the first place.[32] (This situation may arise, more generally, whenever an application of a theorem overlooks a condition on the possibility to do so.)

In his letter to Brouwer, Myhill goes on to propose a repair. Suppose that for every $f \in J$ the proposition $f \in \mathbb{Q}$ could be decided before $n(e)$ choices have been made in it:

> Now in my formalism this is immediately contradictory, because it implies that the species of all real numbers in [0,1] would be split up into the rational[s] and the irrationals, q.e.d.

The letter unfortunately does not present that formalism itself. For the contradiction, Brouwer's Negative Continuity Theorem (see Appendix B) would suffice, but it is likelier that Myhill was thinking of Brouwer's theorem that fully defined functions on [0,1] are uniformly continuous [24]. That theorem is proved from the Bar Theorem, by way of deriving the Fan Theorem and then applying the latter in a way that respects the condition Myhill refers to in his letter. In 1963 Kreisel had proved the Bar Theorem from three continuity axioms [85, p.IV-20]; Myhill preferred that proof to Brouwer's own two[33], and included these axioms in his published formalism of 1968 [124, p.166–167].

3.7 An argument from 1954

The occasion for the following counterexample, from 'Points and spaces' [41, p.4], is a discussion of the Principle of the Excluded Middle. In constructive mathematics any open problem is a weak counterexample to PEM, but this argument shows that, as long as there

[32] Also Heyting's version of Brouwer's proof [70, p.117-118] would be affected by Myhill's criticism, as Myhill indicates [124, p.175]. Note that in the third edition of Heyting's book this matter is not brought up [74, p.121–122], but Heyting in his note 4 to the reprint of Brouwer's paper in the *Collected Works* [45, p.603] in effect concurs with Myhill's objection, without a reference, although Myhill's papers are included in the bibliography, and without suggesting a way to save the theorem. The remainder of Heyting's note there contains a related remark from Brouwer, part of which will be discussed, in a different context, in subsection 7.9 below.

[33] See footnote 27 above.

are open problems at all, in whatever domain of mathematics, a garden-variety instance of quantified PEM is not valid either:

Weak counterexample 30 ([41, p.4])**.** There is no hope of showing that

$$\forall x \in \mathbb{R}(x \in \mathbb{Q} \vee x \notin \mathbb{Q}) \tag{40}$$

Brouwer's theorem from 1927 that all full functions on the unit continuum are uniformly continuous entails the stronger result that that statement is contradictory, but uses the Bar Theorem as a lemma. Veldman has shown that such machinery is not needed for the weaker theorem that all full functions on the unit continuum are continuous [183]. Brouwer did in his paper also establish, without the Bar Theorem, the again weaker Negative Continuity Theorem: Every full function on the unit continuum is negatively continuous [24, p.62]. That would still yield this weak counterexample. Some details of the Negative Continuity Theorem are presented in Appendix B, so as to make some remarks on the resemblance between Brouwer's proof of it and his 1954 plausibility argument for Weak counterexample (30); the former may well have inspired the latter, which runs as follows.

Plausibility argument 31. Let A be a proposition that is at present untestable, and γ a drift with rational counting numbers $\langle c_n(\gamma)\rangle_n$ and an irrational kernel $c(\gamma)$. Then truth of A and rationality of the conditional checking number $C(\gamma, A)$ are equivalent: the choices in $C(\gamma, A)$ become a fixed rational as soon as a construction for A has been found, and only then. So as long as the proposition A is not testable, the proposition $C(\gamma, A) \in \mathbb{Q}$ is not testable, and, as decidability implies testability, the proposition $C(\gamma, A) \in \mathbb{Q} \vee C(\gamma, A) \notin \mathbb{Q}$ is not provable.

For the discussion of BKS^+ it will be not the counterexample itself that is important, but that the observation that $A \leftrightarrow C(\gamma, A) \in \mathbb{Q}$ is also made in Brouwer's own text (see subsection 6.2 for the relevance of this).

4 BKS and CS

It is characteristic of the philosophical instinct of Kripke, Kreisel, and Myhill that they decided to take Brouwer's arguments at face value and analyse them. In the remark that is the motto for the present paper, Kreisel says that 'Brouwer's views may be wrong or crazy (e.g. self-contradictory), but one will never find out without looking at their more dubious aspects' [98, p.159]. I read this with the emphasis on 'looking'. The attitude Kreisel is countering here is that of Heyting and Kleene, described on p.6 above. This is also seen in Kreisel's review of Kleene and Vesley's *Foundations of Intuitionistic Mathematics*: 'Chapter IV is handicapped by the author's obvious wish to avoid the dubious notions used in Brouwer's refutation of $*$ $[= \forall\alpha(\neg\neg\exists x(\alpha x = 0) \to \exists x(\alpha x = 0))]$ instead of studying them' [87, p.261]. The next two subsections discuss the respective analyses of Brouwer's Creating Subject arguments by Kripke and Kreisel.

4.1 BKS

The schema Kripke introduced around 1965 to reconstruct Brouwer's Creating Subject arguments is

$$\exists\alpha \begin{bmatrix} \forall n(\alpha(n) = 0 \vee \alpha(n) = 1) \\ \wedge \\ \forall n(\alpha(n) = 0) \leftrightarrow \neg A \\ \wedge \\ \exists n(\alpha(n) = 1) \to A \end{bmatrix} \tag{BKS$^-$}$$

There is also a strong version, BKS$^+$ (see below).[34] The traditional tag for the schema has 'KS' instead of 'BKS'; but, as will be explained in section 6, the latter, for 'Brouwer-Kripke Schema' is more appropriate.

Thus formulated, a witness for $\exists n(\alpha(n) = 1)$ need not be unique. But it is means no loss of generality to assume that it is: if α possibly contains more than one 1, then α^* contains at most one, while preserving the properties guaranteed by BKS$^-$:

$$\alpha^*(n) = \begin{cases} \alpha(n) & \text{if } \forall m < n(\alpha(m) = 0) \\ 0 & \text{otherwise} \end{cases} \tag{41}$$

Since the assumption of uniqueness seems not often needed, I will not treat it as the default, and instead be explicit when invoking it. If BKS is understood as formulated by Kripke in his Amsterdam lecture of 2016 (see subsection 7.4 below), then it will not be unique, and the same holds for Myhill's formulation [122, p.151].

Kripke did not publish the schema, but it circulated widely [121, p.336]. In particular, Kripke stated it in a letter to Kreisel [101, footnote 8], who went on to introduce the alternative CS, which will be discussed in subsection 4.2. The first references to BKS in print are made in 1967, by Kreisel [98, p.174] and Myhill [123, p. 295].

[34] Note that the notation BKS, BKS$^+$, BKS$^-$ does not correspond to KS, KS+, KS- as used by Dragálin [55, p.132]: his KS is BKS$^-$, his KS+ is BKS$^+$, his KS- is only the bi-implication in BKS$^-$. I use BKS as a general term.

As a special case of BKS$^-$, one obtains for species X [89, p.128]:

$$\forall x\,\exists\alpha\begin{bmatrix}\forall n(\alpha(n)=0\vee\alpha(n)=1)\\ \wedge\\ \forall n(\alpha(n)=0)\leftrightarrow x\notin X\\ \wedge\\ \exists n(\alpha(n)=1)\rightarrow x\in X\end{bmatrix}$$

The quantifier order can be reversed, by coding the sequences obtained for the various values of x into one. This can be done because the infinitely many infinite sequences $\langle\alpha(x)\rangle_k$ may be constructed step by step in a zigzag manner: $\alpha(0)(0), \alpha(1)(0), \alpha(0)(1), \alpha(2)(0), \ldots$. Using the pairing function j to code 'the k-th value of the x-th sequence' into one number, define the sequence β by $\beta(j(x,k)) = \alpha(x)(k)$. This yields

$$\exists\beta\,\forall x\begin{bmatrix}\forall n(\beta(n)=0\vee\beta(n)=1)\\ \wedge\\ \forall n(\beta(j(x,n))=0)\leftrightarrow x\notin X\\ \wedge\\ \exists n(\beta(j(x,n))=1)\rightarrow x\in X\end{bmatrix}\tag{BKS$^-$S}$$

As the way β is constructed from the sequences $\alpha(x)$ does obviously not depend on the exact values that the latter may take, one concludes to the correctness of a more general choice principle [e.g. 81, p.14;

Principle 32.

$$\forall n\,\exists\alpha A(n,\alpha)\rightarrow\exists\beta\,\forall n A(n,(\beta)_n)\tag{AC-NF}$$

with $(\beta)_n := \lambda m.\beta(j(n,m))$.

For inhabited species X, one derives from BKS$^-$S

$$\exists f\,\forall x\begin{bmatrix}\neg\,\exists n(f(n)=x)\rightarrow x\notin X\\ \wedge\\ \exists n(f(n)=x)\rightarrow x\in X\end{bmatrix}\tag{BKS$^-$SF}$$

by putting $f(j(x,k)) = a$ if $\beta(j(x,k)) = 0$, and $f(j(x,k)) = x$ if $\beta(j(x,k)) = 1$. BKS$^-$SF was first published (as the general weak version of BKS) by Kreisel [89, p.128], who does not derive it from BKS$^-$S but remarks that it follows from CS; see Theorem 36 and its proof below.

As an example of the use of BKS$^-$, here is an alternative to Plausibility argument 10 for Weak counterexample 9.

Plausibility argument 33. (In the style of [75, p.245]) Let A be a proposition that is at present not testable. Applying BKS$^-$ to A and to $\neg A$ gives

$$\exists\alpha[\forall n(\alpha(n) = 0) \leftrightarrow \neg A) \wedge \exists n(\alpha(n) = 1) \rightarrow A] \tag{42}$$
$$\exists\beta[\forall n(\beta(n) = 0) \leftrightarrow \neg\neg A) \wedge \exists n(\beta(n) = 1) \rightarrow \neg A] \tag{43}$$

Define the real numbers r

$$r(n) = \begin{cases} 0 & \text{if } \forall m \leqslant n(\alpha(m) = 0) \\ 2^{-k} & \text{if } k \leqslant n \wedge \forall m < k(\alpha(m) = 0) \wedge \alpha(k) = 1 \end{cases} \tag{44}$$

and s

$$s(n) = \begin{cases} 0 & \text{if } \forall m \leqslant n(\beta(m) = 0) \\ -2^{-k} & \text{if } k \leqslant n \wedge \forall m < k(\beta(m) = 0) \wedge \beta(k) = 1 \end{cases} \tag{45}$$

Finally, define the real number t by

$$t(n) = r(n) + s(n) \tag{46}$$

By (42) and (43),

$$\exists n(\alpha(n) = 1) \wedge \exists n(\beta(n) = 1) \rightarrow A \wedge \neg A \tag{47}$$

hence

$$\exists n(\alpha(n) = 1) \rightarrow \forall n(\beta(n) = 0) \rightarrow t > 0 \tag{48}$$
$$\exists n(\beta(n) = 1) \rightarrow \forall n(\alpha(n) = 0) \rightarrow t < 0 \tag{49}$$

and, by contraposing both and using $t = 0 \rightarrow \neg(t > 0) \wedge \neg(t < 0)$,

$$t = 0 \rightarrow \forall n(\alpha(n) = 0) \wedge \forall n(\beta(n) = 0) \tag{50}$$

In turn, by (42) and (43),

$$\forall n(\alpha(n) = 0) \wedge \forall n(\beta(n) = 0) \rightarrow \neg A \wedge \neg\neg A \tag{51}$$

so

$$t \neq 0 \tag{52}$$

By def. $<$, (49) and then (43),

$$t < 0 \rightarrow \neg(t > 0) \rightarrow \neg\forall n(\beta(n) = 0) \rightarrow \neg\neg\neg A \rightarrow \neg A \tag{53}$$

and similarly, from (48) and then (42),

$$t > 0 \to \neg(t < 0) \to \neg\,\forall n(\alpha(n) = 0) \to \neg\neg A \tag{54}$$

which together yield

$$t < 0 \vee t > 0 \to \neg A \vee \neg\neg A \tag{55}$$

Hence, under the hypothesis that A cannot be tested, $t < 0 \vee t > 0$ cannot be proved, and, by implication, neither can $t \lessdot 0 \vee t \gtrdot 0$.

A reconstruction of Argument 11, yielding a weak counterexample only to $t \lessdot 0 \vee t \gtrdot 0$, would proceed in the same way, but, whereas in (53) we had to make the step $t < 0 \to \neg(t > 0)$, we can now, at the corresponding place, reason that $t \lessdot 0 \to \exists n(\alpha(n) = 1)$; and in the case of $\gtrdot$ this yields a stronger conclusion. So instead we have

$$t \lessdot 0 \to \exists n(\beta(n) = 1) \to \neg A \tag{56}$$

and

$$t \gtrdot 0 \to \exists n(\alpha(n) = 1) \to A \tag{57}$$

which together yield

$$t \lessdot 0 \vee t \gtrdot 0 \to A \vee \neg A \to \neg A \vee \neg\neg A \tag{58}$$

Hence, under the hypothesis that A cannot be tested (or not be decided), $t \gtrdot 0 \vee t \lessdot 0$ cannot be proved.

Reconstructions using BKS$^-$ of other Creating Subject arguments as given by Heyting in his *Intuitionism* [70] (and thus including, of Brouwer's arguments discussed above, the one from 1949 but not that from 1954) were carried out by Hull in 1969 [75]. Hull also presents an alternative argument for Heyting's claim that the virtual order of the continuum cannot be expected to be a pseudo-order (subsection 3.3 above). While Hull appeals to BKS$^-$ and continuity [75, p.244], he indicates that Heyting's argument can be reconstructed using only this variant of BKS$^-$ with an additional clause [75, p.246]:

$$\exists\alpha \left[\begin{array}{c} \forall n(\alpha(n) = 0 \vee \alpha(n) = 1) \\ \wedge \\ \forall x(\alpha(n) = 0) \leftrightarrow \neg A \\ \wedge \\ \exists x(\alpha(n) = 1) \to A \\ \wedge \\ \forall x\,\exists y(t(y) > x) \to (\forall x(\alpha(t(x)) = 0) \to A \vee \neg A) \end{array}\right] \tag{BKS$^-$W}$$

where t is an arbitrary term with infinite range (the formalism had no variables for functions). The new clause is an expression of the idea that, since the Creating Subject does not know beforehand where, if anywhere, a 1 will occur in α, knowing that A will not be implied by the presence of a 1 in α at a position of the form $t(x)$ is only possible if A has been decided already. The informal explanation Hull provides is that 'the only way we could know something will not be proved on a Wednesday is that either it has already been proved on some other day, or that its negation has already been proved' [75, p.246]; this also explains my choice of the tag.[35] Now, even if some proposition has already been proved on some other day, the Creating Subject surely has the freedom to prove it again on Wednesday; what is meant here clearly is that the first time the Subject proves A is not a Wednesday. For the additional clause to reflect that idea accurately, it must be assumed that α takes the value 1 at most once. By the remark on p.24 above, this assumption can be made safely.

Hull does not go on to give a reconstruction of Heyting's argument with $\mathrm{BKS}^{-}\mathrm{W}$, but one is as follows.

Plausibility argument 34 (for Weak counterexample 14)**.** Let A be a proposition that is not yet testable. Applying $\mathrm{BKS}^{-}\mathrm{W}$ to A and $t(y) = 2y$, together with the fact that $\forall x\, \exists y(2y > x)$ is true in the domain of the natural numbers, gives

$$\exists\alpha \begin{bmatrix} \forall n(\alpha(n) = 0 \vee \alpha(n) = 1) \\ \wedge \\ \forall n(\alpha(n) = 0) \leftrightarrow \neg A \\ \wedge \\ \exists n(\alpha(n) = 1) \rightarrow A \\ \wedge \\ \forall n(\alpha(2n) = 0) \rightarrow A \vee \neg A \end{bmatrix} \tag{59}$$

[35] Hull's advisor Myhill generalised $\mathrm{BKS}^{-}\mathrm{W}$ by using a decidable subspecies of choice sequences instead of just a term containing x, and called it the 'Never-on-Sunday schema' [122, p.158].

and, similarly,

$$\exists\beta\begin{bmatrix}\forall n(\beta(n)=0\vee\beta(n)=1)\\ \wedge\\ \forall n(\beta(n)=0)\leftrightarrow\neg A\\ \wedge\\ \exists n(\beta(n)=1)\to A\\ \wedge\\ \forall n(\beta(2n+1)=0)\to A\vee\neg A\end{bmatrix} \tag{60}$$

As mentioned (p.24), we may assume that α and β take the value 1 at most once.

Define the real numbers r and s by

$$r(n)=\begin{cases}2^{-n} & \text{if } \forall m\leqslant n(\alpha(m)=0\wedge\beta(m)=0)\\ 2^{-k} & \text{if } k\leqslant n\wedge(\alpha(k)=1\vee\beta(k)=1)\end{cases} \tag{61}$$

and

$$s(n)=\begin{cases}2^{-n} & \text{if } \forall m\leqslant n(\alpha(m)=0\wedge\beta(m)=0)\\ 2^{-k} & \text{if } k\leqslant n\wedge \mathrm{Even}(k)\wedge(\alpha(k)=1\vee\beta(k)=1)\\ 2^{-n} & \text{if } \exists k\leqslant n(\mathrm{Odd}(k)\wedge(\alpha(k)=1\vee\beta(k)=1))\end{cases} \tag{62}$$

Then $0<r$ because $\neg\forall m(\alpha(m)=0\wedge\beta(m)=0)$.

If $s\ \#\ 0$, then by the second clause $s=r$, so $r\ \#\ 0\to s=r$ and therefore, by $s<r\to s\neq r$ and (8), $s<r\to s=0$. By definition, $0<s\to s\neq 0$. Combining the two gives

$$s<r\vee 0<s\to s=0\vee s\neq 0 \tag{63}$$

If $s=0$, then

$$\neg\,\exists k(\mathrm{Even}(k)\wedge\alpha(k)=1\vee\beta(k)=1) \tag{64}$$

hence

$$\neg\,\exists k(\mathrm{Even}(k)\wedge\alpha(k)=1) \tag{65}$$

so

$$\forall n(\alpha(2n)=0) \tag{66}$$

By the last clause in (59),

$$A \vee \neg A \tag{67}$$

hence

$$\neg A \vee \neg\neg A \tag{68}$$

If $s \neq 0$, then

$$\neg\neg\, \exists k(Even(k) \wedge \alpha(k) = 1 \vee \beta(k) = 1) \tag{69}$$

and therefore

$$\neg\, \exists k(Odd(k) \wedge \alpha(k) = 1 \vee \beta(k) = 1) \tag{70}$$

by the assumption that α and β take the value 1 at most once. So

$$\neg\, \exists k(Odd(k) \wedge \beta(k) = 1) \tag{71}$$

and hence

$$\forall n(\beta(2n+1) = 0) \tag{72}$$

By the last clause in (60),

$$A \vee \neg A \tag{73}$$

hence

$$\neg A \vee \neg\neg A \tag{74}$$

Together, the arguments for the cases $s = 0$ and $s \neq 0$ yield

$$s = 0 \vee s \neq 0 \to \neg A \vee \neg\neg A \tag{75}$$

which shows that, as long as A cannot be tested, $s = 0 \vee s \neq 0$ cannot be proved; by (63), this means that, as long as A cannot be tested, $0 < s \vee s < r$ cannot be proved. On the other hand, $0 < r$ can be proved; hence $0 < r \to 0 < s \vee s < r$ cannot be proved, and this means that axiom (33) does not hold for $<$.

