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Bargaining Foundation for Ratio Equilibrium in Public Good Economies*

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Abstract

We provide a bargaining foundation for the concept of ratio equilibrium in public good economies. We define a bargaining game of alternating offers in which players bargain to determine their cost shares of public good production and a level of public good. We study the stationary subgame perfect equilibrium without delay of the bargaining game. We demonstrate that when the players are perfectly patient, they are indifferent between the equilibrium offers of all players. We also show that every stationary subgame perfect equilibrium without delay in which the ratios offered by all players are the same leads to a ratio equilibrium. In addition, we demonstrate that all equilibrium ratios are offered by the players at some stationary subgame perfect equilibrium without delay. We use these results to discuss the case when the assumption of perfectly patient players is relaxed and the cost of delay vanishes.

JEL Classification: H41, D7

Keywords: ratio equilibrium, public good economy, bargaining game, stationary subgame perfect equilibrium

1 Introduction

Ratio equilibrium in a public good economy is defined by a profile of cost ratios (one for each player) and a configuration consisting of a level of private good consumption for every player and a level of public good for the economy. A player's cost ratio specifies that player's share

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of the cost of public good production for any level of public good and in a ratio equilibrium all players agree on the level of public good when each chooses their[1] utility-maximizing consumption of public good within the budget constraint created by their ratio.

The current paper provides a bargaining foundation for ratio equilibrium. This addresses how the players in a public good economy can negotiate over cost ratios and a level of public good to be commonly provided and paid for. We provide a simple and natural bargaining procedure that implements ratio equilibrium in subgame perfect Nash equilibrium. Because the determination of a level of public good and the sharing of the cost of its production go hand-in hand, the bargaining procedure must and does determine both. We define a n-player bargaining procedure in which players take turns proposing cost ratios and, if and when a proposal is approved by all remaining players, the last player to approve chooses a level of public good. This reflects the spirit of ratio equilibrium, in which the players agree on their most preferred level of public good given their budget set as determined by their cost ratios.

We consider stationary subgame perfect equilibria (SSPE) without delay of the bargaining game. We demonstrate that when the players are perfectly patient, they are indifferent between the equilibrium offers of all players. We also show that every stationary subgame perfect equilibrium without delay in which the ratios offered by all players are the same leads to a ratio equilibrium. In addition, we demonstrate that all equilibrium ratios are offered by the players at some stationary subgame perfect equilibrium without delay. We use these results to discuss the case when the assumption of perfectly patient players is relaxed and the cost of delay vanishes.

Related literature Ratio equilibrium was introduced in (Kaneko 1977a, [18]) as an alternative to Lindahl equilibrium. Whereas Lindahl equilibrium is based on players paying personalized prices per unit of the public good, ratio equilibrium is based on players paying personalized shares of the cost of public-good production. Lindahl equilibrium is commonly accepted to have been introduced by Lindahl (1919, [21]) and later formalized by Samuelson (1954, [25]), Johansen (1963, [17]), and Foley (1970, [14]). However, van den Nouweland et al. (2002, [28]) use an axiomatic approach to demonstrate that not Lindahl equilibrium, but ratio equilibrium accurately represents the cost-share ideas expressed in Lindahl (1919, [21]).[2] Kaneko (1977a, [18]) addresses the existence of a ratio equilibrium and Kaneko (1977a, [18]) and (1977b, [19]) address the relationship between ratio equilibrium and the core of a voting game, in which the ratios are exogenous or endogenous, respectively.

The bargaining procedure that we propose is in the spirit of the bargaining game with alternating offers (Rubinstein, 1982, [24]; Fishburn and Rubinstein, 1982, [13]; Osborne and Rubinstein, 1990, [23]) and the unanimity game with n players (Binmore, 1985, [4]; Chatterjee and Sabourian, 2000, [8]). One of the issues in this context is multiplicity of subgame perfect equilibria (SPE) when the number of bargainers is larger than 2. Several

[1] We use the gender-neutral "they" and "their" for both singular and plural pronouns.

[2] Van den Nouweland (2015, [27]) describes how the literature developed from Lindahl (1919, [21]) to Lindahl equilibrium and discusses the relation between ratio equilibrium and the ideas in Lindahl (1919, [21]).

authors investigate n-person bargaining procedures that lead to the uniqueness of SPE, e.g., exit games (Chae and Yang (1988, [6]) and (1994, [7]); Yang (1992, [31]); Krishna and Serrano (1996, [20]), sequential share bargaining based on the analysis of one-dimensional bargaining problems (Herings and Predtetchinski (2010, [15]) and (2012, [16])). Our bargaining game differs from the existing bargaining models in that the players have to determine cost ratios and a level of public good, rather than determining a division of available units of a good.

