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Jean-Pierre Drugeon† & Thai Ha-Huy‡

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†Paris School of Economics and Centre National de la Recherche Scientifique. E-mail: jp-drg@psemail.eu

‡Université Paris-Saclay, Univ Evry, EPEE, 91025, Evry-Courcouronnes; TIMAS, Thang Long University. E-mail: thai.hahuy@univ-evry.fr

Abstract

This article builds an axiomatization of inter-temporal trade-offs that takes an explicit account of the distant future. The focus is on separable representations and the approach is completed following a decision-theory index based approach that is applied to utility streams understood as the well-being of future generations. The introduction of some new axioms is herein shown to lead to the emergence of two distinct orders that respectively relate to the *distant future* and *close future* components of some utility stream. This enlightens the limits of the commonly used fat tail intensity requisites for the evaluation of utility streams. These are replaced by an axiomatic approach to myopia degrees.

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1. Introduction

1.1 Motivations & Concerns

The long-run concerns for the well-being of remote generations of offsprings nowadays widely overstep the boundaries of academic circles and promptly come into the fore for most public agendas. It is however not the least surprising that only limited efforts have tried to achieve an understanding of the actual meaning of the concerns for an arbitrarily remote horizon.

Brown & Lewis [11] initiated an axiomatic approach to the topic based on the notion of myopia according to which the addition of some utility sufficiently far into the future would not modify the preferences order. This approach, however, received little attention at the time, perhaps because the identification of the weight of the distant future and the importance of remote generations was considered an oddity.

That study however raises a number of questions that may not have received sufficient attention. Is an arbitrarily large finite future a satisfactory proxy for an unbounded horizon? Does the very fact of having some remote low orders *tail* for a stream of utils mean that it is negligible in not exerting any influence for finite dates? More precisely, are there some specificities attached to arbitrarily remote horizon streams and is it reasonable to compare these through the same apparatus that is used for the *head* and finite parts of these streams?

1.2 The Approach

The purpose of this article is to more generally provide an integrated appraisal of myopia and the valuation of utility streams. The literature criteria for comparing utility streams commonly rest upon intuitive properties such as completeness, monotonicity, continuity, positive homogeneity and constant additivity. Being combined with continuity, these properties imply that the preferences order can be represented by an index function. As a matter of illustration, one can consider these following families of index functions that would satisfy such fundamental properties.

$$I_1(x) = (1-\delta)\sum_{t=0}^{\infty}\delta^t x_t \text{ for some } 0<\delta<1,$$

$$I_2(x) = \liminf_{t\to\infty} x_t.$$

The first index function I_1 represents an order which is *highly myopic* in that the value of each utility stream is essentially defined, for $0<\delta<1$, through a finite number of dates or generations. In opposition to this, the second index-order I_2 belongs to another, *highly non-myopic*, orders kind: the evaluation of the entire utility stream would not vary if only the values of a *finite* number of dates got modified. It is worth noticing that fundamental properties can be preserved through usual operators, *e.g.*, maximisation, minimisation, or by any convex combination of these. For example,

$$I(x) = \min_{\underline{\chi}\le\chi\le\overline{\chi}}\big[(1-\chi)I_1(x)+\chi I_2(x)\big] \text{ for given } 0\le\underline{\chi}\le\overline{\chi}\le 1.$$

In this example, the parameter χ can be understood as a *degree of myopia* that measures the weight of the *distant future*—the well-being of remote generations—and the potential importance of the tail of the utility stream. Observe that this parameter can vary as a function of the utility stream.

A usual way of studying the effect of the distant future proceeds by considering constant gains or losses. Given an order as represented by some index function I, the weight of the distant future could, *e.g.,* be measured through two simple parameters, *i.e.*, $\chi_g = \lim_{T\to\infty} I(0,0,\cdots,0,1,1,1,\dots)$, and $\chi_\ell = -\lim_{T\to\infty} I(0,0,\cdots,0,-1,-1,-1,\dots)$, that would respectively depict *remote constant gains* and *remote constant losses*. Building on such coefficients, two ranges of questioning naturally arise.

First, is a configuration where both the gains and losses distant future coefficients are equal to zero ($\chi_g=\chi_\ell=0$) associated with some *tail-insensitivity* property, a situation where the distant future becomes negligible? Second, is there some scope for systematically decomposing the evaluation of inter-temporal streams between its distant future value—the *tail* of the utility stream—and its close future value—the *head* of the utility stream. Assuming this is the case, what shape would such a function take?

In order to gain more understanding of such a potentiality, supplementary structures have to be added. Two new axioms are here introduced. The first distant future sensitivities axiom states that given a utility stream and a constant stream, the decision maker can always say about his/her preference between the distant future components of these two streams. The second close future sensitivities axiom is similar but relates to a comparison that takes place between some close future components of the streams.

The introduction of this structure leads to the emergence of two distinct orders that respectively relate to the *distant future* and the *close future*. Both new orders satisfy fundamental properties and can respectively be represented by *distant future and close future* index functions.

The key result of this article is then that the evaluation of a utility stream can be decomposed into a convex combination of its distant future and its close future components. The parameters of this convex combination change as a function of the utility streams and lie between χ_g and χ_ℓ.

Interestingly, these two values play a decisive role in the characterization of the eventual *myopia degrees*. They may represent two different sorts of behaviours in the consideration of the distant future that respectively relate to *optimism* and *pessimism*.

1.3 Related literature

The closest contribution to this article is a recent work due to Lapied & Renault [23]. They consider a decision maker facing alternatives that are defined on a very distant future, *i.e.,* a time horizon that exceeds his life-time horizon. This work emphasizes the emergence of an *asymptotic patience* property, meaning that, for some remote date, no time tradeoff between alternative any longer prevails.

In a pair of influential contributions, Chichilnisky [13, 14] adds *linearity* to the structure of a preferences order. She then imposes *no dictatorship of the present* and *no dictatorship of the future* properties on this order, the weighting parameters corresponding to the present and the future being consequently strictly positive. In the context of this article, this means that, while χ_g and χ_ℓ assume positive values, they are also, as a result of the linearity of the order, to satisfy $\chi_g = \chi_\ell$. The question of equity in evaluating the welfare of generations can be understood in different ways. Chichilnisky's works and this article follow the line where the evaluation of utility streams considers both present future and distant future.

The notion of *strong myopia*, due to Brown & Lewis [12] coincides with the *upward myopia* notion of Sawyer [28] and means, in its version presented by Becker & Boyd [10], that, for any $x \succ y$, one has for any z, $x \succ (y_0, y_1, \ldots, y_T, z_{T+1}, z_{T+2}, \ldots)$ for sufficiently large T. In the context of this article, these cases are equivalent to the *downward myopia* of Sawyer [28]. This corresponds to an extreme occurrence where $\chi_g = \chi_\ell = 0$.

Another extreme was considered by the *completely patient and time invariant* preferences of Marinacci [24], the Banach limits[1] corresponding to the case $\chi_g = \chi_\ell = 1$. In parallel to this, Araujo [4] proves that, in order for a set of non trivial Pareto allocations to exist, consumers must exhibit some impatience in their preferences. Otherwise stated, this excludes the possibility of preferences being represented by Banach limits: at least one of the two values χ_g, χ_ℓ is to be strictly smaller than 1.

Relying upon a distinct recent strand of the experimental literature, a more recent contribution due to Gabaix & Laibson [18] presents a subtle articulation between forecasting accuracy, discounting and myopia in an imperfect information environment that relies

Following parallel roads with a Gilboa & Schmeidler [20] approach but relying upon a different system of axioms based upon *time-variability aversion,* Wakai [29] provides an insightful account of smoothing behaviours where the optimal discount assumes an maximin recursive representation.

Formerly related with Wakai [29] and the current study with an analysis completed over the set of bounded real sequences ℓ_∞, Chambers & Echenique [15] have recently put forth an axiomatic approach to multiple discounts. The current approach com-

[1] For a definition of Banach limits, see page 55 in Becker & Boyd [10].

plements theirs in relaxing *tail insensitivity* and focusing instead on myopia dimensions that *precede* discounting concerns.

Diamond [16] and Basu & Mitra [7] prove that there is no index function which represents a preference order satisfying sensitivity (equivalent to *monotonicity*) and equity (also known as *anonymity*), even when continuity is relaxed. Basu & Mitra [8] show the existence of an order satisfying these two properties but not *archimedeanity*. In the context of this article, even though the *archimedeanity* axiom is admittedly strong, this property is important not only because it ensures continuity, but also because it plays a crucial role in the determination of the index function. Worth mentioning is also the study of Zuber & Asheim [30] who, following related equity concerns, introduce an *anonymity* axiom under which the evaluation of a utility stream does not change after a permutation of generations.

1.4 Contents

The article is organised as follows. Section two introduce basic axioms and emphasize the role of the distant future in the evaluation of utility streams over time. Section three presents different facets of myopia and introduces a decomposition for the future that is based upon closeness versus remoteness. The proofs are given in the Appendix.

2. Some basic Axioms and a Role for the Distant Future in the Evaluation of the Utility Streams

This study contemplates an axiomatization approach to the evaluation of infinite utility streams. This section will introduce some basic axioms, build an index function and emphasise the scope for a non-negligible influence for the remote parts of the utility stream.

2.1 Fundamentals, Basic Axioms & the Construction of an Index Function

Time is discrete. The notation $\mathbb{1}$ presents the constant unitary sequence $(1, 1, \dots)$. Letters like x, y, z will be used for sequences (of utils) with values in $\mathbb{R}$ while a notation $c\mathbb{1}$, $c'\mathbb{1}$, $c''\mathbb{1}$ will be used for constant sequences. In parallel to this, Greek letters λ, η, μ will be used for constant scalars.

Recall first that the ℓ_∞ space[2] is defined as the set of real sequences $\{x_s\}_{s=0}^{\infty}$ such that $\sup_{s\geq 0} |x_s| < +\infty$. For every $x \in \ell_\infty$ and $T \geq 0$, let $x_{[0,T]} = (x_0, x_1, \dots, x_T)$ denote its *head* $T+1$ first components and $x_{[T+1,\infty[} = (x_{T+1}, x_{T+2}, \dots)$ its *tail* starting from date $T+1$. Given two sequences x and y, $(x_{[0,T]}, y_{[T+1,+\infty[})$ denotes the sequence $(x_0, x_1, \dots, x_T, y_{T+1}, y_{T+2}, \dots)$. The following axiom introduces some fundamental properties on ℓ_∞ for the order $\succeq$.

Axiom F. The order $\succeq$ satisfies the following properties:

(i) *Completeness* For every $x, y \in \ell_\infty$, either $x \succeq y$ or $y \succeq x$.