The same thought that justifies BKS$^-$W also justifies a version of BKS that I here call BKS$^-$R and which was proposed and defended by Erdélyi-Szabó and Scowcroft [58, p.1027–1028]. If, upon proving A, the Creating Subject chooses a random positive number, then nothing can be said about that number until A has been proved:

$$\exists\alpha \begin{bmatrix} \exists n(\alpha(n) > 0) \rightarrow A \\ \wedge \\ \neg\exists n(\alpha(n) > 0) \rightarrow \neg A \\ \wedge \\ \forall k > 0(\neg\exists n(\alpha(n) = k) \rightarrow A \vee \neg A) \\ \wedge \\ \forall k > 0\,\forall n((\alpha(n) = k) \rightarrow \forall m \geqslant n(\alpha(m) = k)) \end{bmatrix} \tag{BKS$^-$R}$$

There is a strengthening of BKS$^-$ that is not only of mathematical but also of philosophical significance [123, p.295; 145, p.96]:

$$\exists\alpha \begin{bmatrix} \forall n(\alpha(n) = 0 \vee \alpha(n) = 1) \\ \wedge \\ \exists n(\alpha(n) = 1) \leftrightarrow A \end{bmatrix} \tag{BKS$^+$}$$

In subsection 7.4, Kripke's reason for accepting BKS$^-$ but not BKS$^+$ will be discussed.[36]

As in the case of BKS$^-$, for species X there is [145, p.104]:

$$\exists\beta\,\forall x \begin{bmatrix} \forall n(\beta(n) = 0 \vee \beta(n) = 1) \\ \wedge \\ \forall n(\beta(j(x,n)) = 0) \leftrightarrow x \notin X \\ \wedge \\ \exists n(\beta(j(x,n)) = 1) \leftrightarrow x \in X \end{bmatrix} \tag{BKS$^+$S}$$

Classically, BKS$^+$ is just a weak comprehension principle [167, p.74]. In the setting of classically defined models for formal intuitionistic theories, a consistency proof of BKS$^+$ was given by Scott [133]; his conjecture that BKS$^-$ is weaker than BKS$^+$ was proved, again for a classically defined model, by Krol' [104].

[36] Note that it is not the case that in reconstructions of Creating Subject arguments in terms of BKS$^+$ and BKS$^-$, for strong counterexamples the strong schema is used, and for weak counterexamples the weak.

4.2 CS

The axiom schemata that make up (what became known as) the 'Theory of the Creating Subject' (CS) are due to Kreisel [98], who had received a letter from Kripke stating BKS$^-$ [101, footnote 8].[37] In 1969 Anne Troelstra, in his early and influential treatment of Kreisel's schemata, added a schema (see below) and spoke of 'Brouwer's theory of the creative subject' [145, ch.16] where Kreisel had used the term 'thinking subject'. Since Brouwer had indeed used the term 'creative subject', or rather 'creating subject'[38], but had not presented an explicit theory such as Kreisel proposed, overall I think it is best to speak of 'the Kreisel-Troelstra Theory of the Creating Subject'.

The basic notion in Kreisel's original schemata was $\Sigma \vdash_n A$, where Σ is a variable ranging over Creating Subjects and which means that 'the (thinking) subject Σ has evidence for asserting A at stage m' [98, p.159]. Troelstra assumes the existence of only one Creating Subject, as do most later presentations. I will do the same here, and say something about the reasons why in subsection 5.3.

Troelstra writes $\vdash_n A$ with the meaning 'the creative subject has evidence for A at stage m' [145, p.95], but he is sensitive [145, p.105–106] to a possible ambiguity in the phrase 'to have evidence for': must the Creating Subject be aware of this at stage m, or is evidence closed under, say, trivial (but unrealised) deducibility? It seems to me that, to the extent that the aim is a reconstruction of Brouwer's ideas, a construction can only contribute to making some proposition evident once the Creating Subject has made the connection between the two in an appropriate act, so that evidence is not closed under unrealised deductions (trivial or not). Thus, to use Troelstra's phrase, I take the Creating Subject to establish one conclusion at a time.[39]

Instead of the turnstile of Kreisel and early Troelstra, too reminiscent of formal derivability relations, I prefer to use the propositional operator $\Box_n$ as in Troelstra and Van Dalen's [148], and understand $\Box_n A$ as 'By stage n the Creating Subject has made A evident'.

Like BKS, CS comes in a weak and a strong version. In the original presentation [98] there is only the weak version CS$^-$.

$$\forall n(\Box_n A \vee \neg\Box_n A) \tag{CS$^-$1}$$

That is, for any stage, it is decidable for the Creating Subject whether by that stage it has made A evident.

$$\forall n\,\forall m(\Box_n A \rightarrow \Box_{n+m} A) \tag{CS$^-$2}$$

[37] Kreisel showed his schemata in a letter to Gödel of July 6, 1965 [67, item 011182].

[38] See p.4 above.

[39] For Troelstra [145] this question comes up because of its relation to the paradox in the Theory of the Creating Subject that he there devises. See subsection 7.3 below.

The Creating Subject never forgets what it has made evident. (This axiom was added by Troelstra [145, p.95].)

$$(A \to \neg\neg\exists n \Box_n A) \wedge (\exists n \Box_n A \to A) \tag{CS$^-$3}$$

If A is true, then it is not possible that the Creating Subject will never make it evident; and the Creating Subject makes no mistakes. Kreisel named CS$^-$3 'the principle of Christian Charity'.

In the stronger version [145, p.96], CS$^+$1 = CS$^-$1 and CS$^+$2 = CS$^-$2, but CS$^-$3 is strengthened to

$$\exists n \Box_n A \leftrightarrow A \tag{CS$^+$3}$$

A proposition A is true if and only if the Creating Subject has made A evident by some stage.

Splitting CS$^-$3:

$$A \to \neg\neg\exists n \Box_n A \tag{CS$^-$3a}$$

$$\exists n \Box_n A \to A \tag{CS$^-$3b}$$

and CS$^+$3:

$$A \to \exists n \Box_n A \tag{CS$^+$3a}$$

$$\exists n \Box_n A \to A \tag{CS$^+$3b}$$

In terms of CS$^-$, Plausibility argument 10 for Weak counterexample 9 could be recast as follows:

Plausibility argument 35. Let A be a proposition that is, at present, not testable. The choice sequence α is defined as follows:

$$\alpha(n) = \begin{cases} 0 & \text{if } \forall m \leqslant n(\neg\Box_m A \wedge \neg\Box_m \neg A) \\ 2^{-k} & \text{if } k \leqslant n \wedge \forall m < k(\alpha(m) = 0) \wedge \Box_k A \\ -2^{-k} & \text{if } k \leqslant n \wedge \forall m < k(\alpha(m) = 0) \wedge \Box_k \neg A \end{cases} \tag{76}$$

With CS$^-$2 and CS$^-$3b one shows that for no n both $\Box_n A$ and $\Box_n \neg A$, then with CS$^-$1 that α converges and therefore determines a real number r, and with the contraposition of CS$^-$3a that $r \neq 0$. Then, analogously to the reasonings (20) and (21),

$$\begin{array}{lr} r < 0 & \text{(assumption)} \\ \neg(r > 0) & \text{(def. } <) \\ \neg\exists n \Box_n A & \text{(def. } r) \\ \neg A & (\text{CS}^-3\text{a}) \\ A \text{ has been tested} & \end{array} \tag{77}$$

and, symmetrically,

$$\begin{array}{ll} r > 0 & \text{(assumption)} \\ \neg(r < 0) & \text{(def. >)} \\ \neg\,\exists n \Box_n \neg A & \text{(def. r)} \\ \neg\neg A & (\mathrm{CS}^{-}3\mathrm{a}) \\ A \text{ has been tested} & \end{array} \tag{78}$$

An immediate consequence of CS^{+} that will be used further on is:

Theorem 36 ([145, p.103])**.** Let the species $X \subseteq \mathbb{N}$ be inhabited. Then it is enumerable by the Creating Subject.

Proof 37. Let $j\colon \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ be the Cantorian pairing function

$$j(n,k) = \frac{(n+k)^2 + n + 3k}{2} \tag{79}$$

As X is inhabited, the Creating Subject has at some stage k proved that $a \in X$ for some a. Without loss of generality, it may be assumed that $k = 0$ and $a = 0$, i.e., $\Box_0 0 \in X$. Set $f(0) = a = 0$ and define, more generally,

$$f(j(n,m)) = \begin{cases} f(0) & \text{if not } \Box_m n \in X \\ n & \text{if } \quad\ \Box_m n \in X \end{cases} \tag{80}$$

Then, by $\mathrm{CS}^{+}3$, $n \in X \leftrightarrow \exists m \Box_m n \in X$, and, by the definition of f, $\exists m \Box_m n \in X \leftrightarrow \exists m(f(j(n,m)) = n)$; hence,

$$n \in X \leftrightarrow \exists m(f(j(n,m)) = n) \tag{81}$$

For the Creating Subject this function f is computable, as, by $\mathrm{CS}^{+}1$, for any given m and n, $\Box_m n \in X$ is decidable.

Kreisel had become interested in the Creating Subject after his work on the Theory of Constructions [84, 86]; the Theory of the Creating Subject was meant as an enrichment of that theory [98, p.180]. The particular use to which Kreisel wanted to put his Theory of the Creating Subject is related to the question of completeness of Heyting's predicate logic. Gödel had shown, and in 1962 Kreisel had published [96], a theorem to the effect that a completeness proof of Heyting's predicate logic (relative to a Tarski-style notion of validity but, as was shown thereafter, also to Beth and Kripke models) would entail the validity of MP in the form

$$\forall\alpha\neg\neg\,\exists x\phi(\alpha,x) \to \forall\alpha\,\exists x\phi(\alpha,x)$$

where ϕ is a primitive recursive predicate, and α ranges over choice sequences but not lawless ones.[40] So a rejection of this principle entails the incompleteness of Heyting's predicate logic. As noted in subsection 3.5, Brouwer's Creating Subject argument for the conclusion

$$\neg \forall x \in \mathbb{R}(x \neq 0 \rightarrow x \# 0)$$

entails a strong counterexample to a strong version of MP, and Kreisel was interested to see if an argument like Gödel's could be used also to derive incompleteness of Heyting's predicate logic from that strong counterexample. He did not succeed without invoking Church's Thesis or something of a similar nature [98, p.182], which Brouwerian intuitionists do not accept (see subsection 7.1 below).

Be that as it may, in preparing the final version of 'Informal rigour', Kreisel discovered that he could obtain Brouwer's strong counterexample *without* using the Theory of the Creating Subject [98, p.180–182] (Vesley in 1968 devised a Schema for the purpose; see subsection 7.8 below). It seems he then lost interest in the topic; when in 1982 Van Dalen showed that the Theory of the Creating Subject is conservative over Heyting Arithmetic [166],[41] Kreisel, in his *Zentralblatt* review [93], took that to be 'further evidence for the mathematical sterility of CS'. By the list in subsection 7.1 below, this, seems not borne out by the (mostly later) facts. But I agree when he adds that considerations of CS remain of interest 'as a philosophical object lesson'.

4.3 Comparing BKS and CS

A conspicuous difference between BKS and CS is that the former is extensional, whereas the latter is, with its reference to the Creating Subject and the stages of its activity over time, highly intensional. On the other hand, there seems to be no justification of BKS except its derivation from CS.

These points were highlighted by Myhill [123, p.295–296] and Kreisel [89, p.128]. Myhill has a preference for BKS^- over CS^- because 'we think the loss of extensionality too high a price to pay in technical facility' [124, p.175].[42] To illustrate this, he points out that the mistake in Brouwer's argument from 1949 (which he refers to in Heyting's version [70,

[40]The way Gödel treated negation in his argument – negation as absence of models – is decidedly un-Brouwerian, as he will have been fully aware of. Veldman developed a notion of model in which negation is handled differently, which allowed him to give an intuitionistically correct completeness proof while blocking Gödel's argument. But Veldman did not think that his semantics (nor similar ones) shed much light on the notion 'intuitionistically true sentence' [188, p.159].

[41]But Krivtsov [102] sees the need for an alternative proof, which he supplies (for a more limited result).

[42]I am not sure why Myhill did not include BKS^+ in his system, the occurrence of which in Brouwer's work he had been the one to find (see subsection 6.2); perhaps because Brouwer made no further use of it. Note that in later applications of BKS, it is the strong version that is used; see subsection 4.4.

p.118]) would have been avoided if BKS had been used instead of CS (see subsections 3.5 and 3.6 above). Thus, in his formalisation of intuitionistic analysis, Myhill chooses to adopt BKS$^-$ instead of CS$^-$. But the reason is pragmatic; he held that CS$^-$ is correct, and that it provides a 'deeper analysis' than BKS$^-$ [123, p.296]. Van Dalen [165, p.19] showed that the addition of CS$^+$ to a theory of intuitionistic analysis that contains BKS$^+$ is conservative.

4.4 Applications beyond Brouwerian counterexamples

In the discussion after Kreisel's presentation of 'Informal rigour' in 1965, Heyting had warned that the method of the Creating Subject 'is not central in intuitionistic mathematics. It can only be applied to show that certain propositions of which nobody believed that they could be true, are actually false' [98, p.173]; and we saw above (p.35) that eventually also Kreisel came to consider CS mathematically sterile. Various theorems from BKS (and hence also from CS) show that CS and BKS do have applications beyond Brouwerian counterexamples:

1. (Van Dalen) BKS$^+$ can be used to construct a topological model for the theory of species of natural numbers [163]).
2. (Van Dalen) BKS$^+$ can be used to construct an intuitionistic analogue of the powerset of $\mathbb{N}$ [164].[43]
3. (Burgess) If the 'basic system' of principles common to intuitionistic and classical analysis as defined by Kleene and Vesley [81, p.8] is extended with BKS$^+$, a theorem can be obtained that is classically equivalent to Souslin's Theorem [49].[44]
4. (Van Dalen) Using BKS$^+$, one shows that every negatively defined dense subset $X \subset \mathbb{R}$ is unsplittable, i.e., if X is the disjoint sum of Y and Z, either $Y = X$ or $Z = X$ [168]. (See also subsection 7.8 below.)
5. (Lubarsky, Richman, and Schuster) In the presence of countable choice, BKS$^+$ is equivalent to 'Every open subspace of a separable space is separable' and also to 'Every open subset of a separable metric space is a countable union of open balls' [108].
6. (Kachapova) There is an intuitionistic formal theory SLP for higher types and lawless

[43] That paper was inspired by Troelstra [145, p.104] and Kreisel [89, p.128].

[44] Gielen, Veldman, and De Swart [63, p.134] do not accept this proof because they argue that BKS$^+$ can be justified only for 'determinate' propositions A; see subsection 7.7 below.

sequences, together with a Beth model for it, that includes CS^{+}[45] and which is equiconsistent with TI, a subtheory of classical typed set theory that is stronger than classical second-order arithmetic [77].[46]

Here one should also mention Van Rootselaar's suggestion to use CS also for defining various notions of choice sequence by characterising the Creating Subject's knowledge of them [179, p.196], the analysis by Friedrich and Luckhardt of the role of BKS^{+} in establishing certain uniformity principles [60], and Schuster and Zappe's use of versions of BKS^{+} to classify countability statements [132].

5 A Brouwerian justification of CS and BKS

5.1 A remark on Brouwerian logic

As my primary interest in this paper is in questions of justification and use of Creating Subject arguments in a Brouwerian framework, as opposed to other forms of constructivism, and CS and BKS were both intended to clarify those arguments, I will insist here on construing the logic in a Brouwerian way. Brouwer's conception of logic can be found in his dissertation from 1907 [10, ch.3] and in 'The unreliability of the logical principles' from 1908 [11, 161]. It differs considerably from that embodied in the now ubiquitous combination of Natural Deduction and what has become known as 'the BHK interpretation' [139, 159]. For the present purpose, the main points are:

1. Intuitionistic mathematics consists primarily in the act of effecting mental constructions of a certain kind; its objects, relations, and proofs exist only in so far as they have been constructed in these acts. Neither these acts nor the resulting constructions are of a linguistic nature, but when series of construction acts and their results are described in a language, the descriptions may come to exhibit linguistic patterns. Intuitionistic logic is the mathematical study of these patterns, and in particular of those that characterise correct inferences.[47]

[45]The review in the *Zentralblatt* opines that 'The purist will be disappointed since SLP proves some principles, like weak continuity or the Kripke's schema which are debatable in a predicative setting; also, the formal Church's thesis is inconsistent with SLP' [6]. By the main theme of the present paper, from the Brouwerian point of view, this is all as it should be.

[46]This makes a considerable step towards fulfilling (also) Brouwer's prediction that Hilbert's program for a constructive consistency proof of classical mathematics would succeed, although not based, as he thought, on the consistency of PEM, but on his ideas about the Creating Subject. Brouwer thus was not, before Gödel, sceptical about the possibility of such a consistency proof; but he doubted its value for the foundations of mathematics [22, p.252n4; 25, p.377; 26, p.164].

[47]As Heyting put it, 'every logical theorem [...] is but a mathematical theorem of extreme generality' [70, p.6]. Note that intuitionistic logic, thus conceived, is not the logic 'underlying' intuitionistic mathematics, as is sometimes said; quite the opposite.

2. Sundholm and I suggest the following characterisation of intuitionistically correct inference [161, p.26]: 'A correct inference is one where the construction required by its conclusion can be found from hypothetical actual constructions for its premisses. That is, we assume that constructions for the premisses *have been* effected. The hypotheses here are epistemic ones, in that the premisses are *known*. Thus, they differ from assumptions of the usual natural deduction kind, which merely assume that propositions are true. For Brouwer's conception of truth, however, only these epistemic assumptions play a role, since for him to assume that a proposition is true is to assume that one has a demonstration of it, that is, that one knows that it is true. [...] We will write $A \to B$ for "A (hypothetical) actual construction for A can be continued into a construction for B".'

3. The valid logical principles then are rules under which all of mathematics (however it develops) is closed.

Sundholm has claimed that $CS^{+}3b$ 'is simply wrong' if implication is understood as a relation between a non-epistemic assumption and a consequent [138, p.19]. I am inclined to agree with this, and refer to pages 18–20 of his paper for further discussion of this point.[48] In the early discussions of CS and BKS, I have found no one whose explanation of these principles involves implications with non-epistemic assumptions;[49] it seems that it was not realised at the time that this meant that the treatment of assumptions (and hence of implication) in Natural Deduction renders the latter inappropriate for a faithful modelling of reasoning in Brouwerian mathematics.

5.2 BKS is derivable from CS

Theorem 38 ([123, p.295-296; 179, p.191-192; 145, p.96]). CS^{-} implies BKS^{-}, and CS^{+} implies BKS^{+}.

Proof 39. For both cases, define

$$\alpha(n) = \begin{cases} 0 & \text{if } \neg\Box_n A \\ 1 & \text{if } \Box_n A \end{cases} \tag{82}$$

By $CS^{-}1$ we have $\forall n(\alpha(n) = 0 \vee \alpha(n) = 1)$.

Assume $\forall n(\alpha(n) = 0)$. Then, by the definition of α, $\forall n \neg\Box_n A$, hence $\neg\exists n \Box_n A$, and so, by the contraposition of $CS^{-}3a$ and $\neg\neg\neg A \leftrightarrow \neg A$, $\neg A$. Conversely, assume $\neg A$. Then,

[48] In conversation, Sundholm emphasised that that discussion does not target the Creating Subject arguments in Brouwer's own setting.

[49] In particular Myhill is explicit about his epistemic conception [123, p.295 121, p.326].

by the contraposition of $CS^{-}3b$, $\neg \exists n \Box_n A$, hence $\forall n \neg \Box_n A$, and so, by the definition of α, $\forall n(\alpha(n) = 0)$.