Several authors propose Nash implementation of Lindahl allocations and of the ratio correspondence, see Tian (1989, [26]), Corchón and Wilkie (1996, [9]), and Duggan (2002, [12]). The current paper uses a different approach because we are interested in subgame perfect implementation of the ratio equilibrium. The current paper is more related to Dávila et al. (2009, [11]) who investigate a two-agent bargaining procedure whose subgame perfect equilibria converge to Lindahl allocations as the cost of bargaining vanishes. Their results are similar to those obtained in Dávila and Eeckhout (2008, [10]), which provides a bargaining foundation for Walrasian equilibria in a two-agent exchange economy. The bargaining procedures in these two papers are in the spirit of those in Binmore (1987, [5]) and Yildiz (2003, [32]). Dávila et al. (2009, [11]) consider SSPE without delay and show that for infinitely patient agents, every SSPE leads to the Lindahl allocation and every Lindahl equilibrium allocation is offered by the two agents at some SSPE without delay. The current paper differs from the work of Dávila et al. (2009, [11]) in two regards: We investigate a procedure that leads to ratio equilibrium and we allow for more than two players.

The remainder of the paper is structured as follows. In Section 2 we describe the ratio equilibrium. In Section 3 we introduce the bargaining game and study the implementation of ratio equilibrium through SSPE without delay. In Section 4 we provide concluding remarks.

2 Description of the ratio equilibrium

We consider economies with one public good and one private good. The non-empty finite set of individuals (*players*) in an economy is denoted by N. A *public good economy* is a list

$$E = \langle N, (w_i)_{i \in N}, (u_i)_{i \in N}, c \rangle ,$$

where w_i is the positive endowment of player $i \in N$ of a private good, $u_i : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ is the utility function of player i, and $c : \mathbb{R}_+ \to \mathbb{R}_+$ is the cost function for the production of public good. If player i consumes an amount $x_i \in \mathbb{R}_+$ of the private good and an amount $y \in \mathbb{R}_+$ of the public good, then their utility is equal to $u_i(x_i, y)$. We assume that for every player i, their utility u_i is strictly increasing (in both x_i and y), continuous, and quasi concave. We also assume that a player's utility is higher when they consume positive amounts of both goods than when they consume no private good or no public good. The cost of producing y units of the public good equals $c(y)$ units of the private good and we assume that the cost function c is strictly increasing and strictly convex in y, and that $c(0) = 0$.

A *configuration* in public good economy E with players N is a vector $(\mathbf{x}, y) = ((x_i)_{i \in N}, y)$ that specifies for each player i the level x_i of private good consumption, and the level of public good y consumed by all players. A configuration $(\mathbf{x}, y)$ is *feasible* for E if $c(y) \leq \sum_{i \in N}(w_i - x_i)$, so that the private good contributions by the players cover the cost of public good production.

In a ratio equilibrium, the players agree to share the cost of public good production according to personalized percentages of the cost and they also agree on a level of public good to be provided. The personalized percentages of the cost are modeled by means of ratios $\mathbf{r} = (r_i)_{i \in N}$ with $r_i \in [0, 1]$ for each player $i \in N$, and $\sum_{i \in N} r_i = 1$. A player i with ratio r_i pays a share r_i of the cost of public good production. Thus, if y units of the public good are produced, then player i pays $r_i\, c(y)$ for public good production and has utility $u_i(w_i - r_i\, c(y), y)$.

A ratio equilibrium consists of a cost ratio for each player and a level of public good in the economy that satisfy the requirements that the players agree on their most preferred level of public good given their budget set as determined by their personalized percentage of the cost of the provision of various levels of that public good. Formally, a *ratio equilibrium* in public good economy $E = \langle N, (w_i)_{i \in N}, (u_i)_{i \in N}, c \rangle$ is a pair $(\mathbf{r}, (\mathbf{x}, y))$ consisting of ratios $\mathbf{r} = (r_i)_{i \in N}$ and a configuration $(\mathbf{x}, y) = ((x_i)_{i \in N}, y)$ such that for each player $i \in N$ the following two requirements are satisfied:

(a) $r_i\, c(y) + x_i = w_i$

(b) for all $(x_i', y') \in \mathbb{R}_+ \times \mathbb{R}_+$ satisfying $r_i c(y') + x_i' \leq w_i$, it holds that $u_i(x_i', y') \leq u_i(x_i, y)$.

Kaneko (1977a, [18]) proved the existence of a ratio equilibrium (see Theorem 1 in [18]). Van den Nouweland and Wooders (2017, [30]) proved that every ratio equilibrium allocation for an economy E is in the core of that economy, meaning that there is no feasible configuration in the economy that is strictly preferred by at least one player while not being worse for any other player (see Theorem 1 in [30]).

3 Bargaining over ratios

We consider the following bargaining model of alternating offers with the set of players $N = \{1, 2, \ldots, n\}$. Without loss of generality we assume that the ordering of the players in the bargaining process corresponds to the players' labeling in N. Also, player n is followed by player 1 and the numbering of the players is modulo n, so that player $i+1$ refers to player 1 when $i = n$, etc.

Further, we define for any vector of ratios $\mathbf{r}$ the set of *feasible* levels of public good

$$F(\mathbf{r}) = \{y \in \mathbb{R}_+ \mid r_i\, c(y) \leq w_i \text{ for all } i \in N\}. \tag{1}$$

These are the levels of public good that are such that each player can afford their share of the cost as determined by $\mathbf{r}$. Clearly, any level of public good not in $F(\mathbf{r})$ cannot be provided while adhering to the ratios $\mathbf{r}$ of the players.