(ii) *Transitivity* For every $x, y, z \in \ell_\infty$, if $x \succeq y$ and $y \succeq z$, then $x \succeq z$. Denote as $x \sim y$ the case where $x \succeq y$ and $y \succeq x$. Denote as $x \succ y$ the case where $x \succeq y$ and $y \not\succeq x$.

(iii) *Monotonicity* If $x, y \in \ell_\infty$ and $x_s \geq y_s$ for every $s \in \mathbb{N}$, then $x \succeq y$.

(iv) *Non-triviality* There exist $x, y \in \ell_\infty$ such that $x \succ y$.

(v) *Archimedeanity* For $x \in \ell_\infty$ and $b\mathbb{1} \succ x \succ b'\mathbb{1}$, there are $\lambda, \mu \in]0, 1[$ such that

$$(1-\lambda)b\mathbb{1} + \lambda b'\mathbb{1} \succ x \text{ and } x \succ (1-\mu)b\mathbb{1} + \mu b'\mathbb{1}.$$

(vi) *Weak convexity* For every $x, y, b\mathbb{1} \in \ell_\infty$, and $\lambda \in]0, 1]$,

$$x \succeq y \Leftrightarrow (1-\lambda)x + \lambda b\mathbb{1} \succeq (1-\lambda)y + \lambda b\mathbb{1}.$$

[2]Following the argument of Bewley [11], when one considers the *distant future* behavior of inter-temporal utility streams, one is to remember that the earth resources are limited and it is not something arbitrary to impose that utility levels are generated by bounded consumption streams.

All of the properties (i), (ii), (iii) and (iv) are common in decision theory. The *archimedeanity* property (v) ensures that the order is continuous in the sup-norm topology of ℓ_∞. Moreover, *archimedeanity* plays a crucial role in the determination of the index function[3].

The *Weak convexity* property (vi) is admittedly less immediate. It is referred to as *certainty independence* in the decision theory literature, contains the *positive homogeneity* property and ensures that direction $\mathbb{1}$ is *comparison neutral*: following that direction, the comparison between two sequences does not change.

Under these conditions, the order $\succeq$ can be represented by an index function which is homogeneous of degree one and constantly additive.

LEMMA **2.1.** *The order $\succeq$ satisfies axiom* **F** *if and only if it is represented by an index function I satisfying:*

(i) *For $x \in \ell_\infty$, $\lambda > 0$, $I(\lambda x) = \lambda I(x)$.*

(ii) *For $x \in \ell_\infty$, constant $b \in \mathbb{R}$, $I(x + b\mathbb{1}) = I(x) + b$.*

This statement directly compares with the results in Gilboa & Schmeidler [20], and Ghirardato & al [19]. It is worth emphasizing that, while these works specialize their argument to the space of simple acts—equivalent to sequences in ℓ_∞ which take a finite number of values—this article considers the whole space ℓ_∞.

2.2 Non-negligible distant future and non-negligible close future

In the literature, the notions of *impatience* or *delay aversion*[4] are generally understood through the convergence of $(0\mathbb{1}_{[0,T]}, \mathbb{1}_{[T+1,\infty[})$ to zero and as T tends to infinity[5]. More generally, it is commonly assumed that the value of the distant future converges to zero. In the current framework and under Lemma 2.1, this suggests the convergence to zero of $I\big(0\mathbb{1}_{[0,T]}, -\mathbb{1}_{[T+1,\infty[}\big)$ and $I\big(0\mathbb{1}_{[0,T]}, \mathbb{1}_{[T+1,\infty[}\big)$[6]. To check

[3] In particular, Basu & Mitra [8] show the existence of an order in which, for an increasing stream x converging to some constant b, there exists some $b' < b$ such that $b\mathbb{1} \succ x \succ b'\mathbb{1}$, while $(1-\lambda)b + \lambda b' \succ x$ for any $\lambda \in]0,1[$. In their configuration, *Archemedeanity* is not satisfied.

[4] See Bastianello & Chateauneuf [6].

[5] A very careful account of these notions was recently brought by Bastianello [5].

[6] Observe that these two properties are not equivalent.

upon this, first consider the two following coefficients that are respectively defined for asymptotically constant gains and losses:[7]

$$\begin{aligned}\chi_g &= \lim_{T\to\infty} I\big(0\mathbb{1}_{[0,T]}, \mathbb{1}_{[T+1,\infty[}\big),\\ \chi_\ell &= -\lim_{T\to\infty} I\big(0\mathbb{1}_{[0,T]}, -\mathbb{1}_{[T+1,\infty[}\big)\\ &= 1-\lim_{T\to\infty} I\big(\mathbb{1}_{[0,T]}, 0\mathbb{1}_{[T+1,\infty[}\big).\end{aligned}$$

These two limit values χ_g and χ_ℓ will be considered extensively in the course of this study and will play an important role in the definition of the *myopia degrees.* The equality $\chi_g = \chi_\ell = 0$ is indeed similar to the usual *negligible-tail* or *tail-insensitivity* conditions in the literature. Under this condition, one could formulate a conjecture about the poor relevance of the tail of the sequence, namely and for any $x, z \in \ell_\infty$, the holding of:

$$\lim_{T\to\infty} I\big(x_{[0,T]}, z_{[T+1,\infty[}\big) = I(x), \tag{2.1}$$

Otherwise stated, for sufficiently large values of T, the tail of the sequence z would become irrelevant and the whole evaluation of the utility stream would proceed from the sequence x.

The following counter example however provides an illustration where, in spite of a valuation of the distant future that could be zero,[8] this future remote component could continue to play a significant role in the evaluation of the whole sequence.

Example 2.1. Consider two probability measures belonging to the set[9] ℓ_1, namely ω and $\hat\omega$, and satisfying $\omega \neq \hat\omega$. Define the index function I as:

$$I(x) = \min\Big\{\hat\omega\cdot x, \max\Big\{\omega\cdot x, \liminf_{s\to\infty} x_s\Big\}\Big\}, \text{ for } x \in \ell_\infty.$$

[7] From the *monotonicity* property, $I\big(0\mathbb{1}_{[0,T]}, \mathbb{1}_{[T+1,\infty[}\big)$ and $1-I\big(\mathbb{1}_{[0,T]}, 0\mathbb{1}_{[T+1,\infty[}\big)$ are decreasing as a function of T, so that both of these limits are well defined.

[8] One can prove that $\lim_{T\to\infty} I\big(0\mathbb{1}_{[0,T]}, x_{[T+1,+\infty[}\big) = 0$ for any $x \in \ell_\infty$.

[9] ℓ_1 is the set of real sequences $\{\omega_s\}_{s=0}^{\infty}$ such that $\sum_{s=0}^{\infty}|\omega_s| < +\infty$. For $\omega \in \ell_1$ and $x \in \ell_\infty$, the scalar product is defined as $\omega\cdot x = \sum_{s=0}^{\infty}\omega_s x_s$. The notion of *probability measures* in the statement means that $\omega_s, \hat\omega_s$ are non-negative for any s and that $\sum_{s=0}^{\infty}\omega_s = \sum_{s=0}^{\infty}\hat\omega_s = 1$. Remark that since $\omega_s, \hat\omega_s \geq 0$, the *monotonicity* property is satisfied.

This representation can be understood as a social welfare function for an economy with two agents. While the first agent would be highly myopic and only consider the close future of the utility stream $\hat{\omega} \cdot x$, the second one would rely on a weaker form of myopia by considering the maximum between the close future value $\omega \cdot x$ and the infimum limit of the distant future value of the stream. The social planer maximizes, in the same spirit as the classical maximin criteria of Rawls [27], the welfare of the least favored agent.

It is easy to verify that the index I satisfies the fundamental axiom **F**. Further observe that, for large enough values of T, both $\omega \cdot \big(0\mathbb{1}_{[0,T]}, \mathbb{1}_{[T+1,\infty[}\big)$ and $\hat{\omega} \cdot \big(0\mathbb{1}_{[0,T]}, \mathbb{1}_{[T+1,\infty[}\big)$ are bounded above by 1. This implies that χ_g is defined as the limit of

$$I\big(0\mathbb{1}_{[0,T]}, \mathbb{1}_{[T+1,\infty[}\big) = \min\left\{\sum_{s=T+1}^{\infty} \hat{\omega}_s, 1\right\},$$

that is equal to zero. By a similar argument, $\chi_\ell = 0$.

However remark that there exist $x, z \in \ell_\infty$ such that $\lim_{T\to\infty} I\big(x_{[0,T]}, z_{[T+1,\infty[}\big) \neq I(x)$. The two sequences $\hat{\omega}$ and ω being different, there exists $x \in \ell_\infty$ such that $\hat{\omega} \cdot x > \omega \cdot x > \liminf_{s\to\infty} x_s$. Consider now z satisfying $\hat{\omega} \cdot x > \liminf_{s\to\infty} z_s > \omega \cdot x > \liminf_{s\to\infty} x_s$. This implies that:

$$\lim_{T\to\infty} I\big(x_{[0,T]}, z_{[T+1,\infty[}\big) = \liminf_{s\to\infty} z_s,$$

that differs from $I(x) = \omega \cdot x$.

Example 2.1 makes clear that the sole occurrence of two zero values for the myopia parameters χ_g and χ_ℓ is not sufficient for ensuring the negligibility of the distant future. Likewise, the following example illustrates how, for a configuration $\chi_g = \chi_\ell = 1$, the close future can continue to play a role in the evaluation of the distant future.

Example 2.2. Consider an order being represented by the following index function

$$\hat{I}(x) = \min\left\{\limsup_{s\to\infty} x_s, \max\left\{\omega \cdot x, \liminf_{s\to\infty} x_s\right\}\right\},$$

for ω a probability measure in ℓ_1. Along the interpretation of Example 2.1, while the first agent in this economy is extremely non-myopic and evaluates utility streams by the sole consideration of the supremum of its asymptotic values, the second one is only partially myopic. Relying on the same arguments as for Example 2.1,

$$\lim_{T\to\infty} \hat{I}(o\mathbb{1}_{[0,T]}, \mathbb{1}_{[T+1,\infty[}) = 1,$$
$$-\lim_{T\to\infty} \hat{I}(0\mathbb{1}_{[0,T]}, -\mathbb{1}_{[T+1,\infty[}) = -\Big(\lim_{T\to\infty} \hat{I}(\mathbb{1}_{[0,T]}, 0\mathbb{1}_{[T+1,\infty[}) - \hat{I}(\mathbb{1})\Big) = 1,$$

that implies that $\chi_g = \chi_\ell = 1$.[10]

Consider x, z satisfying $\liminf_{s\to\infty} x_s < \omega \cdot x < \omega \cdot z < \limsup_{s\to\infty} x_s$. One has

$$\lim_{T\to\infty} \hat{I}(z_{[0,T]}, x_{[T+1,\infty[}) = \omega \cdot z,$$

that differs from $\hat{I}(x) = \omega \cdot x$.