Assume $\exists n(\alpha(n) = 1)$. Then, by the definition of α, $\exists n \Box_n A$, and hence, by $CS^{-}3b$, A.

To obtain moreover BKS^{+}, assume A. Then, by $CS^{+}3a$, $\exists n \Box_n A$, and therefore, by the definition of α, $\exists n(\alpha(n) = 1)$.

5.3 A justification of CS

As will now be explained, the conception of the Creating Subject that suits Brouwer's foundational thought is that of an ideal subject, who can, by itself, do whatever mathematics can in principle be done, and whose activity is structured as an ω-sequence. It remains a schematic subject in the sense that it does not have a particular history, but is the correlate of possible histories: in each possible history, the subject as thought of in that history is an instantiation of the schematic Creating Subject.

The idea that the Creating Subject carries out its constructive activities in an ω-sequence of stages is not made explicit by Brouwer, but is implied whenever he states the view that the individual mathematical objects that can be built up from the basic intuition of two-ity are either finite or or potentially infinite constructions.[50] At any given moment, only finitely many constructions will have been carried out, with an open horizon for further ones. To see its mathematical activity as an ω-sequence of stages, the Creating Subject first looks back at its earlier activity (reflection), and projects an initial segment of an ω-sequence (e.g., that of the natural numbers) onto its various earlier acts. The Creating Subject also sees that it can do this again and again in the future. It is clear that actual human beings carry out only finitely many mathematical acts in their finite lives; but in the study of Brouwerian constructibility the choice to allow for potentially infinite time is as reasonable as it is in the study of computability. Indeed, ituitionism is a theory about an idealised mathematician in the same sense as Turing's theory of computable numbers is a theory about an idealised (human or artificial) computer, and Chomsky's theory about grammar is a theory about an idealised speaker. (For further discussion of these parallels, see 64, esp. p.234–235; 61, p.148-151; 156, ch.6; 153.)

The idealisation involved in such theoretical accounts also includes the idea that we never make mistakes. As Troelstra aptly put it at the beginning of the chapter on Creating Subject arguments in *Principles of Intuitionism*,

> The central idea is that of an idealized mathematician (consistent with the

[50] Early examples of statements to that effect are found in the dissertation [10, p.142–143] (1907) and in 'Die möglichen Mächtigkeiten' [12] (1908); for a late one, see the quotation from 'Guidelines of intuitionistic mathematics' (1947) on p.7.7 below.)

> subjectivistic viewpoint of intuitionism, we may think of ourselves; or even better, to obtain the required idealization, we may think of ourselves as we should like to be). [145, p.95]

Van Dantzig [175] turned the fact that such idealisations are required to make Brouwer's Creating Subject arguments work into an objection to them; but this is misguided in view of the nature of the theoretical model of our mathematical activity that intuitionism presents.[51]

Another idea common to these analyses of constructibility, computability, and grammaticality is that whatever falls under these notions can in principle be fully mastered by a single subject. The Creating Subject is essentially singular.[52] This may seem to be contradicted by Brouwer's remark that one and the same untestable proposition can give rise to different Creating Subject sequences [43, p.204n2], or, as he put it more vividly in the 1934 Geneva lectures:

> If I would give the definition of [the Creating Subject sequence] s to one hundred different persons, who are all going to work in a different room, it is possible that one of these one hundred persons at one time will choose an interval not covered by an interval chosen by one of the others. [31, lecture 2, trl. 127, p.45]

But I read this not as an acknowledgement that the notion of Creating Subject would be fundamentally plural, in such a way that mathematical truth would not depend on a single one of them, but rather as an acknowledgement of the schematic character of the notion, and that different real or imagined instantiations may proceed differently from one another.[53] This again corresponds to the theoretical role of the Universal Turing Machine. The latter is defined in general terms, and then in thought experiments actual human beings theorise about different runs of it with different programs and different inputs, so that each of these runs could be said to present a possible history of the Machine's computations, without any of them being the actual history, because, being schematic, the Machine has no history.

With this conception of the Creating Subject in place, the schemata for CS^+ are justified as follows, in keeping with the conception of logic described in subsection 5.1.

[51] Markov in effect shared Van Dantzig's objection that the assumption that the Creating Subject never proves a proposition A does not imply that A is false; perhaps the Creating Subject just lost interest in the problem! (Markov makes this remark in his Russian translation [72, p.192] of Heyting's *Intuitionism* [70]; the remark is translated into French by Margenstern [110, p.290].) However, allowing for such a possibility is already to restrict the Creating Subject [153, section 3].

[52] Compare on this point [50, p.109–110n15].

[53] This may also be behind Brouwer's objection [28, p.11] to Fraenkel's remark that all who agree that a given mathematical question is meaningful will give the same answer to it, but Brouwer would perhaps say that already out of a distrust of language.

$CS^{+}1$ is correct because, if n is in the past, the Creating Subject inspects its perfect memory [30, section 3]; and if n is in the future, the Creating Subject postpones its decision for the finite number of stages required and then sees again.

$CS^{+}2$ is justified by the Creating Subject's perfect memory.

$CS^{+}3b$ can be read as an assertion of the correctness of the Creating Subject's thought.[54] Martino [113, section 5.5] argues that for $CS^{+}3b$ it does not matter whether the proof that provides evidence for A comes from within the Creating Subject or is given to it by someone else; and that there is therefore nothing solipsistic about it. However, it seems to me that the Creating Subject cannot simply accept a proof from outside, and must go through the steps to reconstruct the evidence. In the end, then, its own ability to make A evident is what really matters. The Creating Subject must take a proof that is communicated to it not as itself evidence, but as a set of instructions to obtain that evidence [10, p.169].

$CS^{+}3a$ does not mean, as it may seem at first sight, 'If A is true, whether the Creating Subject knows it or not, a time will come when the Creating Subject proves it'. On the Brouwerian conception of logic, it rather gives expression to the fact that whenever the Creating Subject has effected a construction for A, and thereby experienced the truth of A, it has done so at a particular moment in time, and that moment can be made explicit [123, p.295; 113, section 5.5; 56, p.242].

In a discussion of BKS^{+} that in effect concerned $CS^{+}3a$, Kleene and Moschovakis suggested to Myhill that 'this presupposes a linear ordering of all possible proofs, so that the apparent reference to time is really a reference to Gödel-numbers of (informal) proofs' [123, p.295].[55] I agree with Myhill that this idea does not accord with Brouwer's speaking of 'experiencing the truth', and would explain that by the fact that for Brouwer the number is extracted from that experience but not from its content. Moreover, a Brouwerian understanding of the species of 'all possible proofs' would itself depend on time, as it must

[54] In a draft version of the 'stellingen' for his thesis defense, Brouwer pointed out that in mathematics there should be no hypothesis 'I reason correctly': 'First of all, reasoning is an *act*, in which the self is not objectivated; secondly, these words have meaning, and, a fortiori, meaning as a foundation for something else, only on the basis of already existing mathematical systems, and therefore of already existing logic; and, thirdly, in particular the word "correctly" means nothing but "mathematically correct", and therefore presupposes mathematics and logic.' [48, p.147, trl. MvA]. Creating Subject arguments clearly are an example of what Brouwer (in 1907) called 'second-order mathematics' [10, p.119n.], which allows one to describe, and to reason about, mathematical acts mathematically. In such an objectivation, there certainly is a place for descriptions of aspects of that activity such as $CS^{+}3b$; Brouwer's point, applied here, is just that accepting $CS^{+}3b$ is in no way a condition for engaging in mathematical activity as such. Another example (one that Brouwer remarks on in his dissertation) is that of mathematical induction as a construction act (as opposed to an axiom (schema)) [151].

[55] Webb [192, p.211] writes of CS that ' it implies that all possible proofs can be arranged in an ω-sequence, which acutely conflicts with the impredicative nature of intuitionistic implication.' I argue in [152] that intuitionistic implication is predicative.

be understood as an essentially growing species, for which no single construction method can be given:

> As further examples of denumerably unfinished sets we mention: the totality of definable points on the continuum, and a fortiori the totality of all possible mathematical systems. [10, p.148–149; trl. 45, p.82]

and, in a notebook around 1907,

> The totality of mathematical theorems is, among other things, also a set that is denumerable, but never finished. [9, Notebook VIII, p.44, trl. MvA][56]

The idea also occurs in Kreisel:

> First of all, very little of the 'thinking subject' is used in the derivation [in a reconstructed Creating Subject argument], Instead of writing $\Sigma \vdash_n A$, I could write $\Sigma_n \vdash A$ and read it as: the n-th proof establishes A. In other words, the essential point would not be the individual subject, but the idea of proofs arranged in an ω-order. [...] Also, the sequence Σ_n is not itself considered to be given by a rule. [98, p.179]

The question is whether the existence of the sequence Σ_n could be justified constructively if the ω-order it depends on is not the one that is induced by the fact that the Creating Subject's activities unfold over time, so that, genetically, the individual subject is essential after all.

A further analysis of the reference to time in CS^+3a than has been advanced so far, and which will be important also for the discussion of Kripke's own objection to BKS^+ in subsection 7.4, depends on two distinctions.

The first is that between tokens and types. The other is a distinction between three meanings of the term 'construction', drawn attention to by Sundholm [140, p.164]:

1. process of construction (as it unfolds in time),

2. object obtained as the result of such a process,

3. construction-process as object (the objectification of a process of construction).[57]

[56] 'Het aantal wiskundige stellingen is o.a. ook een Menge, die aftelbaar is, maar nooit af.'

[57] There is a fourth sense [137; 139, p.68]: 'by abstracting of the objectified act with respect to subject and time, [one obtains a construction in the sense of] a prescription or blueprint for construction acts'. I will not be concerned with the abstraction from a (particular) subject, because my discussion concerns only the one Creating Subject; whereas abstraction from (particular) time is dealt with by applying the type-token distinction.

When Brouwer in his dissertation writes that 'strictly speaking the construction of intuitive mathematics in itself is an *act* and not a *science* 10, p.99n; trl. 45, p.61n1, modified][58] he is thinking of constructions in the first sense; and it is clear that constructions in the other two senses presuppose for their existence a construction process in the first sense. Constructions in the first sense are ontologically prior to the others. The objectification of a process happens in an act of reflection; this possibility to reflect on our acts will turn out to be crucial to a Brouwerian argument for BKS.

This can be connected to the type-token distinction as follows. At the most concrete level, construction processes occuring at different times are for that reason different processes. But we may come to see that processes that are different in this sense have various things in common, and we may therefore see them as instantiations or tokens of the same type of construction process. The same can be done for constructions in the other two senses, constructed objects and the objectified processes. For example, this allows us to observe that an act in which we construct the number 2 and an act in which we construct the number 3 have in common that the objects constructed in them are of the same type, that of natural number. In the extreme case, we may even come to identify processes with one another, and identify the objects constructed in them. This is the sense in which we can say, for example, that when constructing the number 2 time and again, each time we carry out the same construction process in which we construct the same object.

The notion of proof is related to that of a construction in a straightforward way. A clear statement of this relation was made by Heyting:

> If mathematics consists of mental constructions, then every mathematical theorem is the expression of a result of a successful construction. The proof of the theorem consists in this construction itself, and the steps of the proof are the same as the steps of the mathematical construction. [71, p.107]

So for proofs the same threefold distinction can be made as was introduced for constructions, and we may consider each case as a type or as a token, depending on our purpose.[59]

The question what counts as a proof of a proposition A must be a question about the proof type; but whenever a proof of A is given to us, what is given to us is primarily a proof token, as intuitionistically types only exist as abstractions from tokens. The parameter t is part of the proof token, as the specific time at which a concrete process occurs is constitutive of its identity; but it is not part of the proof type, as proofs of the same type may be constructed at different times.

Logical principles are formulated at the level of proof types, so as to allow these principles to express the relevant kind of generality: such a principle is of the form 'Whenever

[58] I here translate 'daad' by 'act' instead of 'action'.

[59] This is closely related to Martin-Löf's notion of a 'proof-trace' [137].

I have a proof of the premises, I can obtain from it a proof of the conclusion'. But observe that whenever I have a proof of the premises, this is given to me first of all as a proof token, and only abstractively as a type. This is because, intuitionistically, types do not exist independently from their tokens.

This difference creates room for the following argument. When claiming that an implication $A \to B$ holds, what is claimed is that a certain relation holds between the type 'proof of A' and the type 'proof of B', namely, that there is a construction method to convert *any* token proof of A into a token proof of B. So whenever the Creating Subject wishes to apply that method to a token proof of the antecedent (modus ponens), the parameter t is always available among the data. It is *not* claimed that t is among the specifications of a proof type; nor is that required to make good on the claim $A \to B$. In other words, the idea is *not* that a value for t can somehow be extracted from the propositional content of A.

Perhaps it is objected that, even if we acknowledge this distinction between construction types and tokens, and that givenness of a token proof of the antecedent of an implication includes givenness of t, this does not suffice to consider t part of the *mathematical* data we have on which to base a construction for the consequent.

First note that this objection, if correct, would apply to BKS^- as well. That schema too requires that the time parameter t be a *mathematical* datum. Otherwise the sequence that witnesses the existential quantifier in BKS^-, a witness the constructibility of which Kripke in effect justifies in terms of t, would not exist as an object of pure mathematics. (This again illustrates the theme from Kreisel and Myhill that BKS is an extensional principle but is justified by consideration of intensional aspects.)

A reply to the objection itself must begin with the observation that for Brouwer the Creating Subject is an idealised intuitionistic mathematician, and this includes a property of consciousness that Husserl calls 'inner time awareness'. Brouwer does not mention Husserl in his dissertation, nor elsewhere, but he does confirm there that inner time awareness is what he has in mind:

> Of course we mean here *intuitive time* which must be clearly distinguished from *scientific time*. Very much a posteriori, only by means of experience it becomes clear that the latter can suitably be introduced as a one-dimensional coordinate equipped with a one-parameter group for cataloguing phenomena. [10, p.99n; trl. 45, p.61n2].

To see why the time parameter in this sense is *mathematically* relevant for the Creating Subject, recall Brouwer's statement in his dissertation that mathematics is first of all an *act*. In a much later passage, from 1947, he said the same thing more specifically:

> Intuitionistic mathematics is a mental construction, essentially independent

> of language. It comes into being by self-unfolding of the basic intuition of mathematics, which consists in the abstraction of two-ity. [33, p.339, trl. 45, p.477]

By 'self-unfolding' is meant that in our mathematical acts we first construct certain basic objects, the nature of which Brouwer specifies but is not relevant now, and then to those apply the same mathematical acts to construct further objects. This activity thus has an iterative structure, and induces a linear order on the constructions that the Creating Subject has effected.

Extracting the time parameter t associated to a token construction that makes a proposition A true is just making the position of the token in that induced ordering explicit. To do this, the Creating Subject needs to reflect on its mathematical activity so far and apply mathematics to it. This is an instance of what Brouwer, in his dissertation, describes as 'viewing mathematical activity mathematically', a form of self-reflexivity which he calls 'second-order mathematics' and which is itself of a mathematical nature [10, p.98n; trl. 45, p.61n1]. As the linear ordering of the Creating Subject's constructions is not only the order in which it becomes aware of the objects, but indeed the order in which they come into being, this order is a mathematical fact. It is this consideration that justifies the strong Brouwer-Kripke Schema in a Brouwerian setting. Indeed, more generally, the schemata CS are pieces of general self-knowledge of the Creating Subject, obtained by applying mathematics to mathematical activity as given in reflection.[60]

There are different ways of relating the ω-ordering of the inner time parameter t to the ω-ordering of the elements of a sequence required for a witness of the existential quantifier in BKS. The first element of the witnessing sequence may be correlated to the beginning of all of the Creating Subject's activity, or to the moment at which the Creating Subject has indeed begun working towards a decision of A. However, either of these sequences can be mapped to the other in an order-preserving way. Similarly, one master sequence tracking all of the Creating Subject's other mathematical activity would be sufficient, given that from it any sequence pertaining only to the activity related to a given proposition A may be extracted as needed.

It was remarked above that the number 2 could be constructed at many different times. Would it then in principle be possible, in intuitionistic mathematics, to use different numbers 2 and the times that they occur?[61] Indeed, by the same reasoning as applied to proofs in the present account of BKS^+, one could use the time parameter in the Creating Subject's activity to assign a different number to each token construction of the number 2. But, as emphasised in the quotation from Heyting (p.43 above), any successful construction

[60] Metschl [116] analyses the role of epistemic obligations in the formation of the Creating Subject's knowledge (in general; not with an eye on Creating Subject arguments in particular).

[61] I thank Saul Kripke for raising this question, in conversation.

process may be considered to be a proof (whether subsequently expressed in a theorem or not); to construct a token of the number 2 then is, in that sense, to construct a token of the proof (type) that the number 2, considered as a construction type, exists. It therefore seems to me that even though the notions of proof and object are different, assigning numbers also to objects would not yield mathematical possibilities beyond those given by BKS^+.

6 BKS in Brouwer

BKS is found in Brouwer's published work: on several occasions, he reasons from the antecedent of an instance of BKS^- to its consequent, and he once establishes an equivalence for untested A from which BKS^+ for such propositions follows immediately and hence, it can be argued, for other propositions as well. These occurrences are discussed below. Brouwer does not make any fanfare about these particular inferences, and in particular he does not pause to formulate the general principles. On any interpretation of Brouwer according to which he in effect accepted CS on the grounds given in subsection 5.3, this silence is not surprising, because Brouwer will then have carried out his reasoning in the corresponding terms directly. After all, the derivations of weak and strong BKS from the respective versions of CS are very short, and Brouwer never quite bothered to make the latter explicit either.

When Kripke isolated BKS^-, he was not aware of occurrences of BKS^- and BKS^+ in Brouwer.[62] Clearly, the schema was picked up on in the literature because of Kripke's rediscovery, not because of Brouwer's earlier and implicit use. So the name 'Brouwer-Kripke Schema', which seems to have been introduced in print by De Swart in 1977 [53, p.578],[63] is appropriate.

For the special case of lawless sequences, the connection between the strong counterexample to $\forall x \in \mathbb{R}(x \neq 0 \rightarrow x \# 0)$ and MP had been made by Kripke in 1965, when he proved that MP implies Brouwer's theorem [99, p.103–104]. In that paper $\mathrm{BKS}^-$is not yet stated, but comes near the surface; especially when on p.104 it is suggested to replace lawless sequences by sequences based on the solving of problems. For lawless sequences,

[62] Personal communication.

[63] He had used it in 1976 in his Nijmegen dissertation [52, p.34-35]. Gielen, Veldman and De Swart in 1981 speak of 'the Brouwer-Kripke Axiom' [63, p.122,126]. In a manuscript 'The trustworthiness of intuitionistic principles' (1983), Gielen [62] uses 'Brouwer's Scheme' for a version of BKS with a restriction on the content A which is discussed in subsection 7.7 below. He may have come to do this because the Nijmegen intuitionists considered that restriction to be a necessary condition for the schema's validity, and to be motivated by their interpretation of Brouwer, whereas it is absent from Kripke's formulation. (De Swart's dissertation [52, p.35] presents a result on BKS from Gielen's 'Verzamelingen'. I have not seen that unpublished manuscript from 1976 at the latest, and do not know what term he used there.)

the arguments are simpler and BKS^- is not needed. Kripke's reference for the Creating Subject arguments there is Heyting's *Intuitionism* [70], which includes Brouwer's argument from 1949, but not that of 1954 (subsections 3.5 and 3.7 above); and there is no paper by Brouwer among the references. Kripke remarks that 'I think it probable that such treatments in FC [= Kreisel's formal theory of lawless sequences [83]] will extend to all the counterexamples to classical theorems which Brouwer gives by this method; but I have not made a survey of the literature' [99, p.104].

