Was Frege a Logicist for Arithmetic?

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1 Introduction

Was Frege a logicist? At first glance, the question has an obvious answer: Frege was not only a logicist; he was (possibly together with Dedekind, but in another sense)[1] the very founder of logicism, and remains (one of) the most emblematic representatives of it.

Eva would have possibly agreed. Still, supposing I had insisted in asking, her intellectual curiosity and love for philosophical and historical discussion would have certainly made her reply with another question: 'How could you think that he wasn't?' Answering this other question in detail would have taken more than a discussion. Possibly, we would have decided to tackle the question brick by brick, which would have been the occasion of more than a dinner together. Space limitation forces me to offer her, here, only a quite summary version of what I would have said her during the first of these dinners.

In particular, I shall limit my attention to arithmetic, only. I hope to have room to expound these remarks and extend my scrutiny to real analysis in some other occasion[2]. On this matter, let me only say that what I shall say about the former seems to me to nicely fit with what Andrea Sereni (a friend which I met thanks to Eva) and I have argued concerning Frege's conception of the application constraint both for natural and real numbers ([17]). Taken together, with those advanced in this last paper, my following considerations are aimed to offer a quite dissident picture of Frege's foundational purpose, depicting him more as a mathematician aiming at providing mathematics with an appropriate architecture, than as a philosopher aiming at securing its epistemic grounds. It is this picture that I hope to be able to refine and complete elsewhere.

My present claim is that Frege's primary foundational purpose concerning arithmetic was neither that of making natural numbers logical objects, nor that of making arithmetic a part of

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[1] Cf. [1], ch. 1.

[2] Here and throughout the paper, I use the terms 'arithmetic' and 'real analysis' to specifically refer, for short, to the theories of natural and real numbers, respectively.

logic, but rather that of assigning to it an appropriate place in the architectonics of mathematics and knowledge.

2 A Terminological Remark

Let me begin with an obvious but relevant terminological remark. Frege never termed himself a logicist, and never used any German word for 'logicism' and its cognates.

In its contemporary sense, such a term appeared quite late ([13], pp. 479, 501). It was almost simultaneously and independently firstly used by Fraenkel in his *Einleitung in die Mengenlehre* ([8]) and by Carnap, in his *Abriss der Logistik* ([4]). The former used 'logistische Schule' and 'Logizismus', to refer to Russell and Whitehead's views, as opposed to those of "intuitionists" and "formalists" ([8], p. 263). The latter used 'logizistich' and 'Lozicism' to characterize "a philosophical direction with a strong or even excessive emphasis on the logical point of view" ([4], p. 3).

On September 5th, 1930, Carnap, Heyting and von Neumann delivered their well-known talks about the "*logizstische*", "*intuitionistische*" and "*formalistische*" foundation of mathematics ([5]; [14]; [19]). Carnap's lecture definitively codified the use of 'logicism [*Logizismus*]' in our present sense: forty-four years after the publication of *Grundlagen* ([9]), and twenty-seven after that of second volume of *Grundgesetze* ([10]).

Following Fraenkel (rather than his own earlier use), Carnap called thus "the thesis that mathematics is reducible to logic", is "nothing but a part of logic", and assigned to Russell the role of "chief proponent" of it, while adding that "Frege was the first to espouse this view" ([5], p. 91 [6], p. 31). This is openly false, if mathematics is intended to include geometry, as it should be, in Russell's view. Hence, if Frege was "the first to espouse" it, the view cannot be but that a piece of mathematics is reducible to, or is a part of logic. But which piece? Arithmetic is the only candidate I'll consider here.

3 Some (Dissonant) Quotes

Frege's argument about arithmetic goes in three stages: in *Grundlagen*, alternative conceptions are criticized; in *Grundlagen*, again, *Anzahlen* are informally identified with extensions of second-level concepts, and natural numbers are singled out among them[3]; in *Grundgesetze*, this informal treatment is turned into a formal definition. What is the point Frege aimed to make, by following this tripartite strategy? Different passages drawn both from *Grundlagen* and *Grundgesetze* suggest different responses.

Here are some from *Grundlagen* ([9], *Einleitung*, pp. IV, XI, §§ 87; [11], pp. xvi, xxi, 99, 102)[4]:

[3]Frege pervasively uses two German terms that might be translated in English with 'number': 'Zahl' and 'Anzahl'. The former is more generic, the latter more specific, and it is especially used to denote those numbers which provide an answer to how-many questions. This makes this latter term be often translated with 'cardinal number' (as in [12]). There are, however, several reasons—which, for sake of brevity, I cannot discuss, here—that make me think that this translation is improper (though not importantly misleading). This is why I prefer leaving this term untranslated. What is especially important to retain, here, about *Anzahlen*, is that Frege takes them to include natural numbers (and this is possibly his main point about the latter numbers). This makes al the claims about the former made in this §straightforwardly extend to the latter.

[4]My quotes from Frege's works are substantially taken from to the current English translations of these works. Still, I will locally modifying them, if need be, in order the be more faithful to the original.

> The present work will make it clear that even an inference like that from n to $n+1$, which on the face of it is peculiar to mathematics, is based on the general laws of logic.
>
> I have felt bound to go back [...] further into the general logical foundations [of mathematics][...].
>
> I hope to have made it plausible that the laws of arithmetic are analytic judgements[5] and consequently *a priori*. Arithmetic thus becomes nothing but further pursued logic, and every arithmetic statement becomes a law of logic.
>
> I do not claim to have made the analytic character of arithmetical statements more than plausible, because it can still always be doubted whether their proof only proceeds from purely logical laws [...]. This misgiving [...] can only be removed by producing a gapless chain of inferences [...].

Part II of *Grundgesetze* is intended to achieve this last task. Here is how Frege announces it ([10], *Einleitung*, vol. I, p. 1; [12], p. 1_1):

> In my *Grundlagen der Arithmetik* [...] I aimed to make it plausible that arithmetic is a branch of logic [...]. In the present book this is now to be established by deduction of the simplest laws of *Anzahlen* by logical means alone.

Some lines later, Frege terms this view his 'thought' ([10], *Einleitung*, vol. I, p. 3; [12], p. 3_1):

> If my thought, that arithmetic is a branch of pure logic, is correct, then [...].

This confirms what he implies in the Foreword ([10], *Vorwort*, vol. I, p. VIII; [12], p. VIII_1):

> Mr Dedekind too is of the opinion that the theory of numbers is a part of logic.

In none of these passages, the positive claim is made that arithmetic is a part of logic. Frege maintains to have made it plausible, and to have designed his proofs in order to support it; but he does not take it as an established fact; he merely advances it as a "thought" or "opinion". Still the source of uncertainty is identified. It pertains to Basic Law V (BLV, from now on: [10], *Vorwort*, vol. I, p. VII; [12], p. VII_1):

> [...], a dispute can arise only concerning my basic law of value-ranges (V) [...]. I take it to be purely logical. At any rate, the place is hereby marked where there has to be a decision.