The consideration of Examples 2.1 and 2.2 suggests the need for a deeper understanding of the problem at stake, *i.e.,* the precise influence of the remote components of a utility stream. As this shall be argued in the next section, a clear picture becomes available when appropriate supplementary structures are added on the preferences order.

3. A Decomposition for the Future: Closeness vs Remoteness

3.1 Distant Future Order

The following axiom assumes that there exists an evaluation of the distant future components of the utility stream which is independent from the starting components—the *close future*—of that utility stream.

Axiom G1. For any $x \in \ell_\infty$ and any constant $d \in \mathbb{R}$, either, for any $\varepsilon > 0$, there

[10] This implies that, for any $x \in \ell_\infty$, $\lim_{T\to\infty} \hat{I}(x_{[0,T]}, 0\mathbb{1}_{[T+1,\infty[}) = 0$.

exists $T_0(\varepsilon)$ such that, for any $z \in \ell_\infty$ and for every $T \geq T_0(\varepsilon)$:

$$\big(z_{[0,T]}, x_{[T+1,\infty[}\big) \succeq \big(z_{[0,T]}, d\mathbb{1}_{[T+1,\infty[}\big) - \varepsilon\mathbb{1},$$

or, for any $\varepsilon > 0$, there exists $T_0(\varepsilon)$ such that, for any $z \in \ell_\infty$ and for every $T \geq T_0(\varepsilon)$:

$$\big(z_{[0,T]}, d\mathbb{1}_{[T+1,\infty[}\big) \succeq \big(z_{[0,T]}, x_{[T+1,\infty[}\big) - \varepsilon\mathbb{1}.$$

For any sequence x and a constant sequence $d\mathbb{1}$ and for every $\varepsilon > 0$, this axiom postulates the existence of a date $T_0(\varepsilon)$. Beyond that date, the distant future component of the sequence x will either overtake the sequence $(d-\varepsilon)\mathbb{1}$ or be overtaken by the sequence $(d+\varepsilon)\mathbb{1}$. This takes place independently from the initial components—the *close future*—of the sequence z. Otherwise stated, either x or $d\mathbb{1}$ dominates in the distant future. Such a *distant future sensitivities* axiom interestingly contradicts with the usual *negligible-tail* or *tail-insensitivity* axioms of the literature.

As a matter of illustration, the most intuitive order satisfying both **F** and **G1** would be the infimum limit and be represented by $I(x) = \liminf_{s\to\infty} x_s$. It is also associated with the occurrence of unitary values for both of the myopia parameters, $\chi_g = \chi_\ell = 1$.

The satisfaction of axiom **G1** assumes a direct and intuitive corollary. Indeed, and under the fundamental axiom **F**, this axiom ensures the existence of the limit $\lim_{T\to\infty} I\big(z_{[0,T]}, x_{[T+1,\infty[}\big)$ for any $x, z \in \ell_\infty$. Moreover, and for any $x, y \in \ell_\infty$, the comparison between the limits corresponding to x and y does not depend on initial components brought by z.

LEMMA **3.1.** *Assume that the order $\succeq$ satisfies axioms **F** and **G1**.*

(i) *For any $z \in \ell_\infty$, $\lim_{T\to\infty} I\big(z_{[0,T]}, x_{[T+1,\infty[}\big)$ is well defined.*

(ii) *For any $x, y \in \ell_\infty$, if, for some z, z',*

$$\lim_{T\to\infty} I\big(z_{[0,T]}, x_{[T+1,\infty[}\big) \geq \lim_{T\to\infty} I\big(z_{[0,T]}, y_{[T+1,\infty[}\big),$$

then, for any $z' \in \ell_\infty$,

$$\lim_{T\to\infty} I\big(z'_{[0,T]}, x_{[T+1,\infty[}\big) \geq \lim_{T\to\infty} I\big(z'_{[0,T]}, y_{[T+1,\infty[}\big).$$

Even though the comparison between two limits corresponding to x and y does not depend on z, the limit $\lim_{T\to\infty} I\big(z_{[0,T]}, x_{[T+1,\infty[}\big)$ can keep on depending on z.

DEFINITION 3.1. *Define the order $\succeq_d$ as, for any $x, y \in \ell_\infty$, $x \succeq_d y$ if and only if, for any $\varepsilon > 0$, there exists $T_0(\varepsilon)$ such that, for any $z \in \ell_\infty$ and for every $T \geq T_0(\varepsilon)$:*

$$\big(z_{[0,T]}, x_{[T+1,\infty[}\big) \succeq \big(z_{[0,T]}, y_{[T+1,\infty[}\big) - \varepsilon\mathbb{1}.$$

EXAMPLE 3.1. Consider again the order represented by the index function I in Example 2.1. First observe that the order represented by I satisfies axiom **F**. The following arguments prove that the axiom **G1** is also satisfied.

Fix any $x \in \ell_\infty$ and some scalar $d \in \mathbb{R}$. Consider the case $\liminf_{s\to\infty} x_s \geq d$. Fixing any $\varepsilon > 0$, select $T_0(\varepsilon)$ such that, for any $T \geq T_0(\varepsilon)$:

$$\sum_{s=T+1}^{\infty} \omega_s x_s \geq d \sum_{s=T+1}^{\infty} \omega_s - \varepsilon,$$
$$\sum_{s=T+1}^{\infty} \hat{\omega}_s x_s \geq d \sum_{s=T+1}^{\infty} \hat{\omega}_s - \varepsilon.$$

For any $z \in \ell_\infty$ and any $T \geq T_0(\varepsilon)$, this implies that:

$$\begin{aligned} I\big(z_{[0,T]}, x_{[T+1,\infty[}\big) \geq \min\Big\{ & \hat{\omega} \cdot (z_{[0,T]}, d\mathbb{1}_{[T+1,\infty[}) - \varepsilon, \\ & \max\Big\{\omega \cdot \big(z_{[0,T]}, d\mathbb{1}_{[T+1,\infty[}\big) - \varepsilon, \liminf_{s\to\infty} x_s - \varepsilon\Big\}\Big\} \\ \geq {} & I\big(z_{[0,T]}, d\mathbb{1}_{[T+1,\infty[}\big) - \varepsilon. \end{aligned}$$

Similar arguments may be used for the remaining configuration $\liminf_{s\to\infty} x_s \leq d$. Moreover, even though $\chi_g = \chi_\ell = 0$, the order $\succeq_d$ is not trivial. Select indeed z^*

satisfying $\hat{\omega} \cdot z^* > 0 > \omega \cdot z^*$. It derives that:

$$\lim_{T\to\infty} I\big(z^*_{[0,T]}, 0\mathbb{1}_{[T+1,\infty[}\big) = \min\big\{\hat{\omega} \cdot z^*, \max\{\omega \cdot z^*, 0\}\big\} = 0,$$
$$\lim_{T\to\infty} I\big(z^*_{[0,T]}, \mathbb{1}_{[T+1,\infty[}\big) = \min\big\{\hat{\omega} \cdot z^*, \max\{\omega \cdot z^*, 1\}\big\} = \min\big\{\hat{\omega} \cdot z^*, 1\big\} > 0.$$

Hence $\mathbb{1} \succeq_d 0\mathbb{1}$ and $0\mathbb{1} \not\succeq_d \mathbb{1}$, that establishes the non-triviality of the order $\succeq_d$.

Proposition 3.1 proves that if one of the two myopia parameters χ_g and χ_ℓ differs from zero, then the order $\succeq_d$ satisfies axiom **F**.

PROPOSITION 3.1. *Assume that the initial order $\succeq$ satisfies axioms* **F** *and* ***G1*** .

(i) *The order $\succeq_d$ is complete.*

(ii) *If at least one of the two myopia values χ_g and χ_ℓ differs from zero, then the order $\succeq_d$ is not trivial. In this configuration, it satisfies axiom* **F** *and can be represented by an index function I_d, which is positively homogeneous, constantly additive, and satisfies:*

$$I_d\big(z_{[0,T]}, x_{[T+1,\infty[}\big) = I_d(x) \text{ for any } x, z \in \ell_\infty, T \in \mathbb{N}.$$

Proposition 3.1 also implies that there also exists an index function satisfying the properties of Lemma 2.1. The value of the index function does not depend upon the starting——*close future*—components of the sequence z: upon a change in a mere finite number of values of the inter-temporal stream, the distant future evaluation of that stream remains the same.

3.2 Close future order

In order to enable a decomposition between the *distant future* and the *close future*, consider a *close future sensitivities* axiom **G2**, that is to be understood as the complement of axiom **G1**. This axiom assumes that there exists an evaluation of the close future components of the utility stream which is independent from the tail—the *distant future*—of that utility stream.

AXIOM G2. For any $x \in \ell_\infty$, a constant $c \in \mathbb{R}$, either, for any $\varepsilon > 0$, there exists $T_0(\varepsilon)$ such that, for any $z \in \ell_\infty$ and for every $T \geq T_0(\varepsilon)$,

$$\big(x_{[0,T]}, z_{[T+1,\infty[}\big) \succeq \big(c\mathbb{1}_{[0,T]}, z_{[T+1,\infty[}\big) - \varepsilon\mathbb{1},$$

or there exists $T_0(\varepsilon)$ such that, for any $z \in \ell_\infty$ and for every $T \geq T_0(\varepsilon)$,

$$\big(c\mathbb{1}_{[0,T]}, z_{[T+1,\infty[}\big) \succeq \big(x_{[0,T]}, z_{[T+1,\infty[}\big) - \varepsilon\mathbb{1}.$$

This assumption reads as follows: for any sequence x and a constant sequence $c\mathbb{1}$, either the sequence x will overtake the sequence $(c-\varepsilon)\mathbb{1}$ or it will be dominated by the sequence $(c+\varepsilon)\mathbb{1}$, both of these occurrences being defined whatever the distant future behaviour of that sequence. Otherwise stated, either x or $c\mathbb{1}$ dominates in the close future.

Conditions in the literature commonly assume that the effect of the distant future converges to zero—*e.g.*, the *Continuity at infinity* of Chambers & Echenique [15], or the axioms ensuring some sort of *negligible tail* for the distribution. In opposition to this, the *close future sensitivities* Axiom **G2** merely assumes that the evaluation of the close future is unaltered by the distant future components of the utility stream.