The bargaining proceeds as follows:

Period 0: In period 0, player 1 proposes ratios $\mathbf{r}^1 = (r_1^1, \ldots, r_n^1)$ for all players and the remaining players from 2 to n respond sequentially by saying either 'yes' or 'no' to the proposal. If all players accept the offer, then the last replying player, player n, chooses a level of public good $y^n(\mathbf{r}^1) \in F(\mathbf{r}^1)$ that is feasible given the agreed ratios $\mathbf{r}^1$. In this case the game terminates with the agreement $(\mathbf{r}^1, y^n(\mathbf{r}^1))$ reached in period 0, which is denoted $((\mathbf{r}^1, y^n(\mathbf{r}^1)), 0)$. Under this agreement, $y^n(\mathbf{r}^1)$ of the public good is provided and its cost is shared among the players according to the ratios $\mathbf{r}^1$, so that each player i consumes $x_i = w_i - r_i^1\, c(y^n(\mathbf{r}^1))$ of the private good. If any of the players rejects $\mathbf{r}^1$, then bargaining proceeds to period 1.
Period 1: In period 1, the next player in the given order, player 2, becomes the new proposer and proposes ratios $\mathbf{r}^2 = (r_1^2, \ldots, r_n^2)$. The remaining players respond sequentially, starting with players 3 to n and then 1, by saying either 'yes' or 'no' to the proposal. If all players accept the offer, then the last replying player, player 1, chooses a level of public good $y^1(\mathbf{r}^2) \in F(\mathbf{r}^2)$ and the game terminates with the agreement $(\mathbf{r}^2, y^1(\mathbf{r}^2))$ reached in period 1, or $((\mathbf{r}^2, y^1(\mathbf{r}^2)), 1)$, and each player i consumes $x_i = w_i - r_i^2\, c(y^1(\mathbf{r}^2))$ of the private good. If $\mathbf{r}^2$ is rejected by any of the players, bargaining proceeds to the next period.

$\vdots$

Period k: In period k, the next player in the order (namely player $k+1$, modulo n) proposes ratios $\mathbf{r}^{k+1} = (r_1^{k+1}, \ldots, r_n^{k+1})$ and the remaining players respond in order, starting with player $k+2$, by either accepting or rejecting the proposal. In case the offer is accepted by all players, the last responding player (player k) chooses the feasible level of public good and the game terminates with the agreement $((\mathbf{r}^{k+1}, y^k(\mathbf{r}^{k+1})), k)$. If any player rejects the proposed ratios, then the game passes to period $k+1$.

In general, an agreement on ratios requires the approval of all players, i.e., bargaining terminates if the ratios proposed by one player are accepted by all remaining players, and the last responder chooses a feasible level of public good given the accepted ratios. A rejection of the proposed ratios by any player moves the game to the next period, in which the next player in the order proposes ratios.

The players can either reach an agreement or bargain to infinity. The *result* of the bargaining game is either $((\mathbf{r}, y(\mathbf{r})), t)$ if the players reach the agreement $(\mathbf{r}, y(\mathbf{r}))$ in period $t \in \mathbb{N}$, or a *disagreement* denoted by $\mathcal{D}$ if they never reach any agreement. The utility to player i from $((\mathbf{r}, y(\mathbf{r})), t)$ is given by $\delta_i^t u_i(w_i - r_i\, c(y(\mathbf{r})), y(\mathbf{r}))$, where $0 < \delta_i \leq 1$ is the discount factor of player i determining i's level of impatience. The disagreement $\mathcal{D}$ is assumed to be the worst possible result of the game, i.e., its utility is lower than the utility of any result $((\mathbf{r}, y(\mathbf{r})), t)$.

We consider stationary subgame perfect equilibria (SSPE) without delay of the bargaining game. Stationarity refers to the players' proposals of cost shares as well as their 'yes'/'no' replies to cost shares proposed by other players. Also, the choice $y^i(\mathbf{r})$ of the level of public good depends only on the agreed-upon ratios and the player i who makes this choice. Neither the proposed ratios, nor the (dis)approval of those ratios or the proposed level of public good depend on the period of the bargaining game. They may, of course, depend on the player who is proposing or replying. In a SSPE without delay, moreover, no player has an incentive to reject a proposal of ratios that is made by another player.

We introduce some notation. Fix a player $i \in N$ and let r_i be the player's ratio. Then $BS(r_i) := \{(x_i, y) \in \mathbb{R}_+ \times \mathbb{R}_+ \mid x_i + r_i\, c(y) \leq w_i\}$ denotes player i's budget set as determined by their ratio. $D_i(r_i) := \{(x_i, y) \in BS(r_i) \mid u_i(x_i, y) \geq u_i(\tilde{x}_i, \tilde{y}) \text{ for all } (\tilde{x}_i, \tilde{y}) \in BS(r_i)\}$ denotes the player's utility-maximizing bundle among those that they can afford, and $y^i(r_i)$ denotes the player's desired level of public good:

$$y^i(r_i) \text{ is a solution to } \max_{y \in \mathbb{R}_+} u_i(w_i - r_i\, c(y), y) \text{ subject to } r_i\, c(y) \leq w_i. \tag{2}$$

Stated differently, $y^i(r_i)$ is the public-good component of some $(x_i, y) \in D_i(r_i)$.