These passages, coming from *Grundlagen* and the first volume of *Grundgesetze*, suggest that Frege certainly endorsed that arithmetic is a part of logic, but admitted to have not established it. What he considered to have established was only that this is so, if BLV is a logical law. Still, in this first volume, he makes no effort to argue that BLV is actually a logical law. He takes it to be so, but offers no real argument for it. This suggests that his purpose was less to establish that arithmetic is a part of logic than to ascertain that it belongs to the same field as BLV, namely it is part of a more general theory of concepts and their extensions.

[5] Remember that for Frege, a "truth" is analytic if (and only if) it can be proved from logical laws and explicit definitions alone ([9], § 3).

Things seem to change in the second volume. Though he never explicitly argues, yet, that arithmetic is a part of logic, he has no reticence, there, in claiming that *Anzahlen* are logical objects. While discussing Cantor's views, he distinguishes "physical [...] [from] logical objects", and advances that the latter "include our *Anzahlen*" ([10], § II.74; [12], p. 86_2). Later, while discussing Dedekind's views, he observes that ([10], § II.147; [12], p. 149_2):

> If there are logical objects at all—and the objects of arithmetic are such—then there must also be a means to grasp them, to recognize them. [...] [BLV] serves for this purpose. Without such a means, a scientific foundation of arithmetic would be impossible.

After having become aware of Russell's paradox, he maintained the same view ([10], *Nachwort*, vol. II, pp. 253 and 265; [12], pp. 253_2 and 265_2):

> I have never concealed [...] that [...][BLV] is not as obvious as the others, nor as must properly be required of a logical law. [...]. I would gladly have dispensed with this foundation if I had known of some substitute for it. Even now, I do not see how arithmetic can be founded scientifically, how the numbers can be apprehended as logical objects [...], if it is not—at least conditionally—permissible to pass from a concept to its extension. [...] This question may be viewed as the fundamental problem of arithmetic: how are we to apprehend logical objects, in particular, the numbers? What justifies us to acknowledge numbers as objects?

The end of this passage intimates that Frege was overall concerned with numbers being objects, and making these objects logical was a means for arguing for that, without involving intuition. The beginning suggests this is so also true for extensions: rather than aiming at making arithmetic ensue from BLV, he seems here to take extensions as the most obvious objects to identify with numbers for making them objects.

Is this a shift occurring after the reception of Russell's letter? Other passages from *Grundlagen* suggest that this is not so. Better, that no shift occurred, in fact; that Frege was constantly oscillating among different perspectives. Here is one ([9], § 107; [11], p. 117):

> [...] the sense of the expression 'extension of a concept' is taken as already known. This way of getting over the difficulty cannot be expected to meet with universal approval [...]. By the way, I attach no decisive importance to the use of extensions of concepts.

Coming back to *Grundgesetze* in the light of this claim and comparing it with the previous passages make one think that Frege considered both extensions and BLV as convenient, but replaceable, tools to define *Anzahlen*. Yes, but with which purpose?

Before this passage, no mention is made of logic. Frege discusses ([9], § 104-105; [11], p. 114-115) the possibility of extending his views on *Anzahlen* to "other numbers", and observes that also for them,

> everything will in the end come down to the search of a judgement-content, which can be transformed into an identity whose sides precisely are the new numbers. [...] If we

> proceed as [...][for *Anzahlen*], then the new numbers will be given to us as extensions of concepts [...][, as], objects which we do not come to know as something alien coming from outside, through the medium of the senses, but with objects given directly to our reason and, as its nearest kin, utterly transparent to it.

What Frege seems here to grant is less having made arithmetic a part of logic, than having made *Anzahlen* "given directly to our reason [...], utterly transparent to it", rather than "known as something alien coming from outside, through the medium of senses".

In Kantian language, this is the same as having made knowledge of *Anzahlen* and judgements about them *a priori.* Frege's notion of *a priori* is not coincident with Kant's. For him, a truth (rather than a judgment) is *a priori* if "its proof proceeds as a whole from general laws, which neither need nor admit of proof" ([9], § 3; [11], p. 4). I have no room here to discuss the differences among the two notions. What is relevant is that also for Frege himself his achievement might well be outlined in term of apriority as that of having made truths about *Anzahlen*, and, consequently, arithmetical truths *a priori.* It is then relevant to observe that Frege's explanation of apriority makes logical laws *a priori*, but not *a priori* truths logical. Moreover, Frege's doubts about the logicality, but not the truth of BLV, suggest that he admitted non-logical *a priori*, as it is also confirmed by the following passage, already quoted in part:

> We can distinguish physical from logical objects, by which [...] no exhaustive classification is [...] given.

Hence, making truths about *Anzahlen* and/or knowledge of them *a priori* was possibly not the same, for Frege, as making them logical objects.

* * *

Far from being concordant, when taken all together, the previous passages suggest three different possible understandings of Frege's purpose.

According to the first understanding, Frege's primary aim was to make arithmetic a part of a general theory of extensions of concepts.

According to the two others, extensions should be rather seen as nothing but a convenient tool to be used for reaching an independent purpose.

According to the second understainding, this purpose was that currently ascribed to the logicist Frege, i.e. to make *Anzahlen* logical objects and arithmetic a part of logic.

According to the third understanding, this propose was rather that of making truths about *Anzahlen*, and/or knowledge of them *a priori*, namely transparent to reason without the medium of senses and intuition.

This variety of understandings invites us to eschew general claims, and to search rather an answer to our question within Frege's definitions of *Anzahlen* and, among them, of natural numbers.

4 Concepts and Extensions

It is enough to parse, even quite cursorily (as I shall do below, in § 5), the (informal) definition of natural numbers Frege offers in *Grundlagen* to realize that his treatment of arithmetic could have left extensions aside, if Hume Principle (HP, from now on) had been independently admitted. The (formal) definition of *Grundgesetze* makes this a little bit less evident (because of the technical apparatus used to avoid second-order quantification as much as possible, which makes value-ranges appear at face value virtually in any definition and theorem), but could certainly not have concealed it to Frege's own eyes. There is, then, no doubt that he was perfectly aware of this possibility. But, then, why did he not only adopt BLV and give a so crucial role to value-ranges and, particularly, extensions, but even refrain from giving them up after having become aware of Russell's paradox?

Appealing to Caesar problem does not set the question. Since, taken as such, Caesar problem also arises for BLV. Another answer comes from an argument Frege advances, in the second volume of *Grundgesetze* to deny that BLV results in a creative definition ([10], §§ II.146-147; [12], pp. 147_2-148_2):

> [...] it could be pointed out that [...] we ourselves created new objects, namely value-ranges. [...] We did not list properties and say: we create a thing that has them. Rather, we said: if one function [...] and a second function are so constituted that both always have the same value for the same argument, then one may say instead: the value-range of the first function is the same as the that of the second. We recognize something in common to both functions and this we call the value-range both of the first and of the second function. That we have the right to the acknowledging of what is common, and that, accordingly, we can convert the generality of an equality into an equality (identity), must be regarded as a basic law of logic.