Along Lemma 3.1, the subsequent Lemma 3.2 provides some intuition for axiom **G2**.

LEMMA **3.2.** *Assume that the order $\succeq$ satisfies axioms **F** and **G2**.*

(i) *For any $z \in \ell_\infty$, $\lim_{T\to\infty} I\big(x_{[0,T]}, z_{[T+1,\infty[}\big)$ is well defined.*

(ii) *For any $x, y \in \ell_\infty$, if, for some z,*

$$\lim_{T\to\infty} I\big(x_{[0,T]}, z_{[T+1,\infty[}\big) \geq \lim_{T\to\infty} I\big(y_{[0,T]}, z_{[T+1,\infty[}\big),$$

then, for any $z' \in \ell_\infty$,

$$\lim_{T\to\infty} I\big(x_{[0,T]}, z'_{[T+1,\infty[}\big) \geq \lim_{T\to\infty} I\big(y_{[0,T]}, z'_{[T+1,\infty[}\big).$$

As an illustration, consider the order represented by the index function $I(x) = (1-\delta)\sum_{s=0}^{\infty}\delta^s x_s$, for some $0 < \delta < 1$. This order satisfies both **F** and **G2**, both of its myopia coefficients reducing to zero, $\chi_g = \chi_\ell = 0$. Following the leading Example 2.2, a more elaborated formulation is however conceivable:

EXAMPLE 3.2. Consider the order represented by the index function $\hat{I}$ in Example 2.2. Observe that the order being represented by $\hat{I}$ satisfies axiom **F**. The following arguments are going to prove that the axiom **G2** is also satisfied. Fix indeed any $x \in \ell_\infty$ and some constant $c \in \mathbb{R}$. Firstly, consider the case $\omega \cdot x \geq c$. For any given $\varepsilon > 0$, select a date $T_0(\varepsilon)$ such that, for any $T \geq T_0(\varepsilon)$, one has:

$$\sum_{s=0}^{T} \omega_s x_s \geq c \sum_{s=0}^{T} \omega_s - \varepsilon.$$

For any sequence $z \in \ell_\infty$ and any date $T \geq T_0(\varepsilon)$, the value of the index $\hat{I}$ satisfies:

$$\begin{aligned} \hat{I}\big(x_{[0,T]}, z_{[T+1,\infty[}\big) &\geq \min\left\{\limsup_{s\to\infty} z_s, \max\left\{\omega \cdot (c\mathbb{1}_{[0,T]}, z_{[T+1,\infty[}) - \varepsilon, \liminf_{s\to\infty} z_s\right\}\right\} \\ &\geq \hat{I}\big(c\mathbb{1}_{[0,T]}, z_{[T+1,\infty[}\big) - \varepsilon. \end{aligned}$$

Whence, and for any $T \geq T_0(\varepsilon)$, the behaviour described by Axiom **G2**:

$$\big(x_{[0,T]}, z_{[T+1,\infty[}\big) \succeq \big(c\mathbb{1}_{[0,T]}, z_{[T+1,\infty[}\big) - \varepsilon\mathbb{1}.$$

For the remaining case with $\omega \cdot x \leq c$, use the same arguments.

One should also emphasise that, even though $\chi_g = \chi_\ell = 1$, the close order $\succeq_c$ is not trivial. Select indeed $z^* \in \ell_\infty$ such that $\liminf_{s\to\infty} z^*_s < 0 < \limsup_{s\to\infty} z^*_s$.

$$\begin{aligned} \lim_{T\to\infty} \hat{I}\big(0\mathbb{1}_{[0,T]}, z^*_{[T+1,\infty[}\big) &= \min\left\{\limsup_{s\to\infty} z^*_s, \max\left\{0, \liminf_{s\to\infty} z^*_s\right\}\right\} \\ &= 0, \\ \lim_{T\to\infty} \hat{I}\big(\mathbb{1}_{[0,T]}, z^*_{[T+1,\infty[}\big) &= \min\left\{\limsup_{s\to\infty} z^*_s, \max\left\{1, \liminf_{s\to\infty} z^*_s\right\}\right\} \\ &= \min\big\{\limsup_{s\to\infty} z^*_s, 1\big\} \\ &> 0. \end{aligned}$$

Whence, $\mathbb{1} \succeq_c 0\mathbb{1}$ and $0\mathbb{1} \not\succeq_c \mathbb{1}$ and the order $\succeq_c$ is not trivial.

Definition 3.2. *Define the* close future *order* $\succeq_c$ *as, for any* $x, y \in \ell_\infty$, $x \succeq_c y$ *if and only if, for any* $\varepsilon > 0$, *there exists* $T_0(\varepsilon)$ *such that, for any sequence* $z \in \ell_\infty$ *and for every date* $T \geq T_0(\varepsilon)$,

$$\big(x_{[0,T]}, z_{[T+1,\infty[}\big) \succeq \big(y_{[0,T]}, z_{[T+1,\infty[}\big) - \varepsilon \mathbb{1}.$$

Proposition 3.2. *Assume that the initial order* $\succeq$ *satisfies axioms* **F** *and* ***G2***.

(i) *The close order* $\succeq_c$ *is complete.*

(ii) *The order* $\succeq_c$ *is not trivial if at least one of the two values* χ_g, χ_ℓ *differs from* 1. *In this configuration, the order* $\succeq_c$ *satisfies axiom* **F** *and can be represented by an index function* I_c *which is positively homogeneous, constantly additive and satisfies:*

$$\lim_{T\to\infty} I_c\big(x_{[0,T]}, z_{[T+1,\infty[}\big) = I_c(x) \text{ for any } x, z \in \ell_\infty.$$

The property (ii) illustrates the ways in which the close future order recovers a *tail-insensitivity* property.

All of the results of Propositions 3.1 and 3.2 as well as the characterizations in Examples 3.1 and 3.2 indicate the need for a more achieved characterization of the configurations where the two myopia parameters obtain boundary values, *i.e.*, $\chi_g = \chi_\ell = 0$, or $\chi_g = \chi_\ell = 1$. The following statement provides a clarified view.

Proposition 3.3. *Assume that the order* $\succeq$ *satisfies axioms* ***F*** *and* ***G1***, ***G2***.

(i) *If* $\chi_g = \chi_\ell = 0$, *then the order* $\succeq_d$ *is trivial: for any* $x, y \in \ell_\infty$, $x \sim_d y$. *Moreover,* $I(x) = I_c(x)$.

(ii) *If* $\chi_g = \chi_\ell = 1$, *then the order* $\succeq_c$ *is trivial: for any* $x, y \in \ell_\infty$, $x \sim_c y$. *Moreover,* $I(x) = I_d(x)$.

Otherwise stated, it is only when the initial order satisfies both axioms **G1** and **G2**, *i.e.*, when the decomposition between the distant and the close components future is fully completed, that the boundary values for the myopia coefficients may result into the triviality of one of the two orders. Proposition 3.3 also provides an indirect proof of the non equivalence between axioms **G1** and **G2**. The index function in Example 2.1 indeed satisfies axiom **G1** but not axiom **G2**. By contrast, the index function in Example 2.2 satisfies axiom **G2** but not axiom **G1**.

From now on, by convention, under axioms **F** and **G1**, **G2**, if $\chi_g = \chi_\ell = 0$, the distant future function will be defined as $I_d(x) = 0$ for any $x \in \ell_\infty$ while, if $\chi_g = \chi_\ell = 1$, the close future function will be defined as $I_c(x) = 0$ for any $x \in \ell_\infty$.

3.3 A decomposition between the distant and close future orders

Under axioms **F**, **G1** and **G2**, there is a clear potential for the index function I to be decomposed into a convex sum of two index functions I_d and I_c, *e.g.*,

$$I(x) = (1 - \chi^*) I_c(x) + \chi^* I_d(x),$$

for some value $\chi^* \in [0, 1]$. It shall however be argued that this is not the only possible representation. Would the selected parameter χ^* not change over time, the prevalence of such a decomposition would imply that:

$$\lim_{T\to\infty} I\big(\mathbb{1}_{[0,T]}, 0\mathbb{1}_{[T+1,\infty[}\big) + \lim_{T\to\infty} I\big(0\mathbb{1}_{[0,T]}, \mathbb{1}_{[T+1,\infty[}\big) = 1,$$

which is equivalent to $\chi_\ell = \chi_g$, and therefore to $\chi^* = \chi_g = \chi_\ell$. Under axioms **F**, **G1** and **G2**, the holding of such an equality is anything but obvious. In order to progress in this direction, consider the following axiom, which strengthens the *weak convexity* property introduced in axiom **F**(vi).

Axiom A1. Consider $x, y \in \ell_\infty$ such that $x \succeq y$, constants $c, d \in \mathbb{R}$, $0 \le \lambda \le 1$. For every $\varepsilon > 0$, there exists $T_0(\varepsilon)$ such that, for every $T \ge T_0(\varepsilon)$

$$(1-\lambda)x + \lambda(c\mathbb{1}_{[0,T[}, d\mathbb{1}_{[T+1,+\infty[}) \succeq (1-\lambda)y + \lambda(c\mathbb{1}_{[0,T[}, d\mathbb{1}_{[T+1,+\infty[}) - \varepsilon\mathbb{1}.$$

This axiom ensures that the comparison between two streams does not change when one follows directions that are constant, though possible distinct, in the close and distant futures. These directions are thus *comparison neutral.*

THEOREM 3.1. *Assume that the initial order* $\succeq$ *satisfies axioms* **F** *and* **G1**, **G2**. *Axiom* **A1** *is satisfied if and only if one of the following equivalent properties holds:*

(i) *The two myopia parameters are equal:* $\chi_g = \chi_\ell = \chi^*$.

(ii) *There exists* $0 \leq \chi^* \leq 1$ *such that, for any* $x \in \ell_\infty$ *and any constants* $c, d \in \mathbb{R}$,

$$\lim_{T\to\infty} I\big(x + (c\mathbb{1}_{[0,T[}, d\mathbb{1}_{[T+1,+\infty[})\big) = I(x) + (1-\chi^*)c + \chi^* d.$$

(iii) *There exists* $0 \leq \chi^* \leq 1$ *such that, for any* $x \in \ell_\infty$,

$$I(x) = (1-\chi^*)I_c(x) + \chi^* I_d(x).$$

As this shall emerge from the above statement, under axiom **A1**, the two myopia parameters χ_g and χ_ℓ happen to coincide and the index function satisfies a generalized form of *constant additivity.* The evaluation of a given inter-temporal stream can now be decomposed into a convex combination of its close future and distant future components, the convexity parameter being constant and also synthetising the myopia parameters.