6.1 BKS^-

As observed by Gielen, De Swart, and Veldman [63, p.128], in the proof from 1949 discussed as Proof 29 above, Brouwer reasons along the lines of BKS^-. This occurs where, from the hypothesis that for a direct checking number $D(\gamma, p_f) \mathrel{\circ\!\!>} 0$ holds, a proof is constructed for a previously defined real number f that $f \in \mathbb{Q} \vee f \notin \mathbb{Q}$. This reasoning clearly shows the pattern that makes it amenable to reconstruction with BKS^-: from the (hypothetical) existence of a value distinct from 0 in a certain infinite sequence, the truth of a priorly given proposition can be concluded to.[64] In this argument, no witnessing sequence for BKS^- is actually constructed; rather, the hypothesis (towards a contradiction) that $>$ implies $\mathrel{\circ\!\!>}$ amounts to a hypothesis that such a sequence has been constructed. (To accept the implication of $\mathrel{\circ\!\!>}$ by $>$ would in fact come down to accepting Markov's Principle for real numbers given by arbitrary kinds of converging choice sequences (see page 11 above).)

Brouwer also argues in this manner in a weak counterexample in 'Points and spaces' [41]. Some definitions are needed first.

Definition 40 ([41, p.8-9], simplified). A *spread direction* is a tree over the natural numbers such that each node p allows either all natural numbers as its immediate descendants, or all natural numbers $\leqslant m_p$ for some m_p.

A *spread* is a species of infinite paths through a spread direction.

A subspecies of a spread direction is *thin* if no node in it is a descendant of any of the other nodes.

A subspecies of a spread direction such that no infinite path through the spread direction can fail to intersect it is called a *crude block*.

A decidable, thin subspecies of a spread direction that is a block is called a *proper block* or simply a *block*.[65]

[64] Note that Myhill's criticism and repair of the proof (see subsection 3.6) concerns only the part that comes after this.

[65] The terminology in the *Cambridge Lectures*, held from 1946 to 1951, was different [46, p.21–22]. The crude block of 'Points and spaces' was there called a *barrage*; what was there called a *crude block* is the positive counterpart of a barrage, i.e. a subspecies of a spread direction such that every infinite path through

Definition 41 ([e.g. 41, p.10; 46, p.44]). Let R be a finite or infinite sequence of species N_ν, such that the N_ν are all disjoint and each completely ordered by a respective $<_{N_\nu}$. Then the *ordinal sum* of the N_ν is their union species M equipped with a complete order $<_M$ such that, for $x \in N_{\nu 1}$ and $y \in N_{\nu 2}$,

$$x <_M y \equiv N_{\nu 1} <_R N_{\nu 2} \vee (N_{\nu 1} = N_{\nu 2} = N_k \wedge x <_{N_k} y) \tag{83}$$

The *well-ordered species* are defined inductively:

1. A species containing exactly 1 element is a *basic species*. Basic species are well-ordered species.

2. The ordinal sum of an infinite sequence of previously acquired disjoint well-ordered species is again a well-ordered species.

3. The ordinal sum of a non-empty finite sequence of disjoint previously acquired well-ordered species is again a well-ordered species.

Weak counterexample 42 ([41, p.12]). There is no hope of showing that

$$K \text{ is a block } \rightarrow K \text{ is a well-ordered block} \tag{84}$$

for arbitrary spread directions K.

Plausibility argument 43. Let K be a spread direction, and X_n the species of nodes of K at depth n. An n-union is a union of species X_n; it is constructed step by step from a choice sequence α in which choice $\alpha(n)$ determines whether the n-th species X_n will be included in the n-union or not.

Let A be a proposition that is at present not testable. The Creating Subject constructs the n-union n_A by making its sequence of choices α as follows:

- As long as, by the choice of $\alpha(n)$, the Creating Subject has obtained evidence neither of A nor of $\neg A$, $\alpha(n)$ is chosen to be negative.

- If between the choice of $\alpha(r-1)$ and $\alpha(r)$, the Creating Subject has obtained evidence either of A or of $\neg A$, $\alpha(r)$ is chosen to be positive.

- For all $n > r$, $\alpha(n)$ is chosen to be negative again.

the spread direction intersects it. The notions of *proper block* or simply *block*, defined as that of thin and decidable crude block, likewise were positive.

Then the n-union n_A is a block of K because n_A cannot be empty: its being empty would imply $\neg(A \vee \neg A)$, which is contradictory. Note that, by definition, the elements of n_A cannot reside at different depths. On the other hand, for it to be a well-ordered block, it would have to be known what the nodes in n_A are, and hence what their depth is, and this can only be known if $A \vee \neg A$ is known. So as long as A has not been decided, n_A is a block, but not a well-ordered block.

The argument depends on the fact that the sequence α satisfies the properties

$$\begin{gathered} \forall n(\alpha(n) = 0 \vee \alpha(n) = 1) \\ \wedge \\ \forall n(\alpha(n) = 0) \leftrightarrow \neg(A \vee \neg A) \\ \wedge \\ \exists n(\alpha(n) = 1) \rightarrow A \vee \neg A \end{gathered} \tag{85}$$

and these are precisely the properties guaranteed by BKS^-. MP, if it were valid for sequences like α, would allow one to accept n_A as a well-ordered block.[66]

A third place where Brouwer proceeds thus is in 'Intuitionistic differentiability', also from 1954 [42]. Brouwer had studied intuitionistic differentiation only in 1923 [16]; this counterexample may have been motivated by Van Rootselaar's then recent work [177, 178].[67]

Definition 44 ([42]). An interval $[r, s] \subset \mathbb{R}$ is *substantial* if $r \# s$. Let $f: \mathbb{R} \to \mathbb{R}$ be a total function, r an arbitrary $x \in \mathbb{R}$, $s = i_{s1}, i_{s2}, \ldots$ an infinite sequence of substantial intervals containing r that converges positively to r. Associated to each $i_{s\nu} = [i_{s\nu 1}, i_{s\nu 2}]$ is the *difference quotient*

$$d_{s\nu} = \frac{f(i_{s\nu 2}) - f(i_{s\nu 1})}{i_{s\nu 2} - i_{s\nu 1}}$$

f is *strongly differentiable* in $x = r$, and has the *strong differential quotient* $c \in \mathbb{R}$ *in* $x = r$, if

$$\forall n\, \exists m\, \forall s\, \forall \nu(|i_{s\nu 2} - i_{s\nu 1}| < 2^{-m} \rightarrow |d_{s\nu} - c| < 2^{-n}) \tag{86}$$

[66] The referee pointed out that there is a much simpler argument:

> An example as sought for is already given by $B := \{s \mid \alpha(\text{length}(s)) \neq 0\}$, where α satisfies: $\neg\neg\exists n[\alpha(n) \neq 0] \wedge \forall m \forall n[(\alpha(m) \neq 0 \wedge \alpha(n) \neq 0) \rightarrow m = n]$, and one is unable to prove: $\exists n[\alpha(n) \neq 0]$.

The referee adds that the notion of a block is unlikely to be fruitful in constructive mathematicsl. I mention these points for their intrinsic interest; for the discussion of Brouwer's willingness to reason along the lines of BKS^-, both are moot.

[67] The latter reference is to Van Rootselaar's dissertation, supervised by Heyting and defended in 1954. In his *Intuitionism* Heyting does not treat the topic, and refers [70, p.96] to Van Rootselaar [178, ch.5].

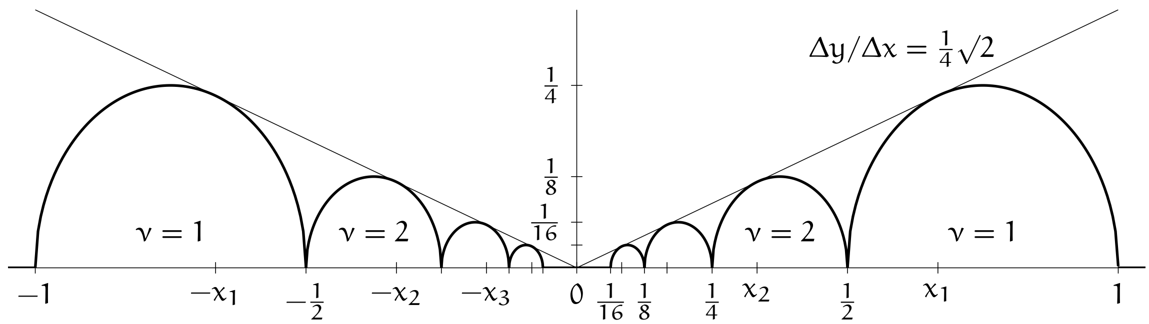

Figure 1: $\sum\limits_{\nu} \omega_\nu$ for the first four values of ν.

f is *weakly differentiable* in $x = r$, and has the *weak differential quotient* $c \in \mathbb{R}$ *in* $x = r$, if

$$\neg\, \exists s\, \exists m\, \forall \nu (|d_{s\nu} - c| > 2^{-m}) \tag{87}$$

Weak counterexample 45 ([42])**.** There is no hope of showing that

$$\text{f is weakly differentiable in } x = r \rightarrow \text{f is strongly differentiable in } x = r \tag{88}$$

Plausibility argument 46. Define the family of functions $\omega_{\nu \in \mathbb{N}^+} \colon \mathbb{R} \rightarrow [0, \frac{1}{4}]$ (Figure 1) by

$$\omega_\nu(x) = \begin{cases} 0 & \text{if } x \leqslant -2^{-\nu+1} \\ \sqrt{-3 \cdot 2^{-\nu}x - x^2 - 2^{-2\nu+1}} & \text{if } -2^{-\nu+1} \leqslant x \leqslant -2^{-\nu} \\ 0 & \text{if } -2^{-\nu} \leqslant x \leqslant 2^{-\nu} \\ \sqrt{3 \cdot 2^{-\nu}x - x^2 - 2^{-2\nu+1}} & \text{if } 2^{-\nu} \leqslant x \leqslant 2^{-\nu+1} \\ 0 & \text{if } x \geqslant 2^{-\nu+1} \end{cases} \tag{89}$$

Let A be a proposition that is at present not decidable. The Creating Subject constructs a choice sequence ζ of total real functions $\zeta(n) \colon \mathbb{R} \rightarrow [0, \frac{1}{4}]$:

- As long as, by the choice of $\zeta(n)$, the Creating Subject has obtained evidence neither of A nor of $\neg A$, $\zeta(n)$ is chosen to be ω_n.
- If between the choice of $\zeta(m-1)$ and $\zeta(m)$, the Creating Subject has obtained evidence of $A \vee \neg A$, $\zeta(n)$ for all $n \geqslant m$ is chosen to be the constant function $\lambda x.0$.

Define the function $Z\colon \mathbb{R} \to [0, \frac{1}{4}]$ by

$$Z(x) = \sum_{\nu=1}^{\infty} \zeta(\nu)(x) \tag{90}$$

From the definition of the $\zeta(n)$ we have

$$\neg \exists n(\zeta(n) = \lambda x.0) \to \neg(A \vee \neg A) \tag{91}$$

which together with $\neg\neg(A \vee \neg A)$ gives

$$\neg\neg \exists n(\zeta(n) = \lambda x.0) \tag{92}$$

So the function Z cannot fail to diverge from $\sum_\nu \omega_\nu$.

In the remainder we only consider $r = 0$.

We have

$$\exists n(\zeta(n) = \lambda x.0) \to \neg \exists s \, \exists m \, \forall \nu(|d_{s\nu}| > 2^{-m}) \tag{93}$$

because if $\exists n(\zeta(n) = \lambda x.0)$ then for any s, once ν has become large enough and i_ν small enough, all $d_{s\nu}$ vanish. From (92) and (93) we have

$$\neg\neg\neg \exists s \, \exists m \, \forall \nu(|d_{s\nu}| > 2^{-m}) \tag{94}$$

which is equivalent to

$$\neg \exists s \, \exists m \, \forall \nu(|d_{s\nu}| > 2^{-m}) \tag{95}$$

So Z is weakly differentiable in $x = 0$ and there has the weak differential quotient 0.

Now suppose that Z is also strongly differentiable in $x = 0$ and there has the strong differential quotient 0.

From the definition of the functions ω_ν it can be derived that there are two lines from the origin that are tangent to all of them and bound all of them from above. These lines have slopes $\sqrt{2}/4$ and $-\sqrt{2}/4$ (Figure 1), and pass through each ω_ν at the points with the respective x coordinates

$$x_\nu = \frac{2^{(-2\nu+1)/2}}{3} \tag{96}$$

and $-x_\nu$. For any $m \in \mathbb{N}$, we can find a ν such that all positive values of ω_ν lie in the interval $(-2^{-m}, 2^{-m})$. For such a ν, the difference quotients relative to the function $\sum_\nu \omega_\nu$

of the intervals $[-x_\nu, 0]$ and $[0, x_\nu]$ attain the absolute value $\frac{1}{4}\sqrt{2}$; and these intervals can be used in the construction of sequences s. In contrast, for $c = 0$, (86) gives

$$\forall n\, \exists m\, \forall s\, \forall \nu (|i_{s\nu 2} - i_{s\nu 1}| < 2^{-m} \to |d_{s\nu}| < 2^{-n}) \tag{97}$$

so for each s the difference quotients relative to Z of any of its intervals will for all n eventually become $< 2^{-n}$ in absolute value, and therefore $\lessdot \frac{1}{4}\sqrt{2}$. It follows that on such intervals none of the values $Z(x)$ will have been generated by an ω_ν anymore. It must then be the case that

$$\exists n(\zeta(n) = \lambda x.0) \tag{98}$$

and hence, by the definition of the $\zeta(n)$,

$$A \vee \neg A \tag{99}$$

By hypothesis, A is not yet decidable; therefore, Z is not strongly differentiable.

In this argument, the argument from the assumption that Z is strongly differentiable in $x = 0$ amounts to an argument that, once A is given, there exists a sequence ζ^* defined by

$$\zeta^*(n) = \begin{cases} 0 & \text{if } \forall k \leqslant n(\zeta(k) \neq \lambda x.0) \\ 1 & \text{otherwise} \end{cases} \tag{100}$$

which has, in particular, the property that $\exists n(\zeta^*(n) = 1) \to A \vee \neg A$; hence it is, in effect, a use of an instance of BKS^-.[68]

A (mis)application of MP would have led one to conclude from (92) to (98); see on this point also subsection 7.2 below.

6.2 BKS$^+$

That Brouwer had asserted BKS$^+$ himself was seen by Myhill [123, p.295], with reference to the point in 'Points and spaces' where Brouwer indeed can be said to demonstrate

[68] The referee pointed out that there is a considerably simpler plausibility argument for Weak counterexample 45:

> Let f be the function from $\mathbb{R}$ to $\mathbb{R}$ such that $\forall x \in \mathbb{R}[f(x) \# 0 \leftrightarrow \exists n[\alpha(n) \neq 0 \wedge \exists a \in (0,1)[x = a/(n+1) + (1-a)/n \wedge f(x) = \inf(a, 1-a)/n]]]$. If one assumes: $\neg\neg\exists n[\alpha(n) \neq 0] \wedge \forall m \forall n[(\alpha(m) \neq 0 \wedge \alpha(n) \neq 0) \to m = n]$, and is unable to prove: $\exists n[\alpha(n) \neq 0]$, then f is weakly differentiable at 0 (with outcome 0) but one is unable to prove: f is strongly differentiable at 0.

The referee adds that the example shows that the notion of weak differentiability is not very useful. I mention these points for their intrinsic interest.

BKS$^+$ when he proves the equivalence $A \leftrightarrow C(\gamma, A) \in \mathbb{Q}$ (see Plausibility argument 31 in subsection 3.7 above). From the equivalence an explicit construction for a sequence α witnessing BKS$^+$ is obtained immediately by setting

$$\alpha(n) = \begin{cases} 1 & \text{if } \exists k \leqslant n(C(\gamma, A)(k) = C(\gamma, A)(k+1)) \\ 0 & \text{otherwise} \end{cases} \tag{101}$$

A hypothesis in Brouwer's argument is that A has been recognised neither as tested nor as testable. Strictly speaking, Brouwer therefore demonstrates BKS$^+$ only for A satisfying that condition; but there is nothing in the construction of the checking number $C(\gamma, A)$ itself that hinges on this. For Brouwer's purpose in the argument at hand, a weak counterexample to the quantified form of the Principle of the Excluded Middle, there naturally is no use for greater generality. There is, however, in his argument from 1949, discussed in subsection 3.5, because various real numbers may well have already been proved to be rational before the construction of the drift γ has even begun. Indeed, as was noticed in Definition 16 in subsection 3.4, Brouwer's notion of a checking number of a drift through A imposes no condition on the status of A, and explicitly allows for the case that A has been decided before the construction of the sequence begins. As a general principle, then, BKS$^+$ will be proved in the same way. There are, however, no later counterexamples in which Brouwer exploits the equivalence again, and which could therefore be reconstructed as uses of BKS$^+$. (Brouwer stopped publishing altogether the next year.) Brouwer's construction dependent on a proposition A of a number that is rational if and only if A is true will have been Brouwer's closest approximation to truth values.[69]

7 Discussion of objections to CS and BKS

From the beginning, several objections have been raised to both the weak and strong versions of CS and BKS. Here I present a survey, including only arguments that either were made within a broadly Brouwerian framework or would (prima facie) be translatable into one. For example, it may be observed that it is highly unlikely that a meaning explanation of the propositional operator $\Box_n$ will be found in Martin-Löf's Constructive Type Theory: in that theory there are no choice sequences as standard objects,[70] mathematical truth is not tensed, and first-person knowledge plays a role, but not in the content of mathematics. But to advance an observation of this difference as an objection would be question-begging.

[69] Joachim Lambek recounts [106, p.62] how when he met Brouwer in 1953 (at the conference where Brouwer presented 'Points and spaces' as a lecture series), the latter expressed doubts whether Wittgenstein had made any contributions to logic. When Lambek suggested that he had, because he had come up with truth tables, Brouwer asked: 'What are truth tables?'

[70] As Martin-Löf shows [112], a simple kind of choice sequence can be defined as a nonstandard type.

The real discussion between the opposing views of Brouwer and Martin-Löf must be about more general matters than any specific principle in mathematics or logic. The most interesting and potentially most effective objections to CS and BKS are those that aim to show that accepting them goes against the philosophical intentions, accepted evidence, or standards of Brouwer's program itself.[71] (The objection of conflict with Church's Thesis in the next subsection is, in this respect, best seen as a boundary case.) I will attempt to show that none of the objections discussed below succeeds. However, several objections point to delicate matters of mathematical or philosophical interest. The grouping of the objections is loosely thematic.

7.1 BKS^- contradicts Church's Thesis

The following result is due to Kripke:

Theorem 47. BKS^- entails that there exists an effective but non-recursive function.

Kripke did not publish his argument, but it was presented by Kreisel in 1970. It uses the formulation for species $\mathrm{BKS}^-\mathrm{SF}$ (p.25 above), which for convenience is repeated here:

$$\exists f \, \forall x \begin{bmatrix} \neg \exists n(f(n) = x) \to x \notin X \\ \wedge \\ \exists n(f(n) = x) \to x \in X \end{bmatrix} \tag{$\mathrm{BKS}^-\mathrm{SF}$}$$

Proof 48 ([89, p.145n10])**.** CT is formulated as

$$\forall f \, \exists e \, \forall n \, \exists p[T_1(e, n, p) \wedge U(p) = f(n)] \tag{102}$$

where T is Kleene's T-predicate.