Hence, for Frege, passing from recognizing something in common to two functions to asserting an identity is licensed by a "basic law of logic". If this were admitted and BLV merely licensed this passage, it would, then, be a logical law. At the best of my knowledge, this is the only substantial argument Frege ever advanced for the logicality of BLV. But, so conceived, it is openly unsound. Since BLV does much more than this. It licenses passing from recognizing that two functions bear to each other an equivalence relation, to associating one same object to both of them, and, vice versa, from recognizing—or, better, admitting—that two functions are associated to the same object in the same canonical way, to concluding that they bear to each other this same equivalence relation.

This makes the crucial point appear. This is not so much that Frege does not consider the left-to-right implication involved in BLV (an allegation that, by the way, would be rightful only if it were admitted that he actually considered this argument as an argument in favor to the logicality of BLV). Rather it is that he seems to take values-ranges to be intrinsically and openly there within functions, and, in particular, extensions to be intrinsically and openly there within concepts. This makes BLV merely make explicit a necessary and sufficient identity condition for value-ranges of first-level one-argument functions (a condition that, in the previous argument, is in fact only regarded as sufficient). If it were so, BLV would essentially differ from HP. Since, whereas values-ranges would be intrinsically and openly there within functions, would come directly together with them, *Anzahlen* certainly do not come directly together with first-level concepts. In §§ 5 and 6, below, I shall shortly account for Frege's definitions of *Anzahlen* in *Grundlagen* and *Grundgesetze*. For the time being, only some clues are in order. According to both definitions, whatever *Anzahl* is the *Anzahl* of a first-level concept. The former definition identifies it with the extension of the second-level concept of being (a first-level concept) equinumerous with this first-level concept.

Hence, conforming to it, an *Anzahl* comes with the concept which it is the *Anzahl* of only insofar as this concept is associated to a second-level one, of which this *Anzahl* is the extension. The latter definition identifies the *Anzahl* of a certain first-level concept with the extension of the first-level concept of being the value-range of a first-level function which takes the True as its value for as many objects as argument as those which fall under this first-level concept. Hence, conforming to it, an *Anzahl* comes with the concept which it is the *Anzahl* of only insofar as this concept is associated to another first-level concept, namely with the concept of being the value-range of an appropriate function. It is, then, clear that, for Frege, HP does much more than making explicit a necessary and sufficient identity condition for something which is intrinsically and openly there within concepts, that comes directly together with them. By asserting that the same *Anzahl* is ascribed to two first-level concepts if and only if these concepts are equinumerous[6], it ensures that the fact that these first-level concepts are associated to two other concepts in the same canonical way makes the extensions of these other concepts to be the same, and, then, to count as the common *Anzahl* of the former concepts if and only if this last condition obtains. Hence, HP requires a proof, whereas BLV "neither needs nor admits of" it.

But should we grant that values-ranges are intrinsically and openly there within functions, that they come directly together with them? If yes, and only if yes, BLV would have a chance of being a law of logic. This is openly questionable, however, and this is just where the residual doubt about BLV's logicality seems to come from, for Frege. To settle this doubt, Frege should have elucidated the notion of a value-range. But this would have made him run the risk of explaining extensions through sets, thus making the latter come logically before the former: a view he was strongly opposed ([1], *Introduction* and ch. 2).

In *Grundlagen*, Frege merely takes for granted "what the extension of a concept is" ([9], § 69; [11], p. 80, footnote 1). In *Grundgesetze*, he recognizes that BLV offers no way to identify value-ranges (which seems, by the way, in contrast with his taking them to openly come directly together with functions), and merely makes a stipulation, identifying the extensions of two concepts under which only the True and the False respectively fall with the True and the False themselves ([10], § I.10; [12], pp. 16_1-18_1). This makes his view unstable. Since not explaining what extensions are entails making impossible to explain what *Anzahlen* are. And merely stipulating that two value-ranges are the True and the False makes the nature of *Anzahlen* subject to stipulation.

What could have, then, made natural numbers logical objects? Identifying these numbers with some extensions of concepts would not have been enough, since logic does not encompass particular concepts; it merely deals with concepts in general. Hence, only the particular nature of the concepts these numbers were taken to be extensions of could have ensured this. To answer the question, we have, then, no other resources than look at these concepts, namely parse, even shortly, Frege's definitions of *Anzahlen*, and natural numbers, among them.

5 The definition of *Grundlagen*

Let us begin with the definition of *Grundlagen* and, for sake of simplicity, render it through a compact notation (still remembering that it is informal, so that this notation is nothing but a

[6] This is what HP *prima facie* asserts in *Grundlagen* ([9], § 73). To see that this is also what HP asserts in *Grundgesetze* is instead necessary to compare the (formal) statement of this principle ([10], §§ I.65, th. 32, and I.69, th. 49) with the informal explanations given earlier (*ibidem*, § I.40).

suitable tool to abbreviate sentences written in natural, though codified language).

Let P and Q be whatever first-level concepts, $\mathcal{P}$ whatever second-level concept, and 'X' and 'x' variables ranging on first-level concepts and objects, respectively. Read: '$\#P$' as 'the *Anzahl* of P'; '$\mathcal{EP}$' as 'the extension of $\mathcal{P}$', '$P \approx Q$' as 'P is equinumerous with Q', and '$\mathbf{N_A}x$' as 'x is an *Anzahl*'. Then, Frege ([9], §§ 68 and 72) begins by stating that:

$$\text{GL}.i) \quad \#P =_{df} \mathcal{E}\left[X : X \approx P\right]$$

$$\text{GL}.ii) \quad \forall x \left[\mathbf{N_A} x \Leftrightarrow \exists X \left[x = \#X\right]\right]$$

This is enough to make him able to informally prove HP (*ibidem*, § 73). After having proved it (*ibidem*, §§ 74 and 76), Frege states that:

$$\text{GL}.iii) \quad 0 =_{df} \#\left[x : x \neq x\right]$$

and

$$\text{GL}.iv) \quad \forall x, y \left[x \mathbf{F_I} y \Leftrightarrow \exists X \exists z \left[x = \#X \wedge Xz \wedge y = \#\left[w : Xw \wedge w \neq z\right]\right]\right],$$

where 'y', 'z' and 'w' are other variables ranging on objects, and '$x\mathbf{F_I}y$' is to be read as 'x follows immediately after y in the natural sequence of numbers'[7]. This allows him to observe (*ibidem*, § 77) that from HP it follows that

$$\#\left[x : x = 0\right] \mathbf{F_I} 0,$$

which suggests stating that

$$\text{GL}.v) \quad 1 =_{df} \#\left[x : x = 0\right].$$

Insofar as from (GL.iv) and (GL.v) it also follows that $\#\left[x : x = 0 \vee x = 1\right] \mathbf{F_I} 1$, Frege might have continued by stating that $2 =_{df} \#\left[x : x = 0 \vee x = 1\right]$, etc. Still, this would have not allowed him to explicitly define natural numbers in general, but only to recursively define indefinitely many such numbers. Hence, by leaving (GL.v) aside, he rather defines the new first-level binary relation of following after in the natural sequence of numbers—which I shall shortly designate by '$\mathbf{F}$'— as the the strong ancestral of $\mathbf{F_I}$ (*ibidem*, § 79 and 81), namely:

$$\text{GL}.vi) \quad \forall x, y \left[x \mathbf{F} y \Leftrightarrow \forall X \left[\left[\forall z \left[z \mathbf{F_I} y \Rightarrow Xz\right] \wedge \forall z, w \left[\left(Xz \wedge w \mathbf{F_I} z\right) \Rightarrow Xw\right]\right] \Rightarrow Xx\right]\right],$$

where '$x\mathbf{F}y$' is to be read as 'x follows after y in the natural sequence of numbers'. Then, based on this relation, he defines (*ibidem*, §§ 81) another first-level binary relation, to be conceived as the relation of belonging to the natural sequence of numbers either beginning or ending with, the difference of the two relations merely depending on the order of relata. The definition is this:

$$\text{GL}.vii) \quad \forall x, y \left[x \mathbf{S} y \Leftrightarrow x \mathbf{F} y \vee x = y\right],$$

where '$x\mathbf{S}y$' is to be read either as 'x belongs to the natural sequence of numbers beginning with y' or 'y belongs to natural sequence of numbers ending with x'.

Finally ([9], §§ 79 and 82-83, [3]), he sketches a proof of

$$\forall x \left[\mathcal{C}(x) \Rightarrow \#\left[z : x \mathbf{S} z\right] \mathbf{F_I} x\right],$$

[7] As odd as this phrase might appear, it is just Frege's own phrase translated literally (*ibidem*, § 76).

where: '$\mathcal{C}(x)$' stands from 'x meets a condition to be specified', by so showing that this holds if

$$\forall x \left[\mathcal{C}(x) \Leftrightarrow x\mathbf{S}0\right],$$

which suggests abbreviating this condition, namely the condition that x belongs to a succession beginning with 0, with 'x is a finite *Anzahl*', or, by short, '$\mathbf{N}x$', from which it follows that

$$\text{GL.}viii) \quad \forall x \left[\mathbf{N}x \Leftrightarrow x\mathbf{S}0\right]$$

and

$$\forall x \left[\mathbf{N}x \Rightarrow \exists y \left[\mathbf{N}_{\mathbf{A}}y \wedge y\mathbf{F}_{\mathbf{I}}x\right]\right]. \qquad (\text{SUCC}_{\text{GL}})$$

What is crucial for us, namely definition (GL.*viii*), is presented as quite marginal by Frege: he does not only merely qualify it as a "convenient abbreviation", but he comes to it while establishing a sufficient condition for it to hold that an *Anzahl* be immediately followed by another *Anzahl*. In contemporary terminology: Frege recognizes that only certain *Anzahlen* meet Peano's successor axiom, and identify finite ones—or natural numbers, as we currently call them—as some of those which do; but he seems much more interested in enquiring about the properties of *Anzahlen* taken as such, than in singling out natural numbers among them. This is all the more clear that, after having established this abbreviation, he quickly leaves natural numbers aside to deal with *Endloss*: the *Anzahl* of the concept of being a finite *Anzahl*.

His main point seems, then, that of establishing that *Anzahlen* are numbers of first-level concepts—which allows studying them within a general theory of concepts and objects—and that some of them satisfy the fundamental theorems of arithmetic. His purpose, in other terms, seems more that of immersing arithmetic within a theory of numbers of first-level concepts, than that of developing it there.

This being said, some remarks on definitions (GL.*i*-*viii*) are in order. The first one is that extensions explicitly appear only in the first of these definitions and in the proof of HP. This suggests that Frege was considering them useful only to make this proof possible, so as to show that *Anzahlen* come together with fist-level concepts, though not directly.

The adverb 'only' in this claim is not to be intended as aiming to undermine the role of extensions. To see how crucial this role is, notice that (SUCC_{GL}) is all that Frege offers in *Grundlagen* concerning the existence of *Anzahlen*. On the standard interpretation of the existential quantifier, this theorem asserts that natural numbers exist, if 0 does. Still, granting this interpretation is not enough, yet, to allow the proof of this theorem to establish this. To this purpose, it is also required to grant that Frege's definitions have a model, that is, that there are objects complying with them. From this, it also follows that 0 exists, but this condition makes it flagrantly circular to appeal to this theorem to conclude that natural numbers exist.

Though Frege would have certainly not reasoned in terms of models, he could have not ignored that alleging that his proof of (SUCC_{GL}) provides an argument for the existence of natural numbers would have been circular. This suggests that he was taking the question of the existence, not only of natural numbers, but, more generally, of *Anzahlen*, settled by his definition of them as extensions of openly identified concepts: he was possibly taking *Anzahlen* to exist, just in virtue of the fact that these concepts are given, and their extensions come directly with them.

Still, Russell's paradox apart—that is, also admitting to adopt an appropriate variant of Frege's framework, where the relevant extensions could be consistently handled[8]—taking *Anzahlen* to be identified with extensions (of second-level) concepts has a price, which is often unnoticed.

If the general nature of *Anzahlen* merely depended on HP and definition (GL.*ii*), one could think that the particular nature of the *Anzahl* of a certain concept only depends on the nature of this concept, namely that the particular nature of $\#P$ only depends on the nature of P. If this were granted, it would be enough to admit that this last concept is logical, to conclude that $\#P$ is either a logical object, provided that HP were taken to be logical, or an object having with logic the same sort of relation as HP. But if $\#P$ is identified with the extension of the second-level concept $[X : X \approx P]$, things change. Since, then, taking $\#P$ as a logical object just means taking this last extension as a logical object, and this is highly questionable, indeed. The reason is obvious: it seems hard to deny that the nature of this extension depends on which concepts are equinumerous with P, which does not seem to be a question of logic, even if P is a concept like $[x : x \neq x]$ or $\left[x : \bigvee_{i=0}^{i=n} x = i\right]$ $(n = 0, 1, \ldots)$, which are the only relevant ones for deciding whether natural numbers are logical objects, and might be plausibly taken as logical ones (if the latter are recursively defined, of course).

There are two obvious ways to resist this objection: *i*) either admitting that the nature of $\mathcal{E}[X : X \approx P]$, or at least its logicality, is not affected, after all, by which concepts are equinumerous with P; *ii*) or that, among all concepts equinumerous with $[x : x \neq x]$ or $\left[x : \bigvee_{i=0}^{i=n} x = i\right]$, the only relevant ones for this matter are logical ones. The problem is that either option seems unavailable to Frege.

Against (*ii*) does not only militate the fact that, in the informal framework of *Grundlagen*, it is hard to establish which concepts are logical, but also, and overall, the obvious circularity of deciding whether $\mathcal{E}[X : X \approx [x : x \neq x]]$ or $\mathcal{E}\left[X : X \approx \left[x : \bigvee_{i=0}^{i=n} x = i\right]\right]$ are logical objects, based of this admission.