While a convex combination between Examples 3.1 and 3.2 could have been conjectured to provide an interesting illustration of this decomposition, such a formulation is inappropriate since it would satisfy neither Axiom **G1** nor Axiom **G2**. The following illustration, that first appeared in Chichilnisky [13], will provide an intuitive picture of the properties at stake.

EXAMPLE 3.3. Consider the order represented by the following index function: for some $0 < \chi^* < 1$, the index function

$$I(x) = (1-\chi^*)\sum_{s=0}^{\infty}(1-\delta)\delta^s x_s + \chi^* \liminf_{s\to\infty} x_s,$$

is such that $I_c(x) = (1-\delta)\sum_{s=0}^{\infty}\delta^s x_s$ and $I_d(x) = \liminf_{s\to\infty} x_s$. Fix any sequence $x \in \ell_\infty$, a constant $c \in \mathbb{R}$ and any scalar $\varepsilon > 0$. Consider the case $(1-\delta)\sum_{s=0}^{\infty}\delta^s x_s \geq$

c and fix a date $T_0(\varepsilon)$ such that, for any date $T \geq T_0(\varepsilon)$, one has $(1-\delta)\sum_{s=0}^{T} \delta^s x_s \geq c\sum_{s=0}^{T} \omega_s - \varepsilon$. This implies that, for any $z \in \ell_\infty$, the following inequality is satisfied:

$$(1-\chi^*)\left(\sum_{s=0}^{T}(1-\delta)\delta^s x_s + \sum_{s=T+1}^{\infty}(1-\delta)\delta^s z_s\right) + \chi^* \liminf_{s\to\infty} z_s$$
$$\geq (1-\chi^*)\left(c\sum_{s=0}^{T}\omega_s + \sum_{s=T+1}^{\infty}\omega_s z_s\right) + \chi^* \liminf_{s\to\infty} z_s - \varepsilon.$$

It derives that, for any $T \geq T_0(\varepsilon)$ and $z \in \ell_\infty$, the index I satisfies:

$$I\big(x_{[0,T]}, z_{[T+1,\infty[}\big) \geq I\big(c\mathbb{1}_{[0,T]}, z_{[T+1,\infty[}\big) - \varepsilon.$$

The case $\sum_{s=0}^{\infty}(1-\delta)\delta^s x_s \leq c$ can be analyzed with a similar argument. The order here hence satisfies the *close future sensitivities* axiom **G2**, its close future order being represented by the function $I_c(x) = \sum_{s=0}^{\infty}(1-\delta)\delta^s x_s$. Relying on the same arguments, this order also satisfies the *distant future sensitivities* axiom **G1**, its distant future index function being given by $I_d(x) = \liminf_{s\to\infty} x_s$.

More generally, and as soon as axiom **A1** is relaxed, results become more involving but also uncover interesting subtler facets of the distant future - close future discrepancy. In point of fact, while the evaluation of a inter-temporal stream still expresses as a convex combination of its close future and distant future values, the decomposition parameter must now change as a function of the involved sequence x. The configuration

$$\lim_{T\to\infty} I\big(\mathbb{1}_{[0,T]}, 0\mathbb{1}_{[T+1,\infty[}\big) + \lim_{T\to\infty} I\big(0\mathbb{1}_{[0,T]}, \mathbb{1}_{[T+1,\infty[}\big) \leq 1,$$

which is equivalent to $\chi_g \leq \chi_\ell$, can first be understood as a *pessimistic*, or a mainly *myopia-bending* occurrence: the value brought by the distant future is not sufficiently large to compensate the loss that is incurred in the close future. Likewise, the configuration

$$\lim_{T\to\infty} I\big(\mathbb{1}_{[0,T]}, 0\mathbb{1}_{[T+1,\infty[}\big) + \lim_{T\to\infty} I\big(0\mathbb{1}_{[0,T]}, \mathbb{1}_{[T+1,\infty[}\big) \geq 1,$$

which is equivalent to $\chi_g \geq \chi_\ell$, can be understood as an *optimistic*, or an essentially *non myopia-bending* situation: the gain in the distant future is valued more than the loss that is incurred in the close future.

The following theorem eventually shows that there exists a multiplicity of possible myopia degrees.

THEOREM 3.2. *Assume that the initial order* $\succeq$ *satisfies axioms* **F** *and* **G1**, **G2**.

(i) *For any* $x \in \ell_\infty$,

a) *If* $I_c(x) \leq I_d(x)$, *then*

$$I(x) = (1 - \chi_g)I_c(x) + \chi_g I_d(x).$$

b) *If* $I_c(x) \geq I_d(x)$, *then*

$$I(x) = (1 - \chi_\ell)I_c(x) + \chi_\ell I_d(x).$$

(ii) *Let* $\underline{\chi} = \min\{\chi_g, \chi_\ell\}$, *and* $\overline{\chi} = \max\{\chi_g, \chi_\ell\}$.

a) *If* $\chi_g \leq \chi_\ell$, *then and for every* $x \in \ell_\infty$,

$$I(x) = \min_{\underline{\chi} \leq \chi \leq \overline{\chi}} \big[(1 - \chi)I_c(x) + \chi I_d(x)\big].$$

b) *If* $\chi_g \geq \chi_\ell$, *then and for every* $x \in \ell_\infty$,

$$I(x) = \max_{\underline{\chi} \leq \chi \leq \overline{\chi}} \big[(1 - \chi)I_c(x) + \chi I_d(x)\big].$$

This theorem also clarifies the choice of the myopia degree χ that determines an optimal share between the close future and the distant future. Remark that, from Theorem 3.2(i), the evaluation can be expressed as a function of the distant and close future values. The weight of the convex combination being provided by the remote gains myopia coefficient χ_g for the case where the close future is less valued than the distant future and by the remote losses myopia coefficient χ_ℓ in the opposite case. Theorem 3.2(ii) is a direct consequence of Theorem 3.2(i). For $\chi_g \leq \chi_\ell$, the decision maker will always assign the highest possible parameter to the smallest value between $I_c(x)$ and $I_d(x)$, she or her indeed always selects the minimum value of a convex combination whose weight is given by χ. For $\chi_g \geq \chi_\ell$, the decision maker exhibits an opposite behaviour, she or he chooses the highest possible parameter to the bigger value between $I_c(x)$ and $I_d(x)$, she or he always selects the maximum value of a convex combination whose weight is given by χ.

It should finally be pointed out that, while the operator min cannot appear under $\chi_g > \chi_\ell$, the same prevails for the operator max under $\chi_g < \chi_\ell$. The optimistic or pessimistic dimension of the decision maker is hence appropriately described by the comparison between χ_g and χ_ℓ.

3.4 Relation with the non-dictatorship criteria

In a rang of well-known contributions, Chichinilsky [13, 14] introduces the influential ideas of *no dictatorship* properties, in order to capture the intuition of *sustainable preferences*. The *no dictatorship of the present* states that there exist two utility streams that the comparison can be reversed by a careful changes in distant future of these streams. By contrast, the *no dictatorship of the future* imposes a similar property, requiring changes in the present future.

Under the assumption that the rate of substitution between any two generations is independent from their levels, a *sustainable criterion* is linear, and can be represented as[11]

$$I(x) = (1-\chi)\sum_{s=0}^{\infty}\omega_s x_s + \chi\phi \cdot x,$$

where ω is a probability measure, χ belongs to $[0,1]$, and ϕ is a *charge*, *i.e.*, a linear function with the property that, if one merely changes a finite number of values in the sequence x, the evaluation of that sequence x under ϕ, $\phi \cdot x$, does not change.

Now, the *no dictatorship of the present* ensures that the convex parameter χ is positive, *i.e*, the value of the distant future cannot be ignored in the evaluation of the utility stream. By contrast, the *no dictatorship of the future* implies a similar property for the close future part with a value of χ that is strictly smaller than

[11]See Theorem 2 in [13].

1. Otherwise stated, and with a simultaneous holding of a *non-dictatorship of the present* and a *non-dictatorship of the future*, the respective weights of the discounted sum and of the charge are both positive.

It is easy to check that, under axioms **F** *and linearity*, the *no dictatorship of the present* is equivalent to axiom **G1**, and *no dictatorship of the future* is equivalent to axiom **G2**. Note however that, by contrast, the functions I_c and I_d in Theorem 3.2 are potentially nonlinear. As an illustrative example, consider, for the close future part the recent multiple discounts criterion introduced by Chambers & Echenique [15], and for the distant part, a convex combination of the infimum limit and the supremum limit. In these two cases, the resulting chosen functions are not linear.

3.5 Present, future and anonymity

The question of equity in evaluating the welfare of generations can be understood in different ways. Another line of literature considers a stronger version of equity labelled as *anonymity*. Under *anonymity*, the evaluation of a utility stream does not change after a permutation between the generations. To the best knowledge of the authors, and among the available criteria satisfying this property, the closest one from this article is due to Zuber & Asheim [30]. In that article, following the axiomatic foundations of Koopmans [22], the authors went on assuming *anonymity*, a restricting *strong Pareto*, *separability* between the present and the future and *stationarity* on the set of utility streams which are increasing or can be re-arranged as increasing. The preferences order is then represented by the following index function:

$$I(x) = \inf_{\pi \in \Pi} \left[(1-\delta) \sum_{s=0}^{\infty} \delta^s x_{\pi(s)} \right],$$

where $\delta \in (0,1)$ is a discount factor and Π is the set of all permutations of the natural numbers. It is easy to verify that the associated order satisfies axiom **F**.

It is also worth emphasising a crucial difference between these two approaches of equity. *No dictatorship* and axioms **G1**, **G2**, which can be considered as a weaker version of *anonymity*, may lead to criteria which satisfy the *strong Pareto* property. In opposition to this, *strong Pareto* does not hold with criteria fully satisfying *Anonymity*[12].

As a final remark, if one follows an extension of the Zuber & Asheim [30] criterion, by considering the following one,

$$I(x) = \inf_{\delta\in(0,1)} \inf_{\pi\in\Pi} \left[(1-\delta)\sum_{s=0}^{\infty} \delta^s x_{\pi(s)}\right],$$

one may find the famous Rawlsian criterion $I(x) = \inf_{s\geq 0} x_s$. Along the one of these authors, this Rawlsian criterion satisfies axiom **F** and *Anonymity*.

3.6 Perspectives for applications

The idea of a welfare criterion which could offer equal opportunities for the future and the present is very appealing, and the non-dictatorship ideas have motivated a large range of researches in economics[13]. It has however and up to now been confronted with two ranges of difficulties. The first relates to the generic non-existence of an optimal solution to the associated optimisation. This was first established by Heal [21] in a model with renewable resources and got later refined by Ayong le Kama & al [3] in a growth context. The allowance for non-linearity that got introduced by the current close future - distant future dichotomy leaves little hope for any improvement in this regard.

