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Abstract

This paper proposes a stochastic model for describing rational expectations. The context is systemic risk, with interconnected components of a unified system. The evolution dynamics leading to the failure of the system is explored either under a theoretical point of view as well as through an extensive scenario analysis.

Keywords: Rational expectation, stochastic system, systemic risk, evolutionary economics.

1 Introduction

The remaining part of the paper is organized as follows.

Last section concludes.

The model conceptualizes a system with components. Components have different levels of importance. Moreover, they are interconnected at different levels.

2 The model

We consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ containing all the random quantities used throughout the paper. We denote the expected value operator related to the probability measure $\mathbb{P}$ by $\mathbb{E}$.
As said in the Introduction, the scope of the paper is to describe the evolution of rational expectations on the evolution of the risk of failure of an economic/financial system. Such a system is a unified entity composed by individual interconnected components.
We denote the *system* by $\mathbf{S}$, and assume that it is composed by n *components* denoted by $C_1, \dots, C_n$ and collected in a set $\mathcal{C}$.
The *state* of $\mathbf{S}$ is a binary quantity. If the system is *active* and works, then its state is 1. Otherwise, the state of $\mathbf{S}$ is 0, and the system is said to be *failed.* The state of $\mathbf{S}$ evolves in time, and we denote by $Y(t)$ the state of the system at time $t \geq 0$. At the beginning of the analysis (time $t = 0$) the system is naturally assumed to be in state 1.
Analogously, the state of the j-th component C_j at time t is denoted by $Y_j(t)$, and it takes value 1 when C_j is active and 0 when C_j is failed. At time $t = 0$ we have $Y_j(0) = 1$, for each $j = 1, \dots, n$.
The value of $Y(t)$ depends on the states of the components of the system at time t. The way in which such a dependence is conceptualized is grounded also on the arguments of the next Section.

2.1 Main assumptions on the system

We now point out three assumptions of our model which are tailored on the empirical evidence on economic/financial systemic risk: first, the different components of the system are assumed to be not homogeneous in terms of their relevance; second, the components of the system are interconnected and exhibit different levels of interconnection; third, relevance and interconnection levels change over time, according to the change of the status of the components of the system.

We enter the details.

For each $j = 1, \dots, n$ and $t \geq 0$, the *relative importance of the component* C_j over the entire system at time t is measured through $\alpha_j(t)$, where $\alpha_j : [0, +\infty) \to [0, 1]$ and $\sum_{j=1}^{n} \alpha_j(t) = 1$, for each t.

For each $t \geq 0$, we collect the $\alpha(t)$'s in a time-varying vector $\mathbf{a}(t) = (\alpha_j(t))_j$, where

$$\mathbf{a} : [0, +\infty) \to [0, 1]^n \qquad \text{such that} \qquad \mathrm{t} \mapsto \mathbf{a}(\mathrm{t}). \tag{1}$$

If a component is not active at time t, then its relevance for the system is null. Moreover, each active component has positive relative relevance, i.e. the system does not contain irrelevant active components. Formally,

$$\alpha_j(t) = 0 \Leftrightarrow Y_j(t) = 0. \tag{2}$$

Condition (2) is useful, in that it allows to describe the status of the components of the system directly through the α's.

For each $j = 1, \dots, n$, the relative relevance of C_j changes in correspondence of the variation of the state of one the components of the system. Once a component fails, then it disappears from the economic/financial system and the relative relevances of the components of the remaining active ones are modified on the basis of a suitably defined *reallocation rule*.

Next example proposes a way to build a reallocation rule.

Example 1. *Consider a system* $\mathbf{S}$ *whose components set is* $\mathcal{C} = \{C_1, C_2, C_3, C_4, C_5\}$.
Assume that, at time $t = 0$, *we have* $\alpha_1(0) = 0.1$, $\alpha_2(0) = 0.15$, $\alpha_3(0) = 0.3$, $\alpha_4(0) = 0.2$, $\alpha_5(0) = 0.25$.

Now, suppose that the first failure of one of the components of the system occurs at time $t = 7$, *when* C_3 *fails. Of course,* $\alpha_j(t) = \alpha_j(0)$, *for each* $t \in [0, 7)$ *and* $j = 1, 2, 3, 4, 5$. *Moreover,* $\alpha_3(7) = 0$.