Against (*i*) militates the very conception of the extension of a concept coming from the tradition that Frege seems to rely on when taking as "known what the extension of a concept is", for example, Kant's definition of it as a "*sphaera*" which "rises up from the multitude of things that are contained under the concept" ([15], p. 911; [16], p. 354). If straightforwardly applied to second-level concepts, according to an intensional notion of a first-level concept, this conception openly suggests that the extensions of $[X : X \approx [x : x \neq x]]$ and $\left[X : X \approx \left[x : \bigvee_{i=0}^{i=n} x = i\right]\right]$ actually depend on which first-level concepts (intensionally speaking) fall under these concepts, and, consequently, on how many objects fall under whatever finite sortal first-level concept. It suggests, for example, that the extension of $\left[X : X \approx \left[x : \bigvee_{i=0}^{i=13} x = i\right]\right]$ depends on how many objects actually fall under the concept of eight-thousanders on Earth. In our actual word, there are just fourteen eight-thousanders on Earth, which makes the latter concept fall under the former. It is obvious, however, that this is not a matter of logic. Thus, the extension of the former concept can hardly be taken

[8]A natural option would be to adopt a consistent variant of the formal system of *Grundgesetze*, powerful enough to allow us to offer an appropriate definition of natural numbers within it. To the best of my knowledge, among several available such variants, the one which is closest to the original system is the system PE, recently suggested by F. Ferreira ([7]).

to be a logical object.

Many countermoves are possible against this conclusion.

The most obvious one consists in observing that nothing forces us to take Frege's conception of extensions of second-order concepts to be in agreement with Kant's view. This is, however, what is openly implied by Frege's own informal proof of HP ([9], § 73). Since, in conducting it, he supposes that

$$\mathcal{E}\left[X : X \approx P\right] = \mathcal{E}\left[X : X \approx Q\right] \Leftrightarrow \forall X\left[X \approx P \Leftrightarrow X \approx Q\right],$$

for whatever first-level concepts P and Q, which suggests that he was just taking the extensions of $[X : X \approx P]$ and $[X : X \approx Q]$ to depend on which first-level concepts are equinumerous with P and Q.

Another countermove consists in admitting this last point, while denying that concepts are to be understood intensionally. Indeed, according to an extensional notion of a concept, the extension of $\left[X : X \approx \left[x : \bigvee_{i=0}^{i=13} x = i\right]\right]$ does not depend at all, for example, on how many objects fall under the concept of eight-thousanders on Earth. For what identifies this last concept is not the condition of being an eight-thousander on Earth, but rather the fourteen mountains that fall under it, taken as such, which would form the very same concept, invariably falling under $\left[X : X \approx \left[x : \bigvee_{i=0}^{i=13} x = i\right]\right]$, even if the eight-thousander on Earth were other than them. In other words, on this conception, the extension of $[X : X \approx [x : x \neq x]]$ is merely depending (in one way or another) on a single first-level concept, namely the empty one, while that of $\left[X : X \approx \left[x : \bigvee_{i=0}^{i=h} x = i\right]\right]$, for whatever natural number h, merely depends on the totality of $(h+1)$-uples, no matter how each of them might be intensionally conceived. In my view, this is not, and could not have been Frege's conception of a concept[9], but, this apart, the point, here, is that ascribing it to him does not put him in a better position for admitting that the extensions of concepts like $\left[X : X \approx \left[x : \bigvee_{i=0}^{i=n} x = i\right]\right]$ $(n = 0, 1, \ldots)$ are logical objects. Since, even according to this conception, these extensions depend on how the world is, namely on which objects exist (though they do not depend on the way these objects are conceptually classified).

This second countermove suggests many other ways to dismiss the point, by appropriately extending or restricting the relevant first-order domain, or, at least, by making it independent of the objects which actually exist. A similar strategy, directly applied, instead, to the second-order domain is suggested by option (ii), above. The difficulty here is not only that of imagining how to do it without making the argument in favor of the logicality of $\mathcal{E}\left[X : X \approx [x : x \neq x]\right]$ and $\mathcal{E}\left[X : X \approx \left[x : \bigvee_{i=0}^{i=n} x = i\right]\right]$ circular. It is also, and overall, that any similar move would be openly incompatible with Frege's universalist and realist conceptions.

Hence, granted these conceptions, it seems plain that Frege could have hardly admitted that the extensions he identifies with natural numbers are logical objects, simply because their particular nature depends on matters on which logic cannot be taken to have any sort of jurisdiction.

Arguably, senses and intuition are necessary to grasp this particular nature. Still, neither Frege's proof of HP and the theorems of arithmetic, nor his argument for the existence of natural

[9] I have argued for that in [1], ch. 2.

numbers depend on grasping this nature. The previous definitions are sufficient for this. One might, then, think that Frege considered these definitions enough for making the relevant properties of natural numbers transparent to reason without the medium of senses or intuition. For, though they make the nature of the objects they define ungraspable without senses and intuition, they openly identify the concepts which these numbers are the extensions of, and allow to prove HP and these theorems. And so much the worse for logical objects, if they make natural numbers hardly be such objects.

This supports the third of the options listed at the end of § 3, and is compatible with the first: it suggests that Frege's primary purpose concerning arithmetic was that of making truths about *Anzahlen*, and/or knowledge of them *a priori*, namely of showing that these numbers can (or have to) be conceived as objects whose relevant properties, though not their particular nature, are transparent to reason without the medium of senses and intuition, and of doing it by also locating arithmetics within a general theory of concepts and their numbers (where the latter are seen as extensions of the former).

This picture is quite different from that conveyed by the usual description of Frege as a logicist about arithmetic, and is rather close to assigning to him a quasi-structuralist conception. But it seems to be the one which his definition of *Grundlagen* more plausibly suggests.

6 The definition of *Grundgesetze*

Mutatis mutandis, this is also the picture suggested by the definition of *Grundgesetze*. Contrary to that of *Grundlagen*, this is a formal definition, that is, it is supplied within a formal system. This makes a crucial difference, which is relevant in many important respects. Still, this system is formal in a quite different sense than in our modern one: on the one hand, it does require no interpretation, since it already comes with a fixed meaning for its symbols, though not with a model, properly speaking; on the other hand, its variables are supposed to vary on given, universal domains. We know today that it could in no way have a model, since it is inconsistent. This is not the important point here, however. This is not only (and not mainly) because we might always suppose to adopt a consistent variant of Frege's system[10], but overall because the point here is not what Frege's (original) definition actually defines, but what he could have taken it to define. Hence, what is important here is, rather, that Frege thought that his system had a fixed interpretation or, at most, a family of fixed interpretations[11] (which we could regard today as the intended one, or ones). The question is, then, what sort of objects, natural numbers should, or better could, be taken to be according to Frege's definition and to this interpretation or these interpretations.

In the new formal setting, value-ranges are governed by BLV, which is restricted to first-level functions. This makes *ipso-facto* hopeless to define *Anzahlen* as extensions of second-level concepts. Hence, Frege rephrases his definitions, by replacing higher-level concepts with value-ranges of appropriate first-level functions. Once again, it is in order here to merely provide a short outline of his definition.