Perhaps more promisingly, another challenge pertains to the time inconsistency problem for the criteria presented in this article. A possible way to overcome this difficulty is to study the class of *markovian rules* that were presented in Phelps & Pollack [26]. A recent approach following this direction in the context of a linear non-dictatorship criterion was considered in the work of Asheim & Ekeland [1] and delivered interesting results. For a sufficiently high productivity of the initial stock, the distant future played no part in the determination of the solution. By contrast and for the converse case building from a low productivity of stock, the distant fu-

[12] In fact, this Impossibility Theorem, after which there is no social welfare function satisfying simultaneously *strong Pareto* and *anonymity*, is proven in Diamond [16] and Basu & Mitra [7].

[13] For a research about *Sustainable Social Welfare Functions*, see Asheim, Mitra & Tungodden [2].

ture part leads the economy to a larger stock conservation that the one which would have been available following a standard discounted utilitarian configuration[14].

The consideration of a related approach under the light of the results and the extra non-linearity brought by this article would be of interest. A careful appraisal of the associated markovian rules nonetheless represents a real challenge at many levels and should definitely be the object of another research in a (hopefully) close future.

A. Proofs for Section 2

A.1 Proof of Lemma 2.1

For $x \in \ell_\infty$, define $b_x = \sup\{b \in \mathbb{R} \text{ such that } x \succeq b\mathbb{1}\}$. By the *archimedeanity* property, it follows that $x \sim b_x\mathbb{1}$. Let $I(x) = b_x$.

(i) First, consider that for any $\lambda > 0$, $x \succeq y$ is equivalent to $\lambda x \succeq \lambda y$. Indeed, for $0 < \lambda \leq 1$, $x \succeq y$ is equivalent to $\lambda x + (1-\lambda)0\mathbb{1} \succeq \lambda y + (1-\lambda)0\mathbb{1}$.

Considering then the case $\lambda > 1$, $\lambda x \succeq \lambda y$ then prevails if and only if $(1/\lambda)\lambda x \succeq (1/\lambda)\lambda y$, or $x \succeq y$. Hence for any $\lambda > 0$, $I(\lambda x) = \lambda I(x)$.

(ii) Second, for any constant $b \in \mathbb{R}$, $x \succeq y$ is equivalent to $x + b\mathbb{1} \succeq y + b\mathbb{1}$. By the *weak convexity* property, $x \succeq y$ implies $(1/2)x + (1/2)b\mathbb{1} \succeq (1/2)y + (1/2)b\mathbb{1}$. Multiplying the two sides by 2, it follows that $x + b\mathbb{1} \succeq y + b\mathbb{1}$. Finally, and if $x + b\mathbb{1} \succeq y + b\mathbb{1}$, then $x + b\mathbb{1} + (-b\mathbb{1}) \succeq y + b\mathbb{1} + (-b\mathbb{1})$, or $x \succeq y$. Hence $I(x + b\mathbb{1}) = I(x) + b$. QED

[14] In a similar configuration, where the economic agent enjoys consumption *and* natural resource stock, Figuières & Tidball [17] prove the existence of an *optimal restricted optimal program*, which is a convex combination between the solutions of the discounted utilitarianism and of the green golden rule.

B. Proofs for Section 3

B.1 Proof of Lemma 3.1

For $x \in \ell_\infty$, define $D(x)$ as the set of values d such that, for any $\varepsilon > 0$, there exists $T_0(\varepsilon)$ such that, for any $z \in \ell_\infty$ and for any $T \geq T_0(\varepsilon)$, one has

$$\big(z_{[0,T]}, x_{[T+1,\infty[}\big) \succeq \big(z_{[0,T]}, d\mathbb{1}_{[T+1,\infty[}\big) - \varepsilon\mathbb{1}.$$

Since $x \in \ell_\infty$, any d smaller than $\inf_{s\geq 0} x_s$ belongs to $D(x)$, whence $D(x) \neq \emptyset$. Define $d_x = \sup D(x)$[15]. By axiom **G1** and for any $d' < d_x$, $d' \in D(x)$. First observe that

$$\liminf_{T\to\infty} I\big(z_{[0,T]}, x_{[T+1,\infty[}\big) \geq \limsup_{T\to\infty} I\big(z_{[0,T]}, d_x\mathbb{1}_{[T+1,\infty[}\big).$$

Assume the contrary. Then, there exists $\epsilon > 0$ and an infinite number of T such that:

$$\begin{aligned} I\big(z_{[0,T]}, x_{[T+1,\infty[}\big) &< I\big(z_{[0,T]}, d_x\mathbb{1}_{[T+1,\infty[}\big) - \epsilon \\ &< I\Big(z_{[0,T]}, \Big(d_x - \frac{\epsilon}{2}\Big)\mathbb{1}_{[T+1,\infty[}\Big) - \frac{\epsilon}{2}. \end{aligned}$$

This implies that $d_x - \epsilon/2$ does not belong to $D(x)$: a contradiction. Using the similar arguments, one obtains

$$\limsup_{T\to\infty} I\big(z_{[0,T]}, x_{[T+1,\infty[}\big) \leq \liminf_{T\to\infty} I\big(z_{[0,T]}, d_x\mathbb{1}_{[T+1,\infty[}\big).$$

This implies that:

$$\limsup_{T\to\infty} I\big(z_{[0,T]}, x_{[T+1,\infty[}\big) = \liminf_{T\to\infty} I\big(z_{[0,T]}, x_{[T+1,\infty[}\big),$$

hence the existence of a limit $I\big(z_{[0,T]}, x_{[T+1,\infty[}\big)$ when T converges to infinity. QED

[15] Observe that d_x can be infinite. The value of the limit can be independent with the choice of x and depend only in z. This case is another version of *tail-insensitivity* in the literature.

B.2 Proof of Proposition 3.1

(i) Without loss of generality, suppose that $\sup D(x) \geq \sup D(y)$. First consider the case $\sup D(y) < +\infty$. Then define $d_y = \sup D(y)$, that is finite.

Fix any $\varepsilon > 0$: since $d_y + (\varepsilon/2)\mathbb{1}$ does not belong to $D(y)$ and $d_y - (\varepsilon/2)\mathbb{1}$ belongs to $D(y)$, there exists $T_0(\varepsilon)$ such that, for any $z \in \ell_\infty$, for $T \geq T_0(\varepsilon)$:

$$\begin{aligned}\left(\left(z + \frac{\varepsilon}{2}\mathbb{1}\right)_{[0,T]}, \left(d_y + \frac{\varepsilon}{2}\mathbb{1}\right)_{[T+1,\infty[}\right) + \frac{\varepsilon}{2}\mathbb{1} &\succeq \left(z_{[0,T]}, y_{[T+1,\infty[}\right) \\ &\succeq \left(\left(z - \frac{\varepsilon}{2}\mathbb{1}\right)_{[0,T]}, \left(d_y - \frac{\varepsilon}{2}\mathbb{1}\right)_{[T+1,\infty[}\right) - \frac{\varepsilon}{2}\mathbb{1}.\end{aligned}$$

This implies, for $T \geq T_0(\varepsilon)$:

$$\left(z_{[0,T]}, d_y\mathbb{1}_{[T+1,\infty[}\right) + \varepsilon\mathbb{1} \succeq \left(z_{[0,T]}, y_{[T+1,\infty[}\right) \succeq \left(z_{[0,T]}, d_y\mathbb{1}_{[T+1,\infty[}\right) - \varepsilon\mathbb{1}.$$

Since $d_x \geq d_y$, for every $\varepsilon > 0$ and $z \in \ell_\infty$, there exists $T_0(\varepsilon)$ such that

$$\begin{aligned}\left(z_{[0,T]}, x_{[T+1,\infty[}\right) &\succeq \left(z_{[0,T]}, d_y\mathbb{1}_{[T+1,\infty[}\right) - \varepsilon\mathbb{1} \\ &\succeq \left(z_{[0,T]}, y_{[T+1,\infty[}\right) - 2\varepsilon\mathbb{1}.\end{aligned}$$

This implies that $x \succeq_d y$.

Consider now the case $\sup D(y) = +\infty$. This implies that $\sup D(x) = +\infty$. Take $d > \sup_s y_s$. Since $d \in D(x)$, for every $\varepsilon > 0$, there exists $T_0(\varepsilon)$ such that, for any $z \in \ell_\infty$, for $T \geq T_0(\varepsilon)$:

$$\begin{aligned}\left(z_{[0,T]}, x_{[T+1,\infty[}\right) &\succeq \left(z_{[0,T]}, d\mathbb{1}_{[T+1,\infty[}\right) - \varepsilon\mathbb{1} \\ &\succeq \left(z_{[0,T]}, y_{[T+1,\infty[}\right) - \varepsilon\mathbb{1}.\end{aligned}$$

(ii) One must prove the existence of $x, y \in \ell_\infty$ such that $x \succ_d y$. Chose by example $\mathbb{1}$ and $0\mathbb{1}$. By the *monotonicity* of $\succeq$, $\mathbb{1} \succeq_d 0\mathbb{1}$. Suppose now that $0\mathbb{1} \succeq_d \mathbb{1}$. Consider the case $\chi_g > 0$. Then, and for $0 < \varepsilon < \chi_g$, there exists $T_0(\varepsilon)$ such that for

$T \geq T_0(\varepsilon)$,

$$I\big(0\mathbb{1}_{[0,T]}, 0\mathbb{1}_{[T+1,\infty[}\big) \geq I\big(0\mathbb{1}_{[0,T]}, \mathbb{1}_{[T+1,\infty[}\big) - \varepsilon.$$

Letting T tend to infinity, it follows that $0 \geq \chi_g - \varepsilon$, a contradiction.

Consider then the case $\chi_\ell > 0$. For $0 < \varepsilon < \chi_\ell$, there exists $T_0(\varepsilon)$ such that, for $T \geq T_0(\varepsilon)$:

$$I\big(\mathbb{1}_{[0,T]}, 0\mathbb{1}_{[T+1,\infty[}\big) \geq I\big(\mathbb{1}_{[0,T]}, \mathbb{1}_{[T+1,\infty[}\big) - \varepsilon.$$

Letting T tend to infinity, it follows that $\varepsilon \geq \chi_\ell$, a contradiction. Hence, the distant order $\succeq_d$ is not trivial.