We first address the determination of the level of public good once agreement on ratios has been reached. Since the last player to approve the ratios $\mathbf{r}^i$ that are proposed by a player i gets to choose the level of public good, and since this last player is player $i - 1$ (modulo n), the level of public good that this player chooses is clearly the one that maximizes their utility among all levels that are feasible given the agreed-upon ratios. We define for every player $i \in N$ and every vector of ratios $\mathbf{r}$ the feasible level $\overline{y}^i(\mathbf{r})$ of public good that maximizes the utility of player i, given the ratios $\mathbf{r}$:

$$\overline{y}^i(\mathbf{r}) = \arg \max_{y \in F(\mathbf{r})} u_i(w_i - r_i\, c(y), y). \tag{3}$$

The following lemma explains how feasibility might influence the level of public good.

Lemma 1. Suppose that all players have, in turn, agreed on ratios $\mathbf{r}$ in some period t. Then in the subsequent determination of the level of public good provision, it is optimal for a player i to choose the level of public good $y^i(r_i)$ if that is feasible, and the maximum feasible level of public good in $F(\mathbf{r})$ otherwise.

Proof. Given any ratios $\mathbf{r}$, any player i's preferred level of public good $y^i(r_i)$ is the solution to $\max_{y \in \mathbb{R}_+} u_i(w_i - r_i\, c(y), y)$ subject to $r_i\, c(y) \leq w_i$. Because the player's utility function is strictly increasing, it follows that the player's preferred bundle $D_i(r_i)$ is on the boundary of their budget set. Because the cost function for public good production is strictly increasing and strictly convex, the boundary of the player's budget set is strictly concave. Because the player's utility function is quasi concave, the indifference curves are convex. Hence, as illustrated in Figure 1, it follows that player i's utility decreases as their consumption moves (further) away from $D_i(r_i)$ along the boundary of their budget set. Hence, it is optimal for player i to choose the feasible level of public good as close as possible to $y^i(r_i)$, meaning that the player chooses $y^i(r_i)$ if that is feasible (i.e., $y^i(r_i) \in F(\mathbf{r})$) and player i chooses the maximum feasible level of public good otherwise. Thus, the level of public good that player i chooses when given the chance equals $\min\{y^i(r_i), \max\{y \mid y \in F(\mathbf{r})\}\}$. □

Lemma 1 implies that the level $\overline{y}^{i-1}(\mathbf{r}^i)$ of public good equal to the minimum of player $i-1$'s desired level of public good and the maximum feasible level of public good will be provided

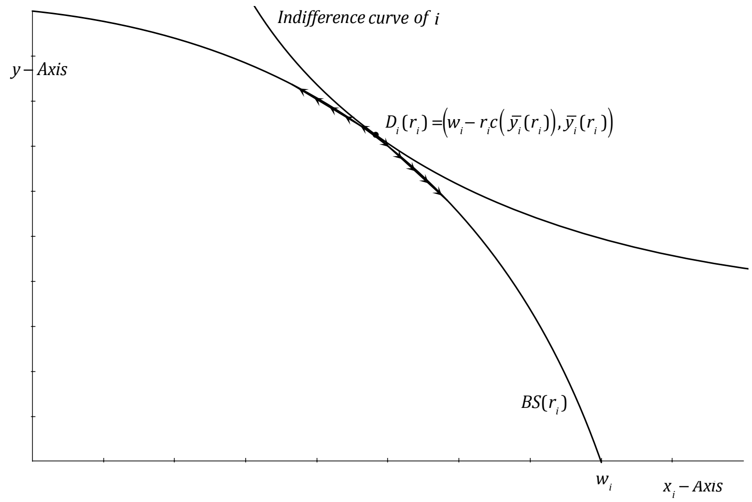

Figure 1: Determination of $\overline{y}^i(\mathbf{r})$

if all players approve the ratios $\mathbf{r}^i$ proposed by player i. We next address the players' proposed ratios in stationary subgame perfect equilibria without delay.

Stationarity implies that a player i proposes the same cost shares in every period in which they are the proposer. We denote these cost shares by $\overline{\mathbf{r}}^i = (\overline{r}^i_1, \ldots, \overline{r}^i_n)$. No delay means that every proposal $\overline{\mathbf{r}}^i$ is accepted by all other players. The SSPE without delay are thus characterized by $(\overline{\mathbf{r}}^i)_{i\in N} = (\overline{\mathbf{r}}^1, \ldots, \overline{\mathbf{r}}^n)$ such that for every $i \in N$, $\overline{\mathbf{r}}^i$ maximizes the utility that i obtains from the remaining players' immediate acceptance, subject to the constraint that it is indeed in the interest of every player to accept the cost shares proposed by player i. Thus, using Lemma 1, the SSPE without delay are characterized by $(\overline{\mathbf{r}}^i)_{i\in N}$ such that for every $i \in N$

$$\overline{\mathbf{r}}^i = \arg\max_{\mathbf{r}} u_i(w_i - r_i\, c(\overline{y}^{i-1}(\mathbf{r})), \overline{y}^{i-1}(\mathbf{r})) \tag{4}$$

subject to

$$u_j(w_j - r_j\, c(\overline{y}^{i-1}(\mathbf{r})), \overline{y}^{i-1}(\mathbf{r})) \geq \delta_j u_j(w_j - \overline{r}^{i+1}_j\, c(\overline{y}^i(\overline{\mathbf{r}}^{i+1})), \overline{y}^i(\overline{\mathbf{r}}^{i+1})) \tag{5}$$

for all players j, with the understanding that the indices i are modulo n (i.e., $n+1$ denotes player 1, and so on).