For this, I will rephrase the basic components of it by using the same notation used above for the

[10]Cf. footnote (8), above.

[11]What I have in mind by admitting this possible pluralization of interpretations is Frege's acknowledgment that BLV is not enough for identifying value-ranges (cf. p. 7, above).

definition of *Grundlagen*, appropriately modified and extended. It is important to notice, however, that this notation is no more to be intended as apt to shorten informal discursive statements, but rather statements (i.e. explicit definitions and theorems) belonging to a formal (deductive) system.

Let Φ and Ψ be whatever first-level one-argument and two-arguments function, respectively, and $\top$ the True (a designed object, for Frege, just as the False, $\bot$). To make a long story short[12], let us denote with '$\mathfrak{C}_\Phi$' the (first-level) concept $\ulcorner$argument for which Φ takes the value $\top\urcorner$', and with '$\mathfrak{R}_\Psi$' the (first-level) relation: $\ulcorner$ordered pair of arguments for which Ψ takes the value $\top\urcorner$, namely:

$$\begin{array}{l} \mathfrak{C}_\Phi =_{abbr.} [x : \Phi(x) = \top] \\ \mathfrak{R}_\Psi =_{abbr.} [x, y : \Psi(x, y) = \top] \end{array},$$

where 'x' and 'y' range on objects. Hence, if Φ is a concept and Ψ a (binary) relation, $\mathfrak{C}_\Phi$ and $\mathfrak{R}_\Psi$ come respectively to coincide with Φ and Ψ. Still, as we shall see, it is important to observe that Frege does not confine himself to consider concepts and relations, but rather extends his concern to functions in general.

This is already clear by the new definition replacing (GL.i). Instead of defining the *Anzahl* of a first-level concept, he now defines the *Anzahl* of an object ([10], § I.40). For whatever object a, let $\mathfrak{T}_a$ be, for short, the (first-level) concept $\ulcorner$argument for which the first-level one-argument function (if any) of which a is the the value-range takes the value $\top\urcorner$, namely:

$$\mathfrak{T}_a =_{abbr.} [x : \exists\varphi\, [a = \mathcal{E}\varphi \wedge \mathfrak{C}_\varphi x]],$$

where 'φ' ranges on first-level one-argument functions, and, for whatever such function Φ, '$\mathcal{E}\Phi$' denotes the value-range of Φ. This makes '$\mathfrak{T}_{\mathcal{E}\Phi}$' designates the same concept as '$\mathfrak{C}_\Phi$'. Frege's new definition can, then, be rendered as follows:

$$\text{GG}.i) \quad \#a =_{df} \mathcal{E}\,[x : \mathfrak{T}_x \approx \mathfrak{T}_a].$$

The fact that '#' applies now to objects, rather than to first-level concepts is, technically speaking, less relevant than it might appear (we shall see later what this entails on an interpretative level). For short, dub 'plain' an object, if any, which is not the value-range of a first-level one-argument function. If a is not plain, then $\#a = \#\mathcal{E}\Phi$ for some such function Φ, so that '#' combines with '$\mathcal{E}$' by yielding the operator '$\#\mathcal{E}$', applying to this function. But what happens, instead, if a is plain? The first thing to be observed on this matter is that nothing, in Frege's system, can ensure that there are plain objects. Though it does not entail it, the stipulation Frege advances in § I.10 rather suggests the contrary. But, even if it were supposed that there be such objects and a were one of them, the concept $\mathfrak{C}_a$ would reduce to $[x : x \neq x]$. Hence, independently whether a is plain or not, definition (GG.i) states that $\#a$ is the extension of the concept of being the value-range of a first-level one-argument function Φ such that $\mathfrak{C}_\Phi$ is equinumerous with $\mathfrak{T}_a$, which allows taking $\#a$ as the *Anzahl* of $\mathfrak{T}_a$[13].

Frege's further definitions can, then, reduce to formal rephrasing of those of *Grundlagen*, under the replacement of first-level concepts with value-ranges of first-level one-argument functions.

[12]Namely to rephrase Frege's function $\xi \frown \zeta$, defined in § I.34 and, then, pervasively used in the whole work.

[13]Cf. p. 7, above. Notice that if 'Δ' denoted, as for Frege, an object whatsoever, '$\mathfrak{T}_\Delta$' would denote, in my notation, the same concept Frege denotes with '$\_\_\xi \frown \Delta$', so that this claim coincides with that made by Frege at the end of § I.40.

First come these four definitions ([10], §§ I.42-43):

$$\text{GG.}ii) \quad \forall x\left[\mathbf{N_A}x \Leftrightarrow \exists z\left[x = \#z\right]\right]$$

$$\text{GG.}iii) \quad 0 =_{df} \#\mathcal{E}\left[x : x \neq x\right]$$

$$\text{GG.}iv) \quad \mathbf{f_I} =_{df} \mathcal{E}\left[x, y : \exists\varphi\exists z\left[\#\mathcal{E}\left[w : \mathfrak{C}_\varphi w \wedge w \neq z\right] = x \wedge \mathfrak{C}_\varphi z \wedge \#\mathcal{E}\varphi = y\right]\right] \quad ,$$

$$\text{GG.}v) \quad 1 =_{df} \#\mathcal{E}\left[x : x = 0\right]$$

where, for whatever first-level two-arguments function Ψ, '$\mathcal{E}\Psi$' denotes the value-range of Ψ, and '$\mathbf{f_I}$' is intended to denote the extension of the relation of being immediately followed by "in the sequence of *Anzahlen*"[14].

Once definition (GG.iv) stated, Frege can easily define the extensions $\acute{-}\mathbf{f_I}$ and $\acute{\smile}\mathbf{f_I}$ of the strong and week ancestral of this last relation ([10], §§ I.45-46):

$$\text{GG.}vi) \quad \acute{-}\mathbf{f_I} =_{df} \mathcal{E}\left[x, y : \exists\psi\left(\mathcal{E}\psi = \mathbf{f_I} \wedge \forall\varphi\left[\left[\left(\begin{array}{l}\forall z\left[x\mathfrak{R}_\psi z \Rightarrow \mathfrak{C}_\varphi z\right] \wedge \\ \forall z, w\left[\left(\begin{array}{c}\mathfrak{C}_\varphi z \wedge \\ z\mathfrak{R}_\psi w\end{array}\right) \Rightarrow \mathfrak{C}_\varphi w\right]\end{array}\right) \Rightarrow \mathfrak{C}_\varphi y\right]\right]\right)\right] \quad .$$

$$\text{GG.}vii) \quad \acute{\smile}\mathbf{f_I} =_{df} \mathcal{E}\left[x, y : \exists\psi\left[\acute{-}\mathbf{f_I} = \mathcal{E}\psi \wedge x\mathfrak{R}_\psi y \vee x = y\right]\right]$$

where 'ψ' ranges on first-level two-argument functions. Provided that $\mathbf{S}^{-1}$ be the (first-level binary) relation whose extension is $\acute{\smile}\mathbf{f_I}$, namely the relation of belonging to the sequence of *Anzahlen* ending with, definition (GG.vii) makes '$0\mathfrak{R}_{\mathbf{S}^{-1}}a$' stand for '$\mathbf{S}^{-1}(0, a) = \top$', whatever object a might be. With the help of this definition, it is, then, finally easy to define the property of being a "finite *Anzahl*", i.e. a natural number ([10], § I.46):

$$\text{GG.}viii) \quad \forall x\left[\mathbf{N}x \Leftrightarrow \exists\psi\left(\mathcal{E}\psi = \acute{\smile}\mathbf{f_I} \wedge 0\mathfrak{R}_\psi x\right)\right]$$