Further observe that, if $x \succeq_d d\mathbb{1}$, then, for every $d' \in \mathbb{R}$, $x + d'\mathbb{1} \succeq_d (d+d')\mathbb{1}$. Indeed, for $\varepsilon > 0$, there exists $T_0(\varepsilon)$ such that, for any $z \in \ell_\infty$, for $T \geq T_0(\varepsilon)$,

$$\big((z - d'\mathbb{1})_{[0,T]}, x_{[T+1,\infty[}\big) \succeq \big((z - d'\mathbb{1})_{[0,T]}, d\mathbb{1}_{[T+1,\infty[}\big) - \varepsilon\mathbb{1}.$$

From the *constantly additive* property, for $T \geq T_0(\varepsilon)$,

$$\big(z_{[0,T]}, (x + d'\mathbb{1})_{[T+1,\infty[})\big) \succeq \big(z_{[0,T]}, (d + d')\mathbb{1}_{[T+1,\infty[})\big) - \varepsilon\mathbb{1}.$$

Hence $x + d'\mathbb{1} \succeq_d (d + d')\mathbb{1}$.

Now, consider $x \in \ell_\infty$ and a constant d such that, for $\varepsilon > 0$, there exists $T_0(\varepsilon)$ with, for any $z \in \ell_\infty$, for $T \geq T_0(\varepsilon)$,

$$\big(z_{[0,T]}, x_{[T+1,\infty[}\big) \succeq \big(z_{[0,T]}, d\mathbb{1}_{[T+1,\infty[})\big) - \varepsilon\mathbb{1}.$$

Fix then any $\lambda > 0$. From axiom **G1**, there exists $T_0'(\varepsilon)$ such that, for $T \geq T_0'(\varepsilon)$,

$$\left(\left(\frac{1}{\lambda}z\right)_{[0,T]}, x_{[T+1,\infty[}\right) \succeq \left(\left(\frac{1}{\lambda}z\right)_{[0,T]}, d\mathbb{1}_{[T+1,\infty[}\right) - \frac{1}{\lambda}\varepsilon\mathbb{1},$$

that in its turn implies, for $T \geq T_0'(\varepsilon)$,

$$\Big(z_{[0,T]}, \big(\lambda x_{[T+1,\infty[}\big)\Big) \succeq \Big(z_{[0,T]}, \lambda d\mathbb{1}_{T+1,\infty[}\Big) - \varepsilon\mathbb{1}.$$

Hence, for $x \succeq_d y$ and for every $\lambda > 0$, one obtains $\lambda x \succeq_d \lambda y$.

Consider now $x, y \in \ell_\infty$ such that $x \succeq_d y$. For every $0 < \lambda < 1$, one has $(1-\lambda)x + \lambda d\mathbb{1} \succeq_d (1-\lambda)y + \lambda d\mathbb{1}$.

The order $\succeq_d$ having been proved to be non trivial, the value $d_x = \sup D(x)$ is finite and, for every $d > d_x > d'$, the relation $d\mathbb{1} \succ_d x \succ_d d'\mathbb{1}$ is to hold. There thus obviously exists $\lambda, \mu \in [0,1]$ such that $(1-\lambda)d + \lambda d' > d_x > (1-\mu)d + \mu d'$ and the order $\succeq_d$ satisfies the *archimedeanity* property.

Since $\succeq_d$ satisfies **F**, there exists an index function I_d which is homogeneous and constantly additive. The last property is a direct consequence of the definition of the order $\succeq_d$. QED

B.3 Proof of Lemma 3.2

The proof of this Proposition uses the same arguments as the proof of Lemma 3.1. QED

B.4 Proof of Proposition 3.2

(i) Using the same arguments as in the proof of Proposition 3.1, the order $\succeq_c$ is complete.

(ii) First, observe $\mathbb{1} \succ_c 0\mathbb{1}$. Indeed, suppose the contrary, $0\mathbb{1} \succeq_c \mathbb{1}$.

Consider the case $\chi_g < 1$. For $0 < \varepsilon < 1 - \chi_g$, there exists $T_0(\varepsilon)$ such that, for $T \geq T_0(\varepsilon)$,

$$I\big(0\mathbb{1}_{[0,T]}, \mathbb{1}_{[T+1,\infty[}\big) \geq I\big(\mathbb{1}_{[0,T]}, \mathbb{1}_{[T+1,\infty[}\big) - \varepsilon.$$

Letting T tend to infinity, one gets $\chi_g \geq 1 - \varepsilon$: a contradiction.

For the case $\chi_\ell < 1$, use of the same arguments. For the proof of the other properties in axiom **F**, follow the arguments in the proof of Proposition 3.1.

Consider any $x \in \ell_\infty$ and fix a constant d. For every $\varepsilon > 0$ and for large enough values of T,

$$I_c\big(c_x \mathbb{1}_{[0,T]}, d\mathbb{1}_{[T+1,\infty[}\big) + \varepsilon \geq I_c\big(x_{[0,T]}, d\mathbb{1}_{[T+1,\infty[}\big) \geq I_c\big(c_x \mathbb{1}_{[0,T]}, d\mathbb{1}_{[T+1,\infty[}\big) - \varepsilon.$$

Letting T tend to infinity and ε converge to zero,

$$\lim_{T\to\infty} I_c\big(x_{[0,T]}, d\mathbb{1}_{[T+1,\infty[}\big) = I_c(x).$$

For every $x, y \in \ell_\infty$, fix then $d \geq \sup_s y_s \geq \inf_s y_s \geq d'$. Whence, for every T,

$$I_c\big(x_{[0,T]}, d\mathbb{1}_{[T+1,\infty[}\big) \geq I_c\big(x_{[0,T]}, y_{[T+1,\infty[}\big) \geq I_c\big(x_{[0,T]}, d'\mathbb{1}_{[T+1,\infty[}\big).$$

Letting T tend to infinity, if eventually follows that $\lim_{T\to\infty} I_c\big(x_{[0,T]}, y_{[T+1,\infty[}\big) = I_c(x)$, the proof is completed. QED

B.5 Proof of Proposition 3.3

First consider the case $\chi_g = \chi_\ell = 0$. Define then the set $D(x)$ as in the proof of Proposition 3.1. Recall that, for every $x, y \in \ell_\infty$, $x \succeq_d y$ if and only if $\sup D(x) \geq \sup D(y)$. Then one must prove that, for every $x \in \ell_\infty$, $\sup D(x) = +\infty$.

Observe that the following holds for any constants c and d:

$$\lim_{T\to\infty} I\big(c\mathbb{1}_{[0,T]}, d\mathbb{1}_{[T+1,\infty[}\big) = c.$$

Indeed, if $c \leq d$, since $\chi_g = 0$,

$$\begin{aligned}
\lim_{T\to\infty} I\big(c\mathbb{1}_{[0,T]}, d\mathbb{1}_{[T+1,\infty[}\big) &= c + \lim_{T\to\infty} I\big(0\mathbb{1}_{[0,T]}, (d-c)\mathbb{1}_{[T+1,\infty[}\big) \\
&= c + (d-c) \lim_{T\to\infty} I\big(0\mathbb{1}_{[0,T]}, \mathbb{1}_{[T+1,\infty[}\big) \\
&= c.
\end{aligned}$$

For $c \geq d$, use the same argument.

This implies that, for every d, d' in $\mathbb{R}$ and for any $\varepsilon > 0$, there exists a large enough

$T(\varepsilon)$ such that

$$\left(c\mathbb{1}_{[0,T]}, d'\mathbb{1}_{[T+1,\infty[}\right) \succeq \left(c\mathbb{1}_{[0,T]}, d\mathbb{1}_{[T+1,\infty[}\right) - \varepsilon\mathbb{1}.$$

Fix now any $x, z \in \ell_\infty$, $d \in \mathbb{R}$ and any $\varepsilon > 0$. Define $c_z = I_c(z)$, that is finite since $\chi_g, \chi_\ell < 1$. Finally fix any d' such that $d' \leq \inf_s x_s$.

There then exists some $T_0(\varepsilon)$ such that, for $T \geq T_0(\varepsilon)$,

$$\begin{aligned}\left(z_{[0,T]}, x_{[T+1,\infty[}\right) &\succeq \left(c_z\mathbb{1}_{[0,T]}, x_{[T+1,\infty[}\right) - \frac{\varepsilon}{3}\mathbb{1} \\ &\succeq \left(c_z\mathbb{1}_{[0,T]}, d'\mathbb{1}_{[T+1,\infty[}\right) - \frac{\varepsilon}{3}\mathbb{1}.\end{aligned}$$

Since, for large enough values of T,

$$\left(c\mathbb{1}_{[0,T]}, d'\mathbb{1}_{[T+1,\infty[}\right) \succeq \left(c\mathbb{1}_{[0,T]}, d\mathbb{1}_{[T+1,\infty[}\right) - \frac{\varepsilon}{3}\mathbb{1},$$

for such values of T, the following also holds:

$$\left(z_{[0,T]}, x_{[T+1,\infty[}\right) \succeq \left(c_z\mathbb{1}_{[0,T]}, d\mathbb{1}_{[T+1,\infty[}\right) - \frac{2\varepsilon}{3}\mathbb{1}.$$

But, by the very definition of c_z and for large enough values of T,

$$\left(c_z\mathbb{1}_{[0,T]}, d\mathbb{1}_{[T+1,\infty[}\right) \succeq \left(z_{[0,T]}, d\mathbb{1}_{[T+1,\infty[}\right) - \frac{\varepsilon}{3}\mathbb{1},$$

that implies

$$\left(z_{[0,T]}, x_{[T+1,\infty[}\right) \succeq \left(z_{[0,T]}, d\mathbb{1}_{[T+1,\infty[}\right) - \varepsilon\mathbb{1}.$$

Hence, for every $x, y \in \ell_\infty$, $\sup D(x) = \sup D(y) = +\infty$, or $x \sim_d y$. Finally and for the remaining case $\chi_g = \chi_\ell = 1$, using of the same arguments, for every $x, y \in \ell_\infty$, one obtains $x \sim_c y$. QED

B.6 Proof of Theorem 3.1

The argument will proceed by proving that **A1** implies (i), (ii) and (iii). First, observe that for any $x \in \ell_\infty$, any constants $c, d \in \mathbb{R}$, the following holds:

$$\lim_{T\to\infty} I\big(x + (c\mathbb{1}_{[0,T]}, d\mathbb{1}_{[T+1,+\infty[})\big) = I(x) + \lim_{T\to\infty} I\big(c\mathbb{1}_{[0,T]}, d\mathbb{1}_{[T+1,+\infty[}\big).$$

Indeed, take $b_x = I(x)$, which is equivalent to $x \sim b_x\mathbb{1}$. Fix any $\varepsilon > 0$. Take $\lambda = \frac{1}{2}$, by axiom **A1**, for sufficiently large values of T,

$$\frac{1}{2}x + \frac{1}{2}(c\mathbb{1}_{[0,T]}, d\mathbb{1}_{[T+1,+\infty[}) \succeq \frac{1}{2}b_x\mathbb{1} + \frac{1}{2}(c\mathbb{1}_{[0,T]}, d\mathbb{1}_{[T+1,+\infty[}) - \frac{\varepsilon}{2}\mathbb{1}.$$

By positive homogeneity property, this implies

$$x + (c\mathbb{1}_{[0,T]}, d\mathbb{1}_{[T+1,+\infty[}) \succeq b_x\mathbb{1} + (c\mathbb{1}_{[0,T]}, d\mathbb{1}_{[T+1,+\infty[}) - \varepsilon.$$

Let T converges to infinity and next, let ε converges to zero,

$$\liminf_{T\to\infty} I\big(x + (c\mathbb{1}_{[0,T]}, d\mathbb{1}_{[T+1,+\infty[})\big) \geq b_x + \lim_{T\to\infty} I\big(c\mathbb{1}_{[0,T]}, d\mathbb{1}_{[T+1,+\infty[}\big).$$

Observe that the limit of $I\big(c\mathbb{1}_{[0,T]}, d\mathbb{1}_{[T+1,+\infty[}\big)$ exists. Indeed, for $c \leq d$, this term is decreasing as a function of T. For the case $c \geq d$, this term is increasing as a function of T.