Combining $\mathrm{BKS}^-\mathrm{SF}$ and CT yields

$$\exists e \, \forall n \begin{bmatrix} \neg \exists m \, \exists p(T_1(e, m, p) \wedge U(p) = n) \to n \notin X \\ \wedge \\ \exists m \, \exists p(T_1(e, m, p) \wedge U(p) = n) \to n \in X \end{bmatrix} \tag{103}$$

Consider the species $X_0 = \{n \mid \forall y \neg T_1(n, n, y)\}$. Kleene has shown that it is not recursively enumerable, which may be paraphrased by saying that if a recursive function is total, it does not have X_0 as its range:

$$\forall m \, \exists p T_1(e, m, p) \to \neg \forall n \begin{bmatrix} \neg \exists m \, \exists p(T_1(e, m, p) \wedge U(p) = n) \to n \notin X_0 \\ \wedge \\ \exists m \, \exists p(T_1(e, m, p) \wedge U(p) = n) \to n \in X_0 \end{bmatrix} \tag{104}$$

[71] This reflects a view on philosophical controversy developed by Henry Johnstone [76].

Kreisel points out that Kleene's theorem has a proof in Heyting Arithmetic. Instantiating (103) with $X = X_0$ yields an e that satisfies the antecedent of (104), so that instance and (104) are contradictory, hence the conjunction of CT and $\mathrm{BKS}^{-}\mathrm{SF}$ is, as Heyting Arithmetic is not contested.[72]

In Kreisel's presentation, the proof applies $\mathrm{BKS}^{-}\mathrm{SF}$ to $X = X_0$ only indirectly, after combining it with CT; it thus leaves implicit that the f that a direct application of $\mathrm{BKS}^{-}\mathrm{SF}$ to $X = X_0$ would give is a counterexample to CT. The existence of a counterexample is more prominent in Myhill's version. It uses

Lemma 49 ([123, p.296–297]). Let $A(n, x, y)$ be an arbitrary predicate with only the free variables n, x, y, all ranging over $\mathbb{N}$. Then BKS^{-} implies that $\exists f \neg \exists n \forall x \forall y (f(x) = y \leftrightarrow A(n, x, y))$.

Myhill does not go on to give a proof of this lemma, but one is given by Dragálin; I here modify it slightly to highlight the fact that it in effect consists in an application of $\mathrm{BKS}^{-}\mathrm{S}$ to the species $X_A = \{n \mid \forall k A(n, j(n, k), 0)\}$.

Proof 50. This is an adaptation of the proof given by Dragálin [55, p.134–135].

Applying $\mathrm{BKS}^{-}\mathrm{S}$ to the species $X_A = \{n \mid \forall k A(n, j(n, k), 0)\}$ yields a sequence β such that

$$\forall x (\forall y (\beta(j(x, y)) = 0) \leftrightarrow x \notin X_A) \tag{105}$$

Set $f = \beta$. Then $\neg \exists n \forall x \forall y (f(x) = y \leftrightarrow A(n, x, y))$. For assume, towards a contradiction, that for some z,

$$\forall x \forall y (f(x) = y = y \leftrightarrow A(z, x, y)) \tag{106}$$

This implies, given that $x = j(\nu, k)$ for unique ν and k, that

$$\forall \nu (\forall k (f(j(\nu, k)) = 0) \leftrightarrow \forall k A(z, j(\nu, k), 0)) \tag{107}$$

Instantiating (105) with $y = z$ and (107) with $\nu = z$, it follows that

$$z \in X_A \leftrightarrow z \notin X_A \tag{108}$$

Proof 51 (of Theorem 47). Set $A(n, x, y) := \exists w (T_1(n, x, w) \wedge U(w) = y)$ and apply Lemma 49.

If CS^{+} is used instead of one of the variants of BKS^{-}, Theorem 47 is proved as follows:

[72] Brouwerians may find the inclusion in its logic of Ex Falso Sequitur Quodlibet objectionable [159], but that is not relevant here.

Proof 52 (of Theorem 47)**.** This proof has been adapted from that given by Van Dalen [169, p.40n3].[73]

Let K be a species that is recursively enumerable, but not recursive. Define

$$f(n,m) = \begin{cases} 0 & \text{if } \neg\Box_m n \notin K \\ 1 & \text{if } \ \ \Box_m n \notin K \end{cases} \tag{109}$$

Then

$$n \notin K \leftrightarrow \exists m f(n,m) = 1 \tag{110}$$

From left to right, this follows from CS^+3a; from right to left, this follows from CS^+3b. For the Creating Subject this function f is computable, as, by CS^+1, for any given m and n, $\Box_m n \notin K$ is decidable. Assume that f is moreover recursive. Then the species $S = \{n \mid \exists m f(n,m) = 1\}$ is recursively enumerable, but as $S = K^c$, this contradicts the hypothesis.[74]

Although Theorem 47 can be, and has been, taken as casting doubt on BKS^-, the real target of any such doubts is of course the notion of constructive non-recursive sequence itself; in its simplest form, it is the doubt that lawless sequences are individual mathematical entities. In subsection 7.7, I argue that Brouwerians have no reason to harbour that doubt. Unsurprisingly, BKS^- is not the only principle in the theory of choice sequences that contradicts CT:

Theorem 53 ([148, p.211])**.** WC-N contradicts CT.

Proof 54. Apply WC-N to CT as formulated in (102); then for every function f, computed by a recursive function with index e, there is an m such that any function g that agrees with f on the arguments $1, \ldots, m$ is likewise computed by the recursive function with index e. But then agreement of two functions on $1, \ldots, m$ implies agreement everywhere, which is absurd.

7.2 BKS^+ and MP together imply DNE and PEM

In 1980, Joan Moschovakis found a connection between BKS^- and MP:

[73] In the exposition I gave of this proof in [158], the appeal to Post's Theorem, while to my mind correct – it is derivable from MP for primitive recursive predicates [148, p.205], which is a highly plausible principle [153, section 2] – is wholly superfluous.

[74] According to Dirk van Dalen (personal communication), this argument was considered common knowledge at the Summer Conference on Intuitionism and Proof Theory, SUNY at Buffalo, 1968. It is, however, not found in either of the two publications that came out of that meeting, Troelstra's *Principles of Intuitionism* [145] and the proceedings edited by Kino, Myhill, and Vesley [78].

Theorem 55 ([119, p.250-251]). Let A be any proposition. Then BKS^- and MP imply Double Negation Elimination, $\neg\neg A \to A$.

Proof 56. Assume $\neg\neg A$. Applying BKS^- gives

$$\exists\alpha \begin{bmatrix} \forall n(\alpha(n) = 0 \lor \alpha(n) = 1) \\ \land \\ \forall n(\alpha(n) = 0) \leftrightarrow \neg A \\ \land \\ \exists n(\alpha(n) = 1) \to A \end{bmatrix} \tag{111}$$

We then have $\neg\neg A \to \neg\forall n(\alpha(n) = 0)$ and therefore $\neg\forall n(\alpha(n) = 0)$ and $\neg\neg\exists n(\alpha(n) = 1)$. By MP, now $\exists x(\alpha(n) = 1)$ and hence A. So $\neg\neg A \to A$.

Troelstra and Van Dalen give the similar direct argument for the conclusion that BKS^+ and MP imply PEM, shown by applying (in effect) BKS^- to $A \lor \neg A$ [148, p.237].[75]

Depending on one's views, these arguments either refute BKS^- and BKS^+, or MP. However, as Troelstra and Van Dalen remark, their refutation of MP from BKS^+ goes through 'in an axiomatic setting' [148, p.237], and the same can be said of the above theorem. What would be needed for contentual arguments are notions of sequence for which both MP and BKS^-, respectively BKS^+ hold. But MP is only plausible (to my mind, highly so) for recursive sequences, whereas the Brouwerian justification of either version of BKS depends on the sequence of acts of the Creating Subject, which, as recalled in the previous subsection, is certainly not recursive.

7.3 CS leads to a paradox

In 1969, Troelstra constructed the following paradox in CS. Assume that at each stage m, the Creating Subject obtains evidence for one proposition, $A^{(m)}$:

> [S]ince it is natural to assume that we know when a conclusion has the form 'a is a lawlike sequence' we have:
>
> $A^{(m)}$ is a conclusion of the form 'a is a lawlike sequence' or
> $A^{(m)}$ is a conclusion of another kind.
>
> Then it is possible for us to enumerate the $A^{(m)}$ of the form 'a is lawlike'; let $A^{(bx)}$ be the x^{th} conclusion of this form, stating 'a_x is a lawlike sequence'. Then
>
> $$\bigwedge x \bigvee a(A^{(bx)} \equiv a \text{ is a lawlike sequence})$$

[75]Compare also Myhill's refutation of MP from 1963 [120], which does not use BKS (which had not been isolated yet) but whose setting is that of Kreisel's Theory of Constructions.

and so we conclude to the existence of a b' such that

$$b'(x,y) \equiv a_x(y),$$

Intuitively $c = \lambda x.b'(x,x) + 1$ is a lawlike sequence, but then we ought to be able to indicate a $z \in \mathbb{N}$ such that

$$A^{(bz)} \equiv c \text{ is a lawlike sequence}$$

which implies:

$$\bigwedge x(b'(x,x) + 1 = b'(z,x))$$

which is contradictory. [145, p.105–106]

However, the steps towards a contradiction depends on Markov's Principle. Associate to each sequence a_n a sequence a_n^* such that $a_n^*(m)$ is 0 if by stage m the construction of a_n has not yet begun, and 1 if it has. Then it can be argued from the essential freedom of the Creating Subject, that

$$\forall n \neg\neg \exists k(a_n^*(k) = 1) \tag{112}$$

because the Creating Subject is free to work towards constructing its n-th lawlike sequence, but that

$$\forall n \, \exists k(a_n^*(k) = 1) \tag{113}$$

is not true, because the same freedom allows the Creating Subject, for given n, not to begin constructing its n-th lawlike sequence by whatever stage k the lawlike sequence a_n^* indicates. MP is not correct for the sequences a_n; but this means that constructively the enumeration b does not exist. For the details of this argument, the reader is referred to [153], where it is also argued that this dependence on Markov's Principle survives in both the solution proposed by Troelstra [145, p.106–107] and that proposed by Niekus [126, section 3], neither of which can therefore be accepted. Briefly sketched, Troelstra's solution consists in a stratification of propositions and the objects they are about, so that the enumeration b will be of a higher level than, and hence not occur among, the a_n. Niekus' solution consists in a limitation of reference to stages to those in the future, and insisting that the conclusion drawn at each stage be new; the combined effect is that when the sequence c is defined, the sequences it depends on will only be constructed in the future, so that c is itself not among them. I will have occasion to discuss Niekus' solution further in subsection 7.10, because it involves questions about how to interpret Brouwer that would have to be discussed also if Troelstra's Paradox had not existed.

Troelstra calls the sequence c 'lawlike'.[76] In view of the non-recursiveness of the Creating Subject's activity (see above), this may be surprising. However, earlier in that chapter Troelstra explained his terminology:

> If we have a definite prescription involving the actions of the creative subject (by means of a relation like $\vdash_n A$) for determining the values of a sequence, we speak of an empirical sequence.
>
> Our idea of lawlike sequence does not exclude empirical sequences, at least not as long as we are willing to consider reference to our own course of activity by means of $\vdash_n$ as 'definite'.
>
> It is clear, however, that e.g. primitive recursive functions are lawlike in a stricter, more objective sense; their values are independent of *future decisions about the order in which we want to make deductions*. [145, p.96–97, emphasis mine]

So Troelstra is clear that Creating Subject sequences ('empirical sequences') do depend on free choices of the Creating Subject, and that they are not lawlike in a sense that would include their being independent of the Creating Subject's free choices. Thus, he introduces a further term to reflect this distinction:

> If a sequence ξ is defined by a complete description from sequences $\chi_1, \chi_2, \ldots$, without reference to the creative subject, we shall call ξ *mathematical* or *absolutely lawlike* in $\chi_1, \chi_2, \ldots$. [145, p.97, emphasis mine]

Troelstra's motivation for the wide conception of lawlikeness would seem to be that also in Creating Subject sequences, once the Creating Subject is about to choose the n-th value, it cannot do so freely, as that value is fully determined by something else. It is just that this something else itself has come about in an exercise of the essential freedom of the Creating Subject: the Creating Subject's activity up to the moment of choosing the n-th value.[77]

To avoid the connotation of predetermination often associated with lawlikeness, a connotation that Troelstra as we saw does not wish to evoke in all cases, in the later presentation of Troelstra's paradox in *Constructivism in Mathematics*, 'lawlike' is replaced by 'fixed by a recipe' [148, p.845]. We will briefly return to this in subsection 7.10.

It is sometimes suggested that the Creating Subject's activity is determined by a law, but just one that it does not know itself. This suggestion, however, is not consistent with the Brouwerian conception of mathematical existence: if some object exists mathematically, this is only because the Creating Subject has defined and constructed it; all

[76] Similarly, Van Dalen [59, p.265] and Troelstra [142, p.126] call a sequence whose existence is guaranteed by BKS 'lawlike'.

[77] This is also remarked on by Niekus [126, p.441].

mathematically relevant truth is brought about in that activity, and therefore known to the Creating Subject [153, section 4].

7.4 Proofs are not different at different times

At the Brouwer memorial symposium in Amsterdam in December 2016, Kripke stated his motivation for BKS^- as follows:[78]

> Intuitionistically, to prove a conditional $A \to B$, one must have a technique so that from any proof of A one can get a proof of B.
>
> Now, the idea of the schema, based on Brouwer's own arguments about the creative subject, is that one imagines a sequence in time that is 0 as long as A has not been proved but is 1 as soon as it has been proved.
>
> Then one claims that a proof that the sequence is always 0 amounts to a proof that A can never be proved, i.e. that it is absurd.
>
> If the sequence gets the value 1, then A has been proved. [100, slides 35 and 36]

Note here the appeal to the intuitionistic notion of truth: since one reasons about one ideal subject, who can carry out whatever mathematical construction can be carried out at all, then if it is known that *it* will never carry out a construction for a certain proposition A this can only be because it is not *possible* to do so.

Kripke then went on to state his objection to BKS^+:

> What would justify the strong form?
>
> Well, as Myhill says,[79] if A is ever proved, it is proved at some definite time, and then one gets a value of α that is not equal to 0 simply by looking at your watch (or calendar, as the case may be)
>
> However:
>
> Is the idea of the intuitionistic conditional (remember that from a proof of A, one can get a proof of B) such that intuitionistic proofs, which indeed must take place at some definite time, have the time of their occurrence as part of the proof, so that the same proof would be different if it happened to occur at a different time?
>
> I think not.
>
> In that case, the strong form does not appear to be justified, but the weak form is. [100, slides 36 and 37]

[78] This corresponds to the end of his contribution to the present volume, [101].

[79] [Note MvA: [123, p.295].]

When Kripke above speaks of 'proofs, which indeed must take place at some definite time', I would argue this must be understood as referring primarily to proofs as token construction processes, and, founded on that, to token constructions in the other two senses; for it is the tokens that are bound to a definite time. And when Kripke asks whether the idea is that 'the same proof would be different if it happened to occur at a different time', I would argue that that phrase is ambiguous but that one of the disambiguations is indeed is correct: the sameness is to be understood as sameness of type, but the difference as difference between tokens.

The conclusion I draw from Kripke's objection to BKS^+ is that he conceives of constructive mathematics, truth and proof in a different way than Brouwer did. But, symmetrically, I don't think that the fact that for Brouwer the strong form is defensible is, by itself, an argument against Kripke's conceptions.

7.5 CS depends on a function that is not unitype

The type-token distinction has been appealed to before in a similar situation, by Timothy Williamson in a discussion not of Kripke's Schema but of Fitch's Paradox [193]. Fitch's Paradox itself, which from the premise that all truths are knowable derives the conclusion that all truths are known, will not concern us here.[80] But in his discussion, Williamson comes to consider the intuitionistic meaning of the principle that if some proposition A is proved, then it is proved at a definite time t; Williamson writes this schematically as $A \to \mathsf{K}A$, and he notes that there are two different forms of the consequent to consider: either t is given explicitly as a parameter, or there is an existential quantifier over moments in time. The latter corresponds to $\mathsf{CS}^+3\mathrm{a}$.

As in my defense of $\mathsf{CS}^+3\mathrm{a}$, Williamson holds that the distinction between types and tokens is pertinent here, but he aims to use it to opposite effect: like Kripke, he argues for the conclusion that the principle does not hold, but for a different reason.

First note the particularity of Williamson's position that the notion of type recognised turns out to be so specific that types cannot be divided into subtypes; the only way the objects of a type can differ is numerically. Applied to proofs, this means that for Williamson two proof tokens are of the same type if and only if these proof tokens have exactly the same intermediate and final conclusions [193, p.431n16].

His argument against $A \to \mathsf{K}A$ then comes down to this.

Let a_1 and a_2 be token proofs of proposition A effected at times t_1 and t_2, respectively. They are both proofs of A, so they are of the same type. Assume that there is a function f that proves the implication $A \to \mathsf{K}A$. Williamson requires that this function be 'unitype', that is, if its arguments are of the same type, then so should its values be.

[80] Congenial discussions are those by Sundholm [138, p.20–21] and Klev [82].

First consider the case where the conclusions of the two token proofs $f(a_1)$ and $f(a_2)$ are given in the first form, with an explicit parameter t. Then function f takes the token proof a_1 to a token proof of the proposition that A is proved at time t_1 and the token proof a_2 to a token proof of the proposition that A is proved at time t_2. These token proofs have different conclusions, so f does not fulfill the requirement of being unitype.

In the other case, the conclusions of the two token proofs $f(a_1)$ and $f(a_2)$ have the form of an existentially quantified statement, and are identical. Williamson points out that in that case these token proofs still differ in another respect, because their intermediate conclusions to which existential generalisation is applied must be different. Then f is still not unitype.[81]

But the notion of type Williamson employs here is far too restrictive, and in his discussion I do not find a motivation for it. Compare for example the Curry-Howard-De Bruijn isomorphism, where a proposition is considered to be the type of all its proofs, however different the latter may be in their inner workings. Similarly, Brouwerian species – species is the word Brouwer would have used for type – may collect construction objects, so in particular proofs, based on whatever similarity we can see between them in reflection upon the underlying construction acts, in second-order mathematics [152, section 3]. In such a setting, Williamson's argument would not succeed. The reason why Williamson actually wants it to succeed is that he takes the principle $A \to KA$ to contradict the idea that there may well be truths that are not ever known. He regards the latter a truism, and perhaps in some or many domains it is. But in pure mathematics as Brouwer conceives of it, as an activity in which the mathematical facts are brought about, 'truth is only in reality, i.e. in the present and past experiences of consciousness' [37, p.1243], and therefore to assume that A is true is to assume that it is known that A is true.

7.6 BKS$^-$ contradicts $\forall\alpha\,\exists\beta$-continuity

A generalisation of C-N is $\forall\alpha\,\exists\beta$-continuity [85, IV-21; 86, p.135; 81, p.73].

Principle 57 ($\forall\alpha\,\exists\beta$-continuity).

$$\forall\alpha\,\exists\beta A(\alpha, \beta) \to \exists\Phi\,\forall\alpha A(\alpha, \Phi(\alpha)) \tag{114}$$

where Φ is a continuous functional.