Mutatis mutandis, what we have noticed above for definition (GL.$viii$) also holds for definition (GG.$viii$): though this is, for us, the crucial definition, Frege does not emphasize it in any way. Rather, he merely states it informally, by confining himself to stipulate that, for whatever object a, 'a belongs to the sequence of *Anzahlen* beginning with 0' (i.e. '$0\mathfrak{R}_{\mathbf{S}^{-1}}a$', in my notation) means the same as 'a is a finite *Anzahl*'.

Based on these definitions, Frege also proves a theorem corresponding to (SUCC$_{\text{GL}}$), which can be rendered as follows[15]:

$$\forall\psi, \chi\forall x\left[\left(\begin{array}{l}\mathbf{N}x \wedge \\ \mathcal{E}\psi = \acute{\smile}\mathbf{f_I} \wedge \\ \mathcal{E}\chi = \mathbf{f_I}\end{array}\right) \Rightarrow x\mathfrak{R}_\chi\#\mathcal{E}\left[z : z\mathfrak{R}_\psi x\right]\right], \qquad (\text{SUCC}_{\text{GG}})$$

[14]Apart from Frege's passing from speaking of the "natural sequence of numbers" to speaking of the "sequence of *Anzahlen*", it is also noteworthy that the new relation is the inverse of the one considered in *Grundlagen*. Remark also that Frege conceives value-ranges of first-level two-arguments functions (including extensions of first-level binary relations) as value-ranges of appropriate first-level one-argument functions associated to these functions. So no extension of BLV is required for definitions (GG.iv) and related to be stated.

[15]This is firstly stated, but not proved in § I.46. Its proof takes up all the first half of part II of Frege's treatise, and only ends in § I.118, where it is stated anew as theorem 155.

where also 'χ' ranges, as 'ψ', on first-level two-argument functions. Still, its proof and that of some corollaries is, again, followed by a long section devoted to *Endloss*, so as to suggest, like in *Grundlagen*, that his aim was less that of defining natural numbers, by singling them out among *Anzahlen*, than studying *Anzahlen* as such.

Despite the parallelism among these definitions and those of *Grundlagen*, the former do not identify natural numbers with the same objects as the latter. Whereas in *Grundlagen* natural numbers are identified with extensions of second-level concepts, in *Grundgesetze* they are identified with extensions of first-level concepts. Possibly, Frege was taking extensions of the former concepts to reduce to extensions of the latter. But, even if it were so, this would not depend on the definitions he offers, but on further informal stipulations, on which the very nature of natural numbers (and, more generally, *Anzahlen*) would, then, depend.

But, what is more relevant is that the definitions of *Grundgesetze* are even less apt to make natural numbers logical objects than those of *Grundlagen*.

On the one hand, similar considerations as those made above (pp. 10-11) about the definitions offered in *Grundlagen* also apply, *mutatis mutandis*, to the definitions of *Grundgesetze*. Indeed, according to these last definitions and BLV, 0 identifies with

$$\mathcal{E}\left[x : \mathfrak{T}_x \approx [z : z \neq z]\right],$$

while each positive natural numbers identifies with one of the following extensions

$$\mathcal{E}\left[x : \mathfrak{T}_x \approx \left[z : \bigvee_{i=0}^{i=n} z = i\right]\right] \qquad (n = 0, 1, \ldots),$$

and, in agreement with BLV and Frege's universalist perspective, these extensions are different in nature according to whether some non-logical facts obtain, unless it is circularly admitted that, among all first-level one-argument functions φ such that $\mathfrak{C}_\varphi$ is equinumerous to $[z : z \neq z]$ or $\left[z : \bigvee_{i=0}^{i=n} z = i\right]$, the only relevant ones for this matter are logical ones. For example, the nature of number 14, namely the extension of $\left[x : \mathfrak{T}_x \approx \left[z : \bigvee_{i=0}^{i=13} z = i\right]\right]$, cannot but depend, in Frege's views, on how many objects actually fall under the concept of eight-thousanders on Earth, or, at least, on which objects exists.

On the other hand, when one goes from the definitions of *Grundlagen* to those of *Grundgesetze* new problems arise for the identification of natural numbers with logical objects.

Consider 0. As we have just seen, it identifies with the *Anzahl* of $[x : x \neq x]$, namely the extension of the concept of being the value-range of a first-level one-argument function that takes the value $\top$ for no argument. Suppose there were plain objects, and that a be one of them. No object would, then, fall under $\mathfrak{T}_a$ so that 0 would be the extension of a (first-level) concept under which a falls. Suppose there were not plain objects. Then, for any object, there would be a first-level one-argument function whose value-range is this very object, so that 0 would be the extension of a (first-level) concept under which fall all and only the value-ranges of a first-level one-argument function φ such that $\mathfrak{C}_\varphi$ is empty. It follows that the nature of 0 depends on whether there are or there are not plain objects. Provided that nothing within Frege's logical system can decide this matter, this makes it clear that this system cannot decide this nature, which makes, in turn, quite odd to take 0 to be a logical object because of the way it is defined within this system.

The supposition that there are no plain objects would not improve the situation. To see this, consider the following (first-level one-argument) function:

$$\mathscr{Q}(x) = \begin{cases} \begin{array}{l}\text{the square constructed}\\ \text{on the diagonal of } x\end{array} & \text{if } x \text{ is a segment} \\ \mathcal{E}[z : z \neq z] & \text{otherwise} \end{cases}$$

Frege could not have taken it as a logical function. Still, unless he had come to identify $\mathcal{E}[z : z \neq z]$ with $\top$ (in opposition to what he says in § I.10), he should have admitted that $\mathfrak{C}_{\mathscr{Q}} \approx [z : z \neq z]$, which makes its value-range $\mathcal{E}\mathscr{Q}$, whatever it might be, fall under the concept whose extension is 0, by so suggesting that neither this concepts not this extension are logical items.