We consider a specific reallocation rule, which states that the relevance is reallocated over the remaining active components proportionally to their α*'s before the failure. This means that*

$$\alpha_1(7) = \frac{0.1}{0.1 + 0.15 + 0.2 + 0.25}, \ \alpha_2(0) = \frac{0.15}{0.1 + 0.15 + 0.2 + 0.25},$$

$$\alpha_3(0) = 0, \ \alpha_4(7) = \frac{0.2}{0.1 + 0.15 + 0.2 + 0.25}, \ \alpha_5(0) = \frac{0.25}{0.1 + 0.15 + 0.2 + 0.25}.$$

In general, if τ_1, τ_2 *are the dates of two consecutive failures, with* $\tau_1 < \tau_2$, *we have*

$$\alpha_j(\tau_2) = \frac{\alpha_j(\tau_1)\mathbf{1}_{\{Y_j(\tau_2)=1\}}}{\sum_{i=1}^{5} \alpha_i(\tau_1)\mathbf{1}_{\{Y_i(\tau_2)=1\}}}, \qquad j = 1, 2, 3, 4, 5.$$

The α*'s are step functions, whose jumps occur in correspondence to the failure of one of the components.*

For what concerns the *interconnections among the components*, we define their time varying relative levels through functions of type $w_{ij} : [0, +\infty) \to [0, 1]$, for each $i, j = 1, \dots, n$, so that $w_{ij}(t)$ is the relative level of the interconnection between C_i and C_j at time $t \geq 0$. We assume that $\sum_{i,j=1}^{n} w_{ij}(t) = 1$, for each t. Moreover, $w_{ii}(t) = 0$, for each i and t, which means that self-connections do not exist in our framework.

For each $t \geq 0$, the $w(t)$'s are collected in a time-varying vector $\mathbf{w}(t) = (w_{ij}(t))_{i,j}$, with

$$\mathbf{w} : [0, +\infty) \to [0, 1]^{n \times n} \qquad \text{such that} \qquad \mathrm{t} \mapsto \mathbf{w}(\mathrm{t}). \tag{3}$$

If C_i is a not active component at time t, then $w_{ij}(t) = 0$, for each $j = 1, \dots, n$. This condition simply states that a failed component is disconnected from the system. This suggests that the failure of a component might generate disconnections among the components of the system.

The behavior of the w's is analogous to that of the α's. Also in this case, the relative levels of interconnections change when one of the components of $\mathbf{S}$ change its state, and there is a reallocation rule for the remaining levels of interconnections.

A natural rewriting of the system is then

$$\mathbf{S} = \{\mathcal{C}, \mathbf{a}, \mathbf{w}\}. \tag{4}$$

2.2 The structure of the system

To capture the dependence of the state of $\mathbf{S}$ on the ones of its components, we simply introduce a function $\phi : \{0,1\}^n \to \{0,1\}$

$$Y(t) = \phi(Y_1(t), \dots, Y_n(t)). \tag{5}$$

In reliability theory, ϕ is usually denoted as the *structure function* of the system.

We denote the elements of $\{0,1\}^n$ as configurations of the states of the components of the system or, briefly, *configurations*.

Function ϕ in (5) has the role of clustering the set of configurations in two subsets: the ones leading to the failure (F) of the system and those associated to the not failed (NF) system. Thus, we say that $K_F \subseteq \{0,1\}^n$ is the collection of configurations such that $\phi(x_F) = 0$, for each $x_F \in K_F$ while $K_{NF} \subseteq \{0,1\}^n$ is the collection of configurations such that $\phi(x_{NF}) = 1$, for each $x_{NF} \in K_{NF}$. By definition, $\{K_F, K_{NF}\}$ is a partition of $\{0,1\}^n$.

In order to describe a systemic risk problem, some requirements on ϕ are needed.

First, $(0, \dots, 0) \in K_F$ and $(1, \dots, 1) \in K_{NF}$. This condition means that when all the components of the system are active (not active), then the system is active (not active) as well.

Second, ϕ is non-decreasing with respect to its components. This has an intuitive explanation: the failure of one of the components of the system might worsen the state of the system and cannot improve it.