[81] Williamson goes on to argue that if A has been decided, a unitype function may still be obtained by mapping every proof token of A to a proof that A was proved at time t for some fixed t (for example, the t at which A was proved for the first time). But he observes that this will not work for as yet undecided A, and in any case, the idea to map a proof token of A obtained at t_1 to the moment t_0 associated with a different proof token goes wholly against the Brouwerian descriptivist conception: there is no intrinsic relation between the act in which the former proof token was created and t_0.

No formulation of it is found in Brouwer, and, as Myhill observed [98, p.174], Brouwer in 1949 [36] in effect constructed a counterexample to it – because, in Proof 29 above, the functional that to each real number f associates the direct checking number $D(\gamma, \alpha_f)$ is discontinuous. Myhill also established

Theorem 58 ([123, p.293–296; 98, p.173–174]). $\forall\alpha\,\exists\beta$-continuity is not consistent with BKS^-

Proof 59 ([98, p.173–174]). Let α range over real numbers. By applying BKS^- to the proposition $\alpha \in \mathbb{Q}$ we have

$$\forall\alpha\,\exists\beta \begin{bmatrix} \forall n(\beta(n) = 0 \vee \beta(n) = 1) \\ \wedge \\ \forall n(\beta(n) = 0) \leftrightarrow \alpha \notin \mathbb{Q} \\ \wedge \\ \exists n(\beta(n) = 1) \rightarrow \alpha \in \mathbb{Q} \end{bmatrix} \tag{115}$$

Now if $\beta = \Phi(\alpha)$ for a continuous functional Φ, we would have

$$\forall\alpha\,\exists m\,\forall\gamma(\bar{\alpha}m = \bar{\gamma}m \rightarrow ((\exists n(\Phi(\alpha)(n) = 1) \rightarrow \gamma \in \mathbb{Q})) \tag{116}$$

then the continuity of Φ contradicts the fact that any initial segment of an $\alpha \in \mathbb{Q}$ can be continued into an irrational number.

The system FIM of Kleene and Vesley includes $\forall\alpha\,\exists\beta$-continuity (there named 'Brouwer's Principle for functions' [81, p.72], which is perhaps unfortunate because, as Kleene and Vesley are aware, Brouwer never formulated it and moreover accepted discontinuous functionals). Dragálin calls the discovery of the resulting inconsistency of BKS^- and FIM 'the most dramatic moment in the history of Kripke's scheme' [55, p.135]. Yet it is clear that FIM, in the form in which it was published, was not designed to be suitable for Brouwer's Creating Subject arguments:

> Another point in the formalization in FIM criticized by Myhill (1967) is that the free choice sequences are extensional.[82] Certainly, extensional free choice sequences are intuitionistically acceptable; for these, one restricts the freedom of the choices only by the choice law adopted in advance. Since in fact the intuitionistic theory of the continuum can be developed using only extensional choice sequences, it seems more interesting to do so. The complication of nonextensional free choice sequences (where at each choice one picks both a

[82][Note MvA: [123, p.286].]

> function value and a new choice law within the preceding one) can be left until a need arises, as perhaps for the formalization of Brouwer's 'historical' arguments. [79, p.138n2]

The inadequacy of FIM if it were meant as a formalisation of Brouwerian foundations is exemplified by the facts that in FIM, MP is formally undecidable [81, p.131], and that if FIM proves the existence of an individual choice sequence with a certain property then it proves the existence of a recursive sequence with that property. [80, p.101][83] At the same time, as Vesley remarks in his very useful retrospective [191, p.324], FIM could be seen as supporting Brouwer's claim, in his 'Second act of intuitionism', that choice sequences bring something new: in FIM it is shown that the Bar Theorem does not hold classically [81, p.87—88].[84] And FIM as published may of course be extended with principles that bring it closer to Brouwerian analysis; Vesley's Schema is an example (see subsection 7.8 below).

The conflict between $\forall\alpha\,\exists\beta$-continuity and BKS^- can be mitigated in two different ways. Troelstra has remarked that there may be interesting subuniverses of choice sequences, such as that described by his own axiomatic theory GC, that are not closed under BKS^-, in which case it does not serve to refute $\forall\alpha\,\exists\beta$-continuity [143, sections 6 and 9; 146, p.137; 144, p.235]. Secondly, the weaker variant $\forall\alpha\,\exists!\,\beta$-continuity, which is $\forall\alpha\,\exists\beta$-continuity except that the antecedent requires uniqueness of β, is often sufficient in analysis, is not inconsistent with KS [122, p.152; 104], and follows from C-N [81, p.89].

7.7 BKS^+ must be restricted to determinate properties

In reaction to the conflict of BKS^- (and hence BKS^+) with $\forall\alpha\,\exists\beta$-continuity, Johan de Iongh proposed (but, characteristically, did not publish) a restriction on the propositions that BKS^+ may be applied to. His students Gielen, Veldman, and De Swart describe it as follows:

> Application of the axiom should be restricted to propositions which are determinate in the sense that they do not depend on infinite objects which still are under construction, and are created more or less freely, not merely being developed from their previously given definition. I may need some more thought, but not more information, in order to know if a determinate proposition is true.
>
> [...]

[83] A phenomenological analysis [157] leads to the conclusion that also non-lawlike choice sequences are individual mathematical objects

[84] Vesley there [191, p.326] also rightly emphasises the fact that Myhill and Kreisel had taken the initiative to see CS and BKS not only as means to formalise Brouwerian counterexamples, but as potentially useful in positive intuitionistic developments. On the latter, see subsection 4.4.

The restriction proposed by J.J. de Iongh, seems rather natural: as long as information about a proposition P has not yet been completed, I cannot really start to think about its truth. [63, p.126-127]

Application of BKS^- to a proposition of the form $\alpha \in \mathbb{Q}$ is then obviously no longer permitted for α that have not been specified by a law.

In the view of De Iongh and some of those inspired by him, a construction process that is not governed by a finite, full definition does not yield a construction object, and remains only a construction project (the term is De Iongh's [51, p.204]). The term 'construction project' itself is considered to be primitive and it is acknowledged that construction projects may involve making more or less free choices [62; 51, p.204]. It then follows that lawless sequences are no construction objects, but remain partially defined construction projects (the Creating Subject knows the choices made so far). But one wonders if an identity criterion for partially defined construction projects that is allowed to depend on the moment in time at which the project is begun would not quickly lead to an identity criterion for lawless sequences [157].

De Swart [51, p.208] explains 'α is lawless' by the negative characterisation 'i.e. there is no finite law that determines α', and then rejects that notion because there seems to be no finite definition of the general notion 'finite law' that it presupposes.[85]

This stands in contrast to the approach of Troelstra and Van Dalen, in which lawless sequences are indeed construction objects, which as such are to be considered individuals, and which can be quantified over. More generally, on their approach the explanation of $\forall\alpha$, where α ranges over sequences of a certain type, requires an analysis of what exactly is given to the Creating Subject when it gives itself a sequence of that type. It does not require a construction method that would generate all sequences of that type.

It seems to me that the approach of Troelstra and Van Dalen captures Brouwer's descriptions such as the following more accurately:

Intuitionistic mathematics is a mental construction, essentially independent of language. It comes into being by self-unfolding of the basic intuition of mathematics, which consists in the abstraction of two-ity. This self-unfolding allows us in the first instance to survey in one act not only a finite sequence

[85] He presents a second argument: propositional functions should be extensional, but 'α is lawlike' yields a true assertion for the sequence defined as $\lambda x.0$ and false for a sequence that 'accidentally' only contains zeros (the quotation marks here are De Swart's). However, if an infinite sequence contains only zeros, this can only be because the Subject had imposed a law on it with that consequence; to suppose otherwise would be to introduce a realist conception. Similarly, I do not understand De Swart's claim on the same page that 'one might imagine lawless sequences α and β which are intensionally different, but extensionally the same'. (See on this point also Martino's comment [113, p.394] on Troelstra's description of his 'abstr' operator as a thought experiment.)

> of mathematical systems, but also an infinitely proceeding sequence, defined by a law, of mathematical systems previously defined by induction. But in the second instance it allows us as well to create a sequence of mathematical systems which infinitely proceeds in complete freedom or is subject to restrictions which may be varied in the course of the progress of the sequence. [33, p.339; trl. 45, p.477]

and

> The first act of intuitionism completely separates mathematics from mathematical language. [...] And the basic operation of mathematical construction is the mental creation of the two-ity of two mathematical systems previously acquired, and the consideration of this two-ity as a new mathematical system.
>
> It is introspectively realized how this basic operation, continually displaying unaltered retention by memory, successively generates each natural number, the infinitely proceeding sequence of the natural numbers, arbitrary finite sequences and infinitely proceeding sequences of mathematical systems previously acquired, finally a continually extending stock of mathematical systems corresponding to 'separable' systems of classical mathematics.
>
> The second act of intuitionism recognizes the possibility of generating new mathematical entities: First, in the form of infinitely proceeding sequences whose terms are chosen more or less freely from mathematical entities previously acquired; in such a way that the freedom existing perhaps at the first choice may be irrevocably subjected, again and again, to progressive restrictions at subsequent choices, while all these restricting interventions, as well as the choices themselves, may, at any stage, be made to depend on possible future mathematical experiences of the creating subject [...]. [41, p.2]

There is no suggestion in these quotations that the infinite sequences described are not on a par and should be divided into those constructions that are proper objects and those that remain construction projects. The sequences described in the second act are generated as mathematical entities just as much as those of the first act.[86] But then so are, in particular, the sequences in which every term is chosen freely; this is a positive characterisation of lawless sequences.[87]

[86]In a lecture from 1951, Brouwer even says that the second act is a special case of the first [46, p.93n]. An analysis of choice sequences as individual, mathematical objects can be carried out phenomenologically [157].

[87]De Swart's negative characterisation seems to be acceptable as an entailment of this positive characterisation on the ground that, whether the extension of the concept of finite law is finitely definable or not, it is certainly part of the meaning of 'law' that it is incompatible with freedom of each choice.

To a description of the 'second act of intuitionism' in another paper than the one just quoted from, but of the same period, Brouwer added the following footnote:

> In former publications I have sometimes admitted restrictions of freedom with regard also to future restrictions of freedom. However this admission is not justified by close introspection and moreover would endanger the simplicity and rigor of future developments. [40, p.142]

This has sometimes been taken to be a rejection of lawless sequences (which may be seen as governed by the second-order restriction that there will be no first-order restrictions). However, the formulation of the second act there, just as the one quoted here in the text, clearly leaves the freedom to make every choice freely. One may phrase this as a second-order restriction, but it is not necessary to bring in that concept.[88] Brouwer's footnote I take to doubt rather the viability of a general theory of higher-order restrictions. Whether that doubt is justified is a question that I will not go into here.[89]

Gielen, De Swart, and Veldman reflect on the reason why De Iongh's suggestion was not widely adopted:

> One may wonder why the restriction [i.e., on BKS^-] we discussed has not found general acceptance among those who work on intuitionism. One reason for this is perhaps that J.J. de longh did not advertise his views, strongly. Besides, Brouwer himself on at least one occasion did not follow this path [9]. He proved a theorem which comes down to
>
> $$\neg\forall\alpha[\neg\neg\exists n[\alpha(n)=0]\rightarrow\exists n[\alpha(n)=0]]$$
>
> by using his principle without any restrictions. [63, p.128]

(Their reference '[9]' is to Brouwer's 1949 paper [36], discussed in subsection 3.5 above.) The distinction between a construction proper and a construction project was well known to Brouwer. It is essential to his notion of denumerably unfinished sets:

> [H]ere we call a set denumerably unfinished if it has the following properties: we can never construct in a well-defined way more than a denumerable subset of it, but when we have constructed such a subset, we can immediately deduce from it, following some previously defined mathematical process, new elements

[88] Also Martino [113, p.396] argues in favour of accepting lawless sequences.

[89] As Kreisel observed (speaking of lawless sequences), the acknowledgement of a higher-order restriction rather simplifies further developments [98, p.180]; a point that Brouwer in writing that footnote may have missed [146, p.131–132]. It can be argued [157, p.41–42] that the so-called 'elimination theorem' for lawless sequences of Kreisel [88] and Troelstra [146, ch.3], does not entail their elimination from the intuitionistic ontology, an entailment that indeed was denied by Kreisel himself [88, p.225–226].

> which are counted to the original set. But from a strictly mathematical point of view this set does not exist as a whole, nor does its power exist; however we can introduce these words here as an expression for a known intention. [10, p.148; trl. 45, p.82]

But in the quotations from 1947 and 1954 above we do not see Brouwer say, analogously, that sequences that are not completely defined do from a strictly mathematical point of view not exist as objects, but that terms for them are introduced as expressions for a known intention (namely, to begin and continue a construction project of a certain kind). This explains the fact noted in the latter half of Gielen, De Swart, and Veldman's reflection.

Still, the distinction at the basis of De Iongh's view between construction processes that are governed by a full definition of the object under construction and those that, as a matter of principle, cannot be thus governed, is a principled one of mathematical relevance, and it is important to realise that, if a proposed axiom turns out not to hold in general, it may still hold for one of these two subclasses.

7.8 A palatable substitute for BKS^-

Vesley [189] proposed a schema that is implied by BKS^- but does not imply it [189, p.199], yet allows alternative arguments for Brouwerian counterexamples without appealing to the Creating Subject. It is also, unlike BKS^-, consistent with FIM and with $\forall\alpha\,\exists\beta$-continuity. Vesley considers it, in the title of his paper, 'a palatable substitute for Kripke's Schema'. Moschovakis showed that in FIM extended with Vesley's Schema it is consistent to hold that all sequences are not not generally recursive [117].

The idea behind Vesley's Schema is the assertion that every continuous function whose domain is a negatively defined dense subset of the continuum can be extended to a continuous function on the full continuum (which classically is the case even without the condition of being negatively defined). This is immediately related to Brouwer's counterexamples because a set such as $\{x \in \mathbb{R} \mid x > 0\}$ is an example of such a negatively defined subset. The formal version of VS does not translate this idea literally, so as to avoid problems that may arise from the notion of partially defined function, but distills:

$$\mathrm{Dense}(\neg A(\alpha), \nu) \wedge \forall\alpha(\neg A(\alpha) \to \exists\beta B(\alpha,\beta)) \to \forall\alpha\,\exists\beta(\neg A(\alpha) \to B(\alpha,\beta)) \qquad \text{(VS)}$$

where ν is the universal spread, and density of the species $\{\alpha \mid \neg A(\alpha)\}$ in ν means that for any initial segment one can find a $\beta \in \nu$ that extends it and for which $\neg A(\beta)$ holds.

Vesley shows that VS is a fragment of BKS^- (his Theorems 1 and 2), and in his Theorem 3 demonstrates various Brouwerian strong counterexamples from it;[90] in fact its

[90] VS has been investigated further by Scowcroft [135].

consequence

$$\text{Dense}(\neg A(\alpha), \nu) \wedge \forall\alpha(\neg A(\alpha) \to \exists b B(\alpha, b)) \to \forall\alpha\, \exists b(\neg A(\alpha) \to B(\alpha, b)) \qquad (117)$$

suffices. Thus, Vesley establishes in an informative and formally precise way that the counterexamples in question do not require the full strength of BKS$^-$ [189, p.198].

Similarly, Van Dalen's result [168] that BKS$^+$ entails that every negative, dense subset $X \subset \mathbb{R}$ is unsplittable (listed in subsection 4.4 above) follows from VS;[91] but the proof from BKS$^-$ (implicit in Vesley, explicit in Van Dalen) is epistemologically to be preferred for the reason given in the previous paragraph.[92]

Epistemologically, on the other hand, it not at all obvious that, once one accepts a Brouwerian setting, VS can be justified on grounds that are clearer than those on which BKS$^-$ is justified. In his 'Autorreferat' in *Zentralblatt* [190], Vesley points out that his paper gives no intuitive motivation for VS; none is given in that review either, nor, it seems, elsewhere. Perhaps such a motivation is found one day; but in the meantime, one seems to have no foundational alternative to accepting VS because of its derivability from BKS$^-$.[93]

Overall, it is clear that there are various contexts in which VS may function as a 'palatable substitute' for BKS, but that the Brouwerian foundational perspective is not one of them. Myhill concluded that 'Vesley's work seems intended as an extremely illuminating technical contribution rather than as a historically accurate rendering of Brouwer's intentions' [124, p.176].

7.9 Brouwer does not want to accept BKS$^-$

In four papers [125–128], Joop Niekus argues that Brouwer avoided the reasoning step that in effect appeals to CS$^-$3b, $\exists n \Box_n A \to A$, and this is taken to indicate that Brouwer had reason to accept neither CS$^-$3b nor, by implication, BKS$^-$.

To motivate his claim that Brouwer avoids the use of CS$^-$3b, Niekus reconstructs the argument from 1948, discussed above in subsection 3.2. In his reconstruction, he defines $<$ as follows:

[91] I thank Joan Moschovakis for pointing this out.

[92] Loeb [107] has shown that Van Dalen's theorem is even equivalent to a weakening of (the assertion motivating) VS, namely to 'Every function $\mathbb{R}\backslash\{0\} \to \mathbb{N}$ is sequentially nondiscontinuously extensible to a function $\mathbb{R} \to \mathbb{N}$'. In another recent formal investigation of VS, Vafeiadou [150, ch.2, sections 7 and 8] investigates VS with $\exists!\, \beta B(\alpha, \beta)$ instead of $\exists \beta B(\alpha, \beta)$.

[93] Vesley [189, p.203] sees some evidence for VS in the penultimate paragraph of Brouwer's 1949 paper on the non-equivalence of the positive and negative order relations [36]; but this is an application of BKS$^-$ (see subsection 6.1). (I thank Joan Moschovakis for discussion of Vesley's remark.) Vesley's reference to p.123 is to the version of Brouwer's paper in the *KNAW Proceedings*, which corresponds to page 38 of the version in *Indagationes Mathematicae* listed in his references.

> All handling of real numbers is done via their generating sequences. For example, for the real numbers a and b, generated by a_n and b_n, $a < b$ holds if $\exists n \, \exists k \, \forall m((b_{n+m} - a_{n+m}) > 2^{-k})$ holds. [127, p.41][94]

The reconstruction proceeds as in Plausibility argument 35 on p.33 above. Niekus notes that Brouwer does not reason thus:

$$\begin{array}{lr} \alpha > 0 & \text{(assumption)} \\ \exists n \Box_n A & \text{(def. } \alpha) \\ A & (\mathrm{CS}^{-}3\mathrm{b}) \end{array} \tag{118}$$

His diagnosis is that

> The use of (4) [=CS$^-$3b] would simplify his argument, and he would not need to resort to an untested proposition, but could have used an undecided one. It seems to us he does not want to use (4). [127, p.36][95]

The simplification, while modest, is there, but it is not so clear that an untested proposition is something to be 'resorted to': the notion of testability is not more complicated than that of decidability, as testability of A is decidability of $\neg A$. And if one's faith that there will always be undecided propositions is based on the faith that there will always be completely open problems, then it is as reasonable to expect that there will always be untested propositions, because a problem is not completely open if for the corresponding proposition A one already possesses a proof of $\neg\neg A$. Two arguments where Brouwer could have used an undecided proposition, but invokes an untested one, can be found in 'Points and spaces' [41, p.533] and in the *Cambridge Lectures* [46, p.50].

Be that as it may, in fact Brouwer in his 1948 paper had a good reason for using an untested proposition: it allows him to draw not only the conclusion that $\neq$ is an essentially negative property, but also that $<$ is, which, unlike the former, could not have been arrived at using an undecided proposition (see the remarks to Plausibility arguments 10 and 11 in subsection 3.2).