The following result concerns the players' utilities in stationary subgame perfect equilibria without delay when all players are perfectly patient.

Lemma 2. Let $\delta_i = 1$ for all $i \in N$. At any SSPE without delay every player is indifferent between the equilibrium offers of all players.

Proof. Using that $\delta_i = 1$ for all $i \in N$, conditions (4) and (5) can be written as, for every $i \in N$

$$u_i(w_i - \overline{r}_i^i\, c(\overline{y}^{i-1}(\overline{\mathbf{r}}^i)), \overline{y}^{i-1}(\overline{\mathbf{r}}^i)) \geq u_i(w_i - r_i\, c(\overline{y}^{i-1}(\mathbf{r})), \overline{y}^{i-1}(\mathbf{r})) \tag{6}$$

subject to, for all j

$$u_j(w_j - r_j\, c(\overline{y}^{i-1}(\mathbf{r})), \overline{y}^{i-1}(\mathbf{r})) \geq u_j(w_j - \overline{r}_j^{i+1}\, c(\overline{y}^i(\overline{\mathbf{r}}^{i+1})), \overline{y}^i(\overline{\mathbf{r}}^{i+1})). \tag{7}$$

Because we assume no-delay, every $\overline{\mathbf{r}}^i$ must be accepted by all players. Thus, for all i and all j

$$u_j(w_j - \overline{r}_j^i\, c(\overline{y}^{i-1}(\overline{\mathbf{r}}^i)), \overline{y}^{i-1}(\overline{\mathbf{r}}^i)) \geq u_j(w_j - \overline{r}_j^{i+1}\, c(\overline{y}^i(\overline{\mathbf{r}}^{i+1})), \overline{y}^i(\overline{\mathbf{r}}^{i+1})). \tag{8}$$

Fixing a player j, repeated application of (8) for all players i gives

$$\begin{aligned} u_j(w_j - \overline{r}_j^{j+1}\, c(\overline{y}^j(\overline{\mathbf{r}}^{j+1})), \overline{y}^j(\overline{\mathbf{r}}^{j+1})) &\geq \\ u_j(w_j - \overline{r}_j^{j+2}\, c(\overline{y}^{j+1}(\overline{\mathbf{r}}^{j+2})), \overline{y}^{j+1}(\overline{\mathbf{r}}^{j+2})) &\geq \\ \vdots \qquad &\geq \\ u_j(w_j - \overline{r}_j^{j}\, c(\overline{y}^{j-1}(\overline{\mathbf{r}}^{j})), \overline{y}^{j-1}(\overline{\mathbf{r}}^{j})) &\geq \\ u_j(w_j - \overline{r}_j^{j+1}\, c(\overline{y}^j(\overline{\mathbf{r}}^{j+1})), \overline{y}^j(\overline{\mathbf{r}}^{j+1})). & \end{aligned} \tag{9}$$

Thus, we have

$$u_i(w_i - \overline{r}_i^i\, c(\overline{y}^{i-1}(\overline{\mathbf{r}}^i)), \overline{y}^{i-1}(\overline{\mathbf{r}}^i)) = u_i(w_i - \overline{r}_i^j\, c(\overline{y}^{j-1}(\overline{\mathbf{r}}^j)), \overline{y}^{j-1}(\overline{\mathbf{r}}^j)) \tag{10}$$

for all $i, j \in N$. Hence, at any SSPE without delay every player is indifferent between the equilibrium offers of all players. □

From now on, whenever this does not give rise to possible misinterpretation, we use the simplified preferences notation

$$(\hat{\mathbf{r}}, \hat{y}) \succeq_i (\tilde{\mathbf{r}}, \tilde{y}) \tag{11}$$

instead of $u_i(w_i - \hat{r}_i\, c(\hat{y}), \hat{y}) \geq u_i(w_i - \tilde{r}_i\, c(\tilde{y}), \tilde{y})$, and we use $\succ_i$ and $\sim_i$ to denote strict preference and indifference, respectively. Using this notation, Lemma 2 tells us that when players do not discount the future, in any SSPE without delay, it holds that

$$(\overline{\mathbf{r}}^i, \overline{y}^{i-1}(\overline{\mathbf{r}}^i)) \sim_i (\overline{\mathbf{r}}^j, \overline{y}^{j-1}(\overline{\mathbf{r}}^j))$$

for all $i, j \in N$.

The following result concerns the relation between ratio equilibrium and SSPE without delay in the proposed bargaining game when all players are perfectly patient.