Third, each component is able to determine the failure of the system. Formally, this condition states that for each $j = 1, \dots, n$ there exists $(y_1, \dots, y_{j-1}, y_{j+1}, \dots, y_n) \in \{0,1\}^{n-1}$ such that $(y_1, \dots, y_{j-1}, 1, y_{j+1}, \dots, y_n) \in K_{NF}$ and $(y_1, \dots, y_{j-1}, 0, y_{j+1}, \dots, y_n) \in K_F$.

2.3 Failure of the system and rational expectations

As said above, time $t = 0$ represents *today* – the starting point of the observation of the evolution of the system –. At time $t = 0$ all the components are active and the system works.

The failure of the system is then a random event, which occurs when the system achieves one of the configurations belonging to K_F.

We define the *system lifetime* as:

$$\mathcal{T} := \inf\{t \geq 0 | \phi(Y_1(t), \dots, Y_n(t)) = 0\}. \tag{6}$$

Analogously, the n-dimensional vector of *components lifetimes* is $\mathbf{X} = (X_1, \dots, X_n)$, where

$$X_j = \inf\{t > 0 \,|\, Y_j(t) = 0\}. \tag{7}$$

We assume that components exhibit also a "social" behavior in failing. Each X_j is composed by two terms: one of them is idiosyncratic, and captures aspects related to the inner life of C_j; the other one depends on the failures of the other components of the system. We denote the former term by X_j^I and the latter one by X_j^S and write

$$X_j = \min\{X_j^I, X_j^S\}. \tag{8}$$

The quantity X_j^S depends naturally on $\mathbf{a}$ and $\mathbf{w}$, while $\{X_1^I, \dots, X_n^I\}$ is a set of independent random variables. In general, $\{X_1, \dots, X_n\}$ are not independent and do not share the same distribution.

However, we can reasonably assume that the failure of the system coincides with the failure of one its components. Furthermore, simultaneous failures of components may occur due to the presence of the terms X^S's in (8).

At each components' failure, the α's and the w's modify and reallocate over the remaining components. Such variations explains the way in which $\mathbf{S}$ fails in accord to the failures of its components.

To fix ideas, we provide an example.

Example 2. *Assume that* $\mathcal{C} = \{C_1, C_2, C_3, C_4, C_5\}$ *and*

$$\mathbf{a}(0) = (0.1, 0.5, 0.2, 0.1, 0.1), \qquad \mathbf{w}(0) = \begin{pmatrix} 0 & 0 & 0.1 & 0.1 & 0 \\ 0 & 0 & 0.1 & 0 & 0 \\ 0.1 & 0.1 & 0 & 0.1 & 0.05 \\ 0.1 & 0 & 0.1 & 0 & 0.05 \\ 0 & 0 & 0.05 & 0.05 & 0 \end{pmatrix}$$

Suppose that the reallocation rules for relative relevance and interconnection levels are of proportional type, as in Example **??**. *Such reallocations are implemented if the system is not failed.*

Furthermore, suppose that if a given component fails, then the components connected only to it fail as well, independently from their level of interconnections. This law drives the definition of the X^S*'s.*

The idiosyncratic term X^I*'s are assumed to be driven by a Poisson Process with parameter* λ *– giving the timing of the failures – jointly with a uniform process over* $\mathcal{C}$ *– which identifies the failed component –.*

Moreover, suppose that the system fails at the first time in which components with aggregated relative relevance greater than 0.4 fail.

Now, suppose that the first failure is observed at time $t = 8$*, when* C_2 *fails. Then, automatically,* C_3 *fails as well, since it is connected only to* C_2*. The aggregate relative relevance before the failures is* $\alpha_2(8^-) + \alpha_3(8^-) = 0.5 + 0.2 > 0.4$*, and the system fails.*

Rational expectations are given, in this context, as the expectation of the time in which the system fails, i.e.

$$RE = \mathbb{E}[\mathcal{T}], \tag{9}$$

where $\mathbb{E}$ is the expected value operator.

In the next Section, we provide some scenario simulations for describing the rational expectations in this framework.

3 Scenario simulations

4 Conclusions