Niekus does not comment on that, but it leads us to the main problem with Niekus' account: his definition of $<$, and his interpretation of that sign in Brouwer's text, correspond not to that of Brouwer's $<$, but to that of Brouwer's $\lessdot$. Yet, Brouwer in this paper uses both $<$ and $\lessdot$, and distinguishes them at the end as the 'the simple negative property $\rho > 0$' and 'the constructive property $\rho \gtrdot 0$';[96] Brouwer evidently trusts that the reader will look up the definitions in his earlier papers. As a consequence, when Niekus observes

[94] Also [126, p.432; 128, p.5].

[95] Similarly, [126, p.434].)

[96] The texts by Brouwer that Niekus discusses in his papers are the second Vienna lecture [27], the second Geneva lecture [31], and 'Essentially negative properties' from 1948 [34]. In each of these, Brouwer

that Brouwer does not make the steps in (118), the explanation is not, as he claims it is [126, p.434; 125, p.230; 127, p.36; 128, p.9, p.13], that Brouwer did not want to appeal to $CS^{-}3b$. It is, at this precise point, even irrelevant whether $CS^{-}3b$ is acceptable; as Brouwer is reasoning in terms of his $>$, not $\gtrdot$, the assumption $\alpha > 0$ does not allow him to derive, towards a subsequent application of $CS^{-}3b$, its antecedent first. That derivation would require accepting the very form of MP that Brouwer's 1948 paper provides a weak counterexample to (Weak counterexample 12 above).[97] Correspondingly, the explanation of Brouwer's use of an untested proposition lies not in an unwillingness to use $CS^{-}3b$, but in the wish to establish a property of the negatively defined $<$. Niekus' observation on Brouwer's reasoning therefore does not lead to a correct argument that Brouwer did not want to accept $CS^{-}3b$ (or BKS^{-}).

The misconstrual of Brouwer's argument leads Niekus to introduce an alternative propositional operator to that in the Theory of the Creating Subject. In the latter the proposition $\Box_n A$ is given the meaning 'By stage n the creating subject has made A evident', where n ranges over all stages, past, present, and future. Niekus offers G_n, which is used to reason about future stages only:

> The G is used in temporal logic to express 'it is going to be the case that', and we shall use it similarly.
>
> We imagine our future to be covered by a discrete sequence of ω stages, starting with the present stage as stage 0, and we define for a mathematical assertion ϕ
>
> $$G_n\phi$$
>
> as: at the n-th stage from now we shall have a proof of ϕ. The introduction of this term enables is to refine the notion of proof.
>
> In intuitionism, stating ϕ means stating the possession of a proof of ϕ. We now demand of such a proof that it can be carried out *here and now*, i.e. all information for the proof is available at the present stage. If future information is involved we use $G_n\phi$. [127, p.37][98]

(For the use of 'we' in the explanation of $G_n\phi$, see subsection 7.10 below.) In one paper [125, p.226] the choice is made to let the values for n at 0, so that $A \leftrightarrow G_0 A$.[99] For

distinguishes between the two orderings. The same contrast is referred to in the opening line of the companion paper on the non-equivalence of the two order relations [36]. That paper is referred to by Niekus [128], but in a different matter, which I discuss in subsection 7.10.

[97] Niekus does not comment on that part of Brouwer's paper.

[98] Correspondingly, [126, p.434; 125, p.226; 128, p.8–9].

[99] To make the parallel to the use of CS as close as possible, one might want to say that, in proving that the sequence α converges, (122) is used for the special case $n = 0$.

this operator, the analogues to CS^+2 and CS^+3a

$$\forall n \forall m (G_n A \to G_{n+m} A) \tag{119}$$

and

$$A \to \exists n G_n A \tag{120}$$

are valid. But we also have

Theorem 60 ([127, p.37–38])**.** The analogues to CS^+3b,

$$\exists n G_n A \to A \tag{121}$$

and to CS^+1,

$$\forall n (G_n A \vee \neg G_n A) \tag{122}$$

are not valid.

Proof 61. Let A be an undecided proposition. Define the sequence α by

$$\alpha(n) = \begin{cases} 0 & \text{if } \neg G_0(A \vee \neg A) \\ 1 & \text{otherwise.} \end{cases} \tag{123}$$

So α will remain 0 until we have decided A, and then it becomes constant with all remaining values being 1. Let $k > 0$ be an arbitrary natural number, and set $B := \alpha(k) = 0 \vee \alpha(k)k = 1$. Then we do not now have B, for that would mean that we already know now whether by the k-th choice from now we shall have a proof of $A \vee \neg A$, which is impossible. But obviously $G_{k+1}(\alpha(k) = 0 \vee \alpha(k) = 1)$, so $G_{k+1}B$. Hence (121) is not valid. Now set $A := \alpha(k) = 1$. If (122) were valid, then we would already know now whether by the k-th choice from now we shall have a proof of $A \vee \neg A$, which is impossible.

Corollary 62 ([127, p.38])**.** BKS^- is not derivable from the schemata for G.

Niekus observes that Brouwer's reasoning in Plausibility argument 10 can be construed in terms of G, analogously to Plausibility argument 35 on p.33, and that this requires (the contraposition of) (120), but neither (121) nor (122). But he goes further and claims [127, p.38-39] that not only is a reconstruction in terms of $G_n A$ instead of $\Box_n A$ possible, this is what Brouwer had in mind. In evidence he cites a handwritten note of Brouwer's, of which he later says that it provides 'an even more decisive argument against 3 [$=CS^-3b$]' [128, p.13] than Brouwer's not appealing to CS^-3b in the 1948 paper (which, as we saw above, actually is not an argument against CS^-3b). This note, in the English translation in the *Collected Works*, runs as follows:

> Further distinctions in connection with the excluded middle.
>
> $\overline{a}$ will mean: a is non-contradictory.
>
> $\underline{a}$ will mean: a is contradictory.
>
> b *implies* a will mean: *from now on* I have an algorithm which enables me to derive a from b.
>
> The principle of testability can assert:
>
> either: *from now on* either $\overline{a}$ or $\underline{a}$ holds, notation: $|a$.
>
> or: from a certain moment in [the] future on either $\overline{a}$ or $\underline{a}$ will hold, notation $a|$.
>
> Then $a|$ is non-contradictory, but $|a$ need not to be non-contradictory. For instance, let p be a point of the continuum in course of development, whose continuation is free at this moment, but may be restricted at any moment in the future; then (p is rational)| is non-contradictory, but |(p is rational) is contradictory, for the complete freedom which exists at this moment makes it impossible to be sure that the rationality of p is contradictory, but also to be sure that it is contradictory that the rationality of p is contradictory. [...][100] However, [...] $|a$ does not seem admissible as a mathematical notion. [45, p.603–604]

Niekus comments:

> In struggling with his new notion of tensed objects he comes up with an explicit distinction, which is the same as we make. For his $a|$ and $|a$ are the same as our $\neg\neg a \vee \neg a$ and $G_n(\neg\neg a \vee \neg a)$. At the end of this note Brouwer expresses doubts about introducing $|a$ as a mathematical notion, without further argument. But we focus here on the logical distinction. That Brouwer, given the distinction, would accept (7) [=(121)] is of course out of question: stating that $\neg\neg a \vee \neg a$ is contradictory and that $G_n(\neg\neg a \vee \neg a)$ is not refutes (7) [=(121)] in a very strong way. We conclude there is no base for KS in Brouwer's creating subject arguments. [127, p.38–39]

Niekus abstracts from Brouwer's unargued reservation about $|a$ being mathematical, in order to concentrate on the logical distinction. This cannot be done, however, because on Brouwer's conception of logic (see subsection 5.1 above), on which it is but an application of mathematics, as opposed to a prior foundation to it, if a proposition is not mathematical then it has no logical significance either.

What may Brouwer's reservation have consisted in? The difference at hand is that between proofs for which all information needed is available now, and proofs for which

[100] In this omitted passage, Brouwer attempts an alternative to Proof 29 in terms of $|a$.

certain information is not yet present but will be generated along the finite way. The latter depend on future activity of the Creating Subject and hence involves its essential freedom. But for testability the only mathematically relevant consideration is that a construction for either $\neg p$ or $\neg\neg p$ be finitely effectible, and then there is no properly mathematical reason to treat those finite processess for which all information is available before they begin differently from the others. A notion of testability that does just that, such as $|a$, would for that reason not be 'admissible as a mathematical notion'. With the notion $a|$, on the other hand, no problem arises as it is inclusive of both kinds. So to the extent that an implication has to relate mathematical propositions, and G is not acceptable as a mathematical notion, neither (120), which Niekus accepts, nor (121), which he rejects, are acceptable.

This consideration obviously applies to provability in general. A division among proofs is not mathematically motivated if defined in terms of G. It is for this reason that Brouwer can write, when discussing the status of mathematical assertions with respect to truth in a lecture manuscript from 1951:

> An immediate consequence [of the introduction of intuitionism] was that for a mathematical assertion α the two cases of truth and falsehood, formerly exclusively admitted, were replaced by the following three:
>
> 1. α has been proved to be true;
> 2. α has been proved to be absurd;
> 3. α has neither been proved to be true nor to be absurd, nor do we know a finite algorithm leading to the statement either that α is true or that α is absurd.†

adding in the footnote †:

> The case that α has neither been proved to be true nor to be absurd, but that we know a finite algorithm leading to the statement either that α is true, or that α is absurd, obviously is reducible to the first and second cases. [46, p.92]

I do take it that a 'finite algorithm' may involve making a specified finite number of choices; to take an uncontroversial example, Newton's Method for converging to an x such that $f(x) = 0$ begins with the instruction 'Choose a starting point x_0'. An example in in Brouwer's writings is found in 'Points and Spaces', when about the depth ('order') at which a choice sequence ('arrow') meets a bar he writes:

> The definition of a crude bar means that for every arrow α of K the order $n(\alpha)$ of the postulated node of intersection with $C(K)$ must be computable, however

> complicated this calculation may be. The algorithm in question may indicate the calculation of a maximal order n_1 at which will appear a finite method of calculation of a further maximal order n_2 at which will appear a finite method of calculation of a further maximal order n_3 at which will appear a finite method of calculation of a further maximal order n_4 at which the postulated node of intersection must have been passed. [41, p.12–13]

The fact that the methods of calculation of the various orders themselves come to appear at various orders, that is, at various points in the choice sequence, indicates the possibility that these methods of calculation depend on the choices made in between the orders in question.

In 2010 Niekus acknowledges that BKS$^+$ is found in Brouwer:

> There is an instance of KS in Brouwer's work, from the last year in which he published, see Brouwer 1975, p. 525, 11th line from below. [127, p.38n3]

(This is the instance discussed in subsection 6.2 above.) But he continues by commenting that

> Whether there are arguments for this specific instance of KS remains an interesting question.

It is not clear to me why Niekus does not say here, on the basis of his own views, that there are no such arguments. As regards his further comment that

> Although there are one or more instances of KS for specific cases in the work of Brouwer, he always carefully avoided its use as a general principle for an unspecified formula. [127, p.38]

However, BKS$^+$ as a fully general principle follows by exactly the same reasoning as Brouwer employs in his weak counterexample (subsection 6); and for Brouwer there was no need to isolate either form of BKS as he could argue directly from (in effect) the three principles CS, of which they are immediate consequences (subsection 5.2).

7.10 Brouwer does not appeal to an ideal subject

Niekus claims that to hold that the Creating Subject is in some sense an ideal mathematician is to hold that the sequences defined in terms of its activity are 'completely determined':

> The method of the creating subject characterizes Brouwer's papers after 1945, when after a long delay he started to publish again. The method has always

> been supposed to be a radically new step in the work of Brouwer. The expression 'creating subject' was then interpreted as 'the idealized mathematician' and the generated sequences by the creating subject as completely determined.
>
> The notion of the idealized mathematician was formalized by Kreisel which resulted in the theory of the idealized mathematician. This theory does not reflect Brouwer's reasoning well and it was struck by a paradox, discovered by Troelstra, that could not be resolved satisfactorily.
>
> We propose a solution of the paradox in which Kreisel's main assumptions are dropped. A consequence of our solution is that the generated sequences are no longer completely determined, they are choice sequences. We will conclude that the method of the creating subject is special, not because of the introduction of an idealized mathematician, but by the systematic application of particular choice sequences. [128, p.2]

For the paradox, see subsection 7.3 above, where it is also mentioned that both the main solution proposed by Troelstra and that by Niekus fail because of their dependence on Markov's Principle. Here the focus will be on the alleged property of complete determination.

That property is embodied, according to Niekus, in $\mathrm{CS}^{-}1$, which asserts the decidability of $\Box_n A$. On the fact that $\mathrm{CS}^{-}1$ is accepted in the Kreisel-Troelstra theory, but not on his alternative, he comments in an earlier paper that

> For the reconstruction of Brouwer this has the consequence that, since A 2.1 [$=\mathrm{CS}^{-}1$] is not valid anymore, we cannot define $(a_n)_n$ completely [...] This is what is to be expected if we interpret Brouwer's method as above, because then $(a_n)_n$ is a choice sequence, all of its values yet undetermined. [126, p.436]

and

> An argument for taking β to be lawlike may be that in the TCS [i.e., The Creating Subject] the stages seem to have a definite description, expressed by (1) [$=\mathrm{CS}^{-}1$]. But in the intuitionistic interpretation, for a disjunction to hold we need a proof of one of the disjunctive parts. In the case of β this seems to be not evident to us.[101]
>
> Let us return to Brouwer's original use of creating subject, let us interpret it as ourselves and let the stages cover our future. We can define β as above. Then its values depend on our future results. We have no way to determine these values, other than going in time to these stages, which are not specified

[101] [Note MvA: β is the Creating Subject sequence in Troelstra's Paradox; see subsection 7.3.]

> at all. We think decidability is questionable, and we do not want to call this sequence lawlike. [127, p.36–37]

More recently, Niekus has commented on Troelstra's changed terminology in 1988 (see subsection 7.3 above):

> In formulating the paradox Troelstra is now more cautious than he was in 1969. He formulates the paradox with $L(\alpha)$ holds iff α 'fixed by a recipe' instead of 'lawlike'. But Troelstra does not abandon his main argument from Troelstra 1969 for calling a CS sequence, and thus the $c(n)$, lawlike. That is the decidability of $\Box_n\phi$ expressed by the TIM [The Ideal Mathematician] Axiom 2: $\Box_n\phi \vee \neg\Box_n\phi$. [128, p.8]

However, for $\Box_n A$, decidability is not questionable at all. It is asserted on the ground that whether $\Box_n A$ is true depends only on the Creating Subject's own activity, which, for any specific n in the future, it can simply carry on for the finitely many stages required to pass stage n, after which simple inspection allows to determine whether in the preceeding acts evidence of A was obtained. This ground clearly is independent of these preceeding acts being lawlike or to any extent free. Niekus is, of course, right that the decision is made by actually proceeding and making choices, but he is wrong in rejecting this as the construction method that proves either one of the disjuncts of $CS^{-}1$ (where the number of choices required is finite). After all, making a finite number of choices can be part of a genuine construction method – the theme of the end of the previous subsection – and so $CS^{-}1$ does not entail that Creating Subject sequences are completely determined sequences, as opposed to choice sequences.

Then Niekus continues and objects that Troelstra's picture is mistaken at an even deeper level:

> Neither does he question the conception underlying the axioms of the TIM: an idealized mathematician, all his mathematical activity covered by a sequence of stages. This questioning is the key of the solution of Niekus 1987. [128, p.8]

But Niekus' own conception just as much entails that all mathematical activity of the Creating Subject is covered by a sequence of stages. For, as we saw above, he writes 'We imagine our future to be covered by a discrete sequence of ω stages, starting with the present stage as stage 0': in particular, then, at the very beginning of all our mathematical activity we imagine our future to be covered by a discrete sequence of ω stages. As we proceed, elements in that sequence that at first corresponded to future stages come to correspond to stages in our past. But if we are intuitionistically entitled to imagine that sequence in the first place, then the systematic change in these correspondences will not turn it into an unacceptable object.

Therefore also in Niekus's framework as he describes it the term 'ideal(ised) mathematician' is called for (with a somewhat weaker meaning than he attaches to it, because it does not include the idea that the sequences it generates are completely determined). He writes that

> According to Brouwer's view, mathematics is a creation of the human mind and by using the expression *creating subject* Brouwer only made explicit his idealistic position; it can be replaced by *we* or *I*.[102] Interpreted in this way, an idealized mathematician is not needed at all for the reconstruction, a simple principle for reasoning about the future is enough. [...] We interpreted the expression *creating subject* as *we*, and anybody else can interpret it as himself. [127, p.32 and p.39, original italics][103]

At the same time, as we just saw, we are asked to imagine our future activity as an ω-sequence of stages; but unidealised human beings (Brouwer, Niekus, myself) do not have such a long future. (See also the beginning of subsection 5.3.)

7.11 BKS$^-$ and CS$^-$ are incompatible with Brouwer's notion of infinite proofs

In Brouwer's Creating Subject arguments, it is presupposed that evidence comes in an ω-sequence. On the other hand, Brouwer also accepted infinite proofs, i.e., proofs in which the conclusion is made evident on the basis of infinitely many elementary inferences. The locus classicus for this is footnote 8 in 'Über Definitionsbereiche von Funktionen' from 1927, the paper in which Brouwer gives his second demonstration of the Bar Theorem. In this demonstration, a well-ordered thin bar is obtained by effecting a transformation on the mental proof that a tree contains a (decidable) bar, which in that footnote is claimed to be infinite:

> Just as, in general, well-ordered species are produced by means of the two generating operations from primitive species [...] so, in particular, mathematical proofs are produced by means of the two generating operations from null elements and elementary inferences [Elementarschlüssen] that are immediately given in intuition (albeit subject to the restriction that there always occurs a last elementary inference). These *mental* mathematical proofs [Beweisführungen] that in general contain infinitely many terms must not be confused with their linguistic accompaniments, which are finite and necessarily inadequate, hence do not belong to mathematics.

[102] [Note MvA: These are cases worth distinguishing [179].]
[103] Also [126, p.435; 125, p.226; 128, p.9].

> The preceding remark contains my main argument against the claims of Hilbert's metamathematics. [24, p.64; trl. 176, p.460n8]

For Brouwer's constructive definition of well-ordered species, see Definition 41 above.

It was Kreisel who noticed the contrast between the ω-sequence of the Creating Subject's activities and the transfinite length of canonical proofs, and he came to see this as an incoherence in Brouwer's thought. Of the two contrasting ideas, Kreisel found the analysis of proofs of the presence of a bar into an infinite canonical form to be 'rather persuasive' [97, p.p.247], so he located the problem squarely in the idea that the stages form an ω-sequence, and made the correspondingobjection to BKS.[104] It is amusing to hear, over a number of years, the crescendo in Kreisel's views:

1. 1969:

   > there is no clear reason to restrict oneself to ω stages when the canonical proofs on p.59 consist of a transfinite sequence. [94, p.61]

2. 1970:

   > The assumptions used in deriving KS-, namely thinking of the body of mathematical evidence as arranged in an ω order, seem arbitrary (though not absurd) if, as in the theory of ordinals, one also thinks of individual proofs as consisting of a transfinite sequence of steps ([3], footnote 8).[105] Therefore the inconsistency of (KS) with Church's thesis does not, I think, refute the latter conclusively. [89, p.128]

3. 1970:

   > [While validity in Kripke models implies validity in Heyting's sense,] the converse is dubious because (some of) the author's counter models allowed on pages 98–99 picture an essentially more elementary process of treating 'evidential situations' than allowed in intuitionistic mathematics. Specifically, the author considers ω-series (in time) of stages of evidence while, at least occasionally, Brouwer considered fully analyzed proofs with a transfinite number of steps. [90, p.331]

4. 1971:[106]

[104] Similarly, [182, p.13].