Theorem 1. Let $\delta_i = 1$ for all $i \in N$. Every SSPE without delay in which the ratios offered by all players are the same leads to a ratio equilibrium. Moreover, all equilibrium ratios are offered by the players at some SSPE without delay.

Proof. Let $(\overline{\mathbf{r}}^i)_{i\in N} = (\overline{\mathbf{r}}^1, \ldots, \overline{\mathbf{r}}^n)$ be the players' proposed ratios in an SSPE without delay. It follows from Lemma 2 that

$$(\overline{\mathbf{r}}^i, \overline{y}^{i-1}(\overline{\mathbf{r}}^i)) \sim_j (\overline{\mathbf{r}}^j, \overline{y}^{j-1}(\overline{\mathbf{r}}^j)) \tag{12}$$

for all $i, j \in N$.

Fix some $i \in N$. By Lemma 1, $\overline{y}^{i-1}(\overline{\mathbf{r}}^i)$ equals $y^{i-1}(\overline{r}^i_{i-1})$ if that is feasible, and it equals the maximum feasible level of public good in $F(\overline{\mathbf{r}}^i)$ otherwise. Note that in the latter case it holds that $\overline{r}^i_j\, c(\overline{y}^{i-1}(\overline{\mathbf{r}}^i)) = w_j$ for at least one player j, and that such a player j consumes no private good. Thus, $(\overline{\mathbf{r}}^i, \overline{y}^{i-1}(\overline{\mathbf{r}}^i)) \sim_j (\overline{\mathbf{r}}^{j+1}, \overline{y}^j(\overline{\mathbf{r}}^{j+1}))$ cannot hold because player j would choose a level of public good that leaves them with positive amounts of both private and public good. Thus, we have derived that for each $i \in N$

$$\overline{y}^{i-1}(\overline{\mathbf{r}}^i) = y^{i-1}(\overline{r}^i_{i-1}), \tag{13}$$

player $i-1$'s desired level of public good at their ratio $\overline{r}^i_{i-1}$. (Note that this also places a restriction on the ratios that can be proposed by the players in SSPE without delay.)

Since player i gets to choose their desired level of public good $y^i(\overline{r}^{i+1}_i)$ at the ratios $\overline{\mathbf{r}}^{i+1}$ proposed by player $i+1$ (and accepted by all other players), it cannot be the case that $\overline{r}^{i+1}_i < \overline{r}^j_i$ for some $j \neq i+1$. This is because when $\overline{r}^{i+1}_i < \overline{r}^j_i$, then i's budget set is smaller under $\overline{\mathbf{r}}^j$ than under $\overline{\mathbf{r}}^{i+1}$, and therefore

$$(\overline{\mathbf{r}}^{i+1}, \overline{y}^i(\overline{\mathbf{r}}^{i+1})) = (\overline{\mathbf{r}}^{i+1}, y^i(\overline{r}^{i+1}_i)) \succ_i (\overline{\mathbf{r}}^j, y^i(\overline{r}^j_i)) \succeq_i (\overline{\mathbf{r}}^j, y^{j-1}(\overline{r}^j_{j-1})) = (\overline{\mathbf{r}}^j, \overline{y}^{j-1}(\overline{\mathbf{r}}^j)),$$

which contradicts (12). Thus, we derive that

$$\overline{r}^{i+1}_i \geq \overline{r}^j_i \tag{14}$$

for all $i, j \in N$.

Now, consider an SSPE without delay in which the ratios offered by all players are the same and let $\overline{\mathbf{r}} := \overline{\mathbf{r}}^1 = \cdots = \overline{\mathbf{r}}^n$. Then, according to Lemma 2, $(\overline{\mathbf{r}}, \overline{y}^i(\overline{\mathbf{r}})) = (\overline{\mathbf{r}}^{i+1}, \overline{y}^i(\overline{\mathbf{r}}^{i+1})) \sim_i (\overline{\mathbf{r}}^{j+1}, \overline{y}^j(\overline{\mathbf{r}}^{j+1})) = (\overline{\mathbf{r}}, \overline{y}^j(\overline{\mathbf{r}}))$ for all $i, j \in N$. Note that $\overline{y}^i(\overline{\mathbf{r}})$ equals player i's desired level of public good at the cost share $\overline{r}_i$ (see (13)). Also, $\overline{y}^j(\overline{\mathbf{r}}) \in F(\overline{\mathbf{r}})$, so that player i could choose the level $\overline{y}^j(\overline{\mathbf{r}})$ of public good at the cost share $\overline{r}_i$. Moreover, player i's utility is strictly increasing in both private and public good consumption. Thus, we conclude that $\overline{y}^j(\overline{\mathbf{r}}) = y^i(\overline{r}_i)$ has to hold.

We have thus demonstrated that $\overline{y}^j(\overline{\mathbf{r}}) = \overline{y}^i(\overline{\mathbf{r}}) = y^i(\overline{r}_i)$ for all $i, j \in N$. Let $\overline{y} := \overline{y}^1(\overline{\mathbf{r}}) = \ldots = \overline{y}^n(\overline{\mathbf{r}})$. Then for each player i it holds that $\overline{y}$ equals the utility-maximizing level of public good within player i's budget at the cost share $\overline{r}_i$. Thus, if we define $\overline{x}_i := w_i - \overline{r}_i\, c(\overline{y})$ for each $i \in N$, then $(\overline{\mathbf{r}}, (\overline{\mathbf{x}}, \overline{y}))$ is a ratio equilibrium.