To avoid these two last difficulties, Frege might have defined 0 as the extension of the concept ⌜extension of a concept equinumerous with $[z : z \neq z]$⌝, namely $\mathcal{E}[x : \exists X [x = \mathcal{E}X \wedge X \approx [z : z \neq z]]]$, under which, in agreement with BLV, only $\mathcal{E}[z : z \neq z]$ falls. This would have made it more plausible to take 0 as a logical object. Why did he not? The answer cannot be that his system involves no straightforward device to restrict quantification on second-order monadic variables to concepts. Since nothing would have forbidden him to invent such a device, or to conceive his system in a slight different way. Possibly the right answer is that he considered important to widen his concern from mere concepts to functions in general. Still, as worthy as this stance might be considered to be, it remains that nobody primarily aiming at identifying natural numbers with logical objects would have preferred the gain of generality it involves to the possibility of defining 0 as such an object.

But this is still not all. After 0, consider 1, and remember Frege's "stipulation" about value-ranges and truth-values of § I.10. According to BLV, this stipulation is equivalent to state that:

$$\top = \mathcal{E}[x : x = \top] \qquad \text{and} \qquad \bot = \mathcal{E}[x : x = \bot]$$

Compared with definitions (GG.i) and (GG.v), namely with

$$1 =_{df} \mathcal{E}[x : \mathfrak{T}_x \approx [x : x = 0]],$$

this makes 1 the extension of a concept under which both $\top$ and $\bot$ fall. Still, if another stipulation were made, namely if $\top$ and $\bot$ were not stipulated to be two value-ranges of a first-level one-argument function, or were stipulated to be value-ranges of such a function that does not takes the value $\top$ for a single argument, this would not be so. Hence, according to Frege's definition, 1 is a different object according to whether some stipulations are made or rejected. This would be quite strange for a logical object.

All this seems to show that the definitions of *Grundgesetze* are quite far from making natural numbers logical objects. By making these numbers, and, more generally, *Anzahlen*, extensions of openly identified concepts, which are, as such, independent of senses and intuition, they rather seem to aim at making truths about *Anzahlen*, and/or knowledge of them *a priori*, to ensure they necessarily exist, and to submit them to HP, so as to allow proving the basic theorems of arithmetic. Hence, as those of the *Grundlagen*, also the definitions of *Grundgesetze* suggest assigning to Frege a quasi-structuralist conception of arithmetic.

This does not mean that these last definitions would have not allowed Frege to argue that arithmetic is a part of logic. Since, if it is admitted that he endorsed such a quasi-structuralist conception, it should also be admitted that he might have considered that arithmetic is a part of logic, even if natural numbers are not logical objects. Indeed, he might have argued that this is so just because these numbers can be defined and arithmetical theorems proved within a system of logic. The informal definitions of *Grundlagen* could have hardly allowed him to argue this way. But the formal ones of *Grundgesetze* make this line of argument possible, provided, however, that a convincing argument for the logicality of BLV were offered.

7 A Last Remark

All this having being said, let me conclude with a counterfactual observation. It is easy to see that, once his system established, Frege could have easily defined natural numbers within it, in a quite different way, apt to make much more plausible taking them as logical objects. He could have defined 0 merely as the extension of the concept $[x : x \neq x]$, then adopted an appropriate definition of the extension of the successor relation, crucially but not greatly different from (GG.*iv*), and finally identified again the natural numbers with the objects that bear with 0 the weak ancestral of this relation, namely, with $\mathcal{E}\,[x : x \neq x]$, $\mathcal{E}\,[x : x = 0]$, $\mathcal{E}\,[z : z = 0 \vee z = 1]$, etc. Surely, the inability of his system to decide what value-ranges of first-level functions actually are would have made it impossible to say what these numbers would have been merely based on considerations internal to this system. Still, none of the previous considerations or other analogous to them would have made it hard to maintain that they are logical objects, provided that the system itself had been taken as logical.

So, why hid he not follow this route? A quite obvious answer is available: defining natural numbers this way would have made natural numbers neither submit to HP, nor be numbers of concepts, and would have so broken their crucial link with the more general family of *Anzahlen*.

This answer is as telling as obvious. Since it confirms that making natural numbers logical objects was not Frege's primary aim: submitting them to HP, and, so, making them numbers of concepts was much more important to him.

References

[1] H. Benis-Sinaceur, M. Panza, and G. Sandu. *Functions and Generality of Logic. Reflections on Frege's and Dedekind's Logicism.* Springer, Cham, Heidelberg, New York, Dordrecht, London, 2015.

[2] G. Boolos. *Logic, Logic and Logic.* Harvard Univ. Press, Cambridge (Mass.) and London, 1998. With introductions and afterword by J. P. Burgess; edited by R. Jeffrey.

[3] G. Boolos and R. G. Heck Jr. Die Grundlagen der Arithmetik, §§ 82-3. In *[18]*, pages 407–428. Oxford. Also in [2], pp. 315-338.

[4] R. Carnap. *Abriss der Logistik [...].* J. Springer, Wien, 1929.

[5] R. Carnap. Die logizistische grundlegung der mathematik. *Erkenntnis*, 2:91–105, 1931.

[6] R. Carnap. The logicist foundation of mathematics. In P. Benacerraf and H. Putnam, editors, *Philosophy of Mathematics*, pages 31–41. Prentice-Hall Inc, Englewood Cliffs, 1964. English translation of [5].

[7] F. Ferreira. Zigzag and Fregean arithmetic. *Forthcoming.* Presented at the Workshop *Ontological Commitment in Mathematics. In Memoriam of Aldo Antonelli*, Paris, IHPST, December 14th-15th 2015.

[8] A. Fraenkel. *Einleitung in die Mengenlehre.* Springer Verlag, Berlin Heidelberg, 1928. Third edition reworked and largely extended.

[9] G. Frege. *Die Grundlagen der Arithmetik.* W. Köbner, Breslau, 1884.

[10] G. Frege. *Die Grundgesetze der Arithmetick*, volume I-II. H. Pohle, Jena, 1893-1903.

[11] G. Frege. *The Foundations of Arithmetic.* Blackwell, Oxford, 1953. Translated by J. L. Austin.

[12] G. Frege. *Basic Laws of Arithmetic.* Oxford University Press, Oxford, 2013. Translated and edited by P. A. Ebert and M. Rossberg, with C. Wright.

[13] I. Grattan-Guinness. *The Search for Mathematical Roots. 1870-1940. Logic, Set Theory and the Foundation of Mathematics from Cantor through Russell to Gödel.* Princeton Univ. Press, Princeton (NJ) ans Oxford, 2000.

[14] A. Heying. Die intuitionistische grundlegung der mathematik. *Erkenntnis*, 2:106–115, 1931.

[15] I. Kant. Wiener Logik. In *Gesammelte Schriften*, volume XXIV.2, pages 785–940. Walter de Gruyter, Berlin, 1966.

[16] I. Kant. *Lectures on Logic.* Cambridge Univ. Press, Cambridge, 1992. translated and edited by J. M. Young.

[17] M. Panza and A. Sereni. Frege's constraint and the nature of Frege's logicism. Forthcoming.

[18] M. Schirn. *Philosophy of Mathematics Today.* Clarendon Press, Oxford, 1998.

[19] J. von Neumann. Die formalistische Grundlegung der Mathematik. *Erkenntnis*, 2:116–121, 1931.