[105] [Note MvA: The reference is to Brouwer's footnote quoted on p.78.]

[106] This is from Kreisel's 'Autorreferat' in *Zentralblatt* of [89]; Kreisel is here referring to his remark quoted in item 2 in this list.

> At the end of §4 (p.128) he considers the schema KS which is inconsistent with CT. (The schema KS was derived by Kripke from Brouwer's assertions about the thinking subject or, better, from the postulate of an ω-ordering of levels of proofs.) The author's objection to KS seems to the reviewer much stronger than the author can have realized, casting doubt on the interest of the papers in the volume which are based on KS.[107] [...] In Section 5 the author apparently expects an (hypothetical) abstract theory of functions and proofs to conflict with CT. Without the kind of implausible ω-ordering of proofs involved in KS, there is no evidence for such a conflict. [91, p.301]

5. 1972:

> the contradiction pointed out at the bottom of p.128 of [6][108], between two well-known assertions of Brouwer; one concerning the transfinite structure of (fully analyzed) proofs, the other concerning an ω-ordering of the body of mathematical evidence as the 'thinking subject' or, equivalently, the idealized mathematician proceeds in time. [92, p.325]

Kreisel's claim of a contradiction can be countered by observing that there are two orderings in play, and that once they are distinguished, the perceived contradiction disappears.

For each element in the ordered species, we distinguish the order it has according to the definition of the species and the order in which it has been generated in time, that is, its genetic order. We can then say, in the case of the Bar Theorem, that the elements of a canonical proof get ordered in two different orderings: in the transfinite well-ordered species that is the demonstration, which order indicates where in the demonstration the element fits in, and in the temporal order of the Creating Subject's acts of mathematical construction, which is an ω-order. While a species may be of a greater order type, our constructive access to it proceeds in an ω-sequence of acts.

That the Creating Subject can indeed construct the elements of a transfinite well-ordering in an ω-sequence of acts is because the inductive definition of well-ordered species gives the freedom to insert the elements in their species-order concurrently. That is to say, of each species that has been used in the construction of the whole species – Brouwer calls these its 'constructive underspecies' –, the elements can be constructed independently of the construction of the elements of the other constructive underspecies. For example, consider a well-ordering of type $\omega + \omega$, say the ordered sum of the species X_0 of the even

[107] [Note MvA: In that volume [78], BKS is discussed in the contributions by Kreisel [89], Myhill [122], Van Rootselaar [179], Vesley [189], and Scott [134].]

[108] [Note MvA: Kreisel's reference '[6]' is to [89]; see item 2 in this list.]

numbers in their natural order and the species X_1 of the odd numbers in their natural order. Then the Creating Subject may first construct 0 in X_0, then 1 in X_1, then turn back to X_0 and construct 2, and so on. More generally, from the definition of well-ordered species one shows by ordinary induction that the elements of the species can be enumerated [13, p.7, p.30]. That a conclusion can be drawn from infinitely many premises that at no point have all been constructed is because this infinity is governed by a finite number of laws to construct it; it is an insight into the construction processes that these laws describe that make the conclusion evident. This is the same as in the case for induction on the natural numbers [151].

8 Concluding remark

The preceding considerations indicate that the fact that Brouwer in 1954 was able to demonstrate BKS^+ was highly dependent on his very specific views on mathematical objects, proofs, truth, and freedom. Even slight changes in these notions or the role they are assumed to play in mathematics may suffice to make all or some versions of BKS or, similarly, CS, implausible or false. But in Brouwerian intuitionism, these principles should be used freely.

A Brouwer's implicit use of MP in 1918

As Joan Moschovakis has observed in her review [118, p.274] of vol. 1 of Brouwer's *Collected Works*, in 1918 Brouwer once implicitly used MP in the form

$$\neg\forall n \neg P(n) \rightarrow \exists n P(n) \tag{124}$$

where P is a decidable predicate [13, p.17]; this fact will now be presented in some detail.

The context is a proof that a certain species is closed. Brouwer defines the species C of choice sequences of positive natural numbers together with a bijective mapping (that I will call f) from C to the dyadically expandable real numbers in $(0,1)$.[109] That is, f maps the choice sequence $a_1, a_2, a_3, \ldots$ ($a_i \in \mathbb{N}^+$) to the real number

$$\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{a_1-1}} + \frac{1}{2^{a_1+a_2}} + \frac{1}{2^{a_1+a_2+a_3}} + \ldots \tag{125}$$

The intention behind Brouwer's somewhat unclear notation here is that the sequence begins by summing the first $a_1 - 1$ values in the series $1/2^n$ ($n \in \mathbb{N}^+$), which, in case $a_1 = 1$, is

[109] The term 'dyadically expandable' translates the term Brouwer uses in 1925 for this class, 'dual entwickelbar' [21, p.251].

0.[110]

In the 1918 paper, Brouwer in fact constructs real numbers in general from C, f, and a bijection from $(0,1)$ to $(-\infty, \infty)$ [13, p.9]; but he quickly and definitively replaced this with a much wider notion where a real number may be any sequence of rationals (alternatively, rational intervals) as long as it converges.[111] It is the latter notion on which Brouwer's weak and strong counterexamples depend.

Through the mapping f, the natural order on the real numbers in $(0,1)$ induces an order (which I notate as) $\sqsubset$ on C. For $x, y \in C$,

$$x \sqsubset y \equiv \exists v \in (0,1)\, \exists w \in (0,1)(x = f^{-1}(v) \wedge y = f^{-1}(w) \wedge v < w) \tag{126}$$

Thus, increasing a_1 in a given element of C yields a greater element, whereas increasing an a_i with $i \geqslant 2$ yields a smaller element.

Brouwer then sets out to prove that C, thus ordered, is a perfect species. A species is perfect if it is dense in itself and closed. He defines closedness as follows:

Definition 63.

> The ordered species M is called *closed*, if there can exist no infinite sequence[112] of closed intervals $i_1, i_2, \ldots$ in it of which $i_{\nu+1}$ is contained in i_ν for every ν, and which have no common element. [13, p.17, trl. MvA]

and then, applying this definition to C and $\sqsubset$, presents this argument (in which I have put the part relevant here in bold):

Proof 64.

> Let us now attempt to determine an infinite sequence of closed intervals $i_1, i_2, \ldots$, of which $i_{\nu+1}$ is contained in i_ν for every ν, and which have no common element. Let $a_1, \ldots, a_n, b_{n+1}, \ldots$ and $a_1, \ldots, a_n, c_{n+1}, \ldots$ $(b_{n+1} > c_{n+1})$ be the end elements of i_1, then the end elements of an arbitrary i_ν have the same initial segment $a_1, \ldots, a_n$, whereas for the later i_ν b_{n+1} cannot increase and c_{n+1} cannot decrease.[113] **Now, as long as $\mathbf{b_{n+1}}$ and $\mathbf{c_{n+1}}$ retain the same**

[110] See Heyting's note 6 in [45, p.591n6]; that concerns the later presentation of the same mapping in [21, p.251], but applies here, too.

[111] On this replacement, see [14, p.3–4, 4n1] and [15, p.955n1].

[112] [Note MvA: 'Fundamentalreihe', defined as any ordered species whose ordering is similar to that of the natural numbers in their natural order [13, p.14]. As Van Dalen [173, p.238n12] observes, Brouwer's actual use of the term wavers a bit. He often but not always means a lawlike infinite sequence. At times Brouwer makes an explicit distinction between 'Fundamentalreihe' and 'unbegrenzt fortgesetzte Folge', such that the latter is the wider notion, e.g. [15, p.202]. But in the definition and proof under discussion here, the wider notion is meant.]

[113] [Note MvA: An increase of b_{n+1} would make the left end element smaller with respect to $\sqsubset$, and a decrease of c_{n+1} would make the right end element greater; but the sequence i is supposed to be decreasing.]

distinct values, the corresponding i_ν contain the element $a_1, \ldots, a_n$, $c_{n+1} + 1, 1, 1, 1, \ldots$; in order to be sure that this element does not belong to all i_ν, it must be possible to indicate a certain i_ν for which either b_{n+1} has decreased or c_{n+1} has increased. As this reasoning can be repeated at will, it must be possible to indicate a later i_ν for which $b_{n+1} = c_{n+1} = a_{n+1}$ will have come to hold, and the end elements of which therefore have the same first $n+1$ numbers. Let these end elements be $a_1, \ldots, a_{n+m}, b_{n+1}, \ldots$ and $a_1, \ldots, a_{n+m}, c_{n+1}, \ldots$. Then in the same way in which we derived from the sequence $a_1, \ldots, a_n$ the sequence $a_1, \ldots, a_{n+m}$, we can obtain from $a_1, \ldots, a_{n+m}$ a further sequence $a_1, \ldots, a_{n+m+p}$; and, continuing this way, we can construct an infinitely proceeding sequence $a_1, a_2, \ldots$. The element of C that this sequence represents belongs to *all* i_ν however, by which we have reached a contradiction, and have recognised that the ordered species in question is *closed.* [13, p.17, original emphasis, trl. MvA]

Write $b(i_\nu)$ and $c(i_\nu)$ for the values of b_{n+1} and c_{n+1} in the endpoints of i_ν, and assume that for a given i_k, $b(i_k) \neq c(i_k)$. Then Brouwer in the bold passage concludes from

$$\neg \forall w \neg (b(i_{k+w}) \neq b(i_k) \vee c(i_{k+w}) \neq c(i_k)) \tag{127}$$

to

$$\exists w (b(i_{k+w}) \neq b(i_k) \vee c(i_{k+w}) \neq c(i_k)) \tag{128}$$

which inference corresponds to that licensed by MP in the form (124).

That Brouwer came to see the problem with this reasoning is strongly suggested by his next presentation of this proof [23, p.461–463]. He there has changed the definition of closedness:

Definition 65.

In a virtually ordered species M an unbounded sequence of closed intervals i_1, $i_2, \ldots$, where each $i_{\nu+1}$ is a subspecies of i_ν, is called a *hollow sequence of nested intervals* [hohle Intervallschachtelung], if for each element p of M a ν_p can be determined such that p cannot belong to i_{ν_p}. [...] If in M there can exist no hollow sequence of nested intervals, M is called *closed.* [23, p.461, trl. MvA]

Thus, the positive information, to produce which from the earlier definition of closedness required MP, has now become part of the definition of closedness itself.[114] In the

[114]The notion of a hollow sequence is related to that of a strong Specker double sequence [1, p.743].

1927 Berlin lectures, Brouwer explicitly remarked on the greater strength of the new definition [47, p.40].

B Brouwer's proof of the Negative Continuity Theorem

Brouwer's argument for the weak counterexample to $\forall x \in \mathbb{R}(x \in \mathbb{Q} \vee x \notin \mathbb{Q})$ from 1954 (subsection 3.7) resembles that for his Negative Continuity Theorem from 1927 [24]. Both arguments turn on the idea that, given the definition of a real number r, one can construct a real number s that starts out like r, but that may come to diverge from it, depending on an event of which it cannot be predicted if and when it will occur. In the case of the weak counterexample, that is the event of proving an as yet untestable A; in the Negative Continuity Theorem, simply the free choice to diverge. The interpretation and correctness of Brouwer's proof of that theorem has been the subject of debate [113, 130, 131, 147, 183, 184], mostly concerned with the question whether a certain negation occurring in the argument is weak or strong and with the exact grounds on which that negation is introduced. I will not reconstruct that debate here in detail, but want to present Brouwer's argument as I read it and make two remarks: one on Carl Posy's criticism of that reading, and one on the importance Brouwer attached to the argument.

The reading of Brouwer's proof below agrees with Heyting's general suggestion to read it in terms of the Creating Subject [73, p.131; 147, p.479]. It also agrees with the (in effect) detailed elaboration of that suggestion by Martino [113, p.383–384], and in particular I agree with the latter [113, p.382] that Brouwer's argument is a proper proof of Theorem 67 (by contradiction), and not only a plausibility argument (by constructing a weak counterexample to its antithesis) as Veldman has suggested it is [185, p.291].[115]

Brouwer introduced the notion of negative continuity in 1924. It was investigated further by Belinfante [2–5] and Dijkman [54].

Definition 66 ([20, p.6])**.** A sequence of real numbers $r_1, r_2, \ldots$ *converges positively* to a real number r_0 if

$$\forall p\, \exists n\, \forall m(m > n \to |r_0 - r_m| < 1/p) \tag{129}$$

A sequence of real numbers $r_1, r_2, \ldots$ *converges negatively* to a real number r_0 if

$$\forall p \neg\, \exists \underline{n}\, \forall m(|r_0 - r_{\underline{n}(m)}| > 1/p) \tag{130}$$

[115]Further on in his paper, Martino remarks that justifications of the continuity principle for lawless sequences have not taken into account their givenness as individuals [113, p.386–390]. I later analysed the individuality of choice sequences, and its relation to WC-N, in my 1999 dissertation [155], published in 2007 [157]; part of it had found its way into a paper with Van Dalen in 2002 [162]. Martino's paper was overlooked in (the work for) each of these three publications. I regret that.

where $\underline{n}$ is a strictly increasing sequence of natural numbers.

A function f is *negatively continuous* at r_0 if for every sequence of real numbers $r_1, r_2, \ldots$ that converges positively to r_0, the sequence $f(r_1), f(r_2), \ldots$ converges negatively to $f(r_0)$.

A function f is *negatively continuous* if it is negatively continuous at all points in its domain.

Theorem 67 ([24, p.62])**.** Let $f\colon [0,1] \to \mathbb{R}$ be a full function. Then f is negatively continuous.

Proof 68. Let $r_0 \in [0,1]$ be arbitrary, and let $r_1, r_2, \ldots$ be a sequence of real numbers that converges positively to r_0. Assume, towards a contradiction, that

$$\exists p\, \exists \underline{n}\, \forall m(|f(r_0) - f(r_{\underline{n}(m)})| > 1/p) \tag{131}$$

where $\underline{n}$ is a strictly increasing sequence of natural numbers.

The Creating Subject constructs a choice sequence r_ω of rational numbers $r_\omega(i)$ as follows.

- $r_\omega(i) = r_0(i)$ if at the choice of $r_\omega(k)$ for some $k < i$ the Creating Subject made the free decision to align all further choices in r_ω with those of r_0.
- $r_\omega(i) = r_{\underline{n}(m)}(i)$ if at the choice of $r_\omega(k)$ for some $k < i$ the Creating Subject made the free decision to align all further choices in r_ω with those of $r_{\underline{n}(m)}$, for some m.
- $r_\omega(i) = r_0(i)$ otherwise.

The decision with which sequence to align the further choices in r_ω is free in respect to both its outcome and the moment at which it is made, if at all. It is a right that the Creating Subject reserves.[116] As assumption (131) guarantees that $f(r_0)$ is co-convergent with no $f(r_{\underline{n}(m)})$, no initial segment of r_ω constructed prior to such a decision provides sufficient information to allow a definition of $f(r_\omega)$; This contradicts the hypothesis of the theorem that f is a full function, according to which a construction method for $f(r_\omega)$ is available from the outset.[117]

[116] Brouwer writes: 'we [...] reserve the right to determine, at any time after the first, second, ..., $m-1$th, and m-th intervals have been chosen, the choice of all further intervals (that is, of the $m+1$th, $m+2$th, and so on) in such a way that either a point belonging to ξ_0 or one belonging to a certain ξ_{p_ν} is generated.' [24, p.62; trl. 176, p.459]. Brouwer's use of 'we' here is conventional, and does not imply that the Creating Subject is not an idealised mathematician; see subsections 5.3 and 7.10, and compare footnote 6.

[117] Such a method may call for first ensuring that at least k choices have been made in r_ω; see subsection 7.10 above.

The negation in the claim '$f(r_\omega)$ is not defined' above is a weak one: it is not excluded that r_ω will be defined later (by making the required decision). But no bound can be given on the stage by which that would have happened. The essential ingredient in Brouwer's proof, then, is the contrast between the condition that f be full and the fact that the Creating Subject cannot be obliged to exercise its right to fix r_ω within any specified finite time.

The reserved right amounts to a restriction on the choices in r_ω that is provisional in that it can be lifted when the Creating Subject chooses to do so [162; 157]. In the latter of these two references I suggested, as I had overlooked before, that Brouwer here is exploiting just that notion [157, p.108–109]. It is a different aspect of the Creating Subject than those described in the axioms of CS.

That suggestion was criticised by Posy [131] who objected that in that case there is something 'introspectively disingenuous' about the Creating Subject's behaviour in the interpretation of the proof in terms of provisional restrictions: 'We know full well that we want $f(r^*)$ [$= f(r_\omega)$] to be undefined, and won't forget that fact in later 2nd and 1st order choices about r^*' [131, p.30–31]. But the ground of the undefinedness of $f(r_\omega)$ is not the Subject's always choosing not to fix r_ω; the crux is that r_ω is a growing sequence that is a genuine real number from the outset, even in absence of a bound on the stage by which r_ω would be fixed, and that the hypothesis of the theorem implies that also r_ω is in the domain of f. I agree, then, with Posy's remark that 'We can say that we don't currently have a grasp of $f(r^*)$; we cannot say that we can't have such a grasp', but add that the point of the argument is that we cannot say that we must have such a grasp by a specified stage.

It seems to me that the fact that the Creating Subject can at no point be obliged to align the sequence r_ω with one of the others is what Brouwer had in mind when he commented that the Negative Continuity Theorem is 'an immediate consequence of the intuitionistic point of view' [24, p.62; trl. 176, p.459], for it is an immediate consequence of the constructive freedom of the Creating Subject. The evidence for this property of the Creating Subject seems to be much easier accessible than that for the canonisability of proofs appealed to in Brouwer's argument for the Bar Theorem, and used towards a demonstration of the positive Continuity Theorem. Indeed, in the 1927 paper [24, p.62-63] Brouwer says that he had been mentioning the Negative Continuity Theorem in lectures and conversations since 1918, but that it was 'much later' – six years – that he could actually prove the Continuity Theorem that is made plausible by it, and refers to his 1924 papers on the topic [18, 19].

Admittedly, when Brouwer again presents a proof of the Negative Continuity Theorem, in the *Cambridge Lectures*, and in a similar way announces it as 'an immediate consequence of the fundamental thoughts of intuitionism without using spread keys or well-ordered species' [46, p.80-81], he has something different in mind, as instead of a Creating Subject sequence with a provisional restriction, he goes on to use a lawlike sequence and a fleeing property (see Definition 5):[118]

Proof 69 (of Theorem 67)**.**

> For, let us suppose that $y = f(x)$ is a full function of U [the unit continuum]; ξ_0 a real number belonging to U; $\xi_1, \xi_2, \ldots$ an infinite sequence of real numbers of U converging to ξ_0; t a natural number; and that $|f(\xi_\nu) - f(\xi_0)| > 1/t$ for every ν.
>
> Let g be a fleeing property and k_g its critical number. We define an infinite sequence of real numbers $q_1, q_2, \ldots$ in the following way: $q_\nu = \xi_\nu$ for $\nu \leqslant k_g$ and $q_\nu = \xi_{k_g}$ for $\nu \geqslant k_g$. This sequence converges to a real number q_0, to which no real number $f(q_0)$ can be assigned. [46, p.80-81]

But the fact that Brouwer now presents this simpler proof is of course no indication that he had come to have second thoughts about the acceptability of that from 1927.

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