Now, let $(\mathbf{r}^*, (\mathbf{x}^*, y^*))$ be a ratio equilibrium. We will define a strategy tuple and demonstrate that it constitutes an SSPE without delay. The strategy of a player $i \in N$ is defined as follows:

Player i always proposes the ratios $\overline{\mathbf{r}}^i := \mathbf{r}^*$, accepts an offer $\mathbf{r}^j$ by an arbitrary player $j \neq i$ if and only if $(\mathbf{r}^j, \overline{y}^{j-1}(\mathbf{r}^j)) \succeq_i (\mathbf{r}^*, \overline{y}^{j-1}(\mathbf{r}^*))$, and chooses their preferred level of public good $\overline{y}^i(\mathbf{r}^{i+1})$ in case an offer $\mathbf{r}^{i+1}$ by player $i+1$ has been accepted by all players (including player i).

Because $(\mathbf{r}^*, (\mathbf{x}^*, y^*))$ is a ratio equilibrium, for each player i it holds that y^* equals the utility-maximizing level of public good within player i's budget at the cost share r_i^*. Thus, $\overline{y}^i(\mathbf{r}^*) = y^*$ for every player i. Therefore, if all players play the described strategy, then in period 0, player 1 proposes the ratios $\mathbf{r}^*$, these ratios are accepted by all players, and player n chooses the level of public good y^*. Hence, the agreement $(\mathbf{r}^*, y^*)$ that corresponds to the ratio equilibrium is reached in period 0.

In order to establish that the strategy tuple constitutes an SSPE without delay, first we observe that Lemma 1 confirms that choosing $\overline{y}^i(\mathbf{r})$, the feasible level of public good that maximizes player i's utility under ratios $\mathbf{r}$, is player i's unique best response after the ratios $\mathbf{r}$ have been agreed upon by all players. Next, we observe that rejecting ratios $\mathbf{r}^i$ proposed by a player i moves bargaining to the next period, in which player $i+1$ is going to propose ratios $\mathbf{r}^*$ which are going to be accepted by all players, and for which player i is going to choose the level of public good $\overline{y}^i(\mathbf{r}^*) = y^*$. Thus, a player j should accept the ratios $\mathbf{r}^i$ if and only if $(\mathbf{r}^i, \overline{y}^{i-1}(\mathbf{r}^i)) \succeq_j (\overline{\mathbf{r}}^{i+1}, \overline{y}^i(\overline{\mathbf{r}}^{i+1})) = (\mathbf{r}^*, y^*)$ and this is exactly what the strategy prescribes. Lastly, suppose that a player deviates and proposes ratios $\mathbf{r}$ different from $\mathbf{r}^*$. If any player rejects this offer, then bargaining moves to the next period, and in that period the next player proposes the ratios $\mathbf{r}^*$, so that the best possible result of the bargaining is agreement on $(\mathbf{r}^*, y^*)$ in the next period. Thus, the deviation to proposing ratios $\mathbf{r}$ can only be profitable if these ratios are accepted by all players. For this to be the case, *all* players must be better off under an ensueing agreement $(\mathbf{r}, y)$ than under the agreement $(\mathbf{r}^*, y^*)$ in the next period. However, the agreement $(\mathbf{r}^*, y^*)$ results in a ratio equilibrium and ratio equilibrium allocations are known to be in the core of an economy (see Theorem 1 in van den Nouweland and Wooders (2017, [30])), so that an agreement that is better for *all* players does not exist. □

We now turn to relaxing the assumption that the players are perfectly patient. When the players' discount factors are strictly smaller than 1, then delay is costly and every SSPE is without delay. A natural follow-up to Theorem 1 thus is to study convergence to the ratio equilibrium as the cost of bargaining vanishes, i.e., as discount factors approach 1.

Remark 1. If the correspondence $\varphi(\delta_1, \ldots, \delta_n)$ that associates with each profile of discount factors all profiles of proposals of ratios that satisfy the SSPE constraints (see (15) below) is continuous w.r.t. $(\delta_1, \ldots, \delta_n)$, and if $\overline{y}^i(\mathbf{r})$ is continuous w.r.t. $\mathbf{r}$ for every player i, then every SSPE without delay in which the ratios offered by all players are the same converges to a ratio equilibrium as $\delta_i \to 1$ for all $i \in N$.

Proof. We refer the reader to Moore (1999, [22], Chapters 9 and 12) and Aliprantis and Border (2006, [1], Chapter 17) for details on correspondences and fixed point theorems used in this proof. The proof is similar to that presented in Dávila et al. (2009, [11]) on a foundation for Lindahl equilibrium allocations. Our need for additional continuity assumptions stems from the fact that we do not limit the economy to two players or to constant returns to scale in production.