Using the same arguments, by changing the role between x and $b_x\mathbb{1}$,

$$b_x + \lim_{T\to\infty} I\big(c\mathbb{1}_{[0,T]}, d\mathbb{1}_{[T+1,+\infty[}\big) \geq \limsup_{T\to\infty} I\big(x + (c\mathbb{1}_{[0,T]}, d\mathbb{1}_{[T+1,+\infty[})\big).$$

Hence, the limit of $I\big(x + (c\mathbb{1}_{[0,T]}, d\mathbb{1}_{[T+1,+\infty[})\big)$ exists and:

$$b_x + \lim_{T\to\infty} I\big(c\mathbb{1}_{[0,T]}, d\mathbb{1}_{[T+1,+\infty[}\big) = \lim_{T\to\infty} I\big(x + (c\mathbb{1}_{[0,T]}, d\mathbb{1}_{[T+1,+\infty[})\big).$$

i. Fix $T_0 \geq 0$. Observe that for any $T \geq T_0$,

$$\big(0\mathbb{1}_{[0,T_0]}, \mathbb{1}_{[T_0+1,+\infty[}\big) + \big(\mathbb{1}_{[0,T]}, 0\mathbb{1}_{[T+1,+\infty[}\big) \succeq \mathbb{1}.$$

Hence for any T_0,

$$\begin{aligned} I\big(0\mathbb{1}_{[0,T_0]}, \mathbb{1}_{[T_0+1,+\infty[}\big) + (1-\chi_\ell) &= I\big(0\mathbb{1}_{[0,T_0]}, \mathbb{1}_{[T_0+1,+\infty[}\big) + \lim_{T\to\infty} I\big(\mathbb{1}_{[0,T]}, 0\mathbb{1}_{[T+1,+\infty[}\big) \\ &= \lim_{T\to\infty} \Big(\big(0\mathbb{1}_{[0,T_0]}, \mathbb{1}_{[T_0+1,+\infty[}\big) + \big(\mathbb{1}_{[0,T]}, 0\mathbb{1}_{[T+1,+\infty[}\big)\Big) \\ &\geq 1. \end{aligned}$$

Let T_0 converges to infinity, this implies $\chi_g + (1-\chi_\ell) \geq 1$, which is equivalent to $\chi_g \geq \chi_\ell$.

Similarly,

$$I\big(0\mathbb{1}_{[0,T_0]}, -\mathbb{1}_{[T_0+1,+\infty[}\big) + \lim_{T\to\infty} I\big(-\mathbb{1}_{[0,T]}, 0\mathbb{1}_{[T+1,+\infty[}\big) \leq -1.$$

Let T_0 converges to infinity, one gets $-\chi_\ell + (\chi_g - 1) \leq -1$, or $\chi_g \leq \chi_\ell$.

ii. Define $\chi^* = \chi_g = \chi_\ell$. For $c \leq d$, one gets

$$\begin{aligned} \lim_{T\to\infty} I\big(c\mathbb{1}_{[0,T]}, d\mathbb{1}_{[T+1,+\infty[}\big) &= c + \lim_{T\to\infty} I\big(0\mathbb{1}_{[0,T]}, (d-c)\mathbb{1}_{[T+1,+\infty[}\big) \\ &= c + (d-c)\lim_{T\to\infty} I\big(0\mathbb{1}_{[0,T]}, \mathbb{1}_{[T+1,+\infty[}\big) \\ &= c + (d-c)\chi^* \\ &= (1-\chi^*)c + \chi^* d. \end{aligned}$$

For $c \geq d$, using the same arguments,

$$\begin{aligned} \lim_{T\to\infty} I\big(c\mathbb{1}_{[0,T]}, d\mathbb{1}_{[T+1,+\infty[}\big) &= (c-d)\lim_{T\to\infty} I\big(\mathbb{1}_{[0,T]}, 0\mathbb{1}_{[T+1,+\infty[}\big) + d \\ &= (1-\chi^*)(c-d) + d \\ &= (1-\chi^*)c + \chi^* d. \end{aligned}$$

Hence

$$\begin{aligned} \lim_{T\to\infty} I\big(x + (c\mathbb{1}_{[0,T]}, d\mathbb{1}_{[T+1,+\infty[})\big) &= I(x) + \lim_{T\to\infty} I\big(c\mathbb{1}_{[0,T]}, d\mathbb{1}_{[T+1,+\infty[}\big) \\ &= I(x) + (1-\chi^*)c + \chi^* d. \end{aligned}$$

iii. Define $c_x = I_c(x)$, $d_x = I_d(x)$. First, observe that

$$I(x) = \lim_{T\to\infty} I\big(c_x \mathbb{1}_{[0,T]}, d_x \mathbb{1}_{[T+1,+\infty[}\big).$$

Indeed, from the definition of c_x and d_x, for large enough values of T,

$$\begin{aligned} x &= \big(x_{[0,T]}, x_{[T+1,\infty[}\big) \\ &\succeq \big(c_x \mathbb{1}_{[0,T]}, x_{[T+1,\infty[}\big) - \varepsilon \mathbb{1} \\ &\succeq \big(c_x \mathbb{1}_{[0,T]}, d_x \mathbb{1}_{[T+1,\infty[}\big) - 2\varepsilon \mathbb{1}. \end{aligned}$$

Therefore

$$I(x) \geq \limsup_{T\to\infty} I\big(c\mathbb{1}_{[0,T]}, d\mathbb{1}_{[T+1,\infty[}\big) - 2\varepsilon\mathbb{1}.$$

This inequality being further true for any arbitrary $\varepsilon > 0$,

$$I(x) \geq \limsup_{T\to\infty} I\big(c_x\mathbb{1}_{[0,T]}, d_x\mathbb{1}_{[T+1,\infty[}\big).$$

Likewise,

$$I(x) \leq \liminf_{T\to\infty} I\big(c_x\mathbb{1}_{[0,T]}, d_x\mathbb{1}_{[T+1,\infty[}\big).$$

Therefore

$$\begin{aligned} I(x) &= \lim_{T\to\infty} I\big(c_x\mathbb{1}_{[0,T]}, d_x\mathbb{1}_{[T+1,\infty[}\big) \\ &= (1-\chi^*)c_x + \chi^* d_x. \end{aligned}$$

It is easy to verify that if one of three properites (i), (ii), (iii) is satisfied, the axiom **A1** is satisfied.

B.7 Proof of Theorem 3.2

(i) First suppose that $\chi_g \leq \chi_\ell$, define $c_x = I_c(x)$ and $d_x = I_d(x)$ and fix $\varepsilon > 0$.

First consider the case $c_x \leq d_x$ or, equivalently, $I_c(x) \leq I_d(x)$. As $d_x - c_x \geq 0$, one obtains:

$$\begin{aligned}
I(x) &= \lim_{T\to\infty} I\big(c_x \mathbb{1}_{[0,T]}, d_x \mathbb{1}_{[T+1,\infty[}\big) \\
&= c_x + \lim_{T\to\infty} I\big(0\mathbb{1}_{[0,T]}, (d_x - c_x)\mathbb{1}_{[T+1,\infty[}\big) \\
&= c_x + (d_x - c_x) \lim_{T\to\infty} I\big(0\mathbb{1}_{[0,T]}, \mathbb{1}_{[T+1,\infty[}\big) \\
&= (1-\chi_g)c_x + \chi_g d_x \\
&= (1-\chi_g)I_c(x) + \chi_g I_d(x).
\end{aligned}$$

For the case $I_c(x) \geq I_d(x)$, using similar arguments,

$$I(x) = (1-\chi_\ell)I_c(x) + \chi_\ell I_d(x).$$

(iia) Consider first the case $\chi_g \leq \chi_\ell$. This implies $\underline{\chi} = \chi_g$, $\overline{\chi} = \chi_\ell$.

For the case $I_c(x) \leq I_d(x)$, for any $\underline{\chi} \leq \chi \leq \overline{\chi}$,

$$I(x) = (1-\chi_g)I_c(x) + \chi_g I_d(x) \leq (1-\chi)I_c(x) + \chi I_d(x).$$

As for the case $I_c(x) \geq I_d(x)$, using of the same arguments

$$\begin{aligned}
I(x) &= (1-\chi_\ell)I_c(x) + \chi_\ell I_d(x) \\
&\leq (1-\chi)I_c(x) + \chi I_d(x),
\end{aligned}$$

for any $\chi \in [\underline{\chi}, \overline{\chi}]$. Whence, finally

$$I(x) = \min_{\underline{\chi} \leq \chi \leq \overline{\chi}} \big[(1-\chi)I_c(x) + \chi I_d(x)\big].$$

(iib) For the other case $\chi_g \geq \chi_\ell$, using the same line of arguments, one obtains:

$$I(x) = \max_{\underline{\chi} \leq \chi \leq \overline{\chi}} \big[(1-\chi)I_c(x) + \chi I_d(x)\big],$$

where $\underline{\chi} = \chi_\ell$, $\overline{\chi} = \chi_g$.

(iii) This is a direct consequence of Propositions 3.1, 3.2 and 3.3. QED

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