Denote players' discount factors by $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_n)$. We define, in several steps, a correspondence whose fixed points correspond to the SSPE. We recall that in a SSPE the ratios $(\mathbf{r}^1, \ldots, \mathbf{r}^n)$ proposed by the players satisfy

$$u_j(w_j - r_j^i \, c(\overline{y}^{i-1}(\mathbf{r}^i)), \overline{y}^{i-1}(\mathbf{r}^i)) \geq \delta_j u_j(w_j - \overline{r}_j^{i+1} \, c(\overline{y}^i(\overline{\mathbf{r}}^{i+1})), \overline{y}^i(\overline{\mathbf{r}}^{i+1})) \tag{15}$$

for all $i, j \in N$ (see (5)). We define, for each $\boldsymbol{\delta}$, the set of profiles of ratios that satisfy these conditions:

$$\varphi(\boldsymbol{\delta}) = \{(\mathbf{r}^1, \ldots, \mathbf{r}^n) \mid \text{the SSPE conditions (15) are satisfied for all } i, j \in N\}. \tag{16}$$

We now add in the maximization required in SSPE (see (4)) and define for each player i

$$\phi_i((\mathbf{r}^j)_{j \in N;\, j \neq i}; \boldsymbol{\delta}) = \arg\max\nolimits_{\mathbf{r}^i : (\mathbf{r}^1, \ldots, \mathbf{r}^n) \in \varphi(\boldsymbol{\delta})} u_i(w_i - r_i^i \, c(\overline{y}^{i-1}(\mathbf{r}^i)), \overline{y}^{i-1}(\mathbf{r}^i))$$

and we define the correspondence

$$\phi(\mathbf{r}^1, \ldots, \mathbf{r}^n; \boldsymbol{\delta}) = \prod_{i \in N} \phi_i((\mathbf{r}^j)_{j \in N;\, j \neq i}; \boldsymbol{\delta}).$$

By construction, for each $\boldsymbol{\delta}$, the fixed points of ϕ correspond to the ratios that are supported in SSPE.

Consider the correspondence Γ that associates with each profile of discount factors the set of fixed points of ϕ:

$$\Gamma(\boldsymbol{\delta}) = \{(\overline{\mathbf{r}}^1, \ldots, \overline{\mathbf{r}}^n) \mid (\overline{\mathbf{r}}^1, \ldots, \overline{\mathbf{r}}^n) \in \phi(\overline{\mathbf{r}}^1, \ldots, \overline{\mathbf{r}}^n; \boldsymbol{\delta})\}.$$

For each $\boldsymbol{\delta}$, the correspondence Γ identifies the SSPE ratios. We know from Theorem 1 that when the players are perfectly patient, i.e., $\boldsymbol{\delta} = (1, \ldots, 1)$, then every symmetric profile of ratios in $\Gamma(1, \ldots, 1)$ supports a ratio equilibrium.

It follows from Berge's Maximum Theorem (see Berge (1959, [2]) and (1963, [3])) that the correspondence $\phi_i((\mathbf{r}^j)_{j \in N;\, j \neq i}; \boldsymbol{\delta})$ is a compact-valued, upper hemicontinuous correspondence for each $i \in N$. These properties also hold for the Carthesian product ϕ. Thus, the correspondence Γ, which associates the fixed points of ϕ with each profile of discount factors, is upper hemicontinuous. In particular, it is upper hemicontinuous at $\boldsymbol{\delta} = (1, \ldots, 1)$, and thus every SSPE in which the ratios offered by all players are the same converges to a ratio equilibrium as $\delta_i \to 1$ for all $i \in N$. □

4 Concluding remarks

In Theorem 1 we consider SSPE in which all players propose the same ratios. It is possible to weaken this assumption, because in the proof of Theorem 1 we show that a necessary condition for the ratios $(\overline{\mathbf{r}}^i)_{i\in N} = (\overline{\mathbf{r}}^1, \ldots, \overline{\mathbf{r}}^n)$ to be proposed in a SSPE without delay is that $\overline{r}_i^{i+1} \geq \overline{r}_i^j$ for all $i, j \in N$. However, without making stronger assumptions, we cannot obtain the inequalities $\overline{r}_i^{i+1} \leq \overline{r}_i^j$ because it is possible to have different vectors of ratios by the players and different levels of public good associated with those ratios in such a way that all players are indifferent between each of the arrangements.

We can consider other natural bargaining games, for instance by having the players propose and accept/reject not only the ratios, but also the level of public good. This leads to results that are qualitatively similar to those that we have explained in this paper. We are not pursuing this here because we think that bargaining over ratios is closer to the spirit of ratio equilibrium, in which the players agree on their most preferred level of public good given their budget set as determined by their cost ratios.

In future research we intend to address the subgame perfect implementation of share equilibrium, which is the extension of ratio equilibrium to local public good economies introduced in van den Nouweland and Wooders (2011, [29]). Because share equilibrium involves the determination of a partition of the player set into jurisdictions, in addition to levels of public good in those jurisdictions, the determination of share equilibria brings with it all the problems associated with coalition formation. The existence result for symmetric economies provided in van den Nouweland and Wooders (2017, [30]) suggests that implementation is possible for such economies.

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